A Simpler Proof of Jensen’s Coding Theorem

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Beller-Jensen-Welch [82] provides a proof of Jensen’s remarkable Coding Theorem, which demonstrates that the universe can be included in \( L[R] \) for some real \( R \), via class forcing. The purpose of this article is to present a simpler proof of Jensen’s theorem, obtained by implementing some changes first developed for the theory of strong coding (Friedman [87]).

The basic idea is to first choose \( A \subseteq ORD \) so that \( V = L[A] \) and then generically add sets \( G_\alpha \subseteq \alpha^+, \alpha \) \( O \) or an infinite cardinal \( (O^+ \) denotes \( \omega \)) so that \( G_\alpha \) codes both \( G_\alpha^+ \) and \( A \cap \alpha^+ \). Also for limit cardinals \( \alpha, G_\alpha \) is coded by \( \langle G_\alpha[\vec{a} < \alpha] \rangle \). Thus there are two “building blocks” for the forcing, the successor coding and the limit coding. We modify the successor coding so as to eliminate Jensen’s use of “generic codes” (this improves an earlier modification of this type, due to Welch and Donder). And we thin out the limit coding so as to eliminate the technical problems causing Jensen’s split into cases according to whether or not \( O^# \) exists.

**Theorem.** (Jensen) There is a class forcing \( P \) such that if \( G \) is \( P \)-generic over \( V \) then \( V[G] \models ZFC + V = L[R], R \subseteq \omega. \) If \( V \models GCH \) then \( P \) preserves cardinals.

It is not difficult to class-generically extend \( V \) to make \( GCH \) true. And any “reshaped” subset of \( \omega_1 \) can be coded by a real via a CCC forcing. (See Section One below for a definition of “reshaped”.) So it suffices to prove that \( V \) can be coded by a “reshaped” subset of \( \omega_1 \), preserving cardinals, assuming the \( GCH \). As a first step, force \( A \subseteq ORD \) such that for each infinite cardinal \( \alpha, L_\alpha[A] = H_\alpha = \) all sets of hereditary cardinality less than \( \alpha \).

**Section One** The Successor Coding \( R^s \).

Fix an infinite cardinal \( \alpha, S_\alpha \) is defined to be a certain collection of “strings” \( s : [\alpha, |s|] \rightarrow 2, \alpha \leq |s| < \alpha^+ \). For \( s \) to belong to \( S_\alpha \) we require that \( s \) is

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"reshaped". This means that for \( \eta \leq |s| \), \( L[A \cap \alpha, s \upharpoonright \eta] \models \text{card}(\eta) \leq \alpha \). The reshaping of \( s \) allows us to code \( s \) by a subset of \( \alpha \), in the manner which we now describe.

For \( s \in S_\alpha \) define structures \( A^0_s = L_{\mu_s^0}[A \cap \alpha, s^*] \), \( A_s = L_{\mu_s}[A \cap \alpha, s^*] \) as follows (where \( s^* = \{ \mu_{s\eta} | s(\eta) = 1 \} \) : If \( |s| = \alpha \) then \( \mu_s^0 = \alpha \). For \( |s| > \alpha \), 
\[
\mu_s^0 = \bigcup \{ \mu_{s\eta} | \eta < |s| \}
\]
and in general \( \mu_s = \text{least p.r. closed ordinal } \mu \text{ greater than } \mu_s^0 \) such that \( L_\mu[A \cap \alpha, s^*] \models \text{card}(s) \leq \alpha \). These ordinals are well-defined due to the reshaping of \( s \).

For \( s \in S_\alpha \) we write \( \alpha(s) = \alpha \). Note that if \( |s| = \alpha(s) \) then \( s = \emptyset \); in this case we think of \( s \) as "labelled" with the ordinal \( \alpha(s) \), so that there are distinct \( s_\alpha \in S_\alpha, \alpha(s_\alpha) = \alpha \).

For later use we also define structures \( \widehat{A}_s \) and \( A'_s \) for \( s \in S_\alpha \) as follows: let \( \mu_s = \text{largest p.r. closed } \mu \text{ such that } \mu = \mu_s^0 \text{ or } L_\mu[A \cap \alpha, s^*] \models |s| \text{ is a cardinal greater than } \alpha \). Then \( \widehat{A}_s = L_{\mu_s}[A \cap \alpha, s^*] \). The ordinal \( \mu'_s \) and structure \( A'_s \) are defined in the same way, except we replace p.r. closure of \( \mu \) by the weaker condition \( \omega \cdot \mu = \mu \).

For \( s \in S_{\alpha^+} \) write \( \bar{s} < s \) to mean that \( \pi(\bar{s}) = s \) where \( \pi : \mathcal{A} \rightarrow A_\alpha \) is an elementary embedding with some critical point \( \alpha(\bar{s}) < \alpha^+ \) and where \( \pi(\alpha(\bar{s})) = \alpha^+ \). Then \( \pi = \pi_{\bar{s}s} \) is unique. Let \( \bar{s} \leq s \) denote \( \bar{s} < s \) or \( \bar{s} = s \). We have the following facts:

(a) \( \{ \alpha(\bar{s}) | \bar{s} < s \} \) is \( CUB \) in \( \alpha^+ \).

(b) If \( \bar{t} \) is a proper initial segment of \( \bar{s} \) then \( \bar{t} < \pi_{\bar{s}s}(\bar{t}) = t \) and \( \pi_{\bar{t}} = \pi_{\bar{s}s} \upharpoonright A_\tau \).

(c) \( A_s = \bigcup \{ \text{Rng}(\pi_{\bar{s}s}) | \bar{s} < s \} \).

Now for \( s \in S_{\alpha^+} \) let \( b_s = \{ \bar{s} | \bar{s} < s \} \). We use the strings \( \bar{s} \upharpoonright i \) with \( \bar{s} < s \upharpoonright \eta \), \( i = 0 \text{ or } 1 \), to code \( s(\eta) \). A condition in the successor coding \( R^s \) is a pair \((u, \bar{u})\) where:

1) \( u \in S_\alpha \)

2) \( \bar{u} \subseteq \{ b_s \eta | s(\eta) = 0 \}, \text{ card } (\bar{u}) \leq \alpha \text{ in } A_s \).

To define extension of conditions, we need a couple of preliminary definitions. We say that \( u \) restricts \( \bar{s} \) to \( 1 \) if \( \bar{s} \in b \) for some \( b \in \bar{u} \) and \( \bar{s} \) lies on \( u \) if \( u(\alpha(\bar{s})) = 1 \) and \( u(\langle \alpha(\bar{s}), \eta \rangle) = \bar{s}(\eta) \) for \( \eta \in \text{Dom}(\bar{s}) \). Also let \( \langle Z_\gamma \rangle < \alpha^+ \) be an \( L_{\alpha^+} \)-definable partition of the odd ordinals less than \( \alpha^+ \) into \( \alpha^+ \) disjoint pieces of size \( \alpha^+ \). We use the \( Z_\gamma \)'s to code \( A \cap \alpha^+ \) into \( G_\alpha \). For \( u \in S_\alpha \), \( u^\text{even} (\delta) = u(2\delta), u^\text{odd} (\delta) = u(2\delta + 1) \).

Extension of conditions for \( R^s \) is defined by: \((u_0, \bar{u}_0) \leq (u_1, \bar{u}_1) \) iff \( u_0 \) extends
\( u_1; \bar{u}_0 \supseteq \bar{u}_1; \bar{u}_1 \) restraints \( \bar{s} \ast 1 \), \( \bar{s} \ast 1 \) lies on \( u_0^{\text{even}} \rightarrow \bar{s} \ast 1 \) lies on \( u_1^{\text{even}} \); \( \gamma < |u_1|; \gamma \notin A; \delta \in Z_{\gamma}; u_0^{\text{odd}}(\delta) = 1 \rightarrow u_1^{\text{odd}}(\delta) = 1 \). Note that \( R^s \in \mathcal{A}_s \).

**Lemma 1.1.** Suppose \( G \) is \( R^s \)-generic over \( \mathcal{A}_s \) and let \( G_\alpha = \bigcup \{|u|(u, \bar{u}) \in G \text{ for some } \bar{u}\} \). Then \( G, A \cap \alpha^+, s \) belong to \( L_{\mu_s}[G_\alpha] \).

**Proof.** We can write \( (u, \bar{u}) \in G \) iff \( u \subseteq G_\alpha \) and \( s \in \bar{b} \in \bar{u}, \bar{s} \ast 1 \) lies on \( G_\alpha^{\text{even}} \rightarrow \bar{s} \ast 1 \) lies on \( u^{\text{even}} \) and \( \gamma < |u|; \gamma \notin A; \delta \in Z_{\gamma}; G_\alpha^{\text{odd}}(\delta) = 1 \rightarrow u^{\text{odd}}(\delta) = 1 \). So \( G \in L_{\mu_s}[A \cap \alpha^+, G_\alpha, s] \). And \( \gamma \in A \cap \alpha^+ \) iff \( G_\alpha^{\text{odd}}(\delta) = 1 \) for unboundedly many \( \delta \in Z_{\gamma} \), so \( G, A \cap \alpha^+ \in L_{\mu_s}[G_\alpha, s] \). Finally note that for any \( \eta < |s|, \bar{s} \) lies on \( G_\alpha^{\text{even}} \) for unboundedly many \( \bar{s} < s \upharpoonright \eta \) by a density argument using the fact that for \( \eta < |s|, (u, \bar{u}) \in R^s, b_{s \eta} \) is almost disjoint from \( \{|u|u \text{ extends some } \bar{s} \ast 1 \text{ retrained by } \bar{u}\} \). So \( s(\eta) = 1 \) iff \( \bar{s} \ast 1 \) lies on \( G_\alpha^{\text{even}} \) for unboundedly many \( \bar{s} < s \upharpoonright \eta \). Thus \( s \upharpoonright \eta \) can be recovered by induction on \( \eta \leq |s|, \) inside \( L_{\mu_s}[G_\alpha] \).

**Lemma 1.2.** \( R^{<s} = \bigcup\{R^t : t \subseteq s, t \neq s\} \) has the \( \alpha^+\text{-}CC \) in \( \mathcal{A}_s \).

**Proof.** If \( \mu_s = \mu_s^0 \) then this is vacuous. Otherwise we need only observe that \( R^{<s} \in \mathcal{A}_s \) and \( (u_0, \bar{u}_0), (u_1, \bar{u}_1) \) incompatible \( \rightarrow u_0 \neq u_1 \) and \( S_\alpha \) has cardinality \( \alpha^+ \) in \( \mathcal{A}_s \).

**Lemma 1.3.** \( R^s \) is \( \leq \alpha \)-distributive in \( \mathcal{A}_s \).

**Proof.** Suppose \( (u_0, \bar{u}_0) \in R^s \) and \( \langle D_i : i < \alpha \rangle \) are predense on \( R^s \), \( \langle D_i : i < \alpha \rangle \in \mathcal{A}_s \).

By induction we define conditions \( (u_i, \bar{u}_i) \) and elementary submodels \( M_i \) of \( \mathcal{A}_s \) with \( (u_i, \bar{u}_i) \in M_{i+1}, \) for \( i \leq \alpha \). Choose \( M_0 \) to contain \( \alpha \) as a subset and to contain \( \langle D_i : i < \alpha \rangle, s, A \cap \alpha^+ \) as elements. Having defined \( (u_i, \bar{u}_i) \) and \( M_i \), choose \( M_{i+1} \) to contain \( M_i \) as a subset and \( (u_i, \bar{u}_i) \) as an element. Choose \( (u_{i+1}, \bar{u}_{i+1}) \) to extend \( (u_i, \bar{u}_i) \), meet \( D_i \), guarantee that if \( s(\eta) = 1, \eta \in M_i \) then \( \bar{s} \ast 1 \) lies on \( u_{i+1}^{\text{even}} - u_{i+1}^{\text{even}} \) for some \( \bar{s} < s \upharpoonright \eta \), guarantee that if \( \gamma \in A \cap (M_i \cap \alpha^+) \) then \( u_{i+1}^{\text{odd}}(\delta) = 1 \) for some \( \delta \notin \text{dom} u_{i+1}^{\text{odd}}, \delta \in Z_{\gamma} \), and finally choose \( \bar{u}_{i+1} \) to contain all \( b_{s \eta} \) with \( \eta = 0, \eta \in M_i \). The last requirement can be imposed because the facts that \( |s| \) has cardinality \( \leq \alpha^+ \) in \( \mathcal{A}_s \), \( H_\alpha^+ \subseteq \mathcal{A}_s \) imply that any subset of \( |s| \) of cardinality \( \leq \alpha \) belongs to \( \mathcal{A}_s \).

