The beginning of the \( P_{\text{max}} \) story is the following result of Woodin:

**Theorem 1** If \( NS_{\aleph_1} \) is saturated and there is a measurable cardinal then \( \delta_2^1 \) equals \( \aleph_2 \).

Here \( NS_{\kappa} \) denotes the ideal of nonstationary subsets of \( \kappa \). The word “saturated” here means “\( \aleph_2 \)-saturated”, i.e., there is no antichain of size \( \aleph_2 \) in the quotient \( P(\omega_1)/NS_{\aleph_1} \). The ordinal \( \delta_2^1 \) is the supremum of the ranks of \( \Delta_2^1 \) definable prewellorderings of the reals. It is not known if \( NS_{\aleph_1} \) can be saturated in the presence of CH; by this result it cannot be if a measurable cardinal exists.

A key step in the proof of the above result is that every element of \( H(\aleph_2) \) belongs to a “generic iterate” of a countable model of ZFC. Woodin used this to define a forcing in \( L(\mathbb{R}) \) called \( P_{\text{max}} \) which when applied for \( L(\mathbb{R}) \) yields a version of \( H(\aleph_2) \) which satisfies AC and has some restricted\(^1\) but attractive absoluteness properties.

In this course we’ll follow Paul Larson’s article in the *Handbook of Set Theory*, which presents the basics of the \( P_{\text{max}} \) theory.

**Iterations**

Suppose that \( I \) is a normal ideal on \( \omega_1 \) containing all countable subsets of \( \omega_1 \). “Normal” means that \( I \) is not all of \( P(\omega_1) \) and whenever \( A \) is an \( I \)-positive set (i.e. a subset of \( \omega_1 \) not belonging to \( I \)), \( f : A \to \omega_1 \) is regressive then \( f \) is constant on an \( I \)-positive set. An example is the ideal of nonstationary subsets of \( \omega_1 \).

If we force with the quotient \( P(\omega_1)/I \) then the result is a \( V \)-ultrafilter on \( \omega_1 \) (i.e., a filter on \( \omega_1 \) which for every \( A \) in \( V \) contains either \( A \) or \( \sim A \)) and this ultrafilter \( U \) is \( V \)-normal (i.e., normal for functions in \( V \)).

\(^1\)One cannot hope for too much absoluteness. Indeed absoluteness for class forcing extensions is not possible, nor is absoluteness for set forcing extensions with regard to arbitrary sentences about \( H(\aleph_2) \).
If we form $\text{Ult}(V,U)$, the ultrapower of $V$ by $U$, we don’t necessarily have a wellfounded model, but the canonical elementary embedding $j : V \rightarrow \text{Ult}(V,U)$ has critical point $\omega_1^V$ and (identifying the wellfounded part of $\text{Ult}(V,U)$ with its transitive collapse) $\omega_2^V$ is an initial segment of the ordinals of $\text{Ult}(V,U)$ (as using $j$ any subset of $\omega_1^V$ in $V$ also belongs to $\text{Ult}(V,U)$). Since $I$ is normal:

$$A \in U \text{ iff } \omega_1^V \in j(A)$$

for subsets $A$ of $\omega_1^V$ in $V$.

Sometimes we will need a weakening of ZFC, denoted by ZFC°. For now we omit the details of the definition of ZFC°. The existence of transitive models of ZFC° is provable in ZFC.

Now we turn to iterated generic ultrapowers. Suppose that $M$ is a model of ZFC°, $I \in M$ is a normal ideal on $\omega_1^M$ and $\mathcal{P}(\mathcal{P}(\omega_1))^M$ is countable. Then there exist generics for $(\mathcal{P}(\omega_1)/I)^M$. Moreover, if $j : M \rightarrow N$ is a resulting generic ultrapower embedding then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is also countable and so there also exist generics for $(\mathcal{P}(\omega_1)/j(I))^N$. We can continue this process for $\omega_1$ stages, as in the following definition.

**Definition 2** Let $M$ be a model of ZFC°, $I$ a normal ideal on $\omega_1^M$ and $\gamma \leq \omega_1$. An iteration of $(M,I)$ of length $\gamma$ consists of models $(M_\alpha \mid \alpha \leq \gamma)$, sets $(G_\alpha \mid \alpha < \gamma)$ and a commuting family of elementary embeddings $(j_{\alpha\beta} : M_\alpha \rightarrow M_\beta \mid \alpha \leq \beta \leq \gamma)$ such that

1. $M_0 = M$
2. $G_\alpha$ is generic for $(\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$
3. $j_{\alpha\alpha}$ is the identity
4. $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by $G_\alpha$
5. For limit $\beta \leq \gamma$, $M_\beta$ is the direct limit of the system $(M_\alpha, j_{\alpha\delta} \mid \alpha \leq \delta < \beta)$ and for $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding into this direct limit.

If in the above iteration $\gamma$ equals $\omega_1$ and each $\omega_1^{M_\alpha}$ is wellfounded then the set of these ordinals forms a club in $\omega_1$. Also note that each of the embeddings $j_{\alpha\beta}$ is cofinal into the ordinals of $M_\beta$.

The models that appear in an iteration of $(M,I)$ are called *iterates* of $(M,I)$. In case $I$ equals $\text{NS}_{\aleph_1}^M$ then we talk about an iteration and iterates of
When we say that \( j : (M, I) \to (M^*, I^*) \) is an \textit{iteration} we mean that \( j \) is \( j_{0}\gamma \) for an iteration of \((M, I)\) as above with \( M_{\gamma} = M^* \) and \( I^* = j(I) \).

\((M, I)\) is \textit{iterable} if every iterate of \((M, I)\) is wellfounded. This is equivalent to saying that every iterate which arises through a countable iteration of \((M, I)\) is wellfounded.

\textit{Conditions for iterability}\n
The basic lemma which yields iterability is the following. An ideal \( I \) is \textit{precipitous} if each of its generic ultrapower is wellfounded.

**Lemma 3** Suppose that \( M \) is a transitive model of enough of ZFC, \( I \) is a normal precipitous ideal on \( \omega_1^M \) in \( M \). Suppose that \( j : (M, I) \to (M^*, I^*) \) is an iteration of \((M, I)\) of length \( \gamma \leq \omega_1 \) and \( \gamma \) belongs to \( M \). Then \( M^* \) is wellfounded.

\textit{2.-3.Vorlesungen}\n
The following provides a sufficient condition for (generic) iterability.

**Lemma 4** Suppose that \( M \) is a transitive model of enough of ZFC, \( I \) is a normal precipitous ideal on \( \omega_1^M \) in \( M \). Suppose that \( j : (M, I) \to (M^*, I^*) \) is an iteration of \((M, I)\) of length \( \gamma \leq \omega_1 \) and \( \gamma \) belongs to \( M \). Then \( M^* \) is wellfounded.

**Proof.** It suffices to show that iterations of length \( \gamma \) of \((H(\kappa)^M, I)\) are wellfounded for each regular \( \kappa \) of \( M \) greater than the \( M \)-cardinality of \( \mathcal{P}(\mathcal{P}(\omega_1))^M \), for \( M^* \) is the union of the \( j(H(\kappa)^M) \) for such \( \kappa \) and \( j \upharpoonright H(\kappa)^M \) results from an iteration of \((H(\kappa)^M, I)\).

If not, let \((\gamma, \kappa, \eta)\) be the lexicographically least triple such that for some iteration \((N_\alpha, G_\beta, j_{\alpha\beta} \mid \beta < \gamma, \alpha \leq \delta \leq \gamma)\) of \((H(\kappa)^M, I)\), \( j_{0\gamma}^\eta(\eta) \) is illfounded. \( \gamma \) is a limit ordinal because \( I \) is precipitous. The triple \((\gamma, \kappa, \eta)\) is definable in \( M \) as it is the least triple \((\gamma, \kappa, \eta)\) for which the existence of such an iteration is forced by the Lévy collapse to \( \omega \) of sufficiently large ordinals of \( M \). Fix such an iteration \((N_\alpha, G_\beta, j_{\alpha\beta} \mid \beta < \gamma, \alpha \leq \delta \leq \gamma)\) and choose \( \gamma^* < \gamma \), \( \eta^* < j_{0\gamma^*}(\eta) \) so that \( j_{\gamma^*}(\eta^*) \) is illfounded. This is possible as both \( \gamma \) and \( \eta \) are limit ordinals. Also note that the above iteration lifts to an iteration \((M_\alpha, G_\beta, j_{\alpha\beta}^* \mid \beta < \gamma, \alpha \leq \delta \leq \gamma)\) of \((M, I)\).
Now by elementarity, \((j^*_{0\gamma},\gamma^*_{\gamma\kappa},j^*_{0\gamma\eta})\) is the lexicographically least triple such that for some iteration of \((H(j^*_{0\gamma})(\kappa))^{M^*}_{\gamma^*}(I)\), the ordinal \(j^*_{0\gamma\eta}(\eta)\) is sent by the iteration into the illfounded part. But there is a lexicographically smaller such triple: Consider the tail of the iteration \((N_\alpha,G_\beta,J_{\alpha\delta} | \beta < \gamma, \alpha \leq \delta \leq \gamma)\) starting at \(N_{\gamma\eta}\). This gives rise to a triple \((\gamma',\kappa',\eta^*)\) whose first component \(\gamma'\) is \(\gamma - \gamma^*\), surely at most \(j^*_{0\gamma\eta}(\gamma)\), whose second component \(\kappa'\) equals \(j^*_{0\gamma\eta}(\kappa)\) and whose third component \(\eta^*\) is strictly less than \(j^*_{0\gamma\eta}(\eta)\). This is a contradiction. \(\square\)

We'll also need the following two little facts.

**Lemma 5** Suppose that \(M\) is a countable transitive model of enough of ZFC and \((M,I)\) is iterable, where \(I \in M\) is a normal ideal on \(\omega_1\) of \(M\). Let \(x\) be a real coding the pair \((M,I)\). Then whenever \(L_\gamma[x]\) models ZFC, iterations of \((M,I)\) of length less than \(\gamma\) yields models of height less than \(\gamma\).

**Proof.** The point is that the set of heights of models which result from a generic iteration of length \(\delta\) is \(\Sigma^1_1\) in \(x\) together with a code for \(\delta\) and therefore bounded by an ordinal admissible in \(x\) together with this code. If \(\delta\) is less than \(\gamma\) and \(L_\gamma[x]\) models ZFC then there is a code for \(\delta\) in \(L_\gamma[x][g]\) where \(g\) is generic for the Lévy collapse to \(\omega\) of \(\delta\) and therefore the heights of models which arise from a generic iteration of length \(\delta\) will be less than \(\gamma\). \(\square\)

**Lemma 6** Suppose that \((M,I)\) is iterable where \(M\) satisfies enough of ZFC. Then \(M\) is closed under \#'s for subsets of \(\omega^M_1\).

**Proof.** If \(j : (M,I) \to (M_1,I_1)\) is a generic ultrapower of \((M,I)\) via the \((\mathcal{P}(\omega_1))^{M}\)-generic ultrafilter \(G_1\) then in \(M[G_1]\) we see that there is an elementary embedding of the \(L[x]\) of \(M\) to itself for each real \(x \in M\). So \(M[G_1]\) thinks that every real of \(M\) has a \# and therefore so does \(M\) (as set-forcing does not create new \#'s). Moreover, if \(A \in M\) is a subset of \(\omega^M_1\) then \(A = j(A) \cap \omega^M_1\) is countable in \(M_1\) and therefore has a \# in \(M_1 \subseteq M[G_1]\), again implying that \(A\) also has a \# in \(M\). The fact that \((M,I)\) is iterable implies that \(M\) is elementarily embeddable into a model containing all countable ordinals and therefore \(M\)'s version of \(A\#\) for \(A \subseteq \omega^M_1\) is the correct \(A\#\). \(\square\)
We now define the $\mathbb{P}_{\text{max}}$ forcing. When we write MA we are referring to Martin’s Axiom for ccc partial orders and collections of dense sets of size $\omega_1$. A condition in $\mathbb{P}_{\text{max}}$ is a pair $((M, I), a)$ such that:

1. $M$ is a countable transitive model of enough of ZFC + MA.
2. $I$ is a normal ideal in $M$.
3. $(M, I)$ is iterable.
4. $a$ belongs to $\mathcal{P}(\omega_1)^M$.
5. For some real $x$ in $M$, $\omega_1^M$ equals $\omega_1^{L[a,x]}$.

