THIN STATIONARY SETS AND DISJOINT CLUB SEQUENCES

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Abstract. We describe two opposing combinatorial properties related to adding clubs to $\omega_2$: the existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ and the existence of a disjoint club sequence on $\omega_2$. A special Aronszajn tree on $\omega_2$ implies there exists a thin stationary set. If there exists a disjoint club sequence, then there is no thin stationary set, and moreover there is a fat stationary subset of $\omega_2$ which cannot acquire a club subset by any forcing poset which preserves $\omega_1$ and $\omega_2$. We prove that the existence of a disjoint club sequence follows from Martin’s Maximum and is equiconsistent with a Mahlo cardinal.

Suppose that $S$ is a fat stationary subset of $\omega_2$, that is, for every club set $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with order type $\omega_1 + 1$. A number of forcing posets have been defined which add a club subset to $S$ and preserve cardinals under various assumptions. Abraham and Shelah [1] proved that, assuming $\text{CH}$, the poset consisting of closed bounded subsets of $S$ ordered by end-extension adds a club subset to $S$ and is $\omega_1$-distributive. S. Friedman [5] discovered a different poset for adding a club subset to a fat set $S \subseteq \omega_2$ with finite conditions. This finite club poset preserves all cardinals provided that there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$, that is, a stationary set $T \subseteq P_{\omega_1}(\omega_2)$ such that for all $\beta < \omega_2$, $|\{a \cap \beta : a \in T\}| \leq \omega_1$. This notion of stationarity appears in [9] and was discovered independently by Friedman. The question remained whether it is always possible to add a club subset to a given fat set and preserve cardinals, without any assumptions.

J. Krueger introduced a combinatorial principle on $\omega_2$ which asserts the existence of a disjoint club sequence, which is a pairwise disjoint sequence $\langle C_\alpha : \alpha \in A \rangle$ indexed by a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$, where each $C_\alpha$ is club in $P_{\omega_1}(\omega_1)$. Krueger proved that the existence of such a sequence implies there is a fat stationary set $S \subseteq \omega_2$ which cannot acquire a club subset by any forcing poset which preserves $\omega_1$ and $\omega_2$.

We prove that a special Aronszajn tree on $\omega_2$ implies there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$. On the other hand assuming Martin’s Maximum there exists a disjoint club sequence on $\omega_2$. Moreover, we have the following equiconsistency result.

Theorem 0.1. Each of the following statements is equiconsistent with a Mahlo cardinal: (1) There does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. (2) There exists a disjoint club sequence on $\omega_2$. (3) There exists a fat stationary set $S \subseteq \omega_2$ such that any forcing poset which preserves $\omega_1$ and $\omega_2$ does not add a club subset to $S$.

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1A similar poset was defined independently by Mitchell [7].
Our proof of this theorem gives a totally different construction of the following result of Mitchell [8]: If \( \kappa \) is Mahlo in \( L \), then there is a generic extension of \( L \) in which \( \kappa = \omega_2 \) and there is no special Aronszajn tree on \( \omega_2 \). The consistency of Theorem 0.1(3) provides a negative solution to the following problem of Abraham and Shelah [1]: if \( S \subseteq \omega_2 \) is fat, does there exist an \( \omega_1 \)-distributive forcing poset which adds a club subset to \( S \)?

Section 1 outlines notation and background material. In Section 2 we discuss thin stationarity and prove that a special Aronszajn tree implies the existence of a thin stationary set. In Section 3 we introduce disjoint club sequences and prove that the existence of such a sequence implies there is a fat stationary set in \( \omega_2 \) which cannot acquire a club subset by any forcing poset which preserves \( \omega_1 \) and \( \omega_2 \). In Section 4 we prove that Martin’s Maximum implies there exists a disjoint club sequence. In Section 5 we construct a model in which there is a disjoint club sequence using an RCS iteration up to a Mahlo cardinal.

Sections 3 and 4 are due for the most part to J. Krueger. We would like to thank Boban Veličković and Mirna Dzamonja for pointing out Theorem 2.3 to the authors.

1. Preliminaries

For a set \( X \) which contains \( \omega_1 \), \( P_{\omega_1}(X) \) denotes the collection of countable subsets of \( X \). A set \( C \subseteq P_{\omega_1}(X) \) is club if it is closed under unions of countable increasing sequences and is cofinal. A set \( S \subseteq P_{\omega_1}(X) \) is stationary if it meets every club. If \( C \subseteq P_{\omega_1}(X) \) is club then there exists a function \( F : X^{< \omega} \to X \) such that every \( a \) in \( P_{\omega_1}(X) \) closed under \( F \) is in \( C \). If \( F : X^{< \omega} \to P_{\omega_1}(X) \) is a function and \( Y \subseteq X \), we say that \( Y \) is closed under \( F \) if for all \( \bar{y} \in Y^{< \omega} \), \( F(\bar{y}) \subseteq Y \). A partial function \( H : P_{\omega_1}(X) \to X \) is regressive if for all \( a \) in the domain of \( H \), \( H(a) \) is a member of \( a \). Fodor’s Lemma asserts that whenever \( S \subseteq P_{\omega_1}(X) \) is stationary and \( H : S \to X \) is a total regressive function, there is a stationary set \( S^* \subseteq S \) and a set \( x \) in \( X \) such that for all \( a \) in \( S^* \), \( H(a) = x \).

If \( \kappa \) is a regular cardinal let \( \text{cof}(\kappa) \) (respectively \( \text{cof}(< \kappa) \)) denote the class of ordinals with cofinality \( \kappa \) (respectively cofinality less than \( \kappa \)). If \( A \) is a cofinal subset of a cardinal \( \lambda \) and \( \kappa < \lambda \), we write for example \( A \cup \text{cof}(\kappa) \) to abbreviate \( A \cup (\lambda \cap \text{cof}(\kappa)) \).

A stationary set \( S \subseteq \kappa \) is fat if for every club \( C \subseteq \kappa \), \( S \cap C \) contains closed subsets with arbitrarily large order types less than \( \kappa \). If \( \kappa \) is the successor of a regular uncountable cardinal \( \mu \), this is equivalent to the statement that for every club \( C \subseteq \kappa \), \( S \cap C \) contains a closed subset with order type \( \mu + 1 \). In particular, if \( A \subseteq \kappa^+ \cap \text{cof}(\mu) \) is stationary then \( A \cup \text{cof}(< \mu) \) is fat.

We write \( \theta \gg \kappa \) to indicate \( \theta \) is larger than \( 2^{\text{hght}(\kappa)} \).

A tree \( T \) is a special Aronszajn tree on \( \omega_2 \) if:

1. \( T \) has height \( \omega_2 \) and each level has size less than \( \omega_2 \),
2. each node in \( T \) is an injective function \( f : \alpha \to \omega_1 \) for some \( \alpha < \omega_2 \),
3. the ordering on \( T \) is by extension of functions, and if \( f \) is in \( T \) then \( f \upharpoonright \beta \) is in \( T \) for all \( \beta < \text{dom}(f) \).

By [8] if there does not exist a special Aronszajn tree on \( \omega_2 \), then \( \omega_2 \) is a Mahlo cardinal in \( L \).

If \( V \) is a transitive model of \( \text{ZFC} \), we say that \( W \) is an outer model of \( V \) if \( W \) is a transitive model of \( \text{ZFC} \) such that \( V \subseteq W \) and \( W \) has the same ordinals as \( V \).
A forcing poset $\mathbb{P}$ is $\kappa$-distributive if forcing with $\mathbb{P}$ does not add any new sets of ordinals with size $\kappa$.

If $\mathbb{P}$ is a forcing poset, $\dot{a}$ is a $\mathbb{P}$-name, and $G$ is a generic filter for $\mathbb{P}$, we write $a$ for the set $\dot{a}^G$.

Martin’s Maximum is the statement that whenever $\mathbb{P}$ is a forcing poset which preserves stationary subsets of $\omega_1$, then for any collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \omega_1$, there is a filter $G \subseteq \mathbb{P}$ which intersects each dense set in $\mathcal{D}$.

A forcing poset $\mathbb{P}$ is proper if for all sufficiently large regular cardinals $\theta > 2^{2^{\aleph_0}}$, there is a club of countable elementary substructures $N(\mathcal{H}(\theta), \in)$ such that for all $p \in N \cap P$, there is $q \leq p$ which is generic for $N$, i.e. $q$ forces $N[\dot{G}] \cap \mathbb{On} = N \cap \mathbb{On}$.

If $\mathbb{P}$ is proper then $\mathbb{P}$ forces $\omega_1$ and preserves stationary subsets of $P_\alpha(\lambda)$ for all $\lambda \geq \omega_1$. A forcing poset $\mathbb{P}$ is semiproper if the same statement holds as above except the requirement that $q$ is generic is replaced by $q$ being semigeneric, i.e. $q$ forces $N[\dot{G}] \cap \omega_1 = N \cap \omega_1$. If $\mathbb{P}$ is semigeneric then $\mathbb{P}$ preserves $\omega_1$ and preserves stationary subsets of $\omega_1$.

If $\mathbb{P}$ is $\omega_1$-c.c. and $N$ is a countable elementary substructure of $H(\theta)$, then $\mathbb{P}$ forces $N[\dot{G}] \cap \mathbb{On} = N \cap \mathbb{On}$; so every condition in $\mathbb{P}$ is generic for $N$.