For \( \lambda \leq \alpha \), limit, \( M_\lambda = \bigcup\{M_i : i < \lambda \} \) and \( u_\lambda = \bigcup\{u_i : i < \lambda \}, \bar{u}_\lambda = \bigcup\{\bar{u}_i : i < \lambda \} \).

By construction, \( u_\lambda \) codes \( A \cap (M_\lambda \cap \alpha^+) \) as well as \( \bar{s} = s \circ \pi^{-1} \) where \( \pi \) is the transitive collapse map for \( M_\lambda \). Thus the sequence of ordinals \( \langle M_i \cap \alpha^+ : i < \lambda \rangle \) is
cofinal in $|u_\lambda|$ and belongs to $L[u_\lambda]$, since the entire sequence $\langle M_i \mid i < \lambda \rangle$ can be recovered in $L[u_\lambda]$. $\mathcal{M}_i$ = transitive collapse ($M_i$). This shows that $u_\lambda$ is reshaped, so $(u_\lambda, \bar{u}_\lambda)$ is a condition. Finally note that $(u_\alpha, \bar{u}_\alpha)$ is an extension of $(u_0, \bar{u}_0)$ meeting each of the $D_i$’s.

**Corollary 1.4.** $R^{<\delta}$ is $\leq \alpha$-distributive in $\hat{A}_s$.

**Proof.** By Lemma 1.2 it suffices to prove $\leq \alpha$-distributivity in $\mathcal{A}_s^0$. This is easily proved by induction on $|s|$, using Lemma 1.3 at successor stages.

**Lemma 1.5.** If $D \subseteq R^{<\delta}, \forall D \in \hat{A}_s$ is predense and $s \subseteq t \in S_{\alpha+}$ then $D$ is predense on $R^t$.

**Proof.** It suffices to show that if $D \subseteq R^t, D \in \mathcal{A}_s$ is predense, $s \subseteq t \in S_{\alpha+}$ then $D$ is predense on $R^t$; for then, as in the proof of Corollary 1.4, we can induct on $|s|$ and use Lemma 1.2.

Suppose $D$ is predense on $R^s, D \in \mathcal{A}_s$ and $(u, \bar{u})$ belongs to $R^t$. We can extend $(u, \bar{u})$ to guarantee that for some $\bar{t} < t, \bar{u} = \{ b_{\eta(t(\eta))} \mid t(\eta) = 0, \eta \in \text{Rng } \pi_{\bar{t}} \}$. $D, s \in \text{Rng } (\pi_{\bar{t}})$ and $|u| = \alpha(\bar{t}) + 1, u(\alpha(\bar{t})) = 0, u \in \mathcal{A}_{\bar{t}} \subseteq \mathcal{A}_s$. Let $(u^*, \bar{u}^*)$ be the least extension of $(u, \bar{u} \cap \mathcal{A}_s) \subseteq R^s$ meeting $D$. We claim that $(u^*, \bar{u}^* \cap \bar{u})$ is an extension of $(u, \bar{u})$, and this will prove the lemma. Clearly $\gamma < |u|, \delta \notin A, \delta \in Z_{\gamma}, u^{*\text{odd}}(\delta) = 1 \rightarrow u^{\text{odd}}(\delta) = 1$, since $(u^*, \bar{u}^*)$ extends $(u, \bar{u} \cap \mathcal{A}_s)$. Suppose $r < t \mid \eta, t(\eta) = 0$ where $\eta \in \text{Rng } \pi_{\bar{t}}$ and $r \ast 1$ lies on $u^{\text{even}}$. If $\eta < |s|$ then $r \ast 1$ lies on $u^{\text{even}}$. As desired, since $(u^*, \bar{u}^*)$ extends $(u, \bar{u} \cap \mathcal{A}_s)$. If $\alpha(r) < \alpha(\bar{t})$ then $|r| < \alpha(\bar{t})$ so again $r \ast 1$ lies on $u^{\text{even}}$ since $|u| > \alpha(\bar{t}) > |r \ast 1|$. If $\alpha(r) = \alpha(\bar{t})$ then $r \ast 1$ cannot lie on $u^{\text{even}}$, by choice of $u$. Finally if $\alpha(r) > \alpha(\bar{t})$ then since $\eta \geq |s|$ we have $\alpha(r) > |u^*|$ by leastness of $(u^*, \bar{u}^*)$. So $r \ast 1$ cannot lie on $u^{\text{even}}$.

**Section Two  Limit Coding.**

We begin with a rough indication of the forcing $\mathcal{P}^u$ for coding $u \in S_{\alpha}, \alpha$ an uncountable limit cardinal, into a subset of $\alpha$. $\mathcal{P}^u \subseteq \mathcal{A}_u$ consists of $\mathcal{P}^{<\alpha} = \bigcup \{ \mathcal{P}^u \mid \xi < |u| \}$ together with certain $p : \text{Card} \cap \alpha \rightarrow V$ such that $p(\beta) = (p_{\beta}, \bar{p}_{\beta}) \in R^{p_{\beta}}$ for $\beta \in \text{dom}(p)$. (We use Card to denote the class of infinite cardinals.) Also for uncountable limit cardinals $\beta < \alpha$ we (inductively) require that $p \upharpoonright \beta \in \mathcal{P}^{p_{\beta}} \subseteq \mathcal{P}^{<p_{\beta}}$. We also insist that $p$ code $u$ in the following sense: For $\xi < |u|$ and $\beta \in \text{Card} \cap \alpha$
define $M^\xi_\beta = \Sigma_1$ Skolem hull of $\beta \cup \{ u \uparrow \xi, A \cap \alpha \}$ in $A_{u|\xi}$ and $b^\xi_\beta = M^\xi_\beta \cap \beta^+$. Then code $u$ by: $u(\xi) = 1$ iff $p_{b^\xi_\beta}^{_{\beta^+}}(b^\xi_\beta) = 1$ for sufficiently large $\beta \in \text{Card} \cap \alpha$. Recall that the successor coding $R^\beta_{\beta^+}$ makes use of odd ordinals (in the $Z_\gamma$'s) so the successor and limit codings do not conflict. For $p, q \in \mathcal{P}^u$ we write $p \leq q$ iff $p(\beta) \leq q(\beta)$ in $R^\beta_{\beta^+}$ for each $\beta \in \text{Card} \cap \alpha$.

To facilitate the proofs of extendibility and distributivity for $\mathcal{P}^u$ we thin out the forcing, in a number of ways. For this purpose we need appropriate forms of $\square$ and $\diamondsuit$ in a relativized form. Jensen observed that his proofs of these principles for $L$ go through when relativized to reshaped strings. Precisely:

**Relativized $\Box$** Let $S = \bigcup\{ S_\alpha | \alpha \text{ an infinite cardinal} \}$. There exists $\langle C_s | s \in S \rangle$ such that $C_s \in A_s$ and:

(a) If $|\alpha(s)| < |s|$ then $C_s$ is closed, unbounded in $\mu^0_s$, ordertype $(C_s) \leq \alpha(s)$.

(b) $\nu \in \text{Lim}(C_s) \rightarrow$ for some $\eta < |s|$, $\nu = \mu^0_{s|\eta}$ and $C_s \cap \nu = C_{s|\eta}$.

(c) Let $\pi : \langle A, \mathcal{C} \rangle \cong \langle A^0_s, C_s \rangle$ and write $\text{crit}((\pi)) = \bar{\alpha}$, $\mathcal{A} = L[\mathcal{A}, \bar{s}^*]$. If $\pi(\bar{\alpha}) = \alpha(s)$ then $L[\mathcal{A}, \bar{s}^*] = |s|$ is not a cardinal $> \bar{\alpha}$ and

\[ (c_1) \quad \mathcal{C} \in L[\mathcal{A}, \bar{s}^*] \text{ where } \mu \text{ is the least p.r. closed ordinal greater than } \bar{\mu} \text{ s.t. } L[\mathcal{A}, \bar{s}^*] \models \text{card}(|s|) \leq \bar{\alpha}. \]

\[ (c_2) \quad \pi \text{ extends to } \pi' : \langle A', \mathcal{A}' \rangle \cong \langle A^0_s, \mathcal{C} \rangle \text{ where } A' = L[\mathcal{A}, \bar{s}^*], \mu' = \text{largest ordinal either equal to } \bar{\mu} \text{ or s.t. } \omega 

\[ \cdot \mu' = \mu' \text{ and } L[\mathcal{A}, \bar{s}^*] = |s| \text{ is a cardinal greater than } \bar{\alpha}. \]

\[ (c_3) \quad \text{If } \bar{\alpha} \text{ is a cardinal and } \pi(\bar{\alpha}) = \alpha \text{ then } \mathcal{A} = A^0_s \text{ and } \mathcal{C} = C_s. \]

**Relativized $\diamondsuit$** Let $E = \{ s \in S \mid |s| \text{ limit and ordertype } (C_s) = \omega \}$. There exists $\langle D_s | s \in E \rangle$ such that $D_s \subseteq A^0_s$ and:

(a) $D \in A_s^0 \neq A^0_s$. $D \subseteq A^0_s \rightarrow \{ \xi < |s| | s \uparrow \xi \in E, D_{s|\xi} = D \cap A^0_{s|\xi} \}$ is stationary in $A_s$.

(b) $D_s$ is uniformly $\Sigma_1$-definable as an element of $A'_s$.

(c) If $A'_s \models \alpha^{++}$ exists then $D_s = \emptyset$.

Now we use these combinatorial structures to impose some further restrictions on membership in $\mathcal{P}^u - \mathcal{P}^{<u}$. First some definitions. For $p \in \mathcal{P}^u$ and $\beta \in \text{Card} \cap \alpha$, $(p)_{\beta}$ denotes $p \uparrow \text{Card} \cap \beta, \alpha$, $D \subseteq \mathcal{P}^{<u}$ is predense if every $p \in \mathcal{P}^{<u}$ is compatible with an element of $D$ and for $\beta \in \text{Card} \cap \alpha, D$ is $\beta$-predense if every condition

5
\( p \in \mathcal{P}^{<u} \) can be extended to some \( q \) such that \( p \upharpoonright \beta = q \upharpoonright \beta \) and \( q \) meets \( D \) (i.e., \( q \) extends an element of \( D \)). And \( p \) reduces \( D \) below \( \beta \) if every \( q \leq p \) can be further extended to \( r \) such that \( r \) meets \( D \) and \( (q)_\beta = (r)_\beta \).

**Requirement A.** (Predensity Reduction) Suppose \( p \in \mathcal{P}^u - \mathcal{P}^{<u} \).

(A1) If \( u \in E \) and \( D_u \subseteq \mathcal{P}^{<u} \) is \( \beta \)-predense for all \( \beta \in \text{Card} \cap \alpha \) then \( p \) meets \( D_u \).

(A2) If \( |u| \) is a successor ordinal, \( D \subseteq \mathcal{P}^{<u} \) is predense and \( D \in \mathcal{A}_u^0 \) then \( p \) reduces \( D \) below some \( \beta < \alpha \).

**Requirement B.** (Restriction) For \( p \in \mathcal{P}^u \) let \( |p| \) denote the least \( \xi \) s.t. \( p \in \mathcal{P}^u \upharpoonright \xi \). If \( p \) belongs to \( \mathcal{P}^u \) and \( \xi < |p| \) then there exists \( r \) s.t. \( p \leq r \) and \( |r| = \xi \).