$((M, I), a) \leq ((N, J), b)$ iff $N$ belongs to $H(\omega_1)^M$ and there exists an iteration $j : (N, J) \to (N^*, J^*)$ such that:

i. $j(b) = a$.
ii. $j, N^*$ belong to $M$.
iii. $I \cap N^* = J^*$.

4.-5. Vorlesungen

We now define the $\mathbb{P}_{\text{max}}$ forcing. When we write MA we are referring to Martin’s Axiom for ccc partial orders and collections of dense sets of size $\omega_1$. A condition in $\mathbb{P}_{\text{max}}$ is a pair $((M, I), a)$ such that:

1. $M$ is a countable transitive model of enough of ZFC + MA.
2. $I$ is a normal ideal in $M$.
3. $(M, I)$ is iterable.
4. $a$ belongs to $\mathcal{P}(\omega_1)^M$.
5. For some real $x$ in $M$, $\omega_1^M$ equals $\omega_1^{L[a,x]}$.

$((M, I), a) \leq ((N, J), b)$ iff $N$ belongs to $H(\omega_1)^M$ and there exists an iteration $j : (N, J) \to (N^*, J^*)$ such that:

i. $j(b) = a$.
ii. $j, N^*$ belong to $M$.
iii. $I \cap N^* = J^*$.

We make some remarks. (1) Suppose that $((M, I), a)$ is a condition. As $M$ is closed under #’s for reals, $a$ cannot be coded by a real and is therefore unbounded in $\omega_1^M$. It follows that if $((M, I), a)$ extends $((N, J), b)$ then the
iteration which shows this has length \( \omega_1^M \).

(2) The requirement (ii) in the definition of extension implies that the ordering on conditions in transitive: If \( j_0 \) witnesses \( ((M_1, I_1), a_1) \leq ((M_0, I_0), a_0) \) and \( j_1 \) witnesses \( ((M_2, I_2), a_2) \leq ((M_1, I_1), a_1) \) then \( j_1(j_0) \) witnesses \( ((M_2, I_2), a_2) \leq ((M_0, I_0), a_0) \).

(3) The requirement of MA will be used to show that any iteration of \((M, I)\) is uniquely determined by the image of \( a \) and therefore there is a unique iteration which witnesses that one condition extends another. The argument will be via almost disjoint coding.

**Lemma 7** Let \( (M, I, a) \) be a \( \mathbb{P}_{\text{max}} \) condition and \( A \) a subset of \( \omega_1 \). Then there is at most one iteration of \((M, I)\) for which \( A \) is the image of \( a \), and if this iteration exists then it belongs to \( L[\mathcal{L}(M, I, a, A)] \).

**Proof.** Choose a real \( x \) such that \( \omega_1^M = \omega_1^{L[a, x]} \). By induction on \( \alpha < \omega_1^M \) choose \( z^*_\alpha \) to be the \( L[a, x]\)-least real distinct from the \( z^*_\beta \), \( \beta < \alpha \), and make the \( z^*_\alpha \)'s almost disjoint by replacing \( z^*_\alpha \) by \( z_\alpha = \) the set of codes for finite initial segments of \( z^*_\alpha \). Suppose that

\[
\mathcal{I} = (M_\alpha, G_\beta, j_\delta \mid \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)
\]

and

\[
\mathcal{I}' = (M'_\alpha, G'_\beta, j'_\delta \mid \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)
\]

are two iterations of \((M, I)\) such that \( j_{0\omega_1} (a) = A = j'_{0\omega_1} (a) \). Then \( j_{0\omega_1} (Z) = j'_{0\omega_1} (Z) \) as well, where \( Z \) is the sequence of \( z^*_\alpha \)'s. Write the latter as \( (z_\alpha \mid \alpha < \omega_1) \).

We show by induction on \( \alpha < \omega_1 \) that \( G_\alpha = G'_\alpha \), which implies that the two iterations are the same. Suppose that \( G_\beta = G'_\beta \) for \( \beta < \alpha \) and we want to show \( G_\alpha = G'_\alpha \). If \( B \) is a subset of \( \omega_1^{M_\alpha} \) in \( M_\alpha = M'_\alpha \) then \( B \) belongs to \( G_\alpha \) if \( \omega_1^{M_\alpha} \) belongs to \( j_{0\alpha+1}(B) \). Since \( M_\alpha \) satisfies MA, there is a real \( y \) in \( M_\alpha \) such that for \( \eta < \omega_1^{M_\alpha} \), \( \eta \) belongs to \( B \) iff \( y, z_\eta \) are almost disjoint. By elementarity, \( \omega_1^{M_\alpha} \) belongs to \( j_{0\alpha+1}(B) \) iff \( y, z_{\omega_1^{M_\alpha}} \) are almost disjoint. As the latter holds also for \( j'_{0\alpha+1} \) we have that \( G_\alpha \) and \( G'_\alpha \) are the same. Moreover this gives a definition of the sequence of \( G_\alpha \)'s in terms of \((M, I), a \) and \( Z \) and hence this sequence belongs to \( L[\mathcal{L}(M, I, a, A)] \). □
A consequence of this lemma is that if $G$ is $\mathbb{P}_{\max}$ generic over $L(\mathbb{R})$ then $L(\mathbb{R})[G] = L(\mathbb{R})[A]$ where $A$ is the union of the $a$ such that $((M, I), a)$ belongs to $G$ for some $(M, I)$.

We say that $(M, I)$ is a precondition iff for some $a$, $((M, I), a)$ is a condition.

**Lemma 8** If $(M, I)$ is a precondition and $J$ is a normal ideal on $\omega_1$ then there exists an iteration $j : (M, I) \rightarrow (M^*, I^*)$ such that $j(\omega_1^M) = \omega_1$ and $I^* = J \cap M^*$.

**Proof.** First note that if $j : (M, I) \rightarrow (M^*, I^*)$ is any iteration of $(M, I)$ of length $\omega_1$ then $I^*$ is contained in $J$. To see this, write the iteration as $(M_\alpha, G_\beta, j_\delta | \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)$ and note that if $E$ belongs to $I^* = j_{\omega_1}(I)$ then $E = j_{\omega_1}(E')$ for some countable $\alpha$ and $E' \in j_{\alpha}(I)$. Then for all countable $\beta \geq \alpha$, $j_{\alpha\beta}(E') \notin G_\beta$ so $\omega_1^{M_\alpha} \notin E$. As the set of such $\omega_1^{M_\alpha}$’s forms a club, it follows that $E$ is nonstationary and therefore belongs to $J$ by the normality of $J$.

Choose a family $(A_{i\alpha} | i < \omega, \alpha < \omega_1)$ of pairwise disjoint members of $\mathcal{P}(\omega_1) \setminus J$. (This is possible as there is no countably additive ideal on $\omega_1$ containing all finite sets which is $\omega_1$-saturated; the proof of this fact uses Ulam matrices.) We describe an iteration $(M_\alpha, G_\beta, j_\delta | \alpha \leq \omega_1, \beta < \omega_1, \delta \leq \mu \leq \omega_1)$ of $(M, I)$ by inductively choosing the $G_\beta$’s. We simultaneously choose enumerations $(B_i^{\alpha} | i < \omega)$ of $\mathcal{P}(\omega_1)^{M_\alpha} \setminus j_{\alpha}(I)$.

Given $(M_\alpha, G_\beta, j_\delta | \alpha \leq \gamma, \beta < \gamma, \delta \leq \mu \leq \gamma)$, if $\omega_1^{M_\alpha}$ belongs to $A_{i\alpha}$ for some $i < \omega$ and $\alpha \leq \gamma$ then we let $G_\gamma$ be any $(\mathcal{P}(\omega_1)/j_{\alpha\gamma}(I))^{M_\gamma}$-generic over $M_\gamma$ which contains $j_{\alpha\gamma}(B_i^{\alpha})$. If $\omega_1^{M_\gamma}$ does not belong to any $A_{i\alpha}$ for $i < \omega$, $\alpha \leq \gamma$ then we let $G_\gamma$ be any $(\mathcal{P}(\omega_1)/j_{\alpha\gamma}(I))^{M_\gamma}$-generic over $M_\gamma$.

Now suppose that $E$ belongs to $\mathcal{P}(\omega_1)^{M_\gamma} \setminus j_{\alpha\gamma}(I)$. We want to show that $E$ does not belong to $J$. Fix $i < \omega$ and $\alpha < \omega_1$ such that $E = j_{\alpha\gamma}(B_i^{\alpha})$. Then $\omega_1^{M_\gamma}$ belongs to $j_{\alpha\gamma+1}(B_i^{\alpha})$ (and therefore to $E$) whenever it belongs to $A_{i\alpha}$. It follows that $E$ contains the intersection of a club with a set not in $J$ and therefore does not belong to $J$. □

We next show that $\mathbb{P}_{\max}$ is homogeneous in the following sense: Any two $\mathbb{P}_{\max}$ conditions $p_0, p_1$ have extensions $q_0, q_1$ such that the suborders of $\mathbb{P}_{\max}$ below $q_0$ and $q_1$ are isomorphic.
Lemma 9  Assume that for every real $x$ there is an inner model $V_0$ containing $x$ and a measurable cardinal $\kappa$ in $V_0$ whose power set in $V_0$ is countable in $V$. (This follows from the existence of “daggers”, less than the existence of two measurable cardinals.) Then $\mathbb{P}_{\text{max}}$ is homogeneous.

Proof. The hypothesis of the lemma implies that any real $x$ belongs to the model $M$ of some precondition $(M, I)$: Let $V_0$ be an inner model containing $x$ with a measurable cardinal $\kappa$ whose power set in $V_0$ is countable in $V$. Then in $V$ there is a generic for the forcing that over $V_0$ that Lévy collapses $\kappa$ to become $\omega_1$ and then forces MA. In this generic extension there is a normal precipitous ideal on $\kappa$ and therefore a precondition $(M, I)$ with $M$ containing $x$.

Now suppose that $p_0 = ((M_0, I_0), a_0)$ and $p_1 = ((M_1, I_1), a_1)$ are $\mathbb{P}_{\text{max}}$ conditions. Fix a precondition $(N, J)$ such that $p_0, p_1$ belong to $H(\omega_1)^N$. Applying the previous lemma in $N$, choose iterations $j_0 : (M_0, I_0) \rightarrow (M_0^*, I_0^*)$ and $j_1 : (M_1, I_1) \rightarrow (M_1^*, I_1^*)$ such that $I_0^* = J \cap M_0^*$ and $I_1^* = J \cap M_1^*$. Let $a_0^* = j_0(a_0)$, $a_1^* = j_1(a_1)$ and consider the conditions $q_0 = ((N, J), a_0^*)$, $q_1 = ((N, J), a_1^*)$. Then $j_0, j_1$ witness that $q_0, q_1$ are extensions in $\mathbb{P}_{\text{max}}$ of $p_0, p_1$.

