Suppose $M$ is a countable transitive model of ZFC
Then $M$ has many set-generic extensions (uncountably many)

Turn this around:
$M$ is a set-generic restriction of $V$ iff
$V$ is a set-generic extension of $M$

Questions:
1. How many set-generic restrictions does a countable $V$ have?
2. Can we “characterise” the set-generic restrictions of $V$?
Laver’s Theorem

In fact, a countable $V$ has only countably many set-generic restrictions:

**Theorem**

*(Laver)* Suppose that $V$ is a set-generic extension of $M$. Then $M$ is a definable inner model of $V$ (with parameters).

**Proof.** Choose a $V$-regular $\kappa$ so that $P$ belongs to $H(\kappa)^M$, where $V$ is $P$-generic over $M$. We need three facts:

1. $M$ $\kappa$-covers $V$: Any subset $X$ of $M$ in $V$ of size $< \kappa$ in $V$ is a subset of such a set in $M$.

This is because if $f$ maps some ordinal $\alpha < \kappa$ onto $X$ then for each $i < \alpha$ there are $< \kappa$ possibilities for $f(i)$, given by the $< \kappa$ different forcing conditions.
2. $M \kappa$-approximates $V$: If $X$ is a subset of $M$ in $V$ all of whose size $< \kappa$ $M$-approximations (i.e., intersections with size $< \kappa$ elements of $M$) belong to $M$, then $X$ also belongs to $M$.

This is because if $\dot{X}$ is forced not to be in $M$ then we can choose for each condition a set in $M$ whose membership in $\dot{X}$ is not decided by that condition; no condition can force the intersection of $\dot{X}$ with the resulting size $< \kappa$ set of elements of $M$ to be in $M$. 
Laver’s Theorem

3. If $N$ is an inner model which $\kappa$-covers and $\kappa$-approximates $V$ such that $M, N$ have the same $H(\kappa^+)$ then $M = N$.

By $\kappa$-approximation it’s enough to show that any set $X$ of ordinals of size $<\kappa$ in $M$ also belongs to $N$ (and vice-versa). Build a $\kappa$-chain $X = X_0 \subseteq X_1 \subseteq \cdots$ of sets of size $<\kappa$ such that $X_{2\alpha+1}$ belongs to $M$ and $X_{2\alpha+2}$ belongs to $N$. If $Y$ is the union of the $X_\alpha$’s then by $\kappa$-approximation, $Y$ belongs to $M \cap N$. But as $M, N$ have the same $H(\kappa^+)$ they also have the same subsets of the ordertype of $Y$ and therefore the same subsets of $Y$. It follows that $X$ belongs to $N$.

Finally: All of this holds with $M, V$ replaced by $H(\lambda)^M, H(\lambda)$ for $V$-regular cardinals $\lambda > \kappa^+$. So $H(\lambda)^M$ is definable in $V$ from $\lambda$, $H(\kappa^+)^M$ uniformly in $\lambda$, so $M$ is $V$-definable.
Another easy consequence of set-genericity is the following.

**Proposition**

Suppose that $V$ is a set-generic extension of $M$. Then $M$ globally covers $V$: For some $V$-regular $\kappa$, if $f : \alpha \to M$ belongs to $V$ then there is $g : \alpha \to M$ in $M$ such that $f(i) \in g(i)$ and $g(i)$ has $V$-cardinality $< \kappa$ for all $i < \alpha$.

To see this define $g(i)$ to be the set of possible values of $f(i)$ given by the different forcing conditions. We can choose any $\kappa$ so that the forcing is $\kappa$-cc.

Surprisingly, we now know enough to characterise set-generic restrictions.
Theorem

(Bukovsky) Suppose that $M$ is a definable inner model which globally covers $V$. Then $V$ is a set-generic extension of $M$.

I’ll give a proof of this and discuss some refinements and open questions.

First suppose that $V = M[A]$ for some set of ordinals $A$; we’ll get rid of this extra hypothesis later.

Fix a $V$-regular $\kappa$ such that $A$ is a subset of $\kappa$ and $M$ globally $\kappa$-covers $V$, i.e., if $f : \alpha \to M$ in $V$ then there is $g : \alpha \to M$ in $M$ so that $f(i) \in g(i)$ and $g(i)$ has $V$-cardinality $< \kappa$ for each $i < \alpha$. 
Bukovsky’s Theorem

The language \( \mathcal{L}^{QF}_\kappa (M) \)

The formulas of \( \mathcal{L}^{QF}_\kappa (M) \) are defined inductively by:

1. Basic formulas \( \alpha \in \dot{\Lambda}, \alpha \notin \dot{\Lambda} \) for \( \alpha < \kappa \).
2. If \( \Phi \in M \) is a size \( < \kappa \) set of formulas then so are \( \bigvee \Phi \) and \( \bigwedge \Phi \).

Each formula can be regarded as an element of \( H(\kappa)^M \). The set of formulas forms a \( \kappa \)-complete Boolean algebra in \( M \), denoted by \( B^M_\kappa \).