**Requirement C.** (Nonstationary Restraint) Suppose \( \mathcal{A}_u \models \alpha \) inaccessible and \( p \in \mathcal{P}^u \). Then there exists a CUB \( C \subseteq \alpha \) s.t. \( C \in \mathcal{A}_u \) and \( \beta \in C \rightarrow \mathfrak{P}_\beta = \emptyset \).

The remaining Requirement D will be introduced at a later point when we discuss strong extendibility at successor stages.

Extendibility and distributivity for \( \mathcal{P}^u \) are stated as follows. Let \( q \leq \beta \) \( p \) signify that \( q \leq p \) and \( q \upharpoonright \beta = p \upharpoonright \beta \). \( (\mathcal{P}^{<u})_\beta \) denotes \( \{(p)_\beta | p \in \mathcal{P}^{<u} \} \), for \( \beta \in \text{Card} \cap \alpha \). \( \Delta \)-distributivity for \( \mathcal{P}^{<u} \) asserts that if \( D_\beta \) is \( \beta^{+} \)-predense on \( \mathcal{P}^{<u} \) for each \( \beta \in \text{Card} \cap \alpha \) then every \( p \in \mathcal{P}^{<u} \) can be extended to meet each \( D_\beta \).

\((*)_u p \in \mathcal{P}^u, \beta \in \text{Card} \cap \alpha \rightarrow \exists q \leq \beta p (q \in \mathcal{P}^u - \mathcal{P}^{<u}) \)

\((**)_u (\mathcal{P}^{<u})_\beta \) is \( \leq \beta \)-distributive in \( \hat{\mathcal{A}}_u \) for \( \beta \in \text{Card} \cap \alpha \).

And if \( \alpha \) is inaccessible in \( \mathcal{A}_u^0 \) then \( \mathcal{P}^{<u} \) is \( \Delta \)-distributive in \( \hat{\mathcal{A}}_u \).

These are proved by a simultaneous induction on \(|u|\). As the base case \(|u| = \alpha \) is vacuous we assume from now on that \(|u| > \alpha \). The following consequences of predensity reduction are needed in the proof.

**Lemma 2.1.** (Chain Condition for \( \mathcal{P}^{<u} \)) Suppose \((**)_u \) holds. Then \( \mathcal{P}^{<u} \) has the \( \alpha^{+} \)-CC in \( \hat{\mathcal{A}}_u \).

**Proof.** We may assume that \( \hat{\mathcal{A}}_u \neq \mathcal{A}_u^0 \). Suppose \( D \subseteq \mathcal{P}^{<u} \) is predense and \( D \in \hat{\mathcal{A}}_u \). Consider \( D^{*} = \{ p \in \mathcal{P}^{<u} | p \) reduces \( D \) below some \( \beta \in \text{Card} \cap \alpha \} \). Then \( D^{*} \in \hat{\mathcal{A}}_u \).
By (**)$_u$ and Lemma 1.2, $D^*$ is $\beta$-predense for all $\beta \in \text{Card} \cap \alpha$. (Use $\leq \beta^+$-distribution of $(P^{<\alpha})_{\beta^+}$ and $\beta^++$-CC of $R^{G_{\beta^+}} \subseteq \beta^+$ denoting the $(P^{<\alpha})_{\beta^+}$-generic, to reduce $D$ below $\beta^+$.) Apply relativized $\Diamond$ to obtain $\xi < |u|$ such that $u \upharpoonright \xi \in E$, $D_u|_{\xi^+} = D^* \cap A^0_{u|_{\xi^+}}$ and $D_u|_{\xi^+}$ is $\beta$-predense for all $\beta \in \text{Card} \cap \alpha$. Thus by predensity reduction and restriction, $D^* \cap A^0_{u|_{\xi^+}}$ is predense on $P^{<\alpha}$ and therefore so is $D \cap A^0_{u|_{\xi^+}}$; a subset of $D$ of $\hat{A}_u$-cardinality $\leq \alpha$.

**Lemma 2.2.** (Persistence for $P^{<\alpha}$) Suppose (**)$_u$ holds, $D \subseteq P^{<\alpha}$ is predense, $D \cap \hat{A}_u$ and $u \subseteq v \in S$. Then $D$ is predense on $P^v$.

**Proof.** By restriction, if $p \in P^v - P^u$ then $p$ extends some $q$ in $P^u - P^{<\alpha}$. By the chain condition for $P^{<\alpha}$ we can assume that $D \in A^0_u$ and hence by induction we can assume that $|u|$ is a successor ordinal. But then by predensity reduction, $q$ reduces $D$ below some $\beta \in \text{Card} \cap \alpha$ and hence so does $p$. In particular $p$ is compatible with an element of $D$.

We can now turn to the proofs of (**)$_u$, (**)$_u$.

**Lemma 2.3.** Assume (**)$_u$ and $|u|$ a limit ordinal. Then (**)$_u$ holds.

**Proof.** We first claim that if $p \in P^{<\alpha}$ and $(D_\beta|_{\beta_0 \leq \beta < \alpha}) \in A^0_u$, $D_\beta \subseteq P^{<\alpha}$ $\beta^+$-predense for each $\beta$ then there is $q \leq \beta_0$ $p$ meeting each $D_\beta$. We prove this with $\alpha$ replaced by $\beta_1 \in \text{Card} \cap \alpha^+$, by induction on $\beta_1$. The base case $\beta_1 = \beta_0^+$ and the case of $\beta_1$ a successor cardinal follow easily, using (**)$_u$. If $\beta_1$ is singular in $A^0_u$ then we can choose $\gamma_0 < \gamma_1 < \cdots$ approximating $\beta_1$ in length $\lambda < \beta_1$ and consider $(E_\delta|_{\delta < \lambda})$ where $E_\delta$ is all $q$ meeting each $D_\beta$, $\lambda \leq \beta < \gamma_\delta$, $|q|$ least so that $(D_\beta|_{\beta_0 \leq \beta < \beta_1}) \in A^0_{u|_{\lambda}}$. Then we are done by induction. If $\beta_1$ is inaccessible in $A^0_u$ then either $\beta_1 = \alpha$, in which case the result follows directly from the second statement of (**)$_u$, or $\beta_1 < \alpha$, in which case we can factor $P^{<\alpha}$ as $(P^{<\alpha})_{\beta_1^+} \ast P^G_{\beta_1^+}$ (where $G_{\beta_1^+}$ denotes $\bigcup \{ p_{\beta_1^+}| p \in G \}$; $G$ the generic for $P^{<\alpha}$). Then choose $(q)_{\beta_1^+} \leq (p)_{\beta_1^+}$ that reduces each $D_\beta$, $\beta_0 \leq \beta < \beta_1$ below $\beta_1^+$, using (**)$_u$ and the $\beta_1^+$-CC of $P^G_{\beta_1^+}$. By induction on $\alpha$, we can extend $q$ to meet all the $D_\beta$'s.

Now write $C_u = \{ \mu^0_{u|_{\xi^+}}| i < \lambda \}$ and choose a successor cardinal $\beta_0 < \alpha$ to be at least as large as $\lambda$ and the $\beta$ given in the statement of (**)$_u$, if $\lambda < \alpha$. Now inductively define a subsequence $(\eta_j| j < \lambda_0)$ of $(\xi_i|i < \lambda)$ and conditions $(p_j| j < \lambda_0)$ as follows. First suppose $\lambda < \alpha$. Let $p$ denote the condition given in the statement
of \((\ast)\). Set \(p_0 = p, \eta_0 = \xi_i \) s.t. \(p \in \mathcal{P}^{<\alpha}[\xi_i]; p_{j+1} = \xi_i \) s.t. for all \(\gamma, \beta_0 \leq \gamma < \alpha, q \) meets all \(\gamma^+\)-predense \(D \subseteq \mathcal{P}^{<\alpha}[\eta_j]; D \in M^{\eta_j}_\gamma = \Sigma_1 \) Skolem hull of \(\gamma^+ \cup \{p, \alpha\} \) in \(\mathcal{A}_0^{\eta_j}[\xi_i]; C_{\eta_j}[\xi_i] \), \(\eta_j+1 = \xi_i \) s.t. \(p_{j+1} \in \mathcal{P}^{<\alpha}[\xi_i]; p_\delta = \text{g.l.b.} \langle p_j | j < \delta \rangle, \eta_\delta = \bigcup(\eta_j | j < \delta \rangle \) for limit \(\delta \leq \lambda_0 \). The ordinal \(\lambda_0 \) is determined by the condition that \(\eta_\lambda_0 \) is equal to \(\eta \). If \(\lambda = \alpha \) then the definition is the same, except in defining \(p_{j+1} \) require \(p_{j+1} \leq \beta \cup \eta_{j+1} \), \(p_j \) where \(\eta_j = \xi_i \) and only require \(p_{j+1} \) to meet \(\gamma^+\)-predense \(D \) as above for \(\gamma \) between \(\beta \cup \eta_i \) and \(\alpha \).

We must verify that \(p_\delta \) as defined above is indeed a condition for limit \(\delta \). (There is no problem at successor stages, using Lemma 2.2 and the first paragraph of the present proof.) First we show that for \(\gamma \in \text{Card} \cap \alpha, p_\delta, \) is reshaped. We need only consider \(\gamma \geq \beta \) and in case \(\lambda = \alpha \) we need only consider \(\gamma \geq \beta \cup \eta_i \) where \(\eta_\delta = \xi_i \).

By construction if \(\gamma \in M^{p_\eta}_\gamma = \Sigma_1 \) Skolem hull of \(\gamma \cup \{p, \alpha\} \) in \(\mathcal{A}_0^{\eta_j}[\xi_i]; C_{\eta_j}[\xi_i] \) then \(p_\delta \) is \(\pi((\mathcal{P}^{<\eta_\delta}[\xi_i]) \) generic over \(\text{TC}(M^{\eta_j}_\gamma) \) where \(\pi : M^{\eta_j}_\gamma \longrightarrow \text{TC}(M^{\eta_j}_\gamma) \) is the transitive collapse. And \(\{p_\delta, \} \) is \(\Sigma_1\)-definably singularized over \(\text{TC}(M^{\eta_j}_\gamma) \). Write \(\text{TC}(M^{\eta_j}_\gamma) = (\mathcal{A}, \mathcal{C}) \). By genericity and cofinality-preservation for \(\pi((\mathcal{P}^{<\eta_\delta}[\xi_i]) \), \(p_\delta \), codes \(\mathcal{A} \) and by Relativized \(\Box \) (c1), \(\mathcal{C} \) is constructible from \(\mathcal{A} \). So \(p_\delta \) is reshaped. If \(M^{\eta_j}_\gamma \cap \alpha = \gamma \) then \(p_\delta \) is again reshaped because of Relativized \(\Box \) (c1), \(\alpha \) (no genericity argument required). Lastly if \(\gamma' = \min(M^{\eta_j}_\gamma \cap (\text{ORD} - \gamma)) < \alpha \) then use the first argument, but with \(\pi((\mathcal{P}^{<\eta_\delta}[\xi_i]) \gamma \) replacing \(\pi((\mathcal{P}^{<\eta_\delta}[\xi_i]) \gamma \).

Next we show that \(p_\delta \upharpoonright \gamma \in A_{p_\delta, \gamma} \). As \(p_\delta \upharpoonright \gamma \) is definable over \(\text{TC}(M^{p_\eta}_\gamma) \in L[A \cap \gamma; p_\delta, \] this amounts to showing that \(\mu_{p_\delta, \gamma} \) is large enough. By \((\ast)\) and Lemma 2.1 we know that \(\mathcal{P}^{<\eta_\delta}[\xi_i] \) has the \(\alpha^+\)-CC in \(\mathcal{A}_{u}[\eta_j] \) and hence (when \(M^{p_\eta}_\gamma \cap \alpha \neq \gamma \)) \(\eta_\delta \) is in fact \(\pi^{-1}((\mathcal{P}^{<\eta_\delta}[\xi_i]) \gamma \)-generic over \(\mathcal{A}' \), where \(\pi' \) (with domain \(\mathcal{A}' \)) is the extension of \(\pi^{-1} \) given by Relativized \(\Box \) (c2) and \(\gamma' = \min(M^{p_\eta}_\gamma \cap (\text{ORD} - \gamma)) \). And thus \(\mathcal{A}'[p_\delta, \gamma \} \models \{p_\delta, \} \) is a cardinal. But by Relativized \(\Box \) (c1), \(\text{TC}(M^{p_\eta}_\gamma) \) appears relative to \(p_\delta \), before the next p.r. closed ordinal after the height of \(\mathcal{A}' \). So \(p_\delta \upharpoonright \gamma \in A_{p_\delta, \gamma} \). If \(M^{p_\eta}_\gamma \cap \alpha = \gamma \) then no genericity argument is required; we only need Relativized \(\Box \) (c1).