We claim that the suborders of $\mathbb{P}_{\text{max}}$ below $q_0$ and $q_1$ are isomorphic. Indeed, suppose that $q'_0 = ((N', J'), a')$ is a condition below $q_0$ and the iteration $j' : (N, J) \rightarrow (N', J')$ witnesses this. Then $a' = j'(a_0^*)$ and $q'_1 = ((N', J'), j'(a_1^*))$ is a condition below $q_1$. Let $\pi$ be the map defined on $\mathbb{P}_{\text{max}}$ below $q_0$ that sends $((N', J'), a')$ to $((N', J'), j'(a_1^*))$ as above. Then $\pi$ is an isomorphism onto $\mathbb{P}_{\text{max}}$ below $q_1$ using the fact that iterations are uniquely determined by where they send the last component of a $\mathbb{P}_{\text{max}}$ condition. □

6.-7. Vorlesungen

$\mathbb{P}_{\text{max}}$ is countably closed

Assume that every real belongs to some $\mathbb{P}_{\text{max}}$ precondition and suppose that for each finite $i$, $p_i = ((M_i, I_i), a_i)$ is a $\mathbb{P}_{\text{max}}$ condition and $j_{i,i+1} : (M_i, I_i) \rightarrow (M_i^*, I_i^*)$ is an iteration witnessing $p_{i+1} < p_i$. We want to find a $\mathbb{P}_{\text{max}}$ condition below all of the $p_i$’s. Let $(j_{ik} \mid i \leq k < \omega)$ be the commuting family of embeddings generated by the $j_{i,i+1}$’s and $a = \cup_i a_i$. Then for each $i$ there is a unique iteration $j_{i\omega} : (M_i, I_i) \rightarrow (N_i, J_i)$ sending $a_i$ to
\( a \) and \( \omega_1^{N_i} = \omega_1^{N_0} \) for all \( i \). We would like to put the \( (N_i, J_i, a) \)'s together to get a \( \mathbb{P}_{\text{max}} \) condition below each of the \( p_i \)'s. To do this we need to discuss “iterations” of the structure \( ((N_i, J_i) \mid i < \omega) \) and prove its “iterability”. Then we can easily generalise our earlier lemma about iterating to the restriction of an arbitrary normal ideal as follows.

**Lemma 10** Suppose that \( I \) is a normal ideal on \( \omega_1 \). Then there is an iteration \( j^* : ((N_i, J_i) \mid i < \omega) \rightarrow ((N_i^*, J_i^*) \mid i < \omega) \) such that \( j^*(\omega_1^{N_0}) = \omega_1 \) and \( J_i^* = I \cap N_i^* \) for each \( i \).

Now to complete the proof of \( \omega \)-closure, choose a \( \mathbb{P}_{\text{max}} \) precondition \( (M, I) \) such that \( H(\omega_1)^M \) contains \( (p_i \mid i < \omega) \) and apply the above lemma in \( M \) to obtain \( j^* \). Then for each \( i \) the embedding \( j^*(j_{i\omega}) \) witnesses that \( ((M, I), j^*(a)) \) is a \( \mathbb{P}_{\text{max}} \) condition below \( p_i \) for each \( i \).

An *iteration* of \( ((N_i, J_i) \mid i < \omega) \) of length \( \gamma \leq \omega_1 \) consists of sequences \( ((N_i^\alpha, J_i^\alpha) \mid i < \omega) \) (\( \alpha \leq \gamma \)) together with normal ultrafilters \( G_\alpha \) on \( \bigcup_i N_i^\alpha \) (\( \alpha < \gamma \)) and a commuting family of embeddings \( j_{\alpha, \delta} : ((N_i^\alpha, J_i^\alpha) \mid i < \omega) \rightarrow ((N_i^\beta, J_i^\beta) \mid i < \omega) \) such that

\[
((N_i^0, J_i^0) \mid i < \omega) = ((N_i, J_i) \mid i < \omega).
\]

\( j_{\alpha, \alpha+1} \) is the embedding resulting by taking the ultrapower of the \( ((N_i^\alpha, J_i^\alpha) \mid i < \omega) \) using \( G_\alpha \).

For limit \( \beta \), \( ((N_i^\beta, J_i^\beta) \mid i < \omega) \) is the direct limit of the \( ((N_i^\alpha, J_i^\alpha) \mid i < \omega) \) for \( \alpha < \beta \) with induced embeddings \( j_{\alpha, \delta} \) (\( \alpha \leq \delta < \beta \)).

We claim that any iterate of \( ((N_i, J_i) \mid i < \omega) \) is wellfounded. It suffices to show that the \( \omega_1 \) of each iterate of \( ((N_i, J_i) \mid i < \omega) \) is wellfounded, as for each iterate \( ((N_i^\alpha, J_i^\alpha) \mid i < \omega) \) of \( ((N_i, J_i) \mid i < \omega) \), the ordinal height of \( N_i^\alpha \) is less than the least \( x_i \)-indiscernible greater than \( \omega_1^{N_0} \) where \( x_i \) is some real in \( N_i^{N_0} \) and hence must be wellfounded assuming that \( \omega_1^{N_0} \) is.

Now we prove that the \( \omega_1 \) of each iterate is wellfounded by induction on the length of the iteration. As the limit case is immediate and the general successor case follows from the case of a single ultrapower we just consider the latter. We want to see that if \( G \) is a normal ultrafilter on \( \bigcup N_i \) and \( j \) the induced ultrapower embedding then \( j(\omega_1^{N_0}) = \sup_i \text{Ord}(N_i) \) and is therefore wellfounded. Note that by choosing reals \( x_i \) in \( N_{i+1} \) with \( \text{Ord}(N_i) \) less than the least \( x_i \)-indiscernible greater than \( \omega_1^{N_0} \), if we let \( f_i(\alpha) \) be the least \( x_i \)-indiscernible above \( \alpha \) then \( j(\omega_1^{N_0}) \geq \sup_i j(f_i)(\omega_1^{N_0}) = \sup_i \text{Ord}(N_i) \). For
the other direction, let $h : \omega_1^{\aleph_0} \to \omega_1^{\aleph_0}$ be a function in some $N_i$. Then the closure points of $h$ contain a final segment of the $x_i$-indiscernibles and therefore $f_i > h$ on a final segment of $\omega_1^{\aleph_0}$; it follows that $[f_i]_G > [h]_G$ so we get $j(\omega_1^{\aleph_0}) = \sup_i \text{Ord}(N_i)$.

**Generalised Iterability**

Let $A$ be a set of reals. We say that a precondition $(M, I)$ is $A$-iterable iff it is iterable, $A \cap M$ is an element of $M$ and for any iteration $j : (M, I) \to (M^*, I^*)$ we have $j(A \cap M) = A \cap M^*$.

We show that if AD holds in $L(\mathbb{R})$ and $A$ is a set of reals in $L(\mathbb{R})$ then there is a $P_{\text{max}}$ precondition $(M, I)$ such that $(H(\omega_1)^M, A \cap M)$ is elementary in $(H(\omega_1), A)$ and $(M, I)$ is $A$-iterable. For this we need the following.

**Lemma 11** Assume AD. Then every set of ordinals belongs to an inner model in which some $V$-countable ordinal is measurable.

*Proof.* Fix a set of ordinals $Z$. For each increasing $f : \omega \to \omega_1$ let $s(f)$ be the sup of the range of $f$ and let $F(f)$ be the filter on $s(f)$ consisting of all subsets of $s(f)$ which contain all but finitely many members of Range $f$. Also let $N(f)$ be the inner model $L[Z, F(f)]$, a model of choice. We claim that for some $f$, $F(f)$ restricted to $N(f)$ is countably complete in $N(f)$, i.e., every function from $s(f)$ to $\omega$ in $N(f)$ is constant on a set in $F(f)$. It then follows that some ordinal at most $s(f)$ is measurable in $N(f)$, which proves the lemma.

Suppose that $F(f)$ is not countably complete in $N(f)$ for each $f$. Notice that if the ranges of $f_0$ and $f_1$ are equal modulo a finite set then $F(f_0)$ equals $F(f_1)$ so the models $N(f_0)$ and $N(f_1)$, as well as their canonical wellorders, are the same. Also note that using the canonical wellorder of $N(f)$ we can choose a function $G$ such that $G(f) : s(f) \to \omega$ is a counterexample to the countable completeness in $N(f)$ of $F(f)$ for each $f$.

We use the following consequence of AD: For every function from the set of increasing $\omega$-sequences through $\omega_1$ to the reals there is an uncountable $E \subseteq \omega_1$ such that this function is constant on the increasing $\omega$-sequences through $E$. 


Now for each increasing \( f : \omega \to \omega_1 \) let \( P(f) : \omega \to \omega \) be defined by \( P(f)(n) = G(f)(f(n)) \). Let \( E \) be an uncountable subset of \( \omega_1 \) such that \( P \) is constant on all increasing \( f : \omega \to E \). Choose \( i : \omega \to \omega \) such that for all increasing \( f : \omega \to E \), \( G(f)(f(n)) = i(n) \) for all \( n \). Then \( i \) must be a constant function, as if \( i(n) \neq i(0) \) and we choose increasing \( f, g : \omega \to E \) so that \( g(m) = f(m + n) \) then \( G(g)(g(0)) = i(0) \neq i(n) = G(f)(f(n)) = G(f)(g(0)) \), contradicting \( F(f) = F(g) \). As \( i \) is a constant function we get that \( G(f) \) is constant on a set in \( F(f) \) for each increasing \( f : \omega \to E \), contradicting the choice of \( G(f) \). \( \square \)

Now we show:

**Theorem 12** Assume \( AD^{L(\mathbb{R})} \) and let \( A \) be a set of reals in \( L(\mathbb{R}) \). Then there is a \( P_{\max} \) condition \((M, I, a)\) such that

1. \( A \cap M \in M \)
2. \((H(\omega_1)^M, A \cap M)\) is elementary in \((H(\omega_1), A)\)
3. \((M, I)\) is \( A \)-iterable
4. If \( M^+ \) is a forcing extension of \( M \) and \( J \) is a normal precipitous ideal on \( \omega_1^{M^+} \) in \( M^+ \) then \( A \cap M^+ \) is an element of \( A^+ \) and \((M^+, J)\) is \( A \)-iterable. Moreover if \( j : (M^+, J) \to (M^*, J^*)\) is an iteration of \((M^+, J)\) then \((H(\omega_1)^{M^*}, A \cap M^*)\) is elementary in \((H(\omega_1), A)\).

**Proof.** Assume that there is a counterexample \( A \). By choosing \( A \) to be definable over \( L_\alpha(\mathbb{R}) \) for the least possible \( \alpha \), we can assume that \( A \) is \( \Delta_2^1 \) definable in \( L(\mathbb{R}) \) (relative to a real parameter). In \( L(\mathbb{R}) \) every \( \Delta_2^1 \) set is the projection of a tree on \( \omega \times \mu \) for some ordinal \( \mu \) and this implies that there are trees \( T_0, T_1 \) such that any transitive model \( N \) with \( T_0, T_1 \) as members satisfies \( A \cap N \in N \) and \((H(\omega_1)^N, A \cap N)\) is elementary in \((H(\omega_1), A)\). Moreover if \( j : N \to N^* \) is elementary then the same holds for \( N^* \) using the trees \( j(T_0), j(T_1) \).

