# CORE MODEL INDUCTION NOTES

Nam Trang
Nam.Trang@unt.edu
Department of Mathematics
University of North Texas, Denton, TX

**Abstract**

The notes contain a fairly complete proof of Steel's theorem that PFA implies the Axiom of Determinacy (AD) holds in $L(\mathbb{R})$. The set-up is somewhat different (in various places); the goal is to set up a framework that allows one to generalize the proof to other situations and possibly go beyond $L(\mathbb{R})$. We also include exercises that help guide the reader through various other similar core model inductions.

## 1. Basic notations

Set up: the following hypotheses are considered

(i) $\kappa$ is singular strong limit and $\square_\kappa$ fails. (CMI below $\kappa$, in some $V^{Col(\omega,\gamma)}$ for some $\gamma < \kappa$ , $\mathrm{cof}(\kappa) < \gamma$, and $\gamma^\omega = \gamma$).

(ii) $\kappa^\omega = \kappa$ and for all $\alpha \in [\kappa^+, (2^\kappa)^+]$, $\square(\alpha)$ fails. (e.g. if PFA holds, can take $\kappa = \omega_2$). (CMI in $V^{Col(\omega,\kappa)}$)

(iii) $\kappa$ is measurable and $\square_\kappa$ fails. (CMI in $V^{Col(\omega,<\kappa)}$).

We want to get AD in $L(\mathbb{R})$ from any of the hypotheses above. In these notes, we will work mainly with the set-up of (i). We will include exercises that help the reader work through the core model induction from the hypotheses of (ii) and (iii).

## 2. Stack of mice

**Definition 2.1.** *Let $A$ be a set of ordinals. An $A$-premouse $\mathcal{M}$ is countably iterable if whenever $\mathcal{N}$ is countable and elementarily embeds into $\mathcal{M}$, $\mathcal{N}$ is $\omega_1 + 1$-iterable.*

*$Lp(A) = \bigcup\{\mathcal{M} : \mathcal{M}$ is a countably iterable $A$-premouse such that $\mathcal{M}$ is sound and $\rho_\omega(\mathcal{M}) = o(A)\}$.*

**Exercise 2.2.** *Show that if $\mathcal{M}, \mathcal{N}$ are countably iterable, sound $A$-premice and both project to $o(A)$, then either $\mathcal{M} \lhd \mathcal{N}$ or $\mathcal{N} \lhd \mathcal{M}$.*

**Lemma 2.3.** *Suppose $\kappa$ is as in $(i)$. Let $A \subseteq \kappa$. Then cof$(Lp(A)) < \kappa$.*

*Proof.* We may as well assume $o(A) = \kappa$. Use the Schimmerling-Zeman construction to see that $\mathrm{Lp}(A) \vDash \square_\kappa$. That $\square_\kappa$ fails in $V$ implies $o(\mathrm{Lp(A)}) < \kappa^+$. Since $\kappa$ is singular, $\mathrm{cof}(\mathrm{Lp(A)}) < \kappa$. $\square$

**Exercise 2.4.** *(a) Suppose $\kappa$ is as in (ii) and $A \subseteq \kappa$. Then cof$(\mathrm{Lp(A)}) \leq \kappa$.*

*(b) Suppose $\kappa$ is as in (iii) and $A \subseteq \kappa$. Then cof$(\mathrm{Lp(A)}) < \kappa$.*

**Remark 2.5.** *The conclusions in Lemma 2.3 and Exercise 2.4 are actually what we need for our proof. These are various forms of the failure of covering for Lp.*