Requirements B, C are easily checked, the latter using the fact that in case of \(\alpha \) inaccessible in \(\mathcal{A}_U \) we required \(p_{j+1} \leq \beta \cup \eta_{j+1} \), \(p_j \) then use diagonal intersection of clubs. To check Requirement (A1) note that if \(M^{p_\eta}_\gamma \cap \alpha \neq \gamma \) then either \(p_\delta \notin E \) or \(D_{p_\delta, \gamma} = \emptyset \), since \(\mathcal{A}'[p_\delta, \gamma \} \models \gamma^+ \) exists and we can apply Relativized \(\Box \) (c1). If \(M^{p_\eta}_\gamma \cap \alpha = \gamma \) then \(p_\delta \in E \) iff \(u \upharpoonright \eta_\delta \in E \) by Relativized \(\Box \)
(c3) and if these hold then by Relativized ⊓ (b), $\pi'[D_{p_\beta}] = D_{u\mid \eta_\delta}$, where $\pi'$ comes from Relativized □ (c2). So all we need to arrange is that our initial condition $p$ be chosen to meet $D_u$, in case $u \in E$, and otherwise choose $\eta_0$ to be at least $\xi_\omega$, so that $u \uparrow \eta_\delta \notin E$ for limit $\delta$.

Lemma 2.4. Assume $|u|$ limit and $(*)_v, (**)_v$ for $v \subseteq u, v \neq u$. Then $(**)_u$ holds.

Proof. We may assume that $\hat{A}_u \neq A_u^0$. We need only make a small change in the construction of the proof of Lemma 2.3. Given predense $D_i | i < \beta$ on $(P^{<u})_{\beta}$ in $\hat{A}_u$ with $\beta < \alpha$, select $\xi < |u|$ of cofinality $>$ $\beta$ such that $D_i \cap (P^{<u}\xi)_\beta$ is predense on $(P^{<u}\xi)_\beta$ for all $i < \beta$ and then choose the continuous sequence $\langle \xi_i | i < \beta \rangle$ from $C_{u\mid \xi}$ by: $\xi_0 = \omega^{th}$ element of $C_{u\mid \xi}$, $\xi_{i+1} = \xi_i + 1$, $\xi \in C_{u\mid \xi}$ greater than $\xi_i$ s.t. $q \in (P^{u\mid \xi}_\beta) \rightarrow \exists r \leq q(r \in (P^{u\mid \xi}_\beta), r \meets D_i)$. $\xi_\lambda = \bigcup \{\xi_i | i < \lambda\}$ for limit $\lambda \leq \beta$. Then $u | \xi_\lambda \notin E$ and $\langle \xi_i | i < \lambda \rangle \in A_{u\mid \xi}$ for limit $\lambda$.

Now repeat the construction of the proof of Lemma 2.3, extending along the $\xi_i$'s instead of along $C_u$, hitting $D_i$ at stage $i + 1$. We can guarantee $\langle D_i \cap (P^{<u}\xi_i)_\beta | i < \lambda \rangle$ is $\Delta_1(A_{u\mid \xi}, C_{u\mid \xi})$ in our choice of $\xi$'s as well, so hitting the $D_i$'s does not interfere with the proof that $p_\beta$ is a condition for limit $\delta$. The proof of $\Delta$-distributivity is similar.

Lemma 2.5. Suppose $(**)_u$ holds and $|u|$ is a successor ordinal. Then $(*)_u$ holds.

Proof. We may assume that the given $p$ belongs to $A_v - A_v^0$ where $v = u \uparrow (|u| - 1)$. Write $C_u = \{\xi_j | j < \omega\}$. Now proceed as in the construction of the proof of Lemma 2.3, making successive $\leq_\beta$-extensions below $p$ (where $\beta$ is given in the statement of $(*)_u$), $p \geq_\beta p_0 \geq_\beta p_1 \geq_\beta \cdots$ so that $p_{j+1}$ meets all $\gamma^+$-predense $D \subseteq P^{<u}$ in $M_{\gamma^+}$, where $M_{\gamma^+} = \Sigma_1$ Skolem hull of $\gamma^+ \cup \{p, \alpha, \xi_0, \cdots, \xi_{j-1}\}$ in $A_v \uparrow \xi_j$, for all $\gamma \in [\beta, \alpha)$.

If we set $\dot{q} = g_{1, b} \langle p_i | i \in \omega \rangle$ then $\dot{q}$ meets the requirements for being a condition at all $\gamma \in \text{Card} \cap \alpha^+$ with the exception of $\gamma$ in $C \cup \{\alpha\}$. $C = \{\gamma | M_\gamma \cap \alpha = \gamma\}$, $M_\gamma = \Sigma_1$ Skolem hull of $\gamma \cup \{p, \alpha\}$ in $\langle A_v, C_u \rangle$. The reason is that for $\gamma \in \alpha - C, T_\gamma = TC(M_\gamma)$ belongs to $A_{q_\gamma}$, since $T_\gamma \equiv |\dot{q}_\gamma|$ is a cardinal and $\dot{q}_\gamma$ is generic over $T_\gamma$.

To make $\dot{q}$ into a condition $q \in P^u$ we must do two things. First extend $\dot{q}_\gamma$ for $\gamma \geq \beta$ so as to code $u(|v|) = 0$ or 1. This is easily done as there are no conflicts between the successor and limit codings. Second for $\gamma \in C$ we extend $\dot{q}_\gamma$ to $q_\gamma = \dot{q}_\gamma \ast u(|v|)$. The only remaining question is whether the restraint $\dot{q}_\gamma$ will allow
us to do this. But $\gamma \in C \rightarrow \overline{\delta}_{\gamma} = \emptyset$ since $C$ is contained in the CUB witnessing Requirement $C$ for $\hat{q}$ at $\alpha$.

Lemma 2.6. Suppose $(\ast)_u$ and $(\ast\ast)_v, v \subseteq u \neq v$ hold and $|u|$ is a successor. Then $(\ast\ast)_u$ holds.

Proof. We must show that if $v = u \upharpoonright (|u| - 1)$ and $p \in (\mathcal{P}^v)_\beta - (\mathcal{P}^{<v})_\beta, \langle D_i | i < \beta \rangle \in \mathcal{A}_v$ are predense on $(\mathcal{P}^v)_\beta$ then there exists $q \leq p$ meeting each $D_i$. For simplicity we assume $\beta = \omega$.

Definition. Suppose $f(\beta) = M_\beta$ is a function in $\mathcal{A}_v$ from $\text{Card}^+ \cap \alpha$ (Card$^+$ denotes all successor cardinals) into $\mathcal{A}_v$ such that $\text{card}(M_\beta) \leq \beta$ for all $\beta \in \text{Dom}(f)$ and suppose $p \in \mathcal{P}^v$. Then $\Sigma^p_f = \{ q \in \mathcal{P}^v | \forall \beta \in \text{Dom}(f), q(\beta) \text{ meets all predense } D \subseteq R^{p\beta}+, D \in M_\beta \}$.

Sublemma 2.7. $\Sigma^p_f$ is dense below $p$ in $\mathcal{P}^v$.

Before proving Sublemma 2.7 we establish the Lemma, assuming it. Choose a limit ordinal $\lambda = \omega^\lambda < \mu_v$ such that $\langle D_i | i < \omega \rangle, C_v \in \mathcal{A}_v \upharpoonright \lambda = L_\lambda[A \cap \alpha, v^*]$ and $\Sigma_1 \text{cof}(\mathcal{A}_v \upharpoonright \lambda) = \omega$. Choose a $\Sigma_1(\mathcal{A}_v \upharpoonright \lambda)$ sequence $\lambda_0 < \lambda_1 < \cdots$ cofinal in $\lambda$ such that $\langle D_i | i < \omega \rangle, C_v, x \in \mathcal{A}_v \upharpoonright \lambda_0$ where $x$ is a parameter defining the $\lambda_i$'s. Set $M_{\gamma} = \text{least } M \prec \Sigma_1 \mathcal{A}_v \upharpoonright \lambda_i$ such that $\gamma \cup \{ x : \langle D_i | i < \omega \rangle, \alpha, C_v \} \subseteq M$, for each $\gamma \in \text{Card}^+ \cap \alpha$. Define $f_i(\gamma) = M_{\gamma}^i$.

Choose $p = p_0 \geq p_1 \geq \cdots$ successively so that $p_{i+1}$ meets $D_i$ and $\Sigma^p_{f_i}$. Set $p^* = \text{g.l.b. } \langle p_i | i \in \omega \rangle$. We show that $p^*$ is a well-defined condition. If $|v| > \alpha$ then thanks to $(\ast\ast)_v$ it will suffice to show that if $D \in M_{\gamma}^i \cap \mathcal{A}_v^0$ is predense on $(\mathcal{P}^{<v})_\gamma$, $\gamma \in \text{Card} \cap \alpha$ then some $p_j$ reduces $D$ below $\gamma$. (For then, $p^*_i$ codes a generic over the transitive collapse of $M_{\gamma}^i \cap \mathcal{A}_v^0$.) If $|v| = \alpha$ then instead of $\mathcal{P}^{<v} = \emptyset$ use $\mathcal{P}^\alpha = \{ p | \beta^+, \beta \in \text{Card} \cap \alpha, p \in \mathcal{P}^v \}$, ordered in the natural way. Note that $\mathcal{P}^\alpha$ is cofinality-preserving, by applying $(\ast\ast)$ at cardinals $< \alpha$.

Choose $j \geq i$ so that for $k > j, p_k$ reduces $D$ no further than $p_j$. Let $\gamma'$ be least so that $p_j$ reduces $D$ below $\gamma'$. Then $\gamma' < \alpha$ by Predensity Reduction for $p$. If $\gamma' \leq \gamma$ then of course we are done. If $\gamma' > \gamma$ is a double successor cardinal then we reach a contradiction since by definition $p_{j+1}$ reduces $D$ further. If $\gamma' = \delta^+$, $\delta$ a limit cardinal then by Predensity Reduction at $\delta$, $D$ is reduced below some $\delta' < \delta$, another contradiction. If $\gamma'$ is a limit cardinal then the same argument applies, replacing $\gamma'$ by $(\gamma')^+$. 

10
Finally we have:

**Proof of Sublemma 2.7.** It suffices to show the following.

**Strong Extendibility** Suppose \( g \in A_v, g(\beta) \in H_{\beta++} \) for all \( \beta \in \text{Card}(\beta_0, \alpha) \) and \( p \in P^v \). Then there is \( q \leq \beta_0 \) such that \( g \upharpoonright \beta \in A_{q\beta} \) for all \( \beta \in \text{Card}(\beta_0, \alpha) \).

For, Strong Extendibility allows us to extend to a condition \( q \) such that for all \( \beta \in \text{Card}(\cap (\beta_0, \alpha), g \upharpoonright \beta \in A_{q \beta} \), where \( g(\beta) = f(\beta) \cap H_{\beta++} \). Then successively extend each \( q(\beta) \) to meet predense \( D \) in \( f(\beta) \).

We now break down Strong Extendibility into the ramified form in which it will be proved. For any \( \mu \) such that \( \mu_0^0 \leq \mu < \mu_v \), \( k \in \omega - \{0\} \) and \( \beta \in \text{Card}(\cap (\beta_0, \alpha) \) let \( M_{\beta}^{\mu,k} = \Sigma_k \) Skolem hull of \( \beta \cup \{\alpha\} \) in \( A_v^* \uparrow \mu = \langle L_\mu[A \cap \alpha, v^*, C_v], A \cap \alpha, v^*, C_v \rangle \). Notice that this structure is \( \Sigma_1 \) projectible to \( \alpha \) without parameter.