By the lemma choose an inner model \( N \) of ZFC and a countable ordinal \( \gamma \) such that \( N \) contains the trees \( T_0, T_1 \) and \( \gamma \) is measurable in \( N \). Let \( \delta \) be a strongly inaccessible cardinal of \( N \) between \( \gamma \) and \( \omega_1^\gamma \). If \( G \) is generic over \( N_\delta \) for the Lévy collapse of \( \gamma \) to \( \omega_1 \) followed by the ccc iteration to make \( \text{MA} \) true, then we obtain an iterable precondition \((N_\delta[G], I)\). It suffices to show that if \( M^+ \) is a forcing extension of \( N_\delta[G] \) in which there is a normal precipitous ideal \( J \) on \( \omega_1^{M^+} \) then \( M^+ \) and \( J \) satisfy conclusion 4 of the theorem.
Let $N^+$ be the corresponding forcing extension of $N[G]$. Then $A \cap N^+$ belongs to $N^+$ and $(H(\omega_1)^{N^+}, A \cap N^+)$ is elementary in $(H(\omega_1), A)$ since $T_0, T_1$ belong to $N$. Fix an iteration $j : (M^+, J) \to (M^*, J^*)$. This lifts to an iteration $j^* : (N^+, J) \to (N^*, J^*)$. Then $A \cap M^* = j(A \cap M^+) \in M^*$ and $(H(\omega_1)^{M^*}, A \cap M^*)$ is elementary in $(H(\omega_1), A)$ as $N^*$ contains $j^*(T_0), j^*(T_1)$. □

8.-9. Vorlesungen

We now prove one of Woodin’s main theorems about the $\mathbb{P}_{\text{max}}$ extension of $L(\mathbb{R})$.

**Lemma 13** Suppose that $V = L(\mathbb{R})$ and AD holds. Let $G$ be $\mathbb{P}_{\text{max}}$-generic over $V$, $A_G = \cup \{a \mid ((M, I), a) \text{ belongs to } G \text{ for some } (M, I)\}$. Then in $V[G]$, if $E$ is a subset of $\omega_1$ then there are $((M, I), a) \in G$ and $e \in \mathcal{P}(\omega_1)^M$ such that $j(e) = E$ where $j$ is the unique iteration of $(M, I)$ sending $a$ to $A_G$. Moreover $E$ is nonstationary iff we can take $e$ to belong to $I$.

**Proof.** For the proof we need two facts. If $p$ is a $\mathbb{P}_{\text{max}}$ condition, $J$ is a normal ideal on $\omega_1$ and $B$ is a subset of $\omega_1$ then let $G_{\omega_1}(p, J, B)$ be the game where Players $I$ and $II$ build a descending $\omega_1$-sequence of $\mathbb{P}_{\text{max}}$ conditions $p_\alpha = ((M_\alpha, I_\alpha), a_\alpha)$ below $p$ where it is $I$’s turn to choose $p_\alpha$ if $\alpha \notin B$ and it is $II$’s turn to choose $p_\alpha$ if $\alpha \in B$; $II$ wins iff, letting $A$ be the union of the $a_\alpha$’s and $j_\alpha : (M_\alpha, I_\alpha) \to (M_\alpha^*, I_\alpha^*)$ the iteration of $(M_\alpha, I_\alpha)$ sending $a_\alpha$ to $A$, $j_\alpha(I_\alpha) = J \cap M_\alpha^*$ for all $\alpha$.

The following is a straightforward generalisation of an earlier lemma.

**Fact 1.** Player $II$ has a winning strategy in $G_{\omega_1}(p, J, B)$ iff $B \notin J$.

We also need:

**Fact 2.** Let $p_0 = ((M, I), a)$ be a $\mathbb{P}_{\text{max}}$ condition in $G$ and $P \in M$ a set of $\mathbb{P}_{\text{max}}$ conditions extended by $p_0$. Let $j$ be the iteration of $(M, I)$ sending $a$ to $A_G$. Then every condition in $j(P)$ belongs to $G$.

**Proof of Fact 2.** Let $(M_\alpha, G_\beta, j_{\alpha\delta}^* \mid \alpha \leq \omega_1, \beta < \omega_1, \alpha \leq \delta \leq \omega_1)$ be the iteration given by $j$ and fix $q = ((N_0, J_0), b_0)$ in $j(P)$. Fix $\alpha_0$ such that $q \in j_{\alpha_0}^*(P)$ and as $q$ is extended by $j_{\alpha_0}^*(p_0)$ we can choose $j_q \in M_{\alpha_0}$ to be the
iteration of \((N_0,J_0)\) sending \(b_0\) to \(j_{\omega_0}^*(a)\). Choose \(p_1= ((N_1,J_1),b_1) \in G\) such that \(p_1 \leq p_0\) and \(\alpha_0 < \omega_1^{N_1}\). Then \((M_\alpha,G_\beta,j_{\alpha,\beta}^*| \alpha \leq \omega_1^{N_1}, \beta < \omega_1^{N_1}, \alpha \leq \delta \leq \omega_1^{N_1})\) is in \(M_{\omega_1}^{N_1}\), and is the unique iteration of \((M,I)\) sending \(a\) to \(b_1\). Since \(j_q(N_0) = j_{\omega_0}^*(I) \cap j_q(N_0)\) and \(j_{\omega_1}^*(I) = J_1 \cap M_{\omega_1}^{N_1}\) it follows that \(j_{\alpha,\omega_1}^*(j_q)\) witnesses \(q \geq p_1\). □

Using these Facts we prove the lemma. Let \(\tau\) be a \(\mathbb{P}_{\max}\)-name for a subset of \(\omega_1\) and let \(A\) be a set of reals coding the set of triples \((p,\alpha,i)\) such that \(p \in \mathbb{P}_{\max}, \alpha < \omega_1, i \geq 2\) and \(p \vDash \alpha \in \tau\) if \(i = 1\), \(p \vDash \alpha \notin \tau\) if \(i = 0\). Let \(p = ((N,J),d)\) be any condition and let \((M,I)\) be an \(A\)-iterable precondition such that \(p\) belongs to \(H(\omega_1)^M\) and \((H(\omega_1)^M,A \cap M) < (H(\omega_1),A)\).

Applying Fact 1 in \(M\) (where \(B\) is the set of countable limit ordinals) we obtain a descending \(\omega_1^M\)-sequence of conditions \(p_\alpha = ((N_\alpha,J_\alpha),d_\alpha)\) such that:

1. \(p_0 = p\)
2. \(p_{\alpha+1}\) decides “\(\alpha \in \tau\”.
3. Letting \(D\) be the union of the \(d_\alpha\)’s, \(j_\alpha(J_\alpha) = I \cap j_\alpha(N_\alpha)\), where \(j_\alpha\) is the iteration of \((N_\alpha,J_\alpha)\) sending \(d_\alpha\) to \(D\).

It follows that \(((M,I),D)\) is a condition below each \(p_\alpha\). Let \(e\) be a subset of \(\omega_1^M\) in \(M\) such that for each \(\alpha, \alpha \in e\) iff \(p_{\alpha+1} \vDash \alpha \in \tau\). Suppose that \(((M,I),D)\) belongs to a generic \(G^e\) and let \((M_\alpha,G_\beta,j_{\alpha,\beta}^*| \alpha \leq \omega_1, \beta < \omega_1, \alpha \leq \delta \leq \omega_1)\) be the iteration of \((M,I)\) sending \(D\) to \(A_{G^e}\). We show that \(j_{\omega_1}^\alpha\) sends \(e\) to \(\tau_{G^e}\). Write \(j_{\omega_1}^\alpha(p_\alpha | \alpha < \omega_1^M)\) as \((q_\alpha | \alpha < \omega_1)\). Then for each \(\gamma < \omega_1\), \(q_\gamma \vDash e\) \(\tau\) if \(\gamma < j_{\omega_1}^\alpha(e)\) and \(q_\gamma \vDash e\) \(\tau\) if \(\gamma \notin j_{\omega_1}^\alpha(e)\). By Fact 2, each \(q_\gamma\) belongs to \(G^e\) so \(j_{\omega_1}^\alpha(e) = \tau_{G^e}\).

Finally note that if \(E = j(e)\) where \(((M,I),a)\) belongs to \(G\), \(e\) belongs to \(I\) and \(j\) is the iteration sending \(a\) to \(A_G\), then \(E\) is disjoint from the critical sequence of the iteration \(j\) and is therefore nonstationary. Conversely, if \(E\) is nonstationary then choose a club \(C\) disjoint from \(E\) and \(((M,I),a)\) \(G\), \(e,c \in \mathcal{P}(\omega_1)^M\) such that \(j(e) = E, j(c) = C\) where \(j\) is the iteration of \((M,I)\) sending \(a\) to \(A_G\); then \(c\) must be a club in \(M\) so \(e\) must belong to \(I\). □

Theorem 14 Suppose that \(V = L(\mathbb{R})\) and \(AD\) holds. Let \(G\) be \(\mathbb{P}_{\max}\)-generic over \(V\). Then in \(V[G]\), \(\delta_2^1 = \omega_2\).
Proof. (a) It suffices to show that for every $\gamma < \omega_2$ there is a real $x$ such that the least $x$-indiscernible above $\omega_1$ is greater than $\gamma$. Working in the $\text{P}_{\text{max}}$ extension $V[G]$, fix a wellorder $\pi$ of $\omega_1$ of length $\gamma$. By the previous lemma we may choose a condition $((M, I), a) \in G$ and $e \in \mathcal{P}(\omega_1)^M$ such that $j(e) = \pi$, where $j$ is the iteration of $(M, I)$ sending $a$ to $A_G$. Then $\gamma$ is in $j(M)$ and so for any real $c$ coding $(M, I)$ is less than the least $c$-indiscernible above $\omega_1$. \Box

10.-11. Vorlesungen

**Theorem 15** Suppose that $V = L(\mathbb{R})$ and AD holds. Let $G$ be $\text{P}_{\text{max}}$-generic over $V$. Then in $V[G]$, $\text{NS}_{\omega_1}$ is $\omega_2$-saturated.

Proof. We show that if $D$ is dense in $\mathcal{P}(\omega_1) \setminus \text{NS}$ then $D$ contains a subset $D'$ of size $\omega_1$ whose diagonal union contains a club.

Let $\tau$ be a name for $D$ and let $A$ be the set of reals coding pairs $(p, e)$ where $p = ((M, I), a)$ is a $\text{P}_{\text{max}}$ condition, $e \in \mathcal{P}(\omega_1)^M \setminus I$ and $p$ forces that $j(e) \in \tau$, where $j$ is the iteration of $(M, I)$ sending $a$ to $A_G$.

Let $p = ((N, J), b)$ be any $\text{P}_{\text{max}}$ condition and let $(M, I)$ be an $A$-iterable precondition such that $p \in H(\omega_1)^M$ and $(H(\omega_1)^M, A \cap M)$ is elementary in $(H(\omega_1), A)$. Fix a partition $\{B^\alpha_i \mid \alpha < \omega_1, i < \omega\}$ in $M$ of $\omega_1^M$ into $I$-positive sets and an injection $g : \omega^M \times \omega \to \omega_1^M$ in $M$ such that $g(\alpha, i) \geq \alpha$ for all $(\alpha, i)$.

Working in $M$ our aim is to build a descending $\omega_1^M$-sequence of conditions $p_\alpha = ((N_\alpha, J_\alpha), b_\alpha)$ (with embeddings $j_{\alpha\beta}$ witnessing $p_\alpha \succeq p_\beta$) together with enumerations $(e^\alpha_i \mid i \in \omega)$ of $\mathcal{P}(\omega_1)^{N_\alpha} \setminus J_\alpha$ and sets $d_\alpha$ such that $p_0 = p$ and:

1. $d_\alpha \in \mathcal{P}(\omega_1)^{N_{\alpha+1}} \setminus J_{\alpha+1}$, $p_{\alpha+1}$ forces that $j(d_\alpha) \in \tau$, where $j$ is the iteration of $(N_{\alpha+1}, J_{\alpha+1})$ sending $b_{\alpha+1}$ to $A_G$, and if $\alpha = g(\beta, i)$ for some $\beta \leq \alpha$ and $i \in \omega$, then $d_\alpha \subseteq j_{\beta, \alpha+1}(e^\beta_i)$.
2. Each $B^\beta_i \setminus j_{g(\beta, i)+1}^{\omega^M}(d_{g(\beta, i)})$ is nonstationary.