We let $<^\omega \text{On}$ denote the class of finite strictly increasing sequences of ordinals. If $\eta$ and $\nu$ are in $<^\omega \text{On}$, write $\eta \leq \nu$ if $\eta$ is an initial segment of $\nu$, and write $\eta < \nu$ if $\eta \leq \nu$ and $\eta \neq \nu$. Let $l(\eta)$ denote the length of $\eta$. A set $T \subseteq <^\omega \text{On}$ is a tree if for all $\eta$ in $T$ and $k < l(\eta)$, $\eta \upharpoonright k$ is in $T$. A cofinal branch of $T$ is a function $b : \omega \to \kappa$ such that for all $n < \omega$, $b \upharpoonright n$ is in $T$.

Suppose $I$ is an ideal on a set $X$. Then $I^+$ is the collection of subsets of $X$ which are not in $I$. If $S$ is in $I^+$ let $I \upharpoonright S$ denote the ideal $I \cap \mathcal{P}(S)$. For example if $I = NS_\kappa$, the ideal of non-stationary subsets of $\kappa$, a set $S$ is in $I^+$ iff $S$ is stationary. In this case $NS_\kappa \upharpoonright S$ is the ideal of non-stationary subsets of $S$ and $(NS_\kappa \upharpoonright S)^+$ is the collection of stationary subsets of $S$.

If $\kappa$ is regular and $\lambda \geq \kappa$ is a cardinal, then $\text{Coll}(\kappa, \lambda)$ is a forcing poset for collapsing $\lambda$ to have cardinality $\kappa$: conditions are partial functions $p : \kappa \to \lambda$ with size less than $\kappa$, ordered by extension of functions.

2. Thin Stationary Sets

Let $T$ be a cofinal subset of $P_\omega(\omega_2)$. We say that $T$ is thin if for all $\beta < \omega_2$ the set $\{a \cap \beta : a \in T\}$ has size less than $\omega_2$. Note that if CH holds then $P_\omega(\omega_2)$ itself is thin. A set $S \subseteq P_\omega(\omega_2)$ is closed under initial segments if for all $a$ in $S$ and $\beta < \omega_2$, $a \cap \beta$ is in $S$.

**Lemma 2.1.** If $S \subseteq P_\omega(\omega_2)$ is stationary and closed under initial segments, then for all uncountable $\beta < \omega_2$, the set $S \cap P_\omega(\beta)$ is stationary in $P_\omega(\beta)$.

*Proof.* Consider $\beta < \omega_2$ and let $C \subseteq P_\omega(\beta)$ be a club set. Then the set $D = \{a \in P_\omega(\omega_2) : a \cap \beta \in C\}$ is a club subset of $P_\omega(\omega_2)$. Fix $a$ in $S \cap D$. Since $S$ is closed under initial segments, $a \cap \beta$ is in $S \cap C$. \qed

**Lemma 2.2.** If there exists a thin stationary subset of $P_\omega(\omega_2)$, then there is a thin stationary set $S$ such that for all uncountable $\beta < \omega_2$, $S \cap P_\omega(\beta)$ is stationary in $P_\omega(\beta)$.

*Proof.* Let $T$ be a thin stationary set. Define $S = \{a \cap \beta : a \in T, \beta < \omega_2\}$. Then $S$ is thin stationary and closed under initial segments. \qed
A set $S \subseteq P_{\omega_1}(\omega_2)$ is a local club if there is a club set $C \subseteq \omega_2$ such that for all uncountable $\alpha$ in $C$, $S \cap P_{\omega_1}(\alpha)$ contains a club in $P_{\omega_1}(\alpha)$ (see [3]). Note that local clubs are stationary.

**Theorem 2.3.** If there is a special Aronszajn tree on $\omega_2$, then there is a thin local club subset of $P_{\omega_1}(\omega_2)$.

**Proof.** Let $T$ be a special Aronszajn tree on $\omega_2$. For each $f$ in $T$ with $\text{dom}(f) \geq \omega_1$, define $S_f = \{f^{-1}_i : i < \omega_1\}$. Note that $S_f$ is a club subset of $P_{\omega_1}(\text{dom}(f))$. For each uncountable $\beta < \omega_2$ define $S_\beta = \bigcup\{S_f : f \in T, \text{dom}(f) = \beta\}$. Then $S_\beta$ has size $\omega_1$. Now define $S = \bigcup\{S_\beta : \omega_1 \leq \beta < \omega_2\}$. Clearly $S$ is a local club. To show $S$ is thin, it suffices to prove that whenever $\beta < \gamma$ are uncountable and $a$ is in $S_\gamma$, then $a \cap \beta$ is in $S_\beta$. Fix $f$ in $T$ and $i < \omega_1$ such that $a = f^{-1}_i$. Then $f \upharpoonright \beta$ is in $T$, so $(f \upharpoonright \beta)^{-1}_i = (f^{-1}_i)^{-1}_i = \beta = a \cap \beta$. □

In later sections of the paper we will construct models in which there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. Theorem 2.3 shows that in such a model there cannot exist a special Aronszajn tree on $\omega_2$, so by [8] $\omega_2$ is Mahlo in $L$. Mitchell [8] constructed a model in which there is no special Aronszajn tree on $\omega_2$ by collapsing a Mahlo cardinal in $L$ to become $\omega_2$ with a proper forcing poset. However, in Mitchell’s model the set $(P_{\omega_1}(\kappa))^L$ is a thin stationary subset of $P_{\omega_1}(\omega_2)$.

**Lemma 2.4.** Suppose $S \subseteq P_{\omega_1}(\omega_2)$ is a local club. Then $S$ is a local club in any outer model $W$ with the same $\omega_1$ and $\omega_2$.

**Proof.** Let $C$ be a club subset of $\omega_2$ such that for every uncountable $\alpha$ in $C$, $S \cap P_{\omega_1}(\alpha)$ contains a club in $P_{\omega_1}(\alpha)$. Then $C$ remains club in $W$. For each uncountable $\alpha$ in $C$, fix a bijection $g_\alpha : \omega_1 \rightarrow \alpha$. Then $(g_\alpha \upharpoonright i : i < \omega_1)$ is a club subset of $P_{\omega_1}(\alpha)$. By intersecting this club with $S$, we get a club subset of $S \cap P_{\omega_1}(\alpha)$ of the form $(a^\alpha_i : i < \omega_1)$ which is increasing and continuous. Clearly this set remains a club subset of $P_{\omega_1}(\alpha)$ in $W$. □

**Proposition 2.5.** (1) Suppose there exists a thin local club in $P_{\omega_1}(\omega_2)$. Then there exists a thin local club in any outer model with the same $\omega_1$ and $\omega_2$. (2) Suppose $\kappa$ is a cardinal such that for all $\mu < \kappa$, $\mu^\omega < \kappa$, and assume $\mathbb{P}$ is a proper forcing poset which collapses $\kappa$ to become $\omega_2$. Then $\mathbb{P}$ forces there is a thin stationary subset of $P_{\omega_1}(\omega_2)$.

**Proof.** (1) is immediate from Lemma 2.4 and the absoluteness of thinness. (2) Let $G$ be generic for $\mathbb{P}$ over $V$ and work in $V[G]$. Since $\mathbb{P}$ is proper, $\omega_1$ is preserved and the set $S = (P_{\omega_1}(\kappa))^V$ is stationary in $P_{\omega_1}(\omega_2)$. We claim that $S$ is thin. If $\beta < \omega_2$ then $\{a \cap \beta : a \in S\} = (P_{\omega_1}(\beta))^V$. By the assumption on $\kappa$, there is $\xi < \kappa$ and a bijection $f : \xi \rightarrow (P_{\omega_1}(\beta))^V$ in $V$. In $V[G]$ there is a surjection of $\omega_1$ onto $\xi$ and hence a surjection of $\omega_1$ onto $\{a \cap \beta : a \in S\}$. □

As we mentioned above, if CH holds then the set $P_{\omega_1}(\omega_2)$ itself is thin. We show on the other hand that if CH fails then no club subset of $P_{\omega_1}(\omega_2)$ is thin. The proof is actually due to Baumgartner and Taylor [2] who proved that for any club set $C \subseteq P_{\omega_1}(\omega_2)$, there is a countable set $A \subseteq \omega_2$ such that $C \cap P(A)$ has size at least $2^\omega$. Their method of proof, which is described in the next lemma, is key to several of our results later in the paper.
Lemma 2.6. Suppose $Z$ is a stationary subset of $\omega_2 \cap \text{cof}(\omega)$ and for each $\alpha$ in $Z$, $M_\alpha$ is a countable cofinal subset of $\alpha$. Then there is a sequence $\langle Z_s, \xi_s : s \in <\omega^2 \rangle$ satisfying:

1. each $Z_s$ is a stationary subset of $Z$,
2. if $s \leq t$ then $Z_t \subseteq Z_s$,
3. if $\alpha$ is in $Z_s$ then $\xi_s$ is in $M_\alpha$,
4. if $\alpha$ is in $Z_{s-0}$ and $\beta$ is in $Z_{s-1}$, then $\xi_{s-0}$ is not in $M_\beta$ and $\xi_{s-1}$ is not in $M_\alpha$.