\( SE(\mu, k) \) Suppose \( p \in P^v \) and \( \beta_0 \in \text{Card}(\cap (\beta_0, \alpha) \). Then there exists \( q \leq \beta_0 \) such that \( \text{TC}(M_{\beta}^{\mu,k}) \in A_{q \beta} \) for all \( \beta \in \text{Card}(\cap (\beta_0, \alpha) \).

It suffices to prove \( SE(\mu, k) \) for all \( \mu, k \) as above. We do so by induction on \( \mu \) and for fixed \( \mu \), by induction on \( k \). To verify the base case of this induction we must impose one last requirement on our conditions.

**Requirement D** Suppose \( p \in P^v - P^<v \) and \( g \in A_v^0 \), \( g(\beta) \in H_{\beta++} \) for all \( \beta \in \text{Card}(\cap (\beta_0, \alpha) \). Then \( g \upharpoonright \beta \in A_{p \beta} \) for sufficiently large \( \beta \in \text{Card}(\cap (\beta_0, \alpha) \).

This requirement is respected by our earlier constructions. Now, if \( k = 1 \) and \( \mu \) is a limit ordinal then we can use a \( \Sigma_1(A_v^* \uparrow \mu) \) approximation to \( \mu \) and induction (or Requirement D if \( \mu = \mu_0^0 \)) to obtain \( q \leq \mu \) satisfying the conclusion of \( SE(\mu, 1) \), using the \( \Sigma_f \)'s for \( f \in A_v^0 \uparrow \mu \). Similarly if \( \mu \) is a successor, \( k = 1 \) then use \( \langle \Sigma_k(A_v^* \uparrow \mu - 1)^j, k \in \omega \rangle \) to approximate \( \Sigma_1(A_v^* \uparrow \mu) \), using the \( \Sigma_f \)'s, \( f \) definable over \( A_v^* \uparrow \mu - 1 \).

Suppose \( k > 1 \). By induction we can assume that \( \text{TC}(M_{\beta}^{\mu,k-1}) \in A_{p \beta} \) for large enough \( \beta \). If \( C = \{ \beta < \alpha | \beta = \alpha \cap M_{\beta}^{\mu,k} \} \) is unbounded in \( \alpha \) then successively extend \( p \upharpoonright \beta \) for \( \beta \in C \) so that \( \text{TC}(M_{\beta}^{\mu,k}) \in A_{q \beta'} \) for \( \beta' < \beta \). There is no problem at limits since \( \text{TC}(M_{\beta}^{\mu,k}), C \cap \beta \in A_{p \beta} \) for \( \beta \in C \).

If \( \alpha \) is \( \Sigma_k(A_v^* \uparrow \mu) \)-singular then choose a continuous cofinal \( \Sigma_k(A_v^* \uparrow \mu) \) sequence \( \beta_0 < \beta_1 < \cdots \) below \( \alpha \) of ordertype \( \lambda_0 = \text{cof}(\alpha) \). Also choose \( \beta_{i+1} \) large enough so that \( M_{\beta_{i+1}}^{\mu,k-1} \models \beta_i \) is defined. This is possible since \( A_v^* \uparrow \mu = \cup\{M_{\beta_{i+1}}^{\mu,k-1} \mid \beta < \alpha \} \) now define \( N_{\beta}^i \) for \( i < \lambda_0, \beta < \beta_i \) to be the \( \Sigma_k \) Skolem hull of \( \beta \)
in \( \mathcal{M}^{\mu,k-1}_{\beta^i} \). Then \( \langle T(C(N)_{\beta^i}) \beta < \beta_i \rangle \in \mathcal{A}_{p_{\beta^i}} \) for \( i < \lambda_0 \) since it is easily defined from \( \mathcal{M}^{\mu,k-1}_{\beta^i} \in \mathcal{A}_{p_{\beta^i}} \). Successively \( \lambda_0 \)-extend \( p \upharpoonright \beta_i \), producing \( p = p_0 \geq \lambda_0 \ p_1 \geq \lambda_0 \cdots \) where \( T(C(N)_{\beta^i}) \in \mathcal{A}_{p_{\beta^i}} \) for \( \beta \in (\lambda_0, \beta_i) \). This is possible by induction on \( \alpha \), and since \( T(C(N)_{\beta^i}) \) is easily defined from \( \langle T(C(N)_{\beta^i}) \beta < \beta_i \rangle \) for limit \( \beta < \beta_i \). (We must also require that \( p_{i+1} \) meets \( \Sigma^1_{\beta_i} \) where \( f_i(\beta) = N_{\beta^i} \), \( p_{\lambda} \) is well-defined for limit \( \lambda \leq \lambda_0 \) and \( \mathcal{A}_{p_{\lambda_{\beta^i}}} \) contains \( \langle T(C(N)_{\beta^i}) \rangle \) \( \beta < \lambda_0 \) and hence \( T(C(M)_{\beta^i}^{\mu,k}) \) for \( \beta > \lambda_0 \). Then use induction to fill in on \( (0, \lambda_0] \) so that \( SE(\mu, k) \) is satisfied.

Lastly, there is the intermediate case where \( \alpha \) is \( \Sigma^1_{\kappa}(\mathcal{A}_0^s \uparrow \mu) \)-regular but \( C = \{ \beta < \alpha | \beta = \alpha \cap M^{\mu,k}_{\beta^i} \} \) is bounded in \( \alpha \). Then \( \Sigma^1_{\kappa+1}(\mathcal{A}_0^s \uparrow \mu) - \text{cof}(\alpha) = \omega \) and we apply induction to produce \( p = p_0 \geq p_1 \geq \cdots \) so that \( p_{i+1} \upharpoonright [\beta_i, \beta_{i+1}] \) obeys \( SE(\mu, k) \) where \( \beta_0 < \beta_1 < \cdots \) is a cofinal \( \omega \)-sequence of successor cardinals below \( \alpha \). Let \( q = \text{g.l.b.} \{ p_i | i \in \omega \} \).

This completes the proof of Sublemma 2.7 and hence of (***)_{\mu}. \]

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**Section Three  Proof of Jensen's Theorem**

A condition in \( \mathcal{P} \) is a function \( p \) from an initial segment of Card into \( V \) such that \( \text{Dom}(p) \) has a maximum \( \alpha(p) \); for any \( \alpha \in \text{Dom}(p), p(\alpha) = (p_0, p_{\alpha}) \); if \( \alpha \in \text{Dom}(p) \cap \alpha(p) \) then \( p(\alpha) \) belongs to \( R^{p_{\alpha}+} \), \( p(\alpha(p)) = (s(p), \emptyset) \) where \( s(p) \in S_{\alpha(p)} \) and for uncountable limit cardinals \( \alpha \in \text{Dom}(p), p \upharpoonright \alpha \in \mathcal{P}^{p_0} \). And \( q \leq p \) in \( \mathcal{P} \) if \( \alpha(p) \leq \alpha(q), s(p) \subseteq q_{\alpha(p)} \) and for \( \alpha \in \text{Dom}(p) \cap \alpha(p), q(\alpha) \leq p(\alpha) \) in \( R^{s_{\alpha(p)}+} \).

For any \( \alpha \in \text{Card}, s \in S_{\alpha}, \mathcal{P}^s \) denotes all \( p \upharpoonright \alpha \) for \( p \in \mathcal{P} \) such that \( \alpha(p) = \alpha \) and \( s(p) = s \). And \( \mathcal{P}^\alpha \) denotes all \( p \in \mathcal{P} \) such that \( \alpha(p) < \alpha \).

Now suppose \( \alpha \) is an uncountable limit cardinal and \( s \in S_{\alpha}, |s| = \alpha + 1 \). By Lemma 2.2, \( \mathcal{P} \)-generic \( \rightarrow G \cap \mathcal{P}^{<s} \) is \( \mathcal{P}^{<s} \)-generic over \( A_0^{\alpha} = L_\mu[A \cap \alpha], \mu \) the least p.r. closed ordinal greater than \( \alpha \). As the forcing relation for \( \mathcal{P}^{<s} \) restricted to sentences of rank \( < \alpha \) belongs to \( L_\mu[A \cap \alpha] \), it follows that the forcing relation \( p \upharpoonright \varphi, p \in \mathcal{P} \) and \( \varphi \) ranked, is \( \langle L[A], A \rangle \)-definable: \( p \upharpoonright \varphi \) iff for some \( \alpha \) as above, \( \varphi \) has rank \( < \alpha \), \( p \in L_\alpha[A] \) and \( L_\mu[A \cap \alpha] \vdash "p \upharpoonright \varphi" ; \mu \) the least p.r. closed ordinal \( > \alpha \).

Now note that \( \mathcal{P} \) preserves cofinalities, as otherwise \( \mathcal{P}^s \) would change cofinalities for some \( s \) as above, contradicting Distributivity (Lemmas 2.4, 2.6) and Chain Condition (Lemmas 1.2, 2.1). If \( G \) is \( \mathcal{P} \)-generic then \( L[G] = L[X] \) where
$X = G_\omega \subseteq \omega_1$. Finally by Jensen-Solovay [68], $X$ can be coded by a real via a CCC forcing. This completes the proof of Jensen’s Coding Theorem, subject to the verification of Relativized $\square$ and $\Diamond$.

**Section Four  Related Square and Diamond**

For completeness, we prove Relativized $\square$ and $\Diamond$. As relativization causes no serious problems, we first establish unrelativized versions, and then afterward indicate what modifications are required. We begin with $\square$.

First we prove $\square$ in the following form:

**Global $\square$** Assume $V = L$. Then there exists $(C_\mu|\mu$ a singular limit ordinal) such that:

(a) $C_\mu$ is CUB in $\mu$

(b) ordertype $(C_\mu) < \mu$

(c) $\vec{\mu} \in \text{Lim} C_\mu \rightarrow C_\vec{\mu} = C_\cap \vec{\mu}$.

In the proof we shall take advantage of Jensen’s $\Sigma^*$ theory, as reformulated in Friedman [94]. For the convenience of the reader we describe that theory here.

For simplicity of notation, for limit ordinals $\mu$ we let $\vec{J}_\mu$ denote $J_\alpha$ where $\omega \alpha = \mu$. So $\text{ORD}(\vec{J}_\mu) = \mu$.

Let $M$ denote some $J_\alpha, \alpha > 0$. (More generally, our theory applies to “acceptable $J$-models”.*) We make the following definitions, inductively. We order finite sets of ordinals by the maximum difference order: $x < y$ iff $\alpha \in Y$ where $\alpha$ is the largest element of $(y - x) \cup (x - y)$.

1) A $\Sigma^*_1$ formula is just a $\Sigma_1$ formula. A predicate is $\Sigma^*_1$ ($\Sigma^*_1$, respectively) if it is definable by a $\Sigma^*_1$ formula with (without, respectively) parameters. $\rho_1^M = \Sigma^*_1$ projection of $M = \text{least } \rho \text{ s.t. there is a } \Sigma^*_1 \text{ subset of } \omega \rho \text{ not in } M$ and $p_1^M = \text{least } p \text{ s.t. } A \cap \rho_1^M \notin M$ for some $A \Sigma^*_1$ in parameter $p$ (where $p$ is a finite set of ordinals). $H_1^M = H_{\omega_1^M} = \text{sets } x \in M \text{ s.t. } M\text{-card (transitive closure (}x\text{))} < \omega \rho_1^M$.

For any $x \in M$, $M_1(x) = \text{First reduct of } M \text{ relative to } x = \langle H_1^M, A_1(x) \rangle$ where $A_1(x) \subseteq H_1^M$ codes the $\Sigma^*_1$ theory of $M$ with parameters from $H_1^M \cup \{x\}$ in the natural way: $A_1(x) = \{\langle y, n\rangle| \text{ the } n^{th} \Sigma^*_1 \text{ formula is true at } \langle y, x\rangle, y \in H_1^M\}$. A good $\Sigma^*_1$ function is just a $\Sigma_1$ function and for any $X \subseteq M$ the $\Sigma^*_1$ hull $(X)$ is just the $\Sigma_1$ hull of $X$.  