These conditions imply that if $B$ is the union of the $b_\alpha$'s then for each $\alpha$, $j_\alpha(J_\alpha) = I \cap j_\alpha(N_\alpha)$ where $j_\alpha$ is the iteration of $(N_\alpha, J_\alpha)$ sending $b_\alpha$ to $B$, and $((M, I), B)$ extends each $p_\alpha$. 

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Assume that we can construct the sequence as above. For each \((\alpha, i)\) let \(d'_{\alpha i}\) be \(j_{g(\alpha, i)}(\omega, (d_{g(\alpha, i)}) \). Then the diagonal union of \(A = \{ d'_{\alpha i} \}_{\alpha} \) is \(I\)-large. Thus if \(((M, I), B)\) belongs to the generic \(G\) and \(j\) is the iteration of \((M, I)\) sending \(B\) to \(A_G\) then the diagonal union of \(j(A)\) contains the critical sequence and therefore a club. We claim that \(j(A)\) is a subset of \(\tau_G\): Write \(j((p_{\alpha} \mid \alpha < \omega_1^M))\) as \((q_{\alpha} \mid \alpha < \omega_1)\). By Fact 2, each \(q_{\alpha}\) belongs to \(G\) and since \(((M, I), I)\) is \(A\)-iterable, each member of \(j(A)\) is forced to be in \(\tau_G\) by some \(q_{\alpha}\), so \(j(A)\) is contained in \(\tau_G\).

It remains to construct the above sequence satisfying conditions (1) and (2). Condition (1) is easily achieved: As \(\tau\) names a dense subset of \(P(\omega_1)\) \(\setminus NS\), for each \(\alpha < \omega_1^M\) there is a pair \((p^*, d^*)\) such that \(p^* \leq p_{\alpha}\) and (1) holds with \((p^*, d^*)\) in the role of \((p_{\alpha+1}, d_{\alpha})\).

To achieve condition (2), fix a ladder system \((h_{\alpha} \mid \alpha < \omega_1, \alpha \text{ limit})\) in \(M\) (i.e., \(h_{\alpha}\) maps \(\omega\) increasingly and cofinally into \(\alpha\) for each limit \(\alpha\)). Assuming that the \(p_{\alpha}\)'s have been constructed below a limit \(\beta\), let \(((N^\beta_i, J^\beta_i) \mid i < \omega), b^\beta_\gamma)\) be the limit sequence corresponding to the sequence \((p_{h_\beta(i)} \mid i < \omega)\) and for each \(i\) let \(j'_i, J^\beta_i)\) be the unique iteration of \((N_{h_\beta(i)}, J_{h_\beta(i)}(i))\) sending \(b_{h_\beta(i)}\) to \(b^\beta_\gamma\). Fix a precondition \((N_\beta, J_\beta)\) in \(M\) with \(((N^\beta_i, J^\beta_i) \mid i < \omega), b^\beta_\gamma)\) \(H(\omega_1)^N_\beta\).

By an earlier lemma we can choose an iteration \(j''_\beta\) of \(((N^\beta_i, J^\beta_i) \mid i < \omega)\) in \(N_\beta\) such that \(j''_\beta(J^\beta_i) = J_\beta \cap j''_\beta(N^\beta_i)\) for each \(i\); also require that \(\omega_1^N_\beta \in B^\gamma_k\) for some \(\gamma < \beta\) and \(k < \omega\) with \(\gamma(\gamma, k) < \beta\). Then, letting \(i''\) be the least \(i\) such that \(h_\beta(i') \geq g(\gamma, k)\),

\[j''_\beta(J_{g(\gamma, k)+1}, h_\beta(i'')(d_{g(\gamma, k)\}))\]

is in the filter corresponding to the first step of the iteration of the sequence \(((N^\beta_i, J^\beta_i) \mid i < \omega)\), ensuring (provided we let \(b_\beta = j''_\beta(b^\beta_\gamma)\)) that \(\omega_1^N_\beta \in j_{g(\gamma, k)+1}, b^\gamma_{\gamma(\gamma, k)}\). As the set of \(\omega_1^N_\beta\)'s for limit \(\beta\) is a club, condition (2) is thereby satisfied.

The \(P_{\max}\) extension of \(L(\mathbb{R})\) is a model of choice:

**Theorem 16** Assume \(AD\) in \(V = L(\mathbb{R})\) and let \(G\) be \(P_{\max}\)-generic. Then \(AC\) holds in \(V[G]\).
Proof. It suffices to show that in $V[G]$, the subsets of $\omega_1$ can be wellordered in ordertype $\omega_2$.

First note that we at least have some choice in $V[G]$; By absoluteness, any $\mathbb{P}_{\text{max}}$ condition can be $\alpha$-iterated for any countable $\alpha$. It follows that if $((M, I), a)$ belongs to $G$ then $(M, I)$ can be $\omega_1$-iterated, by a density argument. Thus there is an embedding $j$ sending $\omega_1^M$ to $\omega_1$; as $M$ satisfies choice, it contains an injection of $\omega_1^M$ into the reals of $M$ and by applying $j$ we get an injection of $\omega_1$ into the reals. This is enough to partition $\omega_1$ into $\omega_1$-many stationary sets.

For any $\gamma < \omega_2$ if $f_\gamma : \omega_1 \to \gamma$ is a surjection then define the “canonical function” $g_\gamma : \omega_1 \to \omega_1$ by $g_\gamma(\beta) =$ ordertype $(f_\gamma(\beta))$. Without choice we cannot choose the $f_\gamma$’s, but the $g_\gamma$’s are unique modulo the nonstationary ideal and so we can choose for each $\gamma$ the equivalence class of $g_\gamma$ modulo NS.

Claim. In $V[G]$, if $A, B$ are stationary, costationary subsets of $\omega_1$ then $A = g_\gamma^{-1}[B]$ mod NS for some $\gamma$.  

Proof of Claim. We first use choice to show:

\[ (*) \text{ If } (M, I) \text{ is a precondition and } A, B \in M \text{ are } I\text{-positive, } co-I\text{-positive subsets of } \omega_1^M \text{ in } M \text{ and } J \text{ is a normal ideal on } \omega_1 \text{ then there are an iteration } j : (M, I) \to (M^*, I^*) \text{ of } (M, I) \text{ of length } \omega_1 \text{ and an ordinal } \gamma < \omega_2 \text{ such that } I^* = J \cap M^* \text{ and } j(A) = g_\gamma^{-1}[j(B)] \text{ mod NS.} \]

Then given any $\mathbb{P}_{\text{max}}$ condition $p_0 = ((M_0, I_0), a_0)$ forcing $\tau_0, \tau_1$ to be stationary, costationary subsets of $\omega_1$ we can choose an $A$-iterable $(M, I)$ (for an appropriate $A$) with $p_0 \in H(\omega_1)^M$, and apply $(*)$ in $M$ with the ideal $I$ to obtain an extension $((M, I), a)$ of $p_0$ forcing the conclusion of the Claim for $\tau_0^G, \tau_1^G$.

To prove $(*)$ let $x$ be a real coding $(M, I)$ and form an iteration of $(M, I)$ so that at stage $\alpha^* = \text{the least } x\text{-indiscernible greater than } \alpha$, $j_{\alpha^*}(B)$ belongs to $G_{\alpha^*}$ iff $j_{\alpha^*}(A)$ belongs to $G_{\alpha}$. This is possible as $A, B$ are both $I$-positive and co-$I$-positive. The result is that for a club of $x$-indiscernible $\alpha$, $\alpha \in j(A)$ iff $\alpha^* \in j(B)$ and therefore $A = g_\gamma^{-1}[B]$ mod NS, where $\gamma$ is the least $x$-indiscernible greater than $\omega_1$. □ (Claim)
Obviously if $A_0 = g_{\gamma_0}^{-1}[B]$ mod NS, $A_1 = g_{\gamma_1}^{-1}[B]$ mod NS and $A_0 \triangle A_1$ is stationary, then $\gamma_0 \neq \gamma_1$. Now fix a stationary, costationary $B$ and let $(A_\alpha \mid \alpha < \omega_1)$ be a partition of $\omega_1$ into stationary sets. For $X$ a subset of $\omega_1$ (other than $\emptyset$ or all of $\omega_1$) choose $\gamma_X$ such that $A_X = g_{\gamma_X}^{-1}[B]$ where $A_X$ is the union of the $A_\alpha$’s for $\alpha$ in $X$. Then the function $X \mapsto \gamma_X$ is an injection of a set of size $P(\omega_1)$ into $\omega_2$. □

12. Vorlesung

$\Pi_2(H(\omega_2))$ Invariance of the $\mathbb{P}_{\text{max}}$ extension

We show, assuming large cardinals, that any $\Pi_2(H(\omega_2))$ sentence that holds in a set-forcing extension of the universe also holds in the $\mathbb{P}_{\text{max}}$ extension of $L(\mathbb{R})$. By “sentence” we of course mean “sentence without parameters” as there are even $\Sigma_1(H(\omega_2))$ sentences with parameters from $H(\omega_2)$ which can be forced over $L(\mathbb{R})[G]$ but do not hold there (just take a stationary, costationary subset of $\omega_1$ and add a club subset of it). However the parameter $\omega_1$ is allowed because any $\Pi_2(H(\omega_2))$ sentence using it is equivalent to one without it.

We will use (but not prove) the following

**Lemma 17** If $\delta$ is a Woodin cardinal then the Lévy collapse $\text{Coll}(\omega_1, < \delta)$ forces that NS is precipitous.

**Theorem 18** Suppose that there is a proper class of Woodin cardinals and $P$ is a set partial order which forces that the $\Pi_2$ sentence $\varphi$ holds in $H(\omega_2)$. Then $\varphi$ holds in the $H(\omega_2)$ of the $\mathbb{P}_{\text{max}}$ extension of $L(\mathbb{R})$.

**Proof.** Write $\varphi$ as $\exists X \forall Y \psi(X, Y)$ where $\psi$ is $\Delta_0$. It suffices to show that for every $\mathbb{P}_{\text{max}}$ condition $p = ((M, I), a)$ and every $x \in H(\omega_2)^M$ there is a $\mathbb{P}_{\text{max}}$ condition $q = ((N, J), b)$ extending $p$ so that if $j : (M, I) \rightarrow (M^*, I^*)$ is the unique iteration sending $a$ to $b$ then

$$H(\omega_2)^N \models \exists y \psi(j(x), y).$$

Given this, for any $X$ in the $H(\omega_2)$ of the $\mathbb{P}_{\text{max}}$ extension we can write $X$ as $j(x)$ where $((M, I), a)$ belongs to the $\mathbb{P}_{\text{max}}$-generic, $j : (M, I) \rightarrow (M^*, I^*)$ is the iteration of $(M, I)$ taking $a$ to $A_G$ and $\psi(X, Y)$ holds in $H(\omega_2)^M$. For
some $Y$; but then $\psi(X,Y)$ also holds in the $H(\omega_2)$ of the $\mathbb{P}_{\text{max}}$ extension because $\psi$ is $\Delta_0$.