Proof. Let $Z() = Z$ and $\xi()$ is undefined. Suppose $Z_s$ is given. Define $X_s$ as the set of $\xi$ in $\omega_2$ such that the set $\{ \alpha \in Z_s : \xi \in M_\alpha \}$ is stationary. A straightforward argument using Fodor’s Lemma shows that $X_s$ is unbounded in $\omega_2$. For each $\alpha$ in $Z_s$ such that $X_s \cap \alpha$ has size $\omega_1$, there exists $\xi < \alpha$ in $X_s$ such that $\xi$ is not in $M_\alpha$.

By Fodor’s Lemma there is a stationary set $Z' \subseteq Z_s$ and $\xi' \in X_s$ such that for all $\alpha$ in $Z' \subseteq Z_s$, $\xi' \in M_\alpha$, which is stationary since $\xi' \in X_s$. Define $Y_s$ as the set of $\xi$ in $\omega_2$ such that $\alpha \in Z'$ so that $Y_s \cap \alpha$ has size $\omega_1$, there is $\xi < \alpha$ in $Y_s$ which is not in $M_\alpha$. By Fodor’s Lemma there is $\xi_1$ in $Y_s$ and $Z_0 \subseteq Z' \subseteq Z_1$ stationary such that for all $\alpha$ in $Z_1$, $\xi_1$ is not in $M_\alpha$. Now define $Z_0$ as the set of $\alpha$ in $Z_1 \subseteq Z_0$ such that $\xi_1$ is in $M_\alpha$.

Theorem 2.7. Assume CH fails. Then for any club set $C \subseteq P_{\omega_1}(\omega_2)$, $C$ is not thin.

Proof. Let $F : \omega^2 \rightarrow \omega_2$ be a function such that any $\alpha$ in $P_{\omega_1}(\omega_2)$ closed under $F$ is in $C$. Let $Z$ be the stationary set of $\alpha$ in $\omega_2 \cap \text{cof}(\omega)$ closed under $F$. For each $\alpha$ in $Z$ fix a countable set $M_\alpha \subseteq \alpha$ such that sup($M_\alpha$) = $\alpha$ and $M_\alpha$ is closed under $F$. Fix a sequence $\langle Z_s, \xi_s : s \in <\omega^2 \rangle$ as described in Lemma 2.6.

For each function $f : \omega \rightarrow 2$ define $b_f = \text{cl}_F(\{ \xi_{f|n} : n < \omega \})$. Then $b_f$ is in $C$. Note that if $n < \omega$ and $\alpha$ is in $Z_f|n$, then $\text{cl}_F(\{ \xi_{f|m} : m \leq n \}) \subseteq M_\alpha$. For by Lemma 2.6(2), for $m \leq n$, $Z_f|n \subseteq Z_f|m$. So $\alpha$ is in $Z_f|m$, and hence $\xi_{f|m}$ is in $M_\alpha$ by (3). But $M_\alpha$ is closed under $F$.

Let $\gamma = \text{sup}(\{ \xi_{s+1} : s \in <\omega^2 \})$. Since $<\omega^2$ has size $\omega$, $\gamma$ is less than $\omega_2$. We claim that for distinct $f$ and $g$, $b_f \cap \gamma \neq b_g \cap \gamma$. Let $n < \omega$ be least such that $f(n) \neq g(n)$. If $b_f \cap \gamma = b_g \cap \gamma$, then there is $k > n$ such that $\xi_{g|(n+1)}$ is in $\text{cl}_F(\{ \xi_{f|m} : m \leq k \})$.

Fix $\alpha$ in $Z_f|k$. By the last paragraph, $\xi_{g|(n+1)}$ is in $M_\alpha$. But $\alpha$ is in $Z_f|(n+1)$ by (2), which contradicts (4).

Let $\kappa$ be an uncountable cardinal. The Weak Reflection Principle at $\kappa$ is the statement that whenever $S$ is a stationary subset of $P_{\omega_1}(\kappa)$, there is a set $Y$ in $P_{\omega_2}(\kappa)$ such that $\omega_1 \subseteq Y$ and $S \cap P_{\omega_1}(Y)$ is stationary in $P_{\omega_1}(Y)$. Martin’s Maximum implies the Weak Reflection Principle holds for all uncountable cardinals $\kappa$ [4].

The Weak Reflection Principle at $\omega_2$ is equivalent to the statement that for every stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is a stationary set of uncountable $\beta < \omega_2$ such that $S \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$. This is equivalent to the statement that every local club subset of $P_{\omega_1}(\omega_2)$ contains a club. The Weak Reflection Principle at $\omega_2$ is equiconsistent with a weakly compact cardinal [3].

Corollary 2.8. Suppose CH fails and there is a special Aronszajn tree on $\omega_2$. Then the Weak Reflection Principle at $\omega_2$ fails.
Proof. By Theorems 2.3 and 2.7, there is a thin local club subset of \( P_{\omega_1}(\omega_2) \) which is not club. Hence the Weak Reflection Principle at \( \omega_2 \) fails. \( \square \)

In Sections 4 and 5 we describe models in which there is no thin stationary subset of \( P_{\omega_1}(\omega_2) \). On the other hand S. Friedman proved there always exists a thin cofinal set.

Theorem 2.9 (Friedman). There exists a thin cofinal subset of \( P_{\omega_1}(\omega_2) \).

Proof. We construct by induction a sequence \( \langle S_\alpha : \omega_1 \leq \alpha < \omega_2 \rangle \) satisfying the properties: (1) each \( S_\alpha \) is a cofinal subset of \( P_{\omega_1}(\alpha) \) with size \( \omega_1 \), (2) for uncountable \( \beta < \gamma \), if \( \alpha \) is in \( S_\beta \), then \( a \cap \beta \) is in \( \bigcup \{ S_\gamma : \omega_1 \leq \alpha \leq \beta \} \), and (3) if \( \beta < \gamma < \omega_2 \), \( a \) is in \( P_{\omega_1}(\gamma) \), and \( a \cap \beta \) is in \( S_\beta \), then there is \( b \) in \( S_\gamma \) such that \( a \subseteq b \) and \( a \cap \beta = b \cap \beta \).

Let \( S_{\omega_1} = \omega_1 \). Given \( S_\alpha \), let \( S_{\alpha+1} \) be the collection \( \{ b \cup \{ \alpha \} : b \in S_\alpha \} \). Conditions (1), (2), and (3) follow by induction. Suppose \( \gamma < \omega_2 \) is an uncountable limit ordinal and \( S_\alpha \) is defined for all uncountable \( \alpha < \gamma \). If \( cf(\gamma) = \omega_1 \) then let \( S_\gamma = \bigcup \{ S_\alpha : \omega_1 \leq \alpha < \gamma \} \). The required conditions follow by induction.

Assume \( cf(\gamma) = \omega \). Fix an increasing sequence of uncountable ordinals \( \langle \gamma_n : n < \omega \rangle \) unbounded in \( \gamma \). Let \( T_\alpha \) be some cofinal subset of \( P_{\omega_1}(\gamma) \) with size \( \omega_1 \). Fix \( n < \omega \). For each \( x \) in \( T_\gamma \) and \( a \) in \( S_{\gamma_n} \) define a set \( b(a, x, n) \) in \( P_{\omega_1}(\gamma) \) inductively as follows. Let \( b(a, x, n) \cap \gamma_m = a \). Given \( b(a, x, n) \cap \gamma_m \) in \( S_{\gamma_m} \) for some \( m \geq n \), apply condition (3) to \( \gamma_m, \gamma_{m+1} \), and the set

\[
(b(a, x, n) \cap \gamma_m) \cup (x \cap [\gamma_m, \gamma_{m+1}])
\]

to find \( y \) in \( S_{\gamma_{m+1}} \) such that \( y \cap \gamma_m = b(a, x, n) \cap \gamma_m \) and \( x \cap [\gamma_m, \gamma_{m+1}] \subseteq y \). Let \( b(a, x, n) \cap \gamma_{m+1} = y \). This completes the definition of \( b(a, x, n) \). Clearly \( b(a, x, n) \cap \gamma_m = a \), \( x \setminus \gamma_m \subseteq b(a, x, n) \), and for all \( k \geq n \), \( b(a, x, n) \cap \gamma_k \) is in \( S_{\gamma_k} \).

Now define \( S_\gamma = \{ b(a, x, n) : n < \omega, a \in S_{\gamma_n}, x \in T_\gamma \} \). We verify conditions (1), (2), and (3). Clearly \( S_\gamma \) has size \( \omega_1 \). Let \( \beta < \gamma \) and consider \( b(a, x, n) \) in \( S_\gamma \). Fix \( k > n \) such that \( \beta < \gamma_k \). Then \( b(a, x, n) \cap \gamma_k \) is in \( S_{\gamma_k} \). So by induction \( b(a, x, n) \cap \beta \) is in \( \bigcup \{ S_\gamma : \omega_1 \leq \alpha \leq \beta \} \). Now assume \( a \) is in \( P_{\omega_1}(\gamma) \), \( \beta < \gamma \), and \( a \cap \beta \) is in \( S_\beta \). Choose \( x \) in \( T_\gamma \) such that \( a \subseteq x \). Fix \( k \) such that \( \beta < \gamma_k \). By the induction hypothesis there is \( a' \) in \( S_{\gamma_k} \) such that \( a \cap \gamma_k \subseteq a' \) and \( a' \cap \beta = a \cap \beta \). Let \( c = b(a', x, k) \). Then \( c \) is in \( S_\gamma \), \( c \cap \beta = (c \cap \gamma_k) \cap \beta = a' \cap \beta = a \cap \beta \), and \( a \subseteq c \).