(2) For $n \geq 1$, a $\Sigma^*_n$ formula is one of the form $\varphi(x) \leftrightarrow M_n(x) \models \psi$, where $\psi$ is $\Sigma_1$. A predicate is $\Sigma^*_n$ ($\Sigma^*_n$ respectively) if it is defined by a $\Sigma^*_n$ formula with (without, respectively) parameters. $\rho^M_n = \Sigma^*_n$ projection of $M$ of least $\rho$ such that there is a $\Sigma^*_n$ subset of $\omega \rho$ not in $M$ and $p^M_n = p^M_n \cup \rho$ where $p$ is least such that $A \cap \rho^M_{n+1} \notin M$ for some $A \Sigma^*_n$ in parameter $p^M_n \cup \rho$. $H^M_n = H^M_\omega \rho^M_{n+1} = \text{sets } x \in M \text{ s.t. } M\text{-card (transitive closure (x))} < \omega \rho^M_{n+1}$. For any $x \in M$, $M_{n+1}(x) = \langle n + 1 \rangle$ st reduce of $M$ relative to $x = \langle H^M_{n+1}, A_{n+1}(x) \rangle$ where $A_{n+1}(x) \subseteq H^M_{n+1}$ codes the $\Sigma^*_n$ theory of $M$ with parameters from $H^M_{n+1} \cup \{x\}$ in the natural way: $A_{n+1}(x) = \langle \langle y, m \rangle \rangle$ the $m^\text{th}$ $\Sigma^*_n$ formula is true at $\langle y, x \rangle, y \in H^M_{n+1}$}. A good $\Sigma^*_n$ function $f$ is a function whose graph is $\Sigma^*_n$ with the additional property that for $x \in \text{Dom}(f), f(x) \in \Sigma^*_n$ hull ($H^M_n \cup \{x\}$). The $\Sigma^*_n$ hull ($X$) for $X \subseteq M$ is the closure of $X$ under good $\Sigma^*_n$ functions.

**Facts.** (a) $\varphi, \psi \Sigma^*_n$ formulas $\rightarrow \varphi \lor \psi, \varphi \land \psi$ are $\Sigma^*_n$ formulas
(b) $\varphi \Sigma^*_n$ or $\prod^*_n$ (= negation of $\Sigma^*_n$) $\rightarrow \varphi$ is $\Sigma^*_n$
(c) $Y \subseteq \Sigma^*_n$ hull ($X$) $\rightarrow \Sigma^*_n$ hull ($Y$) $\subseteq \Sigma^*_n$ hull ($X$)
(d) $f$ good $\Sigma^*_n$ function $\rightarrow$ $f$ good $\Sigma^*_n$ function
(e) $\Sigma^*_n$ hull ($X$) $\subseteq \Sigma^*_n$ hull ($X$)
(f) There is a $\Sigma^*_n$ relation $W(e, x)$ s.t. if $S(x)$ is $\Sigma^*_n$ then for some $e \in \omega, S(x) \leftrightarrow W(e, x)$ for all $x$.

(g) The structure $M_n(x) = \langle H^M_n, A_n(x) \rangle$ is amenable.

(h) $H^M_n = J^{A_n}_\omega M$ where $A_n = A_n(0)$.

(i) Suppose $H \subseteq M$ is closed under good $\Sigma^*_n$ functions and $\pi : \overline{M} \rightarrow M$, $\overline{M}$ transitive, $\text{Range (}\pi) = H$ and $p^M_{n-1} \in H$ (if $n > 1$). Then $\pi$ preserves $\Sigma^*_n$ formulas: for $\Sigma^*_n \varphi$ and $x \in \overline{M}, \overline{M} \models \varphi(x) \leftrightarrow M \models \varphi(\pi(x))$. And (for $n > 1$), $\pi(p^M_{n-1}) = p^M_{n-1}$.

**Proof of (i)** Note that $H \cap M_{n-1}(\pi(x))$ is $\Sigma_1$-elementary in $M_{n-1}(\pi(x))$. And $\pi^{-1}[H \cap M_{n-1}(\pi(x))] = \langle J^A_{\omega p}, A(x) \rangle$ for some $\rho, A, A(x)$. But (by induction on $n$) $A = A^M_{n-1} \cap J^A_{\omega p}, A(x) = A_{n-1}(x) \overline{M} \cap J^A_{\omega p}$. And $\rho = p^M_{n-1}$ using our assumption about the parameter $p^M_{n-1}$. And $\pi^{-1}(p^M_{n-1}) = \rho$ must be $p^\overline{M}_{n-1}$ as $\overline{M} = \Sigma^*_n$ hull of $H^\overline{M}_{n-1} \cup \{p^\overline{M}_{n-1}\}$.

**Theorem 4.1.** By induction on $n > 0$:
1) If \( \varphi(x, y) \) is \( \Sigma^*_n \) then \( \exists y \in \Sigma^*_{n-1} \) hull \((H^M_{n-1} \cup \{x\})\varphi(x, y) \) is also \( \Sigma^*_n \).
2) If \( \varphi(x_1 \cdots x_k) \) is \( \Sigma^*_m, m \geq n \) and \( f_1(x), \cdots, f_k(x) \) are good \( \Sigma^*_n \) functions, then \( \varphi(f_1(x) \cdots f_k(x)) \) is \( \Sigma^*_n \).
3) The domain of a good \( \Sigma^*_n \) function is \( \Sigma^*_n \).
4) Good \( \Sigma^*_n \) functions are closed under composition.
5) (\( \Sigma^*_n \) Uniformization) If \( R(x, y) \) is \( \Sigma^*_n \) then there is a good \( \Sigma^*_n \) function \( f(x) \) s.t. \( x \in \text{Dom}(f) \iff \exists y \in \Sigma^*_{n-1} \) hull \((H^M_{n-1} \cup \{x\})R(x, y) \iff R(x, f(x)) \).
6) There is a good \( \Sigma^*_n \) function \( h_n(e, x) \) s.t. for each \( x, \Sigma^*_n \) hull \((\{x\}) = \{h_n(e, x)|e \in \omega\} \).

Proof. The base case \( n = 1 \) is easy (take \( \Sigma^*_0 \) hull \((X) = M \) for all \( X \)). Now we prove it for \( n > 1 \), assuming the result for smaller \( n \).

1) Write \( \exists y \in \Sigma^*_{n-1} \) hull \((H^M_{n-1} \cup \{x\})\varphi(x, y) \) as \( \exists \bar{y} \in H^M_{n-1} \varphi(x, h_{n-1}(e, \langle x, \bar{y} \rangle)) \) using 6) for \( n - 1 \). Since \( h_{n-1} \) is good \( \Sigma^*_n \) we can apply 2) for \( n - 1 \) to conclude that \( \varphi(x, h_{n-1}(e, \langle x, \bar{y} \rangle)) \) is \( \Sigma^*_n \). Since the quantifiers \( \exists e \exists \bar{y} \in H^M_{n-1} \) range over \( H^M_{n-1} \) they preserve \( \Sigma^*_n \)ness.

2) \( \varphi(f_1(x) \cdots f_k(x)) \iff \exists x_1 \cdots x_k \in \Sigma^*_{n-1} \) hull \((H^M_{n-1} \cup \{x\}) [x_i = f_i(x) \text{ for } 1 \leq i \leq k \land \varphi(x_1 \cdots x_k)] \). If \( m = n \) then this is \( \Sigma^*_n \) by 1). If \( m > n \) then reason as follows: the result for \( m = n \) implies that \( A_n(\langle f_1(x) \cdots f_k(x) \rangle) \) is \( \Delta_1 \) over \( M_{n+1}(x) \). Thus \( A_{m-1}(\langle f_1(x) \cdots f_k(x) \rangle) \) is \( \Delta_1 \) over \( M_{m-1}(x) \). So as \( \varphi \) is \( \Sigma^*_m \) we get that \( \varphi(f_1(x) \cdots f_k(x)) \) is also \( \Sigma_1 \) over \( M_{m-1}(x) \), hence \( \Sigma^*_m \).

3) If \( f(x) \) is good \( \Sigma^*_n \) then \( \text{dom}(f) = \{x|\exists y \in \Sigma^*_{n-1} \) hull \((H^M_{n-1} \cup \{x\}) (y = f(x))\} \) is \( \Sigma^*_n \) by 1).

4) If \( f, g \) are good \( \Sigma^*_n \) then the graph of \( f \circ g \) is \( \Sigma^*_n \) by 2). And \( f \circ g(x) \in \Sigma^*_{n-1} \) hull \((H^M_{n-1} \cup \{x\}) \) since the latter hull contains \( g(x) \), \( f \) is good \( \Sigma^*_n \) and Fact c) holds.

5) Using 6) for \( n - 1 \), let \( \overline{R}(x, \bar{y}) \iff R(x, h_{n-1}(\bar{y})) \land \bar{y} \in H^M_{n-1} \). Then \( \overline{R} \) is \( \Sigma^*_n \) by 2) for \( n - 1 \) and using \( \Sigma_1 \) uniformization on \( (n-1) \) s.t. reduces we can define a good \( \Sigma^*_n \) function \( \overline{f} \) s.t. \( \overline{R}(x, \overline{f}(x)) \iff \exists \bar{y} \in H^M_{n-1} \overline{R}(x, \bar{y}) \). Let \( f(x) = h_{n-1}(\overline{f}(x)) \).

Then \( f \) is good \( \Sigma^*_n \) by 4).

6) Let \( W \) be universal \( \Sigma^*_n \) as in Fact f). By 5) there is a good \( \Sigma^*_n \) \( g(e, x) \) s.t. \( \exists y \in \Sigma^*_{n-1} \) hull \((H^M_{n-1} \cup \{x\}) W(e, \langle x, y \rangle) \iff W(e, \langle x, g(e, x) \rangle) \) and \( g(e, x) \) defined \( \rightarrow W(e, \langle x, g(e, x) \rangle) \). Let \( h_n(e, x) = g(e, x) \). If \( y \in \Sigma^*_{n-1} \) hull \((\{x\}) \) then for some \( e, W(e, \langle x, y' \rangle) \iff y' = y \) so \( y = h_n(e, x) \). Clearly \( h_n(e, x) \in \Sigma^*_n \) hull \((\{x\}) \) since
$h_n$ is good $\Sigma^*_n$. 

Now we are ready to prove Global $\Box$. Assume $V = L$ and let $\mu$ be a singular limit ordinal. Our goal is to define $C_\mu$, a CUB subset of $\mu$. Let $\beta(\mu) \geq \mu$ be the least limit ordinal $\beta$ such that $\mu$ is not regular with respect to $\tilde{J}_\beta$-definable functions, and let $n(\mu)$ be least so that there is a good $\Sigma^*_n(\tilde{J}_\beta(\mu))$ partial function from an ordinal less than $\mu$ cofinally into $\mu$. Note that $\rho^\beta_n(\mu) \leq \mu$ as otherwise such a partial function would belong to $\tilde{J}_\beta(\mu)$, contradicting the leastness of $\beta(\mu)$. Also $\mu \leq \rho^\beta_{n(\mu)-1}$; else we have contradicted the leastness of $n(\mu)$.

For $X \subseteq \tilde{J}_\beta(\mu)$ let $H(X) = \Sigma^*_n(\mu)$ hall of $X$ in $\tilde{J}_\beta(\mu)$. For some least parameter $q(\mu) \in \tilde{J}_\beta(\mu)$; $H(\mu \cup \{q(\mu)\}) = \tilde{J}_\beta(\mu)$, (“Least” refers to the canonical well-ordering of $L$.) Also let $\alpha(\mu) = \bigcup \{\alpha < \mu | \alpha = H(\alpha \cup \{q(\mu)\}) \cap \mu\}$. Then (unless $\alpha(\mu) = \bigcup \emptyset = 0$) $\alpha(\mu) = H(\alpha(\mu) \cup \{q(\mu)\}) \cap \mu$ and $\alpha(\mu) < \mu$. To see the latter note that for large enough $\alpha < \mu$, $H(\alpha \cup \{q(\mu)\})$ contains both the domain and defining parameter for a good $\Sigma^*_n(\mu)$ partial function from an ordinal less than $\mu$ cofinally into $\mu$.