\( A \subseteq \kappa \) satisfies \( \varphi \) iff \( \varphi \) is true when \( \dot{\Lambda} \) is replaced by \( A \).

\( T \models \varphi \) iff for all \( A \subseteq \kappa \) (in a set-generic extension of \( M \)), if \( A \) satisfies all formulas in \( T \) then \( A \) also satisfies \( \varphi \).

The above is expressible in \( M \) for \( T, \varphi \) in \( M \) and by Lévy absoluteness, \( T \models \varphi \) in \( M \) iff \( T \models \varphi \) in \( V \).
Bukovsky’s Theorem

Quotients of $\mathcal{B}_\kappa^M$: Suppose that $T$ is a set of formulas in $\mathcal{B}_\kappa^M(2^{<\kappa})^+$. Then $\mathcal{I}_T$ is the ideal of formulas in $\mathcal{B}_\kappa^M$ which are inconsistent with $T$.

Now we prove the genericity of $A$ over $M$. Recall that $M$ globally $\kappa$-covers $V$. Let $f$ be a function in $V$ from subsets of $\mathcal{B}_\kappa^M$ in $M$ to $\mathcal{B}_\kappa^M$ such that:
If $A$ satisfies some $\psi \in \Phi$ then $A$ satisfies $f(\Phi) \in \Phi$.
Using a wellorder in $M$ we can regard $f$ as a function from some ordinal into $M$. Apply global $\kappa$-covering to get $g$ in $M$ so that $g(\Phi) \subseteq \Phi$ has size $< \kappa$ and $f(\Phi) \in g(\Phi)$ for each $\Phi$.
Consider the following set of formulas $T$ in $\mathcal{B}_\kappa^M(2^{<\kappa})^+$:
$$T = \{ (\forall \Phi \rightarrow \forall g(\Phi)) \mid \Phi \subseteq \mathcal{B}_\kappa^M, \ \Phi \in M \}.$$ Let $P$ be the forcing $(\mathcal{B}_\kappa^M \setminus \mathcal{I}_T)/\mathcal{I}_T$ the set of $T$-consistent formulas modulo $T$-provability.
Claim 1. \( P = (\mathcal{B}_\kappa^M \setminus \mathcal{I}_T)/\mathcal{I}_T \) is \( \kappa \)-cc.

Proof. Suppose that \( \Phi \) is a maximal antichain in \( P \). We show that \( g(\Phi) = \Phi \) (and therefore \( \Phi \) has size \( < \kappa \)). It suffices to show that any \( \varphi \in \Phi \) is \( T \)-consistent with some element of \( g(\Phi) \). Choose any \( B \subseteq \kappa \) which satisfies \( T \cup \{ \varphi \} \) (this is possible because \( \varphi \) is \( T \)-consistent). As \( T \) includes the formula \( \bigvee \Phi \rightarrow \bigvee g(\Phi) \) it follows that \( B \) also satisfies \( \bigvee g(\Phi) \) and therefore \( \psi \) for some \( \psi \in g(\Phi) \). So \( \varphi \) is \( T \)-consistent with \( \psi \in g(\Phi) \). \( \Box \)

Claim 2. Let \( G(A) \) be \( \{ [\varphi]_{\mathcal{I}_T} \mid \varphi \text{ belongs to } \mathcal{B}_\kappa^M \text{ and } A \text{ satisfies } \varphi \} \). Then \( G(A) \) is \( \mathcal{P} \)-generic over \( M \).

Proof. Suppose that \( \Phi \) consists of representatives of a maximal antichain \( X \) of equivalence classes in \( P \). Then \( T \models \bigvee \Phi \), else the negation of \( \bigvee \Phi \) represents an equivalence class violating the maximality of \( X \). As \( A \) satisfies the theory \( T \) it follows that \( A \) satisfies some element of \( \Phi \) and therefore \( G(A) \) meets \( X \). \( \Box \)
It now follows that $M[A]$ is a $P$-generic extension of $M$, as $M[A] = M[G(A)]$.

This proves Bukovsky’s theorem assuming that $V = M[A]$ for some set of ordinals $A$.

But the same proof shows that $M[A]$ is a $\kappa$-cc generic extension of $M$ for any set of ordinals $A \in V$. Choose $A$ so that $M[A]$ contains all subsets of $2^{<\kappa}$ in $V$. Then $M[A]$ must equal all of $V$; Otherwise for some set $B$ of ordinals in $V$, $M[A, B]$ is a nontrivial $\kappa$-cc generic extension of $M[A]$ and therefore adds a new subset of $2^{<\kappa}$ to $M[A]$. 
The above proof shows that for $M$ a definable inner model of $V$:

$V$ is a $\kappa$-cc forcing extension of $M$ iff

$M$ globally $\kappa$-covers $V$

Is there a similar characterisation with “$\kappa$-cc” replaced by “size at most \(\kappa\)”?
Bukovsky’s Theorem: Refinements

\( M \ \kappa \text{-decomposes} \ V \) iff every subset of \( M \) in \( V \) is the union of at most \( \kappa \)-many subsets, each of which belongs to \( M \).