To prove \( S_\gamma \) is cofinal consider \( a \) in \( P_{\omega_1}(\gamma) \). Fix \( x \) in \( T_\gamma \) such that \( a \subseteq x \). By induction \( S_{\gamma_0} \) is cofinal in \( P_{\omega_1}(\gamma_0) \). So let \( y \) be in \( S_{\gamma_0} \) such that \( x \cap \gamma_0 \subseteq y \). Then \( a \) is a subset of \( b(y, x, 0) \).

Now define \( S = \bigcup \{ S_\beta : \omega_1 \leq \beta < \omega_2 \} \). Conditions (1) and (2) imply that \( S \) is thin and cofinal in \( P_{\omega_1}(\omega_2) \). \( \square \)

3. DISJOINT CLUB SEQUENCES

We introduce a combinatorial property of \( \omega_2 \) which implies there does not exist a thin stationary subset of \( P_{\omega_1}(\omega_2) \). This property follows from Martin’s Maximum and is equiconsistent with a Mahlo cardinal. It implies there exists a fat stationary subset of \( \omega_2 \) which cannot acquire a club subset by any forcing poset which preserves \( \omega_1 \) and \( \omega_2 \).

Definition 3.1. A disjoint club sequence on \( \omega_2 \) is a sequence \( \langle C_\alpha : \alpha \in A \rangle \) such that \( A \) is a stationary subset of \( \omega_2 \cap \text{cof}(\omega_1) \), each \( C_\alpha \) is a club subset of \( P_{\omega_1}(\alpha) \), and \( C_\alpha \cap C_\beta \) is empty for all \( \alpha < \beta \) in \( A \).
Proposition 3.2. Suppose there is a disjoint club sequence on \( \omega_2 \). Then there does not exist a thin stationary subset of \( P_{\omega_1}(\omega_2) \).

Proof. Let \( \langle C_\alpha : \alpha \in A \rangle \) be a disjoint club sequence. Suppose for a contradiction there exists a thin stationary set. By Lemma 2.2 fix a thin stationary set \( T \subseteq P_{\omega_1}(\omega_2) \) such that for all uncountable \( \beta < \omega_2 \), \( T \cap P_{\omega_1}(\beta) \) is stationary in \( P_{\omega_1}(\beta) \). Then for each \( \beta \in A \) we can choose a set \( a_\beta \in C_\beta \cap T \). Since \( \text{cf}(\beta) = \omega_1 \), \( \sup(a_\beta) < \beta \).

By Fodor’s Lemma there is a stationary set \( B \subseteq A \) and a fixed \( \gamma < \omega_2 \) such that for all \( \beta \in B \), \( \sup(a_\beta) = \gamma \). If \( \alpha < \beta \) are in \( B \), then \( a_\alpha \neq a_\beta \) since \( C_\alpha \cap C_\beta \) is empty. So the set \( \{a_\beta : \beta \in B \} \) witnesses that \( T \) is not thin, which is a contradiction. \( \square \)

Lemma 3.3. Suppose there is a disjoint club sequence \( \langle C_\alpha : \alpha \in A \rangle \) on \( \omega_2 \). Let \( W \) be an outer model with the same \( \omega_1 \) and \( \omega_2 \) in which \( A \) is still stationary. Then there is a disjoint club sequence \( \langle D_\alpha : \alpha \in A \rangle \) in \( W \).

Proof. By the proof of Lemma 2.4, each \( C_\alpha \) contains a club set \( D_\alpha \) in \( W \). Since \( \omega_1 \) is preserved, each \( \alpha \in A \) still has cofinality \( \omega_1 \). \( \square \)

Theorem 3.4. Suppose \( \langle C_\alpha : \alpha \in A \rangle \) is a disjoint club sequence on \( \omega_2 \). Then \( A \cup \text{cof}(\omega) \) does not contain a club.

Proof. Suppose for a contradiction that \( A \cup \text{cof}(\omega) \) contains a club. Without loss of generality \( 2^{\omega_1} = \omega_2 \). Otherwise work in a generic extension \( W \) by \( \text{Coll}(\omega_2, 2^{\omega_1}) \): in \( W \) the set \( A \cup \text{cof}(\omega) \) contains a club and by Lemma 3.3 there is a disjoint club sequence \( \langle D_\alpha : \alpha \in A \rangle \).

Since \( 2^{\omega_1} = \omega_2 \), \( H(\omega_2) \) has size \( \omega_2 \). Fix a bijection \( h : H(\omega_2) \rightarrow \omega_2 \). Let \( A \) denote the structure \( (H(\omega_2), \in, h) \). Define \( B \) as the set of \( \alpha \in \omega_2 \cap \text{cof}(\omega_1) \) such that there exists an increasing and continuous sequence \( \langle N_i : i < \omega_1 \rangle \) of countable elementary substructures of \( A \) such that:

1. For \( i < \omega_1 \), \( N_i \) is in \( N_{i+1} \).
2. The set \( \{N_i \cap \omega_2 : i < \omega_1 \} \) is in \( P_{\omega_1}(\alpha) \).

We claim that \( B \) is stationary in \( \omega_2 \). To prove this let \( C \subseteq \omega_2 \) be club. Let \( B \) be the expansion of \( A \) by the function \( \alpha \mapsto \text{min}(C \setminus \alpha) \). Define by induction an increasing and continuous sequence \( \langle N_i : i < \omega_1 \rangle \) of elementary substructures of \( B \) such that for all \( i < \omega_1 \), \( N_i \) is in \( N_{i+1} \). Let \( N = \bigcup\{N_i : i < \omega_1 \} \). Then \( \omega_1 \subseteq N \) so \( N \cap \omega_2 \) is an ordinal. Write \( \alpha = N \cap \omega_2 \). Then \( \alpha \) is in \( C \) and \( \{N_i \cap \omega_2 : i < \omega_1 \} \) is club in \( P_{\omega_1}(\alpha) \). So \( \alpha \) is in \( B \cap C \).

Since \( A \cup \text{cof}(\omega) \) contains a club, \( A \cap B \) is stationary. For each \( \alpha \in A \cap B \) fix a sequence \( \langle N_0^\alpha : i < \omega_1 \rangle \) as described in the definition of \( B \). Then \( \{N_0^\alpha \cap \omega_2 : i < \omega_1 \} \cap C_\alpha \) is club in \( P_{\omega_1}(\alpha) \). So there exists a club set \( c_\alpha \subseteq \omega_1 \) such that \( \{N_0^i \cap \omega_2 : i \in c_\alpha \} \) is club and is a subset of \( C_\alpha \). Write \( i_\alpha = \text{min}(c_\alpha) \) and let \( d_\alpha = c_\alpha \setminus \{i_\alpha\} \).

Define \( S = \{N_0^\alpha : \alpha \in A \cap B, i \in d_\alpha \} \). If \( N \) is in \( S \) then there is a unique pair \( \alpha \) in \( A \cap B \) and \( i \) in \( d_\alpha \) such that \( N = N_0^\alpha \). For if \( N = N_0^i = N_0^j \), then \( N \cap \omega_2 \) is in \( C_\alpha \cap C_\beta \), so \( \alpha = \beta \). Clearly then \( i = j \). Also note that if \( N_0^\alpha \) is in \( S \) then \( N_0^\alpha \) is in \( N_0^\beta \).

So the function \( H : S \rightarrow H(\omega_2) \) defined by \( H(N_0^\alpha) = N_0^\alpha \) is well-defined and regressive.

We claim that \( S \) is stationary in \( P_{\omega_1}(H(\omega_2)) \). To prove this let \( F : H(\omega_2)^{<\omega} \rightarrow H(\omega_2) \) be a function. Define \( G : \omega_2^{<\omega} \rightarrow \omega_2 \) by letting \( G(\alpha_0, \ldots, \alpha_n) \) be equal to \( h(F(h^{-1}(\alpha_0), \ldots, h^{-1}(\alpha_n))) \). Let \( E \) be the club set of \( \alpha \) in \( \omega_2 \) closed under \( G \). Fix \( \alpha \in E \cap A \cap B \). Then there is \( i \) in \( d_\alpha \) such that \( N_0^i \cap \omega_2 \) is closed under \( G \). We claim
that $N^\alpha_\beta$ is closed under $F$. Given $a_0, \ldots, a_n$ in $N^\alpha_\beta$, the ordinals $h(a_0), \ldots, h(a_n)$ are in $N^\alpha_\beta \cap \omega_2$. So $\gamma = G(h(a_0), \ldots, h(a_n)) = h(F(a_0, \ldots, a_n))$ is in $N^\alpha_\beta \cap \omega_2$. Therefore $h^{-1}(\gamma) = F(a_0, \ldots, a_n)$ is in $N^\alpha_\beta$.