If $\mu < \beta(\mu)$ let $p(\mu) = \langle q(\mu), \mu, \alpha(\mu) \rangle$ and if $\mu = \beta(\mu)$ let $p(\mu) = \alpha(\mu)$.

We are ready to define $C_\mu$. Let $C^0_\mu = \{\bar{\alpha} < \mu |$ For some $\alpha, \bar{\alpha} = \bigcup (H(\alpha \cup \{p(\mu)\}) \cap \mu)\}$. Then $C^0_\mu$ is a closed subset of $\mu$. If $C^0_\mu$ is unbounded in $\mu$ then set $C_\mu = C^0_\mu$. If $C^0_\mu$ is bounded but nonempty then let $\mu_0 = \bigcup C^0_\mu$ and define $C^1_\mu = \{\bar{\alpha} < \mu |$ For some $\alpha, \bar{\alpha} = \bigcup (H(\alpha \cup \{p(\mu), \mu_0\}) \cap \mu)\}$. If $C^1_\mu$ is unbounded then set $C_\mu = C^1_\mu$. If $C^1_\mu$ is bounded but nonempty then let $\mu_1 = \bigcup C^1_\mu$ and define $C^2_\mu = \{\bar{\alpha} < \mu |$ For some $\alpha, \bar{\alpha} = \bigcup (H(\alpha \cup \{p(\mu), \mu_0, \mu_1\}) \cap \mu)\}$. Continue in this way, defining $C^k_\mu$ for $k \in \omega$ until $C^k_\mu$ is unbounded or empty. Note that $\alpha_0 > \alpha_1 > \cdots$ where $\alpha_k$ is greatest so that $\bigcup (H(\alpha_k \cup \{p(\mu), \mu_0, \cdots, \mu_k-1\}) \cap \mu) = \mu_k$, since $\alpha_k \in H(\alpha_k \cup \{p(\mu), \mu_0, \cdots, \mu_k-1\})$. So for some least $k(\mu) \in \omega$, $C^{k(\mu)}_\mu$ is indeed unbounded or empty. If $C^{k(\mu)}_\mu$ is unbounded then set $C_\mu = C^{k(\mu)}_\mu$.

If $C^{k(\mu)}_\mu = \emptyset$ then we choose $C_\mu$ to be an $\omega$-sequence cofinal in $\mu$, coding approximations to the structure $\tilde{J}_\beta(\mu)$, as follows. (This is necessary to establish Relativized $\Box$ (c).) Note that $H = H(\{p(\mu), \mu_0, \cdots, \mu_{k(\mu)-1}\})$ is cofinal in $\mu$ since $C^{k(\mu)}_\mu = \emptyset$. Assume first that $n(\mu) = 1$, when $C_\mu$ is more easily described. Then $H$ is also cofinal in $\beta(\mu)$, else $H \in \tilde{J}_\beta(\mu)$ and $\mu$ is singular inside $\tilde{J}_\beta(\mu)$. Let $h = h_1(\epsilon, x)$ be the canonical good $\Sigma^*_1$ Skolem function for $\tilde{J}_\beta(\mu)$, so $H = \{h(\epsilon, p) | \epsilon \in \omega\}$ where $p = \{p(\mu), \mu_0, \cdots, \mu_{k(\mu)-1}\}$. Let $\bar{\sigma}_n = \max\{h(\epsilon, p) | \epsilon < n\} \cap \mu$ and $\sigma_n = \max\{h(\epsilon, p) | \epsilon < n\} \cap \mu$. Then $\bar{\sigma}_n = \max\{h(\epsilon, p) | \epsilon < n\} \cap \mu$ and $\sigma_n = \max\{h(\epsilon, p) | \epsilon < n\} \cap \mu$. Then
max\{\{h(e,p)\mid e < n\} \cap \beta(\mu)\}. Then \(C_\mu = \{\delta_0, \delta_1, \cdots\}\) where \(\delta_n\) is an ordinal coding TC(\(\Sigma_1^* \text{ hull } (\sigma_n \cup \{p\})\) restricted to \(\sigma_n\)), where TC denotes “transitive collapse”. By the \(\Sigma_1^*\) hull of \(X\) restricted to \(\sigma_n\) we mean the closure of \(X\) under \(h^{\delta_n}\), obtained by interpreting the \(\Sigma_1^*\) definition of \(h\) in \(\tilde{F}_n\).

Now suppose \(n(\mu) > 1\), \(C_{\mu}^{k(\mu)} = \emptyset\). Then if \(\rho(\mu)\) denotes \(\rho_{n(\mu)-1}\), \(H\) is cofinal in \(\rho(\mu)\), else \(H \in \tilde{F}_{\rho(\mu)}\) and \(\mu\) is singular in \(\tilde{F}_{\rho(\mu)}\). Let \(h\) be the canonical good \(\Sigma_{n(\mu)}^*\) Skolem function for \(\tilde{F}_{\rho(\mu)}\) and let \(p = \{p(\mu), \mu_0, \cdots, \mu_{k(\mu)-1}\}\). Let \(\sigma_n = \max\{\{h(e,p)\mid e < n\} \cap \mu\}, \sigma_n = \max\{\{h(e,p)\mid e < n\} \cap \rho(\mu)\}\). Then \(C_\mu = \{\delta_0, \delta_1, \cdots\}\) where \(\delta_n\) is an ordinal coding TC(\(\Sigma_{n(\mu)}^* \text{ hull } (\sigma_n \cup \{p\})\) restricted to \(\sigma_n\)). The \(\Sigma_{n(\mu)}^*\) hull of \(X\) restricted to \(\sigma_n\) is the closure of \(X\) under \(h^{\sigma_n}\), obtained by replacing the \((n(\mu) - 1)\) st reduction \(M_{n(\mu)-1}(x)\) by \(M_{n(\mu)-1}(x) \upharpoonright \sigma_n\) in the \(\Sigma_{n(\mu)}^*\) definition of \(h\). (Recall that \(M_n(x) = \langle J^{|\omega\times n}_n, A_n(x)\rangle; \) by \(M_n(x) \upharpoonright \sigma\) we mean \(\langle J^{|\omega\times n}_\sigma, A_n(x) \cap J^{|\omega\times n}_\sigma\rangle\).)

Clearly \(C_\mu\) is CUB in \(\mu\), and by the same argument used to justify \(\alpha(\mu) < \mu\), the ordertype of \(C_\mu\) is less than \(\mu\). (These facts are obvious when \(C_\mu^{k(\mu)} = \emptyset\).) So to prove Global \(\Box\) we only need to check coherence: \(\bar{\mu} \in \text{Lim } C_\mu \rightarrow C_{\bar{\mu}} = C_\mu \cap \bar{\mu}\).

**Lemma 4.2.** \(\bar{\mu} \in C_\mu^k \rightarrow C_{\bar{\mu}}^k = C_{\bar{\mu}} \cap \bar{\mu}\).

**Proof.** First suppose that \(k = 0\). Given \(\bar{\mu} \in C_\mu^0\) we can choose \(\alpha < \bar{\mu}\) such that \(\bar{\mu} = \bigcup (H(\alpha \cup \{p(\mu)\}) \cap \mu)\), where \(H\) is the operation of taking the \(\Sigma_{n(\mu)}^*\) hull. Also let \(\rho = \bigcup (H(\alpha \cup \{p(\mu)\}) \cap \rho_{n(\mu)-1})\). Let \(\pi : \tilde{F}_\beta \rightarrow \tilde{F}_{\beta(\mu)}\) be the inverse to the transitive collapse of \(H = \Sigma_{n(\mu)}^* \text{ hull } (\bar{\mu} \cup \{p(\mu)\})\) restricted to \(\rho\). Note that any \(x \in H\) belongs to \(H(\mu', \rho') = \Sigma_{n(\mu)}^* \text{ hull } (\mu' \cup \{p(\mu)\})\) restricted to \(\rho'\), for some \(\mu' < \bar{\mu}, \rho' < \rho\) and \(\rho'\) are chosen to be close to \(\mu\) in \(H\). It follows that \(H \cap \mu = \bar{\mu}\) and therefore when \(\mu < \beta(\mu), \pi(\bar{\mu}) = \mu\). Also note that \(\Sigma_{n(\mu)-1}\) hull \(\rho(\mu) \cup \{p(\mu)\}) \cap \rho_{n(\mu)-1} = \rho\) so \(H\) is closed under good \(\Sigma_{n-1}^*\) functions. It follows that \(\pi : \tilde{F}_\beta \rightarrow \tilde{F}_{\beta(\mu)}\) is \(\Sigma_{n-1}^*\)-regular and \(\bar{\mu}\) is \(\Sigma_{n(\mu)-1}(\tilde{F}_\beta)\)-regular, \(\Sigma_{n(\mu)}^*(\tilde{F}_\beta)\)-singular. So \(\beta = \beta(\bar{\mu}), n(\mu) = n(\bar{\mu})\). Also \(\pi(q(\bar{\mu})) = q(\mu)\). Since \(\alpha(\bar{\mu}) < \alpha\) it must be that \(\alpha(\bar{\mu}) = \alpha(\mu)\). So \(\pi(q(\bar{\mu})) = q(\mu)\). Now it is easy to see that \(C^0_{\bar{\mu}} = C^0_{\mu} \cap \bar{\mu}\).

Now suppose \(k = 1\). The above argument shows that \(\bar{\mu} \in C_{\mu}^1 \rightarrow C_{\bar{\mu}}^0 = C_{\bar{\mu}}^1 \cap \bar{\mu}\) and hence, since \(\mu_0 < \bar{\mu}, \bar{\mu}_0 = \mu_0\). Now again, the above argument shows that \(C_{\mu}^1 = C_{\mu} \cap \bar{\mu}\). The general case \(k \geq 0\) now follows similarly.

Coherence now follows easily: if \(\bar{\mu} \in \text{Lim } C_\mu\) and \(C_{\mu} = C_{\mu}^k\) then by Lemma 4.2, \(C_{\bar{\mu}}^k = C_{\mu}^k \cap \bar{\mu}\) is unbounded in \(\bar{\mu}\) so \(C_{\bar{\mu}} = C_{\bar{\mu}}^k\) and we’re done. If \(C_{\mu}^k = \emptyset\) for some \(k\).
then \( \lim C_{\mu} = \emptyset \) so coherence is vacuous.

To establish the appropriate relativized form of \( \Box \) we need:

**Lemma 4.3.** Suppose \( \pi : \langle \tilde{J}_{\beta} \tilde{C} \rangle \xrightarrow{\Sigma_{\bar{n}}} \langle \tilde{J}_{\mu} ; C_{\mu} \rangle \). Then \( \tilde{C} = C_{\tilde{\mu}} \) and \( \pi \) extends uniquely to a \( \Sigma^{*}_{\bar{n}(\mu)} \)-elementary \( \tilde{\pi} : \tilde{J}_{\beta(\mu)} \rightarrow \tilde{J}_{\beta(\mu)} \) such that \( p(\mu) \in \text{Rng} \tilde{\pi} \).