Let $Z$ be a countable elementary submodel of a large $H(\theta)$ with $((M,I),a)$, $P$ and $\delta$ as members where $\delta$ is a Woodin cardinal such that $P$ belongs to $H(\delta)$. Let $N$ be the transitive collapse of $Z$. We know that any forcing extension of $M$ in which NS is precipitous is iterable with respect to its NS. Let $N[g_0]$ be a $P$-generic extension of $N$ where $P$ is the image of $P$ under the transitive collapse of $Z$ to $N$ and let $j : (M,I) \rightarrow (M^*,I^*)$ be an iteration in $N[g_0]$ such that $I^* = \text{NS}^{N[g_0]} \cap M^*$. As $\varphi$ holds in $H(\omega_2)^{N[g_0]}$ there is $y \in H(\omega_2)^{N[g_0]}$ such that $\psi(j(x),y)$ holds in $H(\omega_2)^{N[g_0]}$. In $N[g_0]$ the image $\delta$ of $\delta$ under the transitive collapse of $Z$ is Woodin; let $N[g_0][g_1]$ be a $\text{Coll}(\omega_1,\delta)^{N[g_0]}$-generic extension of $N[g_0]$ and let $N^* = N[g_0][g_1][g_2]$ be a ccc forcing extension of $N[g_0][g_1]$ in which MA holds. Then $((N^*,\text{NS}^{N^*}),j(a))$ is the desired $\mathbb{P}_{\text{max}}$ condition extending $p$. $\square$

Remarks. (a) The previous result also holds if we replace $H(\omega_2)$ by $(H(\omega_2),A)$ for any set of reals $A$ in $L(\mathbb{R})$. (b) Viale has pointed out the following variant of the previous theorem (perhaps also due to Woodin): Let $(\ast)$ be the axiom that AD holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\text{max}}$ extension of $L(\mathbb{R})$. If $(\ast)$ holds and there is a proper class of Woodin cardinals then set-forcing which preserve $(\ast)$ cannot affect the truth of arbitrary first-order properties of $H(\omega_2)$. This can be viewed as an analogue to the fact that if there is a proper class of Woodin cardinals then no set-forcing can affect the truth of first-order properties of $H(\omega_1)$. These results are part of a general programme of showing that the truth of certain statements about some $H(\lambda)$ is not affected by certain set-forcing which preserve the truth of certain axioms. (c) One should not hope for too much with these “truth-invariance” results. Indeed, they appear to fall apart when replacing set-forcing by class-forcing. And no large cardinal axiom is able to ensure invariance of even $\Sigma_2(H(\omega_1))$ truth with respect to arbitrary (non-generic) extensions which satisfy it.

13. Vorlesung

**Theorem 19** Suppose that $\text{NS}$ is saturated and there is a measurable cardinal. Then $\delta_2^1 = \omega_2$ and therefore CH fails.

**Proof.** Recall that $\delta_2^1$ is the supremum of the $(\omega_1^Y)^+$ of $L[\mathbb{R}]$ for reals $R$. 

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Suppose $\alpha < \omega_2$. Form the structure $A = (H(\mu), <, \{\alpha\})$ where $<$ is a wellorder of $H(\mu)$. Then by virtue of the measurability of $\mu$, there is an $\omega$-sequence $(\iota_n \mid n < \omega)$ of ordinals less than $\mu$ such that:

1. The $\iota_n$’s are indiscernibles for $A$.
2. Let $N$ be the Skolem hull of the $\iota_n$’s in $A$ and for any limit ordinal $\gamma$ let $N_\gamma$ be the “stretch” of $N$ to $\gamma$ indiscernibles, i.e., the structure generated from $\gamma$-many indiscernibles in the same way that $N$ is generated from the $\iota_n$’s. Then $N_\gamma$ is wellfounded and for $\gamma_0 < \gamma_1$, $N_{\gamma_0}$ is isomorphic to an initial segment of $N_{\gamma_1}$.

As $\text{NS}$ is saturated it is precipitous and therefore $N \vDash \text{NS}$ is precipitous. It follows that generic iterations of $(N, \text{NS})$ of length less than ordertype $(N \cap \text{Ord})$ are wellfounded. But for any limit ordinal $\gamma$, generic iterations of $(N, \text{NS})$ lift to generic iterations of $(N_\gamma, \text{NS})$ and therefore generic iterations of $(N, \text{NS})$ of any length are wellfounded.

**Claim.** Let $\bar{N}$ be the transitive collapse of $N$ and $\bar{I} = \text{NS}^\bar{N}$. Then there is a generic iteration $j : (\bar{N}, \bar{I}) \to (\bar{N}_*, I^*)$ of length $\omega_1$ such that $\text{Ord}(\bar{N}_*) > \alpha$.

The Theorem follows from the Claim as if we let $R$ be a real coding the countable model $\bar{N}$ we see that $\alpha$ is less than $(\omega_1^V)^+$ of $L[R]$.

We prove the Claim by inductively defining iterates $\bar{N}_\gamma$ of $\bar{N}$ together with embeddings $j_\gamma : \bar{N}_\gamma \to A$. Suppose that $\bar{N}_\gamma$, $j_\gamma$ are defined.

Let $\delta_\gamma$ be the $\omega_1$ of $\bar{N}_\gamma$ and $U_\gamma$ the ultrafilter on $\delta_\gamma$ derived from $j_\gamma$, i.e., $X \subseteq \delta_\gamma$ belongs to $U_\gamma$ iff $\delta_\gamma \in j_\gamma(X)$.

Then as $\text{NS}^{\bar{N}_\gamma}$ is saturated, $U_\gamma$ is generic for $(\mathcal{P}(\omega_1)/\text{NS})^{\bar{N}_\gamma}$: Indeed, if $\bar{A} \in \bar{N}_\gamma$ is a maximal antichain in this forcing then $\bar{A}$ is a collection $(X_i \mid i < \delta_\gamma)$ of stationary sets whose diagonal union contains a club in $\delta_\gamma$, and therefore the diagonal union of $j_\gamma(\bar{A}) = (X_i \mid i < \omega_1)$ contains a club in $\omega_1$. It follows that $\delta_\gamma$ belongs to this diagonal union and therefore for some $i < \delta_\gamma$, $\delta_\gamma$ belongs to $X_i$. It follows that $X_i$ belongs to $U_\gamma$.

Now let $\bar{N}_{\gamma+1}$ be the ultrapower of $\bar{N}_\gamma$ by $U_\gamma$ and define $j_{\gamma+1} : \bar{N}_{\gamma+1} \to A$ by $j_{\gamma+1}([f]) = j_\gamma(f)(\delta_\gamma)$. At limit stages we take a direct limit and embed it into $A$ in the natural way. Note that if $M_\gamma$ denotes the range of $j_\gamma$, then for
each $\gamma, M_{\gamma+1}$ is the Skolem hull in $A$ of $M_\gamma \cup \{\delta_\gamma\}$. As the union $M^*$ of the $M_\gamma$'s contains $\alpha$ as an element and $\omega_1$ as a subset, it follows that $M^*$ also contains $\alpha + 1$ as a subset and therefore its transitive collapse $N^*$, the direct limit of the $\tilde{N}_\gamma$'s, has ordinal height greater than $\alpha$. $\square$

The Stationary Tower

14.-15. Vorlesungen

We now switch topics from $P_{\text{max}}$ to stationary tower forcing, based on Paul Larson’s book on this topic. This forcing can be used to collapse the successor of a singular cardinal using less than a measurable, to show that if there is a proper class of Woodin cardinals then truth in $L(\mathbb{R})$ is invariant under set forcing and, using Martin-Steel’s work on projective determinacy, show that $\text{AD}$ in $L(\mathbb{R})$ follows from the existence of infinitely many Woodin cardinals with a measurable above.

Generalised Stationarity

If $X$ is any nonempty set, then a subset $C$ of $\mathcal{P}(X)$ is CUB iff it is of the form $\{a \subseteq X \mid F[a^{<\omega}] \subseteq a\}$ for some $F : [X]^{<\omega} \rightarrow X$. And $S \subseteq \mathcal{P}(X)$ is stationary iff it intersects all CUB sets, i.e., iff for any $F : [X]^{<\omega} \rightarrow X$ there exists $a \in S$ such that $F[a^{<\omega}] \subseteq a$.

For any infinite cardinal $\kappa \leq \text{Card} (X)$, the set of subsets of $X$ of cardinality $\kappa$ is a stationary subset of $\mathcal{P}(X)$. If $X = \alpha$ is an ordinal of uncountable cofinality then a subset of $\alpha$ is also a subset of $\mathcal{P}(\alpha)$ and it is stationary in the above sense iff it is stationary in the usual sense.

Another way of expressing stationarity is in terms of structures for a countable language: $S \subseteq \mathcal{P}(X)$ is stationary iff every structure $\mathcal{A}$ with universe $X$ has an elementary substructure with universe in $S$.

The following are left as exercises.

Lemma 20 (Projection and Lifting) Suppose $X \subseteq Y$.

(a) If $S$ is a stationary subset of $\mathcal{P}(Y)$ then $S_X = \{a \cap X \mid a \in S\}$ is stationary in $\mathcal{P}(X)$.

(b) If $S$ is a stationary subset of $\mathcal{P}(X)$ then $S^Y = \{a \subseteq Y \mid a \cap X \in S\}$ is a stationary subset of $\mathcal{P}(Y)$. 

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**Lemma 21** (Fodor) Suppose that $S \subseteq \mathcal{P}(X)$ is stationary and $F : S \to X$ is regressive, i.e., $F(a) \in a$ for each $a \in S$. Then there is $x \in X$ such that $F(a) = x$ for stationary-many $a$ in $S$.

Now we force with the associated ideals of nonstationary sets. For any $X$ let $\mathbb{P}_X$ be the partial order of stationary subsets of $\mathcal{P}(X)$, ordered by inclusion. If $G$ is $\mathbb{P}_X$ generic then $G$ defined an ultrafilter $U$ on $\mathcal{P}(X)^V$ and we can form the ultrapower $j : V \to \text{Ult}(V, U) = (M, E)$. Of course the elements of $M$ are the equivalence classes $[f]_U$ of functions $f : \mathcal{P}(X) \in V$ in $V$. Let $\text{id}$ denote the identity function on $\mathcal{P}(X)$. Then $\text{id}$ “represents” $j[X]$ in $M$, i.e.

**Lemma 22** $j[X]$ equals $\{ m \in M \mid mE[\text{id}]_U \}$.

**Proof of lemma.** Suppose that $x$ belongs to $X$. Then by the definition of $j$, $j(x)$ is $[c_x]_U$ where $c_x$ is the constant function on $\mathcal{P}(X)$ with value $x$. Now $c_x(a) = x \in a = \text{id}(a)$ for CUB-many $a \in \mathcal{P}(X)$ so it follows by Łoś that $j(x) = [c_x]_U E[\text{id}]_U$. Conversely, suppose that $mE[\text{id}]_U$ and write $m = [f]_U$. Then $f(a) \in \text{id}(a) = a$ for a set of $a$ in $U$. By Fodor and genericity, there is $x \in X$ such that $f(a) = x$ for a set of $a$ in $U$. but then $m = [f]_U = [c_x]_U = j(x)$. □

This lemma implies that $j[X] \cap \text{Ord}^M = j[X \cap \text{Ord}]$ is represented in $M$ and therefore so are all of its initial segments. It follows that the ordertype of $X \cap \text{Ord}$ is represented in $M$ and therefore belongs to the wellfounded part of $M$ (if we identify the wellfounded part of $M$ with its transitive collapse).

**Stationary Tower Embeddings**

Note that if $S$ is a stationary subset of $\mathcal{P}(X)$ then $X = \cup S$. So we just say that $S$ is stationary iff $\cup S$ is nonempty and $S$ is stationary in $\mathcal{P}(\cup S)$.

**Definition 23** (The Stationary Tower) Let $\kappa$ be strongly inaccessible. The full stationary tower up to $\kappa$, denoted $\mathbb{P}_{< \kappa}$, consists of stationary $a \in H(\kappa)$, ordered as follows:

$b \leq a$ iff $\cup a \subseteq \cup b$ and $b_{\omega a} \subseteq a$, i.e., $z \cap (\cup a) \in a$ for each $z \in b$. 