Since $S$ is stationary and $H : S \to H(\omega_2)$ is regressive, there is a stationary set $S^* \subseteq S$ and a fixed $N$ such that for all $N^\alpha_\beta$ in $S^*$, $H(N^\alpha_\beta) = N$. The set $S^*$, being stationary, must have size $\omega_2$. So there are distinct $\alpha$ and $\beta$ such that for some $i$ in $d_\alpha$ and $j$ in $d_\beta$, $N^\alpha_\beta$ and $N^\beta_\beta$ are in $S^*$. Then $N = N^\alpha_\beta = N^\beta_\beta$. So $N \cap \omega_2$ is in $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$, which is a contradiction. \hfill \Box

Abraham and Shelah [1] asked the following question: Assume that $A$ is a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$. Does there exist an $\omega_1$-distributed forcing poset which adds a club subset to $A \cup \text{cof}(\omega)$? We answer this question in the negative.

**Corollary 3.5.** Assume that $\langle C_\alpha : \alpha \in A \rangle$ is a disjoint club sequence. Let $W$ be an outer model of $V$ with the same $\omega_1$ and $\omega_2$. Then in $W$, $A \cup \text{cof}(\omega)$ does not contain a club subset.

**Proof.** If $A$ remains stationary in $W$, then by Lemma 3.3 there is a disjoint club sequence $\langle D_\alpha : \alpha \in A \rangle$ in $W$. By Theorem 3.4 $A \cup \text{cof}(\omega)$ does not contain a club in $W$. \hfill \Box

4. Martin’s Maximum

In this section we prove that Martin’s Maximum implies there exists a disjoint club sequence on $\omega_2$. We apply MM to the poset for adding a Cohen real and then forcing a continuous $\omega_1$-chain through $P_{\omega_1}(\omega_2) \setminus V$.

**Theorem 4.1** (Krueger). Martin’s Maximum implies there exists a disjoint club sequence on $\omega_2$.

We will use the following theorem from [1].

**Theorem 4.2.** Suppose $P$ is $\omega_1$-c.c. and adds a real. Then $P$ forces that $(P_{\omega_1}(\omega_2) \setminus V)$ is stationary in $P_{\omega_1}(\omega_2)$.

Note: Gitik [6] proved that the conclusion of Theorem 4.2 holds for any outer model of $V$ which contains a new real and computes the same $\omega_1$.

Suppose that $S$ is a stationary subset of $P_{\omega_1}(\omega_2)$. Following [3] we define a forcing poset $\mathbb{P}(S)$ which adds a continuous $\omega_1$-chain through $S$. A condition in $\mathbb{P}(S)$ is a countable increasing and continuous sequence $\langle a_i : i \leq \beta \rangle$ of elements from $S$, where for each $i < \beta$, $a_i \cap \omega_1 < a_{i+1} \cap \omega_1$. The ordering on $\mathbb{P}(S)$ is by extension of sequences.

**Proposition 4.3.** If $S \subseteq P_{\omega_1}(\omega_2)$ is stationary then $\mathbb{P}(S)$ is $\omega$-distributive.

**Proof.** Suppose $p$ forces $\dot{f} : \omega \to \text{On}$. Let $\theta \gg \omega_2$ be a regular cardinal such that $\dot{f}$ is in $H(\theta)$. Since $S$ is stationary, we can fix a countable elementary substructure $N$ of the model

$$\langle H(\theta), \in, S, \mathbb{P}(S), p, \dot{f} \rangle$$

such that $N \cap \omega_2$ is in $S$. Let $\langle D_n : n < \omega \rangle$ be an enumeration of all the dense subsets of $\mathbb{P}(S)$ in $N$. Inductively define a decreasing sequence $\langle p_n : n < \omega \rangle$ of elements of $N \cap \mathbb{P}(S)$ such that $p_0 = p$ and $p_{n+1}$ is a refinement of $p_n$ in $D_n \cap N$. Write $\bigcup \{p_n : n < \omega \} = \langle b_i : i < \gamma \rangle$. Clearly $\bigcup \{b_i : i < \gamma \} = N \cap \omega_2$. Since $N \cap \omega_2$ is in $S$,
the sequence $\langle b_i : i < \gamma \rangle \cup \{ (\gamma, N \cap \omega_2) \}$ is a condition below $p$ which decides $f(n)$ for all $n < \omega$.

**Theorem 4.4.** Suppose $P$ is an $\omega_1$-c.c. forcing poset which adds a real. Let $\dot{S}$ be a name such that $P$ forces $\dot{S} = (P_{\omega_1}(\omega_2) \setminus V)$. Then $P * P(\dot{S})$ preserves stationary subsets of $\omega_1$.

**Proof.** By Theorem 4.2 and Proposition 4.3, the poset $P * P(\dot{S})$ preserves $\omega_1$. Let $A$ be a stationary subset of $\omega_1$ in $V$. Suppose $p * q$ is a condition in $P * P(\dot{S})$ which forces $C$ is a club subset of $\omega_1$.

Let $G$ be a generic filter for $P$ over $V$ which contains $p$. In $V[G]$ fix a regular cardinal $\theta \gg \omega$ and let

$$A = \langle H(\theta), \in, A, S, q, C \rangle.$$

Fix a Skolem function $F : H(\theta)^{<\omega} \rightarrow H(\theta)$ for $A$. Define $F^* : \omega_2^{<\omega} \rightarrow P_{\omega_1}(\omega_2)$ by letting

$$F^*(\alpha_0, \ldots, \alpha_n) = cl_F(\{\alpha_0, \ldots, \alpha_n\}) \cap \omega_2.$$

Since $P$ is $\omega_1$-c.c. there is a function $H : \omega_2^{<\omega} \rightarrow P_{\omega_1}(\omega_2)$ in $V$ such that for all $\alpha$ in $\omega_2^{<\omega}$, $F^*(\alpha) \subseteq H(\alpha)$. Let $Z^*$ be the stationary set of $\alpha$ in $\omega_2 \cap \text{cof}(\omega)$ closed under $H$.

Working in $V$, since $A$ is stationary we can fix for each $\alpha$ in $Z^*$ a countable cofinal set $M_\alpha \subseteq \alpha$ closed under $H$ with $M_\alpha \cap \omega_1 = A$. By Fodor’s Lemma there is $Z \subseteq Z^*$ stationary and $\delta$ in $A$ such that for all $\alpha$ in $Z$, $M_\alpha \cap \omega_1 = \delta$. Fix a sequence $\langle \xi_s, Z_s : s \in \omega_2 \rangle$ satisfying conditions (1)--(4) of Lemma 2.6.

Let $f : \omega \rightarrow 2$ be a function in $V[G] \setminus V$. For each $n < \omega$ let $M_n$ denote

$$cl_H(\delta \cup \{\xi_{f(m)} : m \leq n\}).$$

Define $M = \bigcup\{M_n : n < \omega\}$. Note that $M$ is closed under $H$ and hence it is closed under $F^*$. Therefore $N = cl_F(M)$ is an elementary substructure of $A$ such that $N \cap \omega_2 = M$.

As in the proof of Theorem 2.7, for all $n < \omega$, if $\alpha$ is in $Z_{f(n)}$, then $M_n \subseteq M_\alpha$. Note that $M \cap \omega_1 = \delta$. For if $\gamma$ is in $M \cap \omega_1$, there is $n < \omega$ such that $\gamma$ is in $M_n$. Fix $\alpha$ in $Z_{f(n)}$. Then $\gamma$ is in $M_\alpha \cap \omega_1 = \delta$.

We prove that $M$ is not in $V$ by showing how to compute $f$ by induction from $M$. Suppose $f \upharpoonright n$ is known. Fix $j < 2$ such that $f(n) \neq j$. We claim that $\xi_{f(\upharpoonright n) \upharpoonright -j}$ is not in $M$. Otherwise there is $k > n$ such that $\xi_{f(\upharpoonright n) \upharpoonright -j}$ is in $M_k$. Fix $\alpha$ in $Z_{f(j)}$. Then $\xi_{f(\upharpoonright n) \upharpoonright -j}$ is in $M_\alpha$. But $\alpha$ is in $Z_{f(j \upharpoonright n+1)}$, contradicting Lemma 2.6(4). So $f(n)$ is the unique $i < 2$ such that $\xi_{f(\upharpoonright n) \upharpoonright -j}$ is in $M$. This completes the definition of $f$ from $M$. Since $f$ is not in $V$, neither is $M$.

Let $(D_n : n < \omega)$ enumerate the dense subsets of $P(S)$ lying in $N$. Inductively define a decreasing sequence $\langle q_n : n < \omega \rangle$ in $N \cap P(S)$ such that $q_0 = q$ and $q_{n+1}$ is in $D_n \cap N$. Write $\bigcup\{q_n : n < \omega\} = \langle b_i : i < \gamma \rangle$. Clearly $\bigcup\{b_i : i < \gamma \} = N \cap \omega_2 = M$, and since $M$ is not in $V$, $r = \langle b_i : i < \gamma \rangle \cup \{ (\gamma, M) \}$ is a condition in $P(S)$. By an easy density argument, $r$ forces that $N \cap \omega_1 = \delta$ is a limit point of $\dot{C}$, and hence is in $\dot{C}$. Let $\dot{r}$ be a name for $r$. Then $p * \dot{r} \leq p * \dot{q}$ and $p * \dot{r}$ forces $\delta$ is in $A \cap \dot{C}$. □

The proof of Theorem 4.4 above is similar to the proof of Theorem 4.2.