**Proof.** First suppose that \( C_{\mu} = C_{\mu}^{k} \) for some \( k \). For \( \mu' \in C_{\mu} \) form \( H(\mu') \) as \( H \) was formed in the proof of Lemma 2 for \( \tilde{\mu} \). Then \( \pi(\mu') : \tilde{J}_{\beta(\mu')} \rightarrow \tilde{J}_{\beta(\mu')} \) with range \( H(\mu') \) is \( \Sigma^{*}_{n(\mu)-1} \)-elementary and \( \tilde{J}_{\beta(\mu)} = \bigcup \{ H(\mu') \mid \mu' \in C_{\mu} \} \). And \( \pi(\mu') \restriction \mu' = \text{id} \restriction \mu' \), \( \pi(\mu')(\mu'(\mu')) = p(\mu) \). Now let \( X = \text{Range}(\pi) \) and form \( \tilde{X} = \Sigma^{*}_{n(\mu)} \) hull \( (X \cup \{ p(\mu) \}) \) in \( \tilde{J}_{\beta(\mu)} \). If \( y \in \tilde{X} \) then for some \( \mu' \in C_{\mu} \), \( y = \pi(\mu')(y') \) where \( y' \in \Sigma^{*}_{n(\mu)} \) hull \( \{(X \cap \tilde{J}_{\mu'}) \cup \{ p(\mu') \}) \). In particular if \( y \in \tilde{J}_{\tilde{\mu}} \) then \( y \in \Sigma^{*}_{\bar{n}} \) hull \( X \) in \( \langle \tilde{J}_{\tilde{\mu}}, C_{\tilde{\mu}} \rangle \). So the inverse to the transitive collapse of \( \tilde{X} = \tilde{\pi} \) is a \( \Sigma^{*}_{n(\mu)} \)-elementary embedding extending \( \pi \), with \( p(\mu) \) in its range. If \( \tilde{\pi} : \tilde{J}_{\beta} \rightarrow \tilde{J}_{\beta(\mu)} \) then \( \bar{\mu} = \tilde{\pi}^{-1}(\mu) \) is singular via a \( \Sigma^{*}_{n(\mu)}(\tilde{J}_{\beta}) \) partial function since either \( \Sigma^{*}_{n(\mu)} \) hull of \( \mu' \cup \{ p(\mu) \} \) in \( \tilde{J}_{\beta(\mu)} \) is unbounded in \( \mu \) for some \( \mu' < \bigcup (\text{Rng}(\tilde{\pi}) \cap \mu) \), in which case we can assume \( \mu' \in \text{Rng} \tilde{\pi} \) and by \( \Sigma^{*}_{n(\mu)} \)-elementary of \( \tilde{\pi} \) we’re done, or if not \( \mu^{*} = \mu \cap \Sigma^{*}_{n(\mu)} \) hull \( \{ p(\mu) \} \) in \( \tilde{J}_{\beta(\mu)} \) where \( \mu^{*} = \bigcup (\text{Rng} \tilde{\pi} \cap \mu) \), contradicting the definition of \( \alpha(\mu) \). Since \( \bar{\mu} = \Sigma^{*}_{n(\mu)-1}(\tilde{J}_{\beta}) \)-regular, we get \( \bar{\beta} = \beta(\bar{\mu}) \), \( n(\bar{\mu}) = n(\bar{\mu}) \). Then the \( \Sigma^{*}_{n(\mu)} \)-elementarity of \( \tilde{\pi} \) guarantees that \( \tilde{C} = C_{\tilde{\mu}} \). The uniqueness of \( \tilde{\pi} \) follows from the fact that \( \tilde{J}_{\beta(\mu)} = \Sigma^{*}_{n(\mu)} \) hull \( \{ \bar{\mu} \cup \{ p(\bar{\mu}) \} \} \) and \( \tilde{\pi} \restriction \bar{\mu} \) is determined by \( \pi \).

If \( C_{\mu}^{k} = \emptyset \) for some \( k \) then \( C_{\mu} \) was defined as a special \( \omega \)-sequence cofinal in \( \mu \).

That definition was made precisely to enable the preceding argument to also apply in this case.

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**Relativized \( \Box \).** Let \( S = \bigcup \{ S_{\alpha} \mid \alpha \text{ an infinite cardinal} \} \). There exists \( \langle C_{s} \rangle_{s \in S} \) such that \( C_{s} \in A_{s} \) and:

(a) \( C_{s} \) is closed, unbounded in \( \mu_{s}^{0} \); ordertype \( (C_{s}) \leq \alpha(s) \).

If \( |s| \) is a successor ordinal then ordertype \( (C_{s}) = \omega \).

(b) \( \nu \in \text{Lim}(C_{s}) \rightarrow \) for some \( \eta < |s| \), \( \nu = \mu_{s}^{0}_{\eta} \) and \( C_{s} \cap \nu = C_{s|\eta} \).

(c) Let \( \pi : \langle A_{s}, C_{s} \rangle \xrightarrow{\Sigma_{\bar{n}}} \langle A^{0}_{s}, C_{s} \rangle \) and write \( \text{crit}(\pi) = \alpha \), \( A = L_{\eta}[A, \bar{\pi}] \). If \( \pi(\alpha) = \alpha(s) \) then \( L_{\alpha}[A, \bar{\pi}] \upharpoonright \bar{\pi} \) is not a cardinal > \( \alpha \) and

(c1) \( \bar{C} \in L_{\mu}[A, \bar{\pi}] \) where \( \mu \) is the least p.r. closed ordinal greater than \( \bar{\mu} \). s.t.
$L_\mu[\mathcal{A}, \mathcal{S}] \models \text{card}(|s|) \leq \alpha$.

(c2) $\pi$ extends to $\pi' : \mathcal{A}' \overset{\Sigma_1}{\longrightarrow} \mathcal{A}'_s$ where $\mathcal{A}' = L_\mu'[\mathcal{A}, \mathcal{S}]$, $\mu' =$ largest ordinal either equal to $\bar{\mu}$ or s.t. $\omega \cdot \mu' = \mu'$ and $L_\mu'[\mathcal{A}, \mathcal{S}] \models |s|$ is a cardinal greater than $\bar{\alpha}$.

(c3) If $\bar{\alpha}$ is a cardinal and $\pi(\bar{\alpha}) = \alpha$ then $\bar{\mathcal{A}} = \mathcal{A}'_s$ and $\bar{\mathcal{U}} = \mathcal{C}_s$.

**Relativized $\Diamond$** Let $E = \{ s \in S \text{ such that ordertype } (C_s) = \omega \}$. There exists $\langle D_s \mid s \in E \rangle$ such that $D_s \subseteq \mathcal{A}'_s$ and:

(a) $D \in \mathcal{A}'_s \neq \mathcal{A}'_s$ and $D \subseteq \mathcal{A}'_s$ $\longrightarrow \{ \xi < |s| \mid s \upharpoonright \xi \in E, D_{s\upharpoonright \xi} = D \cap \mathcal{A}'_{s\upharpoonright \xi} \}$ is stationary in $\mathcal{A}$.

(b) $D_s$ is uniformly $\Sigma_1$-definable as an element of $\mathcal{A}'_s$.

(c) If $\mathcal{A}'_s = \alpha^{++}$ exists then $D_s = \emptyset$.

We now make the necessary modifications to obtain Relativized $\Box$. First, if $\mu$ is a singular limit ordinal and $\check{\mu} \models \alpha$ is the largest cardinal then we thin out $C_\mu$ to give it ordertype $\leq \alpha$ : By induction on limit $\bar{\mu} \leq \mu$ define $C_{\bar{\mu}}^*$ as follows. For $\bar{\mu} \leq \alpha$, $C_{\bar{\mu}}^* = \bar{\mu}$. Otherwise $C_{\bar{\mu}}^* = \{ \text{ith element of } C_{\bar{\mu}} \mid i \in C_{\bar{\mu}}^* \}$ where $\bar{\mu}_0 = \text{ordertype } (C_{\bar{\mu}})$. This defines $C_{\bar{\mu}}^*$. It is easily verified that the $C_{\bar{\mu}}^*$ enjoy all the properties of the $C_{\bar{\mu}}$ except they are only defined when $\mu$ is a singular limit ordinal such that $\check{\mu} \models \text{There is a largest cardinal.}$ In addition, ordertype $C_{\bar{\mu}}^* \leq \alpha(\mu)$, the largest cardinal of $\check{\mu}$.

Now suppose $V = L$, $\alpha$ is a cardinal, $s \in S_\alpha$, $|s| > \alpha$ and $s$ is a 0-string, meaning that $s(\mu) = 0$ for all $\eta \in \text{Dom}(s)$. Then we can choose our predicate $A = \emptyset$, define $C_s = C_{\mu_s^*}$ and Relativized $\Box$ will hold for such 0-strings. The final comment is that all we have done will relativize to arbitrary strings $s \in S_\alpha$, defined relative to an arbitrary predicate $A \subseteq \text{ORD}$. $H_\alpha = L_\alpha[A]$ for all cardinals $\alpha$.

Now we turn to Relativized $\Diamond$. Again we begin with a nonrelativized version. Let $\alpha$ be a cardinal and assume $V = L$.

$\Diamond \text{ on } \alpha^{+}$ Let $E = \{ \mu < \alpha^+ \text{ s.t. } C_\mu \text{ has ordertype } \omega \}$. There exists $\langle D_\mu \mid \mu \in E \rangle$ s.t. $D_\mu \subseteq \check{\mu}$ and:

(a) If $D \subseteq \check{\mu}$ then $\{ \mu \in E \mid D \cap \check{\mu} = D _\mu \}$ is stationary in $\alpha^+$.

(b) $D_\mu$ is uniformly $\Sigma_1$ definable as an element of $\check{\mu}_{\beta(\mu)}$ where $\beta(\mu) =$ largest $\beta$ s.t. either $\beta = \mu$ or $\omega \beta = \beta$ and $\check{\mu} \models \mu$ is a cardinal greater than $\alpha$.

(c) If $\check{\mu}_{\beta(\mu)} \models \alpha^{++}$ exists then $D_\mu = \emptyset$. 

19
Proof. For $\mu \in E$ let $D_\mu = \emptyset$ if $\bar{J}_{\beta(\mu)} \models \alpha^{++}$ exists and otherwise let $\langle D_\mu, F_\mu \rangle$ be least in $\bar{J}_{\beta(\mu)}$ such that $F_\mu$ is CUB in $\mu$ and $\bar{p} \in F_\mu \rightarrow \bar{p} \notin E$ or $D_\mu \neq D_\mu \cap \bar{J}_p$. If $\langle D_\mu, F_\mu \rangle$ doesn’t exist let $D_\mu = \emptyset$. Properties (b), (c) are clear. To prove (a), suppose it fails and let $\langle D, F \rangle$ be least in $\bar{J}_\alpha^{++}$ such that $D \subseteq \bar{J}_\alpha^{++}$. $F$ is CUB in $\alpha^+$ and $\mu \in F \rightarrow \mu \notin E$ or $D_\mu \neq D \cap \bar{J}_\mu$. Let $\sigma$ be least such that $\omega \sigma = \sigma$ and $\langle D, F \rangle \in \bar{J}_\sigma$. Then $\bar{J}_\sigma \models \alpha^+$ is the largest cardinal. Let $H = \Sigma_1$ Skolem hull of $\{\alpha^+\}$ in $\bar{J}_\sigma$ and $\mu = \bigcup (H \cap \alpha^+)$. Then $\bar{J}_{\beta(\mu)}$ is the transitive collapse of the $\Sigma_1$ Skolem hull of $\mu \cup \{\alpha^+\}$ in $\bar{J}_\sigma$; let $\pi : \bar{J}_{\beta(\mu)} \rightarrow \bar{J}_\sigma$ have range equal to the latter hull. Then we have a contradiction provided $\mu \in E$. But the fact that $H$ is unbounded in $\mu$ implies that $C^0_\mu = \emptyset$ so ordertype $(C_\mu)$ is $\omega$ and $\mu \in E$. \[+\]

Now as with Relativized $\Box$, if $V = L$ and $\alpha$ is a cardinal, $s \in S_\alpha$ is a 0-string in $E$, $|s| > \alpha$ then we can choose $A = \emptyset$ and define $D_s = D_\mu^\emptyset$ where the latter comes form $\diamond$ on $\alpha^+$. Finally, relativize everything to arbitrary reshaped strings $s \in S_\alpha$ and an arbitrary predicate $A \subseteq \text{ORD}$, $L_\alpha[A] = H_\alpha$ for all cardinals $\alpha$.

This completes the proof of Relativized $\Box$ and $\diamond$.

References