Can also define the stack $\mathrm{Lp}^+(A)$ of $\mathcal{M}$ such that $\mathcal{M}$ is sound, $\rho_\omega(\mathcal{M}) = o(A)$, $\mathcal{M}$ is $\omega_1 + 1$-iterable after collapsing $A$ to $\omega$ (and other variations like $\mathrm{Lp}_\kappa(A)$ to be the stack of $\mathcal{M}$ such that $\mathcal{M}$ is sound, $\rho_\omega(\mathcal{M}) = o(A)$, and for any $\pi : \mathcal{N} \to \mathcal{M}$ such that $|\mathcal{N}| < \omega_1$ in $V[g]$ where $g \subseteq Col(\omega, < \kappa)$ is $V$-generic, $\mathcal{N}$ is $\omega_1 + 1$-iterable in $V[g]$). It often comes up in CMI applications beyond $L(\mathbb{R})$ that we need to know the relationships between the different stacks.

**Exercise 2.6.** *(i) Does the conclusion in Lemma 2.3 and Exercise 2.4 hold for $Lp^+(A)$?*

*(ii) If $\kappa$ is as in (iii) and $A \subseteq \kappa$. How are the stacks $Lp(A)$, $Lp^+(A)$, $Lp_\kappa(A)$ related?*

## 3. Scales in $L(\mathbb{R})$

Again, we work in an appropriate $V[g]$ where $g$ is defined according to whether we are in case (i), (ii), or (iii). The coarse mouse-capturing hypothesis (at $\alpha$) $W^*_\alpha$ says roughly that given a set $A \subseteq \mathbb{R}$, as soon as a scale on $A$ and $\neg A$ appears in $J_\alpha(\mathbb{R})$, a coarse mouse's iteration strategy that term captures the scale appears in $J_\alpha(\mathbb{R})$. We need to analyze how scales appear in $L(\mathbb{R})$ under appropriate determinacy hypotheses.

Let $\mathsf{LST}^+ = \mathsf{LST} \cup \{\dot{R}\}$. So $\mathsf{LST}^+$ is just the language of set theory augmented by an extra constant symbol $\dot{R}$. This symbol is intended to be interpreted as $\mathbb{R}$.

**Definition 3.1.** *An ordinal $\beta$ is critical just in case there is some $U \subseteq \mathbb{R}$ such that $U, \mathbb{R} \backslash U$ admit scales in $J_{\beta+1}(\mathbb{R})$ but $U$ admits no scales in $J_\beta(\mathbb{R})$.*

**Definition 3.2.** *We say that $J_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_1 J_\beta(\mathbb{R})$ if for any $\Sigma_1$ formula $\varphi(v)$ in $\mathsf{LST}^+$, for any real $x$, $J_\alpha(\mathbb{R}) \vDash \varphi[x] \Leftrightarrow J_\beta(\mathbb{R}) \vDash \varphi[x]$.*

**Definition 3.3.** *$[\alpha, \beta]$ is a $\Sigma_1$ gap in $L(\mathbb{R})$ if*

1. *$J_\alpha(\mathbb{R}) \prec^{\mathbb{R}}_1 J_\beta(\mathbb{R})$,*
2. *for any $\gamma < \alpha$, $J_\gamma(\mathbb{R}) \not\prec^{\mathbb{R}}_1 J_\alpha(\mathbb{R})$,*
3. *for any $\gamma > \beta$, $J_\beta(\mathbb{R}) \not\prec^{\mathbb{R}}_1 J_\gamma(\mathbb{R})$*

In a core model induction through $L(\mathbb{R})$, it suffices to show $W^*_{\beta+1}$ holds for all critical $\beta$ to obtain $W^*_\alpha$ for all $\alpha$. In the next section, we'll see how this has anything to do with proving determinacy in $L(\mathbb{R})$. The scales analysis in [12] gives us the following. If $\beta$ is critical then $\beta+1$ is critical. If $\beta$ is a limit of critical ordinals, then $\beta$ is critical iff $J_\beta(\mathbb{R})$ is not admissible. Suppose $\beta$ is critical, then the following are the possibilities:

(A) $\beta = \eta + 1$ for some $\eta$.