Now we are ready to prove that MM implies there exists a disjoint club sequence on $\omega_2$.

**Proof of Theorem 4.1.** Assume Martin’s Maximum. Inductively define $A$ and $\langle C_\alpha : \alpha \in A \rangle$ as follows. Suppose $\alpha$ is in $\omega_2 \cap \text{cof}(\omega_1)$ and $A \cap \alpha$ and $\langle C_\beta : \beta \in A \cap \alpha \rangle$ are defined. Let $\alpha$ be in $A$ iff the set $\bigcup\{C_\beta : \beta \in A \cap \alpha \}$ is non-stationary in $P_{\omega_1}(\alpha)$.
If $\alpha$ is in $A$ then choose a club set $C_\alpha \subseteq P_{\omega_1}(\alpha)$ with size $\omega_1$ which is disjoint from this union.

This completes the definition. We prove that $A$ is stationary. Then clearly

$\langle C_\alpha : \alpha \in A \rangle$ is a disjoint club sequence. Fix a club set $C \subseteq \omega_2$.

Let $\text{Add}$ denote the forcing poset for adding a single Cohen real with finite conditions and let $\dot{S}$ be an $\text{Add}$-name for the set $(P_{\omega_1}(\omega_2) \setminus V)$. By Theorem 4.4 the poset $\text{Add} \ast \mathbb{P}(\dot{S})$ preserves stationary subsets of $\omega_1$. We will apply Martin’s Maximum to this poset after choosing a suitable collection of dense sets.

For each $\alpha < \omega_2$ fix a surjection $f_\alpha : \omega_1 \rightarrow \alpha$. If $\beta$ is in $A$ enumerate $C_\beta$ as $\langle a_\beta^i : i < \omega_1 \rangle$. For every quadruple $i, j, k, l$ of countable ordinals let $D(i, j, k, l)$ denote the set of conditions $p \ast \dot{q}$ such that:

1. $p$ forces that $i$ and $j$ are in the domain of $\dot{q}$, and for some $\beta_i$ and $\beta_j$, $p$ forces $\beta_i = \sup(\dot{q}(i))$ and $\beta_j = \sup(\dot{q}(j))$.
2. There is some $\zeta < \omega_1$ such that $p$ forces $\zeta$ is the least element in $\text{dom}(\dot{q})$ such that $f_{\beta_i}(\zeta) \in \dot{q}(\zeta)$.
3. There is $\xi$ in $C$ larger than $\beta_i$ and $\beta_j$ such that $p$ forces $\xi$ is the supremum of the maximal set in $\dot{q}$.
4. If $f_{\beta_i}(k) = \gamma$ is in $A$, then there is $z$ such that $p$ forces $z$ is in the symmetric difference $\dot{q}(i) \triangle a_\beta^\gamma$.

It is routine to check that $D(i, j, k, l)$ is dense.

Let $G \ast H$ be a filter on $\text{Add} \ast \mathbb{P}(\dot{S})$ intersecting each $D(i, j, k, l)$. For $i < \omega_1$ define $a_i$ as the set of $\beta$ for which there exists some $p \ast \dot{q}$ in $G \ast H$ such that $p$ forces $i \in \text{dom}(\dot{q})$ and $p$ forces $\beta$ is in $\dot{q}(i)$. The definition of the dense sets implies that $\langle a_i : i < \omega_1 \rangle$ is increasing, continuous, and cofinal in $P_{\omega_1}(\alpha)$ for some $\alpha$ in $C \cap \text{cof}(\omega_1)$. By (4), for each $\gamma$ in $A \cap \alpha$, $\{a_i : i < \omega_1 \}$ is disjoint from $C_\gamma$. Therefore $\bigcup \{C_\gamma : \gamma \in A \cap \alpha \}$ is non-stationary in $P_{\omega_1}(\alpha)$, hence by the definition of $A$, $\alpha$ is in $A \cap C$. So $A$ is stationary.

5. The Equiconsistency Result

We now prove Theorem 0.1 establishing the consistency strength of each of the following statements to be exactly a Mahlo cardinal: (1) There does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. (2) There exists a disjoint club sequence on $\omega_2$. (3) There exists a fat stationary set $S \subseteq \omega_2$ such that any forcing poset which preserves $\omega_1$ and $\omega_2$ does not add a club subset to $S$.

By [5] if there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$ then for any fat stationary set $S \subseteq \omega_2$, there is a forcing poset which preserves cardinals and adds a club subset to $S$. So (2) and (3) both imply (1), which in turn implies there is no special Aronszajn tree on $\omega_2$. So $\omega_2$ is Mahlo in $L$ by [8].

In the other direction assume that $\kappa$ is a Mahlo cardinal. We will define a revised countable support iteration which collapses $\kappa$ to become $\omega_2$ and adds a disjoint club sequence on $\omega_2$. At individual stages of the iteration we force with either a collapse forcing or the poset $\text{Add} \ast \mathbb{P}(\dot{S})$ from the previous section. To ensure that $\omega_1$ is not collapsed we verify that $\text{Add} \ast \mathbb{P}(\dot{S})$ satisfies an iterable condition known as the $\mathbb{I}$-universal property. Our description of this construction is self-contained, except for the proof of Theorem 5.9 which summarizes the relevant properties of the RCS iteration. For more information on such iterations and the $\mathbb{I}$-universal property see [10].
Definition 5.1. A pair $\langle T, I \rangle$ is a tagged tree if:

1. $T \subseteq \omega^{\omega} \text{On}$ is a tree such that each $\eta \in T$ has at least one successor,
2. $I : T \rightarrow V$ is a partial function such that each $I(\eta)$ is an ideal on some set $X_\eta$ and for each $\alpha \in T$ in the domain of $I$, the set $\{\alpha : \eta \upharpoonright \alpha \in T\}$ is in $(I(\eta))^+$, and
3. for each cofinal branch $b$ of $T$, there are infinitely many $n < \omega$ such that $b \upharpoonright n$ is in the domain of $I$.

If $\eta$ is in the domain of $I$, we say that $\eta$ is a splitting point of $T$. It follows from (1) and (3) that for every $\eta$ in $T$ there is $\eta \triangleleft \nu$ which is a splitting point.

Definition 5.2. Let $I$ be a family of ideals and $\langle T, I \rangle$ a tagged tree. Then $\langle T, I \rangle$ is an $\mathfrak{I}$-tree if for each splitting point $\eta$ in $T$, $I(\eta)$ is in $\mathfrak{I}$.

Suppose $T \subseteq \omega^{\omega} \text{On}$ is a tree. If $\eta$ is in $T$, let $T[\eta]$ denote the tree $\{\nu \in T : \nu \subseteq \eta$ or $\eta \subseteq \nu\}$. A set $J \subseteq T$ is called a front if for distinct nodes in $J$, neither is an initial segment of the other, and for any cofinal branch $b$ of $T$ there is $\eta$ in $J$ which is an initial segment of $b$.

Definition 5.3. Suppose $\langle T, I \rangle$ is tagged tree. Let $\theta$ be a regular cardinal such that $\langle T, I \rangle$ is in $H(\theta)$, and let $<_\theta$ be a well-ordering of $H(\theta)$. A sequence $\langle N_\eta : \eta \in T \rangle$ is a tree of models for $\theta$ provided that:

1. each $N_\eta$ is a countable elementary substructure of $(H(\theta), \in, <_\theta, (T, I))$,
2. if $\eta \triangleleft \nu$ in $T$, then $N_\eta \subseteq N_\nu$,
3. for each $\eta$ in $T$, $\eta$ is in $N_\eta$.

Definition 5.4. Suppose $\langle T, I \rangle$ is an $\mathfrak{I}$-tree, and $\theta$ is a regular cardinal such that $H(\theta)$ contains a front in $T[\mathfrak{I}]$. Then $\langle T, I \rangle$ is an $\mathfrak{I}$-suitable tree of models for $\theta$ if it is a tree of models for $\theta$ and for every $\eta$ in $T$ and $I$ in $\mathfrak{I} \cap N_\eta$, the set

$\{\nu \in T[\eta] : \nu$ is a splitting point and $I(\nu) = I\}$

contains a front in $T[\mathfrak{I}]$.

Definition 5.5. Let $\langle T, I \rangle$, $\mathfrak{I}$, and $\theta$ be as in Definition 5.4. A sequence $\langle N_\eta : \eta \in T \rangle$ is an $\omega_1$-strictly $\mathfrak{I}$-suitable tree of models for $\theta$ if it is an $\mathfrak{I}$-suitable tree of models for $\theta$ and there exists $\delta < \omega_1$ such that for all $\eta$ in $T$, $N_\eta \cap \omega_1 = \delta$.

If $\langle N_\eta : \eta \in T \rangle$ is a tree of models and $b$ is a cofinal branch of $T$, write $N_b$ for the set $\bigcup\{N_{\eta[n]} : n < \omega\}$. Note that if $\langle N_\eta : \eta \in T \rangle$ is an $\omega_1$-strictly $\mathfrak{I}$-suitable tree of models for $\theta$, then for any cofinal branch $b$ of $T$, $N_b \cap \omega_1 = N_0 \cap \omega_1$.