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We associate a generic elementary embedding \( j : V \rightarrow (M, E) \) to a \( \mathbb{P}_{<\kappa} \)-generic \( G \) as follows. For each nonempty \( X \in H(\kappa) \) define \( U_X = \{ b_X \mid b \in G \) and \( X \subseteq \bigcup b \} \), where as before \( b_X \) is the projection of \( b \) to \( X \), i.e. the set of all \( z \cap X \) for \( z \) in \( b \).

Claim. \( U_X \) is an ultrafilter on \( \mathcal{P}(X)^V \) extending the CUB filter on \( \mathcal{P}(X)^V \). And for \( X \subseteq Y \), \( S \) belongs to \( U_X \) iff \( S^Y = \{ Z \subseteq Y \mid Z \cap X \in S \} \) belongs to \( U_Y \).

Proof. Any CUB subset of \( \mathcal{P}(X) \) is compatible with each stationary set and therefore belongs to \( G \) and hence to \( U_X \). We must show that \( U_X \) is an ultrafilter on \( \mathcal{P}(X)^V \). It suffices to show that if \( S \subseteq \mathcal{P}(X) \) then any \( b \) can be extended to a \( c \) such that \( c_X \) is contained in or disjoint from \( S \). We may assume that \( X \) is a subset of \( \bigcup b \). Let \( b^+ \) be set of \( z \in b \) such that \( z \cap X \) belongs to \( S \) and \( b^- \) the set of \( z \in b \) such that \( z \cap X \) does not belong to \( S \). Then \( b^+_X \) is contained in \( S \) and \( b^-_X \) is disjoint from \( S \). Let \( c \) be \( b^+ \) if this is stationary and otherwise \( b^- \). The last claim follows easily from the definitions.

Now for each \( X \) form the ultrapower by \( U_X \) to get \( j_X : V \rightarrow (M_X, E_X) \). And for \( X \subseteq Y \) define \( j_{XY} : M_X \rightarrow M_Y \) by \( j_{XY}([f]_{U_X}) = [f_Y]_{U_Y} \) where \( f_Y : \mathcal{P}(Y) \rightarrow V \) is defined by \( f_Y(Z) = f(Z \cap X) \). This defines a direct system of models \( (M_X, E_X) \) with embeddings. Let \( (M, E) \) denote the direct limit of this directed system and \( j \) the corresponding embedding of \( V \) into this direct limit. For each \( a \in G \) and \( f : \mathcal{P}(a) \rightarrow V \) in \( V \) we let \( [f]_G \) denote the member of \( M \) represented by \( f \). The following is a straightforward adaptation of Lemma 22.

Fact. The identity function \( \text{id}_X \) on \( \mathcal{P}(X) \) represents \( j[X] \) in \( M \), i.e., \( j[X] = \{ b \in M \mid bE[\text{id}_X]_G \} \).

Identify the wellfounded part of \( M \) with its transitive collapse. Then by this Fact, \( X \) and \( j \upharpoonright X \) belong to \( M \) for each \( X \in H(\kappa) \) and therefore \( H(\kappa) \) is a subset of \( M \). Also, as \( j[X] \) is an element of \( M \) we obtain the usual description of an ultrafilter in terms of its associated ultrapower embedding: \( U_X = \{ a \subseteq \mathcal{P}(X) \mid j[X] \in j(a) \} \). Thus \( a \in G \) iff \( j[\mathcal{U}a] \in j(a) \) and \( [f]_G = j(f)(j[\mathcal{U}a]) \) when \( f \) has domain \( \mathcal{P}(\mathcal{U}a) \).

As \( j \upharpoonright H(\alpha) \) belongs to \( M \) for each cardinal \( \alpha < \kappa \), it follows that \( G \cap H(\alpha) = \{ a \in H(\alpha) \mid j[\mathcal{U}a] \in j(a) \} \) also belongs to \( M \) for each cardinal \( \alpha < \kappa \).
Fact. For $\alpha < \kappa$, $\alpha$ is represented in $M$ by the function $f : \mathcal{P}(\alpha) \to \alpha$ given by $f(Z) = \text{ot}(Z)$.

It follows that for $\beta \subseteq \bigcup a$:

$a \models j(\alpha) \leq \beta$ iff
\begin{align*}
\text{ot}(Z \cap \beta) &\geq \alpha \text{ for “almost all” } Z \text{ in } a \text{ (i.e., for some CUB } C \text{ in } \mathcal{P}(\bigcup a)),} \\
\text{ot}(Z \cap \beta) &\geq \alpha \text{ for all } Z \text{ in } a \cap C).
\end{align*}

Thus $a$ forces $j(\alpha) = \alpha$ iff $\text{ot}(Z \cap \alpha) = \alpha$ for almost all $Z$ in $a$.

**Completely Jónsson Cardinals**

$\kappa$ is completely Jónsson iff it is strongly inaccessible and for each stationary $a \in H(\kappa)$, the set of $X \subseteq H(\kappa)$ such that $X \cap (\bigcup a) \in a$ and $X$ has cardinality $\kappa$ is stationary in $\mathcal{P}(H(\kappa))$.

Ramsey cardinals are completely Jónsson and measurable cardinals are Ramsey, so these cardinals are not very big. Also, as complete Jónsson-ness is a $\Pi^1_1$ property, it follows that measurable cardinals are also limits of completely Jónsson cardinals.

Completely Jónsson cardinals are relevant for the following reason. Suppose that $\kappa$ is a strongly inaccessible limit of completely Jónsson cardinals. Then any $a \in \mathbb{P}_{<\kappa}$ has an extension $b$ forcing $j(\alpha) = \alpha$ for some $\alpha < \kappa$; choose $\alpha < \kappa$ to be completely Jónsson and such that $a$ belongs to $H(\alpha)$ and let $b$ be $\{Z \subseteq H(\alpha) \mid Z \cap \alpha \text{ has cardinality } \beta \text{ and } Z \cap (\bigcup a) \in a\}$; then $b$ extends $a$ and forces $j(\alpha) = \alpha$. Thus if $\kappa$ is a strongly inaccessible limit of completely Jónsson cardinals, it follows that $j$ has unboundedly many fixed points below $\kappa$. In fact $\kappa$ is also a fixed point of $j$ as it is not hard to show that $j$ is continuous at strongly inaccessibles.

Also note that if $\kappa$ is a strongly inaccessible limit of completely Jónsson cardinals then any set in the $H(\kappa)$ of $V[G]$ belongs to the wellfounded part of $M$, $\kappa$ belongs to the wellfounded part of $M$ (as $j(\kappa) = \kappa$) and so the $H(\kappa)$ of $M$ equals the $H(\kappa)$ of $V[G]$. Thus $j(H(\kappa)) = H(\kappa)$ of $V[G]$.

**Forcing Applications**

**Example 1.** (Universality for set forcing) Suppose that there is a proper class of completely Jónsson cardinals. Let $\mathbb{P}_\infty$ denote the stationary tower class
forcing, using arbitrary stationary sets $a$ as conditions. Then $\mathbb{P}_\infty$ is universal for set-forcing in the sense that any set forcing is a regular subforcing of $\mathbb{P}_\infty$. To see this, suppose that $Q \in V$ is a set forcing and choose a cardinal $\alpha$ such that the power set of $Q$ in $V$ belongs to $H(\alpha)$. Then consider the stationary set $a = \{ \alpha < \lambda \mid \text{cof}(\alpha) = \gamma \}$, a stationary subset of $\mathcal{P}(\lambda)$. Suppose that $a$ belongs to a $\mathbb{P}_\infty$ generic $G$ with associated $j : V \to V[G]$. Since $a$ belongs to $G$, $j[\lambda] \in j(a)$ and since $a$ consists of ordinals, so does $j(a)$. Thus $j[\lambda]$ is an ordinal and therefore $j$ is the identity on $\lambda$. Moreover by elementarity, $j(\alpha)$ consists of those ordinals less than $j(\lambda)$ which have cofinality $j(\gamma) = \gamma$ in $V[G]$, so in fact $j[\lambda] = \lambda$ is an ordinal less than $j(\lambda)$ of cofinality $\gamma$ in $V[G]$. As $j$ is the identity on $\lambda$, cardinals below $\lambda$ are preserved and if $2^\delta$ is less than $\lambda$ then no new subsets of $\delta$ are added.

For example, we could have GCH in $V$ and with $\mathbb{P}_\infty$ add no new bounded subsets of $\aleph_\omega$ but change the cofinality of $\aleph_{\omega+1}$ to $\aleph_\gamma$. By core model theory, such a weird effect cannot be achieved if ZFC is preserved by adding $V$ as an additional predicate, without using more than a Woodin cardinal and probably this would need a supercompact cardinal.

**Wellfoundedness**
Suppose that $G$ is $\mathbb{P}_{<\delta}$ generic with resulting embedding $j : V \to M$. We'll show that if $\delta$ is a Woodin cardinal then $M$ is wellfounded and closed in $V[G]$ under sequences of length less than $\delta$.

Suppose that $D$ is a subset of $\mathbb{P}_{<\delta}$. Then $Y \prec V_{\delta+1}$ captures $D$ iff there is $d \in D \cap Y$ such that $Y \cap (\cup d) \in d$. If $D$ is an antichain then the choice of $d$ is unique; if $d' \in Y \cap D$ is distinct from $d$ there is a function $h$ such that no $Z$ closed under $h$ satisfies both $Z \cap (\cup d) \in d$ and $Z \cap (\cup d') \in d'$; as $h$ may be chosen in $Y$ and $Y$ is closed under such an $h$ one cannot have $Y \cap (\cup d') \in d'$. Also note that if $A \subseteq V_{\delta+2}$ and stationary-many $Y \in A$ capture the antichain $D$ then in the forcing $\mathbb{P}_\infty$, $A$ is compatible with some $d \in D$: By Fodor we can thin $A$ to $A'$ consisting of $Y$ which capture $D$ with the same choice of $d \in D \cap Y$; then $A'$ extends both $A$ and $d$.

We also define $sp(D)$ as follows. For sets $X \subseteq Y$, we say that $Y$ end extends $X$ if $X = Y \cap V_\alpha$ where $\alpha$ is the rank of $X$ (i.e. the least $\alpha$ such that $X$ is a subset of $V_\alpha$). Then $sp(D)$ consists of all $X \prec V_{\delta+1}$ of size $< \delta$ such that $D \in X$ and there exists $Y \prec V_{\delta+1}$ such that:

1. $X$ is a subset of $Y$.
2. $Y$ end extends $X \cap V_\alpha$.
3. $Y$ captures $D$.

$D$ is semiproper iff $sp(D)$ contains a club in $\mathcal{P}_\delta(V_{\delta+1})$.

**Lemma 24** Let $\eta$ be an infinite cardinal less than $\delta$. Suppose that for each sequence $(D_\alpha | \alpha < \eta)$ of predense subsets of $\mathbb{P}_{<\delta}$ there are arbitrarily large strongly inaccessible $\gamma < \delta$ such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper in $\mathbb{P}_{<\gamma}$ for all $\alpha < \eta$. Then the ultrapower $(M, E)$ arising from a $\mathbb{P}_{<\delta}$ generic $G$ is closed under sequences of length $\eta$ in $V[G]$. In particular, this ultrapower is wellfounded.

**Lemma 25** Suppose that $\delta$ is a Woodin cardinal. Then for each sequence $(D_\alpha | \alpha < \delta)$ of predense subsets of $\mathbb{P}_{<\delta}$ there are arbitrarily large strongly inaccessible $\gamma < \delta$ such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper in $\mathbb{P}_{<\gamma}$ for each $\alpha < \gamma$.