(B) $\beta$ is limit of critical ordinals and either

  (a) $\text{cof}(\beta) = \omega$ or

  (b) $\text{cof}(\beta) > \omega$ but $J_\beta(\mathbb{R})$ is not admissible.

(C) $\alpha = sup\{\eta : \eta \text{ is critical}\} < \beta$ and either

  (a) $[\alpha, \beta]$ is a $\Sigma_1$ gap, or

  (b) $\beta - 1$ exists and $[\alpha, \beta - 1]$ is a $\Sigma_1$ gap.

To prove $W^*_{\beta+1}$, we need to feed truth at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R})$ into mice. In cases (A), (B)(a), $\Sigma_1^{J_\beta(\mathbb{R})}$ is a countable union of sets in $J_\beta(\mathbb{R})$; to capture $\Sigma_1^{J_\beta(\mathbb{R})}$, we just need to put together countably many mice given by the inductive hypothesis.

Case (B)(b) requires a new mouse operator to capture $\Sigma_1^{J_\beta(\mathbb{R})}$. We will discuss the so-called *diagonal operator* later.

Case (C)(a) is the *weak gap* case and (C)(b) is the *strong gap* case. We need the following facts to deal with the gap cases.

**Theorem 3.4** (Reflection Theorem, Martin,cf. [2]). *Assume $W^*_\beta$, where Case (C) holds at $\beta$. Then for any $x, y \in \mathbb{R}$, if $x \in OD^\gamma(y)$ for some $\gamma < \beta$, then $x \in OD^\gamma(y)$ for some $\gamma < \alpha$.*

**Theorem 3.5** (Scales Existence Theorem, [12]). *Assume $W^*_\beta$ and $\beta$ is as in Case (C). Then*

1. *every set of reals in $J_\beta(\mathbb{R})$ admits a scale whose individual norms are in $J_\beta(\mathbb{R})$.*
2. *letting $n$ be the least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ and $U$ any boldface $\Sigma_n^{J_\beta(\mathbb{R})}$ set of reals, $U = \bigcup_n U_n$ where $U_n \in J_\beta(\mathbb{R})$ for each $n$.*

**Definition 3.6** (Self-justifying system). *A self-justifying system (sjs) is a countable set $\mathcal{A} \subseteq \wp(\mathbb{R})$ which is closed under complements and scales.*

Thus, if $W^*_\beta$ holds and $\beta$ is as in case (C) then for any $A \in J_\beta(\mathbb{R})$, there is a sjs $\mathcal{A}$ containing $A$. The set coding truth at the bottom of the Levy hierarchy over $J_\beta(\mathbb{R})$ which we feed into our mice will be $\Sigma$, an iteration strategy guided by a sjs $\mathcal{A}$ for a mouse $\mathcal{M}$ with a Woodin cardinal; $\mathcal{M}$ will be full with respect to mice with iteration strategy in $J_\alpha(\mathbb{R})$. The mice which witness $W^*_{\beta+1}$ will be $\mathcal{M}_n^{\Sigma,\sharp}$ for all $n$. Here we need to relativize core model theory for $\Sigma$-mice.

## 4. Capturing

### 4.1. Capturing and determinacy

**Definition 4.1.** *Let $A \subseteq \mathbb{R}$, $(M, \Sigma)$ is a (countable) mouse (pair), and $\delta$ a cardinal in $M$.*

*(a) $(M, \Sigma)$ **term captures $A$ at $\delta$** if there is a term $\tau \in M^{Col(\omega,\delta)}$ such that whenever $i : M \to N$ is according to $N$, and $g \subseteq Col(\omega, i(\delta))$ is $N$-generic, then $A \cap N[g] = i(\tau)_g$.*

*(b) $(M, \Sigma)$ **Suslin captures $A$ at $\delta$** if there is a pair of trees $(T, U) \in M$ such that whenever $i : M \to N$ is according to $N$, and $g \subseteq Col(\omega, i(\delta))$ is $N$-generic, then $A \cap N[g] = p[i(T)]^{N[g]} = \mathbb{R}^{N[g]} \backslash p[i(U)]$.*