Lemma 5.6. Let $\langle T, I \rangle$, $\mathfrak{I}$, and $\theta$ be as in Definition 5.4, and let $\langle N_\eta : \eta \in T \rangle$ be an $\omega_1$-strictly $\mathfrak{I}$-suitable tree of models for $\theta$. Suppose $\eta < \nu$ in $T$ and $(N_\nu \cap \omega_2) \setminus N_\eta$ is non-empty. Let $\gamma$ be the minimum element of $(N_\nu \cap \omega_2) \setminus N_\eta$. Then $\gamma \geq \sup(N_\eta \cap \omega_2)$.

Proof. Otherwise there is $\beta$ in $N_\eta \cap \omega_2$ such that $\gamma < \beta$. By elementarity, there is a surjection $f : \omega_1 \rightarrow \beta$ in $N_\eta$. So $f^{-1}(\gamma) \in N_\nu \cap \omega_1 = N_\eta \cap \omega_1$. Hence $f(f^{-1}(\gamma)) = \gamma$ is in $N_\eta$, which is a contradiction. □

Let $\mathfrak{I}$ be a family of ideals. We say that $\mathfrak{I}$ is restriction-closed if for all $I$ in $\mathfrak{I}$, for any set $A$ in $I^+$, the ideal $I \upharpoonright A$ is in $\mathfrak{I}$. If $\mu$ is a regular uncountable cardinal, we say that $\mathfrak{I}$ is $\mu$-complete if each ideal in $\mathfrak{I}$ is $\mu$-complete.

Definition 5.7. Suppose that $\mathfrak{I}$ is a non-empty restriction-closed $\omega_2$-complete family of ideals and let $\mathbb{P}$ be a forcing poset. Then $\mathbb{P}$ satisfies the $\mathfrak{I}$-universal property
if for all sufficiently large regular cardinals \( \theta \) with \( \mathbb{I} \mid H(\theta) \), if \( \langle N^\eta_\eta ; : \eta \in T \rangle \) is an \( \omega_1 \)-strictly \( \mathfrak{I} \)-suitable tree of models for \( \theta \), then for all \( p \in N^\eta_0 \cap \mathcal{P} \) there is \( q \leq p \) such that \( q \) forces there is a cofinal branch \( b \) of \( T \) such that \( N^\eta_0[\mathcal{G}] \cap \omega_1 = N^\eta_0 \cap \omega_1 \).

Definition 5.7 is Shelah’s characterization of the \( \mathbb{I} \)-universal property given in [10] Chapter XV 2.11, 2.12, and 2.13. Note that in the definition, \( q \) is semigeneric for \( N^\eta_0 \). In 2.12 Shelah proves that there are stationarily many structures \( N \) for which \( N = N^\eta_0 \) for some \( \omega_1 \)-strictly \( \mathfrak{I} \)-suitable tree of models \( \langle N^\eta_\eta ; : \eta \in T \rangle \). So by standard arguments if \( \mathcal{P} \) satisfies the \( \mathbb{I} \)-universal property then \( \mathcal{P} \) preserves \( \omega_1 \) and preserves stationary subsets of \( \omega_1 \). Note that any semiproper forcing poset satisfies the \( \mathbb{I} \)-universal property.

**Theorem 5.8.** Let \( \mathbb{I} \) be the family of ideals of the form \( NS_{\omega_2} \upharpoonright A \), where \( A \) is a stationary subset of \( \omega_2 \cap \text{cof}(\omega) \). Let \( \dot{S} \) be an \( \text{Add} \) name for the set \( \langle P_{\omega_1}(\omega_2) \setminus V \rangle \). Then \( \text{Add} \ast \mathcal{P}(\dot{S}) \) satisfies the \( \mathbb{I} \)-universal property.

**Proof.** Fix a regular cardinal \( \theta \gg \omega_2 \) and let \( \langle N^\eta_\eta ; : \eta \in T \rangle \) be an \( \omega_1 \)-strictly \( \mathfrak{I} \)-suitable tree of models for \( \theta \). Let \( p \ast \dot{q} \) be a condition in \( \langle \text{Add} \ast \mathcal{P}(\dot{S}) \rangle \cap N^\eta_0 \). We find a refinement of \( p \ast \dot{q} \) which forces there is a cofinal branch \( b \) of \( T \) such that \( N^\eta_0[\mathcal{G} \ast \dot{H}] \cap \omega_1 = N^\eta_0 \cap \omega_1 \).

We use an argument similar to the proof of Lemma 2.6 to define a sequence \( \langle \eta_\xi, \xi ; : s \in \langle \omega \rangle \rangle \) satisfying:

1. Each \( \eta_\xi \) is in \( T \), each \( \xi_\zeta \) is in \( N^\eta_\eta \cap \omega_2 \), and \( s \prec t \) implies \( \eta_s < \eta_t \).
2. If \( s \prec 0 \) or \( t \) then \( \xi_\zeta-1 \) is not in \( N^\eta_{\eta_0} \), and if \( s \prec 1 \) or \( t \) then \( \xi_\zeta-0 \) is not in \( N^\eta_{\eta_0} \).

Let \( \eta_0 = \emptyset \) and \( \xi_0 = 0 \). Suppose \( \eta_s \) is defined. Choose a splitting point \( \nu_\zeta \) in \( T \) above \( \eta_s \). Let \( Z \) denote the set of \( \alpha < \omega_2 \) such that \( \nu_\zeta \prec \alpha \) is in \( T \). Since \( \nu_\zeta \) is a splitting point, by the definition of \( I \) the set \( Z \) is a stationary subset of \( \omega_2 \cap \text{cof}(\omega) \). For each \( \alpha \) in \( Z \), \( \alpha \) is in \( N^\eta_{\nu_\zeta-\alpha} \) and has cofinality \( \omega \), so \( N^\eta_{\nu_\zeta-\alpha} \cap \alpha \) is a countable cofinal subset of \( \alpha \). Define \( X_\zeta \) as the set of \( \xi \) in \( \omega_2 \) such that the set

\[ \{ \alpha \in Z : \xi \in N^\eta_{\nu_\zeta-\alpha} \cap \alpha \} \]

is stationary. An easy argument using Fodor’s Lemma shows that \( X_\zeta \) is unbounded in \( \omega_2 \). For all large enough \( \alpha \) in \( Z \), the set \( (X_\zeta \setminus \text{sup}(N^\eta_{\nu_\zeta} \cap \omega_2)) \cap \alpha \) has size \( \omega_1 \). So there is a stationary set \( Z_1' \subseteq Z \) and an ordinal \( \xi_\zeta-0 \) in \( X_\zeta \) such that \( \xi_\zeta-0 \) is larger than \( \text{sup}(N^\eta_{\nu_\zeta} \cap \omega_2) \) and for all \( \alpha \) in \( Z_1' \), \( \xi_\zeta-0 \) is not in \( N^\eta_{\nu_\zeta-\alpha} \cap \alpha \). Let \( Z_1^0 \) be the stationary set of \( \alpha \) in \( Z_1' \) such that \( \xi_\zeta-0 \) is in \( N^\eta_{\nu_\zeta-\alpha} \cap \alpha \). Now define \( Y_\zeta \) as the set of \( \xi \) in \( \omega_2 \) such that the set

\[ \{ \alpha \in Z_1' : \xi \in N^\eta_{\nu_\zeta-\alpha} \cap \alpha \} \]

is stationary. Again we can find \( Z_0 \subseteq Z_1^0 \) stationary and \( \xi_\zeta-1 \) in \( Y_\zeta \) such that \( \xi_\zeta-1 \) is larger than \( \text{sup}(N^\eta_{\nu_\zeta} \cap \omega_2) \) and for all \( \alpha \) in \( Z_0 \), \( \xi_\zeta-1 \) is not in \( N^\eta_{\nu_\zeta-\alpha} \cap \alpha \). Let \( Z_1 \) be the stationary set of \( \alpha \) in \( Z_1' \) such that \( \xi_\zeta-1 \) is in \( N^\eta_{\nu_\zeta-\alpha} \cap \alpha \).

Now define \( \eta_\zeta-0 \) to be equal to \( \nu_\zeta \prec \alpha \) for some \( \alpha \) in \( Z_0 \) larger than \( \xi_\zeta-1 \), and define \( \eta_\zeta-1 \) to be \( \nu_\zeta \prec \beta \) for some \( \beta \) in \( Z_1 \) larger than \( \xi_\zeta-0 \). By definition \( \xi_\zeta-0 \) is in \( N^\eta_{\nu_\zeta-\alpha} \) and \( \xi_\zeta-1 \) is in \( N^\eta_{\nu_\zeta-\alpha} \).