**Proposition**

\( V \) is a size at most \( \kappa \) forcing extension of \( M \) iff \( M \) globally \( \kappa^+ \)-covers and \( \kappa \)-decomposes \( V \).

**Proof.** For the easy direction, suppose that \( V = M[G] \) where \( G \) is \( P \)-generic and \( P \) has size at most \( \kappa \). As \( P \) is \( \kappa^+ \)-cc it follows that \( M \) globally \( \kappa^+ \)-covers \( V \). To show that \( M \ \kappa \)-decomposes \( V \), suppose that \( X \in V \) is a subset of \( M \) and choose \( Y \in M \) that covers \( X \). Let \( \dot{X} \) be a name for \( X \) and for each \( p \in G \) let \( X_p \) consist of those \( x \in M \) such that \( p \) forces \( x \in \dot{X} \). Then the \( X_p \)'s give the desired \( \kappa \)-decomposition of \( X \).
Conversely, suppose that $M$ globally $\kappa^+$-covers and $\kappa$-decomposes $V$. By Bukovsky’s Theorem, $V$ is a $P$-generic extension of $M$ for some $P$ which is $\kappa^+$-cc. We want to argue that $P$ is equivalent to a forcing of size at most $\kappa$. We may assume that $P$ is in fact a complete $\kappa^+$-cc Boolean algebra which we write as $B$.

Write $V$ as $M[G]$ where $G$ is $B$-generic over $M$. Take a $B$-name for a $\kappa$-decomposition $\dot{G} = \bigcup_{i < \kappa} \dot{G}_i$ of $\dot{G}$, where each $\dot{G}_i$ is forced to belong to $M$. For each $i < \kappa$ let $X_i$ be a maximal antichain of conditions in $B$ which decide a specific value in $M$ for $\dot{G}_i$. For each $p$ in $X_i$ let $p(\dot{G}_i)$ denote the value of $\dot{G}_i$ forced by $p$ and $b(p)$ the meet of the conditions in $p(\dot{G}_i)$; $b(p)$ is a nonzero Boolean value because if $G_p$ is generic below $p$ then $G_p$ must contain a condition below each element of $p(\dot{G}_i)$. Let $D$ be the set of $b(p)$ for $p$ in the union of the $X_i$’s.
Claim. $D$ is dense in $B$.

If $q$ belongs to $P$ then some $r$ below $q$ forces that $q$ belongs to $\dot{G}_i$ for some $i$; we can assume that $r$ extends some element $p$ of $X_i$. But then as $p$ decides a value for $\dot{G}_i$, it also forces that $q$ belongs to $\dot{G}_i$ and therefore $q$ is extended by $b(p) \in D$. □

We have characterised $\kappa$-cc generic extensions and size at most $\kappa$ generic extensions in terms of covering and decomposition properties. As a result, these properties are $\Pi_2$ properties of $V$ with a predicate for $M$.

Question. Is the property “$V$ is a set-forcing extension of $M$” a strictly $\Sigma_3$ property of $V$ with a predicate for $M$?
Bukovsky’s Theorem: Refinements

Class Forcing

I don’t know a good version of Laver, Bukovsky for class forcing. Below is a special case.

Morse-Kelley Class Theory MK: Can form new classes by quantifying over classes.

Models of MK (with global choice) correspond to models of:
1. ZFC$^-$ (without Power)
2. There is an inaccessible cardinal $\kappa$
3. Every set has cardinality at most $\kappa$

Call this theory SetMK.
Suppose that $M \subseteq V$ are models of SetMK, $M$ is definable in $V$ and $\kappa$ is the largest cardinal of $V$. Then every element of $V$ is in a $\kappa$-cc set-generic extension of $M$ iff:

$\ast$ For any $V$-definable function $f : M \rightarrow \kappa$ there is an $M$-definable $g : M \rightarrow \kappa$ which dominates $f$.

In terms of models of MK (with global choice) this says:
Bukovsky’s Theorem: Refinements

**Theorem**

Suppose that \((M, C^M) \subseteq (V, C^V)\) are models of MK with global choice and \(C^M\) is definable in \((V, C^V)\) (by a formula which quantifies over classes). Then each class in \(C^V\) belongs to a class-generic extension of \((M, C^M)\) via a class forcing whose antichains are sets iff:

\((*)\) For any \((V, C^V)\)-definable function \(f\) from \(C^M\) to \(M\) there is an \((M, C^M)\)-definable function \(g\) from \(C^M\) to \(M\) such that \(f(x) \in g(x)\) for each \(x \in C^M\).

If one goes beyond class theory to hyperclass theory (hyperclasses of classes) then the situation simplifies greatly. In the other direction, working with a weak class theory like Gödel-Bernays looks very difficult.