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Lemma 26 Let \( \eta \) be an infinite cardinal less than \( \delta \). Suppose that for each sequence \( (D_\alpha \mid \alpha < \eta) \) of predense subsets of \( \mathbb{P}_{<\delta} \) there are arbitrarily large strongly inaccessible \( \gamma < \delta \) such that \( D_\alpha \cap \mathbb{P}_{<\gamma} \) is semiproper in \( \mathbb{P}_{<\gamma} \) for all \( \alpha < \eta \). Then the ultrapower \( (M,E) \) arising from a \( \mathbb{P}_{<\delta} \) generic \( G \) is closed under sequences of length \( \eta \) in \( V[G] \). In particular, this ultrapower is wellfounded.

Lemma 27 Suppose that \( \delta \) is a Woodin cardinal. Then for each sequence \( (D_\alpha \mid \alpha < \delta) \) of predense subsets of \( \mathbb{P}_{<\delta} \) there are arbitrarily large strongly inaccessible \( \gamma < \delta \) such that \( D_\alpha \cap \mathbb{P}_{<\gamma} \) is semiproper in \( \mathbb{P}_{<\gamma} \) for each \( \alpha < \gamma \).

Proof of Lemma 26. Fix \( a_0 \in \mathbb{P}_{<\delta} \) and a term \( \tau \) for an \( \eta \)-sequence of ordinals in \( (M,E) \). For \( \alpha < \eta \) let \( A_\alpha \) be a maximal antichain of conditions \( a \) such that \( a \models \tau(\alpha) = [f]_G \) for some \( f : a \to \text{Ord} \). By the hypothesis of the lemma there is a strongly inaccessible \( \gamma < \delta \) such that:

1. \( a_0 \in V_\gamma, \eta < \gamma \).
2. \( A_\alpha \cap \mathbb{P}_{<\gamma} \) is semiproper for each \( \alpha < \eta \).

Let \( a \) be the set of \( X \prec V_{\gamma+1} \) such that:

- \( X \) has size less than \( \gamma \).
- \( X \cap (\cup a_0) \in a_0 \).
- \( X \) captures \( A_\alpha \) for each \( \alpha \in X \cap \eta \) (i.e., for \( \alpha \in X \cap \eta \) there is \( b \in X \cap A_\alpha \) such that \( X \cap (\cup b) \in b \)).

Claim. \( a \) is stationary in \( \mathcal{P}_\gamma(V_{\gamma+1}) \).

Proof of Claim. Fix \( H : [V_{\gamma+1}]^{<\omega} \to V_{\gamma+1} \). Since \( a_0 \) is stationary we may choose \( X_0 \prec V_\delta \) of size less than \( \gamma \) containing all relevant parameters (including \( H \)) such that \( X_0 \cap (\cup a_0) \in a_0 \). Define an elementary chain \( (X_\alpha \mid \alpha \in X_0 \cap \eta) \) as follows: If \( \alpha \in X_0 \cap \eta \) is a limit ordinal then let \( X_\alpha \) be the union of the \( X_\beta, \beta \in X_0 \cap \alpha \). At successor stages, since \( A_\alpha \cap \mathbb{P}_{<\gamma} \) is semiproper we can choose \( Y \prec V_\delta \) of size less than \( \gamma \) such that:

1. \( X_\alpha \) is a subset of \( Y \).
2. \( Y \cap V_\gamma \) end-extends \( X_\alpha \cap V_\gamma \).
3. \( Y \) captures \( A_\alpha \) (i.e., \( Y \cap (\cup b) \in b \) for some \( b \in A_\alpha \cap Y \)).
(Formally speaking, we only get \( Y < V_{\gamma+1} \) but this can be easily improved to \( Y \triangleleft V_\delta \).) Choose \( X_{\alpha+1} \) to be such a \( Y \). Let \( X \) be the union of the \( X_\alpha \), \( \alpha \in X_0 \cap \eta \). Then \( X \) has size less than \( \gamma \) and as \( X_0 \) contains \( H \), \( X \cap V_{\gamma+1} \) is closed under \( H \) and, for each \( \alpha \in X_0 \cap \eta \), as \( X_{\alpha+1} \) captures \( A_\alpha \) and \( X \cap V_{\gamma} \) end-extends \( X_{\alpha+1} \cap V_\gamma \) it follows that \( X \) captures \( A_\alpha \). Thus \( a \) is stationary in \( P_\gamma(V_{\gamma+1}) \) and the Claim is proved.

Now we define a function \( f : a \rightarrow V \) that is forced to represent \( \tau \). Recall that if we define \( g_\eta : a \rightarrow V \) by \( g_\eta(X) = X \cap \eta \) then \( g_\eta \) represents \( \dot{f}[\eta] \). We define \( f \) so that for each \( X \in a \), \( f(X) \) is a function with domain \( X \cap \eta \), so that \( [f]_\delta \) will be a function in the ultrapower with domain \( j[\eta] \). What we want to have is: \( [f]_\delta(j(\alpha)) = \tau_\delta(\alpha) \) for each \( \alpha < \eta \). For then \( f \) represents the function from \( j[\eta] \) to \( M \) given by \( j(\alpha) \mapsto \tau_\delta(\alpha) \) and therefore \( \tau_\delta \) belongs to \( M \).

Fix \( X \in a \) and \( \alpha \in X \cap \eta \). As \( X \) captures \( A_\alpha \) we can choose \( b \in X \cap A_\alpha \) such that \( X \cap (\cup b) \in b \). The choice of \( b \) is unique as \( A_\alpha \) is an antichain. Now as \( b \) belongs to \( A_\alpha \) we can choose \( f_\alpha \) such that \( b \vdash [f_\alpha]_\delta = \tau(\alpha) \) and we define:

\[
f(X)(\alpha) = f_\alpha(X \cap (\cup b)).
\]

We claim that \( f \) works, i.e., for each \( \alpha < \eta \), \( a \vdash [f]_\delta(j(\alpha)) = \tau(\alpha) \).

Fix \( \alpha < \eta \) and \( G \) generic for \( \mathbb{P}_{\leq \delta} \), \( a \) an element of \( G \). Let \( \bar{a} \in G \) consist of those \( X \in a \) such that \( \alpha \in X \). Now each \( X \in \bar{a} \) captures \( A_\alpha \) with a unique \( b \in A_\alpha \cap X \) such that \( X \cap (\cup b) \in b \). By normality and the genericity of \( G \) we may fix \( a_1 \leq \bar{a} \) and \( b_1 \in A_\alpha \cap \mathbb{P}_{< \gamma} \) such that \( a_1 \in G \) and for all \( Y \in a_1 \), \( b_1 \in Y \cap A_\alpha \) and \( Y \cap (\cup b_1) \in b_1 \). As \( a_1 \) extends \( b_1 \) it follows that \( b_1 \) belongs to \( G \). So since \( b_1 \vdash [f_\alpha]_\delta = \tau(\alpha) \) and \( f(X)(\alpha) = f_\alpha(X \cap (\cup b_1)) \) for each \( X \in a_1 \), it follows that \( [f]_\delta(j(\alpha)) = \tau_\delta(\alpha) \), as desired. \( \square \)

18. Vorlesung

**Lemma 28** Suppose that \( \delta \) is a Woodin cardinal. Then for each sequence \( (D_\alpha : \alpha < \delta) \) of p-finite subsets of \( \mathbb{P}_{< \delta} \) there are arbitrarily large strongly inaccessible \( \gamma < \delta \) such that \( D_\alpha \cap \mathbb{P}_{< \gamma} \) is semiproper in \( \mathbb{P}_{< \gamma} \) for each \( \alpha < \gamma \).

**Proof of Lemma 28.** Recall that \( \delta \) is Woodin iff for each \( f : \delta \rightarrow \delta \) there is an elementary embedding \( j : V \rightarrow M \) with critical point \( \gamma < \delta \) such that \( \gamma \) is closed under \( f \) and \( V_{j(f)(\gamma)} \) is contained in \( M \).
Now fix an $f : \delta \to \delta$ with limit ordinal values such that $\gamma < f(\gamma)$ for all $\gamma < \delta$ and for all strongly inaccessible $\gamma < \delta$ closed under $f$:

(a) For all $\alpha < \gamma$, $D_\alpha \cap \mathbb{P}_{<\gamma}$ is predense in $\mathbb{P}_{<\gamma}$.
(b) If $\alpha < \gamma$ is such that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper in $\mathbb{P}_{<\gamma}$, there exists a condition in $D_\alpha \cap V_{f(\gamma)}$ compatible with

$$a = \{ X < V_{\gamma+1} \mid \text{card}(X) < \gamma \text{ and } X \notin \text{sp}(D_\alpha \cap \mathbb{P}_{<\gamma}) \}.$$

Note that (b) is possible as since $D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper in $\mathbb{P}_{<\gamma}$ the set $a$ above is stationary and therefore compatible with an element of $D_\alpha$ as $D_\alpha$ is predense.

Now apply Woodinness to get $j : V \to M$ with critical point $\gamma < \delta$ closed under $f$ such that $V_{j(f)(\gamma)}$ is contained in $M$. We claim that $\gamma$ works, i.e., $D_\alpha \cap \mathbb{P}_{<\gamma}$ is semiproper for all $\alpha < \gamma$. Fix such an $\alpha$ and suppose that $D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper. Let $a$ be as in (b) above; thus $a$ is stationary. Then $\mathbb{P}_{<\gamma}^M = \mathbb{P}_{<\gamma}$ and $j(D_\alpha) \cap \mathbb{P}_{<\gamma} = D_\alpha \cap \mathbb{P}_{<\gamma}$ is not semiproper in $M$ so by the elementarity of $j$ there exists $b \in j(D_\alpha) \cap V_{j(f)(\gamma)}$ which is compatible with $a^M = a$ in $j(\mathbb{P}_{<\delta})$. Note that $b$ is stationary in $V$ since $V_{j(f)(\gamma)}$ is contained in $M$. Let $c$ be the greatest lower bound of $a, b$.

We may assume that $j(\delta) = \delta$. Choose $X < V_{\delta}$ such that $X \cap (\cup c) \in c$ and $b, j \downarrow V_{\gamma+1}$ and $< \text{ belong to } X$ where $<$ is a wellorder of $j(V_{\gamma+1})$ which belongs to $M$. As $\cup a = V_{\gamma+1}$, $a$ consists of sets of size less than $\gamma$ and $c$ extends $a$, it follows that $X \cap V_{\gamma+1}$ has size less than $\gamma$. So $j(X \cap V_{\gamma+1}) = j[X \cap V_{\gamma+1}]$ and the latter belongs to $j(a)$ and hence not to $j(\text{sp}(D_\alpha \cap \mathbb{P}_{<\gamma}))$. We obtain a contradiction by obtaining a witness $Y$ to the fact that $j[X \cap V_{\gamma+1}]$ does in fact belong to $j(\text{sp}(D_\alpha \cap \mathbb{P}_{<\gamma}))$.

We take $Y$ to be the Skolem hull in $j(V_{\gamma+1})$ of $\{ b \} \cup j[X \cap V_{\gamma+1}] \cup (X \cap (\cup b))$, using the wellorder $< \text{ in } X \cap M$. Note that as all of these sets belong to $M$, so does $Y$. And as all of these sets are subsets of $X$ and $j(V_{\gamma+1})$ is an element of $X$, it follows that $Y$ is a subset of $X$. Now $Y$ contains $j[X \cap V_{\gamma+1}]$ and since $j[X \cap V_{\gamma+1}] \cap V_{f(\gamma)} = j[X \cap V_{\gamma+1}] \cap V_\gamma$, it follows that $Y$ end-extends $j[X \cap V_{\gamma+1}]$ below $j(\gamma)$. Finally, $b$ witnesses that $Y$ captures $j(D_\alpha \cap \mathbb{P}_{<\gamma})$ since $b$ belongs to $Y$ and $Y \cap (\cup b) = X \cap (\cup b) \in b$. $\square$