We claim that if \( \eta_\zeta-0 \leq \nu \) in \( T \), then \( \xi_\zeta-1 \) is not in \( N_\nu \). Since \( \alpha \) is in \( Z_0 \), \( \xi_\zeta-1 \) is not in \( N^\eta_{\nu_\zeta-\alpha} \cap \alpha \). But \( \xi_\zeta-1 < \alpha \), so \( \xi_\zeta-1 \) is not in \( N^\eta_{\nu_\zeta-\alpha} \). By Lemma 5.6 the minimum element of \( N^\eta_{\nu_\zeta} \cap \omega_2 \) which is not in \( N^\eta_{\nu_\zeta-\alpha} \) if such an ordinal exists, is at least \( \text{sup}(N^\eta_{\nu_\zeta-\alpha} \cap \omega_2) \geq \alpha > \xi_\zeta-1 \). So \( \xi_\zeta-1 \) is not in \( N^\eta_{\nu_\zeta} \). Similarly if \( \eta_\zeta-1 \leq \nu \) in \( T \), then \( \xi_\zeta-0 \) is not in \( N^\eta_{\nu_\zeta} \). This completes the definition. Conditions (1) and (2) are now easily verified.
Since \( \mathbb{P} \) is \( \omega_1 \)-c.c., the condition \( p \) itself is generic for each \( N_\eta \). Let \( G \) be a generic filter for \( \text{Add} \) over \( V \) which contains \( p \). Then for all \( \eta \) in \( T \), \( N_\eta[G] \cap n = N_\eta \cap n \).

So for any cofinal branch \( b \) of \( T \) in \( V[G] \), \( N_b[G] \cap n = \bigcup \{ N_b[n] \cap n : n < \omega \} \); in particular, \( N_b[G] \cap \omega_1 = N_0 \cap \omega_1 \).

Let \( f : \omega \to 2 \) be a function in \( V[G] \setminus V \). Define \( f = \bigcup \{ \eta(f[n] : n < \omega) \} \). We prove that \( N_b \cap \omega_2 \) is not in \( V \) by showing how to define \( f \) inductively from this set. Suppose \( f \downarrow n \) is known. Fix \( j < 2 \) such that \( f(n) \neq j \). We claim that \( \xi^* = \xi_{(f(n))} \) is not in \( N_b \cap n \). Otherwise there is \( k > n \) such that \( \xi^* \) is in \( N_{\eta(k)} \). But \( f \uparrow (n + 1) \approx f \uparrow k \). So by condition (2), \( \xi^* \) is not in \( N_{\eta(k)} \), which is a contradiction. So \( f(n) \) is the unique \( i < 2 \) such that \( \xi_{(f(n))} = i \) in \( N_b \cap \omega_2 \).

Let \( \langle D_\alpha : n < \omega \rangle \) enumerate all the dense subsets of \( \mathbb{P}(S) \) lying in \( N_{b_\alpha}[G] \). Inductively define a sequence \( \langle q_n : n < \omega \rangle \) by letting \( q_0 = q \) and choosing \( q_{n+1} \) to be a refinement of \( q \) in \( D_n \cap N_{b_\alpha}[G] \). Let \( \langle b_i : i < \gamma \rangle = \bigcup \{ q_n : n < \omega \} \). Clearly \( \bigcup \{ b_i : i < \gamma \} = N_{b_\alpha} \cap n \). Since \( N_{b_\alpha} \cap n \) is not in \( V \), \( r = \langle b_i : i < \gamma \rangle \) is a \( \mathbb{P}_\alpha \) family of ideals in \( N_{b_\alpha}[G] \) that satisfies the \( \mathbb{P}_\alpha \) condition below \( q \) and \( r \) and \( r' \) forces \( N_{b_\alpha}[G] \uparrow \omega_1 = N_{b_\alpha}[G] \cap n = N_0 \cap n \). Let \( r \) be a name for \( r \). Then \( p * r \leq p * q \) as required. □

We state without proof the facts concerning RCS iterations which we shall use. These facts follow immediately from [10] Chapter XI 1.13 and Chapter XV 4.15.

**Theorem 5.9.** Suppose \( \langle \mathbb{P}_1, \mathbb{Q}_1 : i \leq \alpha, j < \alpha \rangle \) is an RCS iteration. Then \( \mathbb{P}_\alpha \) preserves \( \omega_1 \) if the iteration satisfies the following properties:

1. for each \( i < \alpha \) there is \( n < \omega \) such that \( \mathbb{P}_{i+n} \Vdash \langle \mathbb{P}_i \rangle \leq \omega_1 \),
2. for each \( i < \alpha \) there is an uncountable regular cardinal \( \kappa_i \) and a \( \mathbb{P}_i \)-name \( \dot{I}_i \) such that \( \mathbb{P}_i \) preserves \( \dot{I}_i \) as a name for some non-empty restriction-closed \( \kappa_i \)-complete family of ideals in \( \dot{I}_i \) that is \( \omega_1 \)-universal.

**Theorem 5.10.** Let \( \alpha \) be a strongly inaccessible cardinal. Suppose that \( \langle \mathbb{P}_1, \mathbb{Q}_1 : i \leq \alpha, j < \alpha \rangle \) is a revised countable support iteration such that \( \mathbb{P}_\alpha \) preserves \( \omega_1 \) and for all \( i < \alpha \), \( | \mathbb{P}_i | < \alpha \). Then \( \mathbb{P}_\alpha \) is \( \alpha \)-c.c.

Suppose \( \kappa \) is a Mahlo cardinal and let \( A \) be the stationary set of strongly inaccessible cardinals below \( \kappa \). Define an RCS iteration \( \langle \mathbb{P}_1, \mathbb{Q}_1 : i \leq \alpha, j < \alpha \rangle \) by recursion as follows. Our recursion hypotheses will include that each \( \mathbb{P}_\alpha \) preserves \( \omega_1 \), and is \( \alpha \)-c.c. if \( \alpha \) is in \( A \).

Suppose \( \mathbb{P}_\alpha \) is defined. If \( \alpha \) is not in \( A \) then let \( \mathbb{Q}_\alpha \) be a name for \( \text{Coll}(\omega_1, | \mathbb{P}_\alpha |) \). Suppose \( \alpha \) is in \( A \). By the recursion hypotheses \( \mathbb{P}_\alpha \) forces \( \alpha = \omega_2 \). Let \( \dot{Q}_\alpha \) be a name for the poset \( \text{Add} * \mathbb{P}(S) \).

If \( \alpha \) is not in \( A \) then choose some regular cardinal \( \kappa_\alpha \) larger than \( | \mathbb{P}_\alpha | \), and let \( \dot{I}_\alpha \) be a name for some non-empty restriction-closed \( \kappa_\alpha \)-complete family of ideals on \( \kappa_\alpha \). Then \( \mathbb{P}_\alpha \) is \( \kappa_\alpha \)-c.c., and since \( \mathbb{Q}_\alpha \) is proper, \( \mathbb{P}_\alpha \) forces \( \dot{Q}_\alpha \) satisfies the \( \dot{I}_\alpha \)-universal property. Suppose \( \alpha \) is in \( A \). Then let \( \alpha = \kappa_\alpha \) and define \( \dot{I}_\alpha \) as a name for the family of ideals on \( \omega_2 \) as described in Theorem 5.8. Then \( \mathbb{P}_\alpha \) is \( \kappa_\alpha \)-c.c. and forces \( \dot{Q}_\alpha \) satisfies the \( \dot{I}_\alpha \)-universal property.

Suppose \( \beta \leq \kappa \) is a limit ordinal and \( \mathbb{P}_\alpha \) is defined for all \( \alpha < \beta \). Define \( \mathbb{P}_\beta \) as the revised countable support limit of \( \langle \mathbb{P}_\alpha : \alpha < \beta \rangle \). By Theorem 5.9 and the recursion hypotheses, \( \mathbb{P}_\beta \) preserves \( \omega_1 \). Hence if \( \beta \) is in \( A \cup \{ \kappa \} \), then \( \mathbb{P}_\beta \) is \( \beta \)-c.c. by Theorem 5.10.

This completes the definition. Let \( G \) be generic for \( \mathbb{P}_\kappa \). The poset \( \mathbb{P}_\kappa \) is \( \kappa \)-c.c. and preserves \( \omega_1 \), so in \( V[G] \) we have that \( \kappa = \omega_2 \) and \( A \) is a stationary subset of
\(\omega_2 \cap \text{cof}(\omega_1)\). For each \(\alpha\) in \(A\) let \(C_\alpha\) be the club on \(P_{\omega_2}(\alpha)\) introduced by \(Q_\alpha\). If \(\alpha < \beta\) are in \(A\), then \(C_\alpha\) and \(C_\beta\) are disjoint since \(C_\beta\) is disjoint from \(V[G | G]\). So \(\langle C_\alpha : \alpha \in A \rangle\) is a disjoint club sequence on \(\omega_2\) in \(V[G]\).

We conclude the paper with several questions.

(1) Assuming Martin’s Maximum, the poset \(\text{Add} * P(\dot{S})\) is semiproper. Is this poset semiproper in general?

(2) Is it consistent that there exists a stationary set \(A \subseteq \omega_2 \cap \text{cof}(\omega_1)\) such that neither \(A \cup \text{cof}(\omega)\) nor \(\omega_2 \setminus A\) can acquire a club subset in an \(\omega_1\) and \(\omega_2\) preserving extension?

(3) To what extent can the results of this paper be extended to cardinals greater than \(\omega_2\)? For example, is it consistent that there is a fat stationary subset of \(\omega_3\) which cannot acquire a club subset by any forcing poset which preserves \(\omega_1\), \(\omega_2\), and \(\omega_3\)?

References


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