*In the above, $\Sigma$ is a $\omega_1 + 1$-iteration strategy of $M$. $M$ need not be fine structural.*

*Without $\Sigma$, we say that $M$ term (Suslin) understands $A$ at $\delta$.*

**Lemma 4.2.** *Suppose $\delta_0 < \delta_1$ are Woodin cardinals in $M$ and $A \subseteq \mathbb{R}$ is Suslin captured by $(M, \Sigma)$ at $\delta_1$. Then $A$ is determined.*

*Proof.* Let $(T, U)$ witness $A$ is Suslin captured at $\delta_1$. Then $N \vDash p[T] = A$ is homogeneously Suslin. (Brief proof: by Woodin, $T, U$ are weakly $\kappa$-homogeneous for all $\kappa < \delta_1$. By Martin-Steel, $A$ is $\kappa$-homegenous for all $\kappa < \delta_0$).

So $N \vDash A$ is determined. Say $\tau \in N$ is a winning strategy for player I. We claim that in $V$, $\tau$ is I's winning strategy. Suppose $y$ is a play by II defeating $\tau$ in $V$. Let $i : M \to N$ come from a $y$-genericity iteration according to $\Sigma$. Let $g \subseteq Col(\omega, i(\delta_0))$ be $N$-generic such that $y \in N[g]$. In $N[g]$, $\tau(y) \notin p[i(T)]$ so $\tau(y) \in p[i(U)]$. By absoluteness of wellfoundedness, $N \vDash \exists y\ \tau(y) \in p[i(U)]$. Contradiction. □

**Lemma 4.3** (Neeman, [4, 5])**.** *Suppose $\delta$ is a Woodin cardinal in $M$ and $A \subseteq \mathbb{R}$ is Suslin captured by $(M, \Sigma)$ at $\delta$. Then $A$ is determined.*

**Lemma 4.4** (Neeman)**.** *Suppose $\delta$ is a Woodin cardinal in $M$ and $A \subseteq \mathbb{R}$ is term captured by $(M, \Sigma)$ at $\delta$. Then $A$ is determined.*

*Proof sketch.* The lemmata follow from [5, Lemmata 2.2, 2.5]. **Exercise:** Read the statements of [5, Lemmata 2.2, 2.5] and relevant discussions to verify this. □

**Lemma 4.5.** *Suppose $\delta_0 < \delta_1$ are $M$-cardinals such that $\delta_1$ is a Woodin cardinal in $M$ and $A \subseteq \mathbb{R} \times \mathbb{R}$ is term captured by $(M, \Sigma)$ at $\delta_1$. Then $\exists^{\mathbb{R}} A, \forall^{\mathbb{R}} A$ are term captured by $(M, \Sigma)$ at $\delta_0$.*

*Proof.* Let $\tau \in M$ be a $Col(\omega, \delta_0) \times Col(\omega, \delta_1)$-term that captures $A$. Let $\sigma$ be a $Col(\omega, \delta_0)$ term in $M$ defined as: for any $p \in Col(\omega, \delta_0)$, any $\rho \in H^M_{\delta_0^+}$,

$$(p, \rho) \in \sigma \Leftrightarrow \exists q \in Col(\omega, \delta_1)\ (p, q) \Vdash \exists y (y, \rho_{\dot{g}}) \in \tau_{\dot{g} \times \dot{h}}.$$

$\sigma$ is a term for $\exists^{\mathbb{R}}A$. For $\forall^{\mathbb{R}}A$, note that $\forall^{\mathbb{R}}A = \neg\exists^{\mathbb{R}}\neg A$ and if $M$ captures some set $B$ by $\tau$ then $M$ captures $\neg B$ by a term easily definable from $\tau$.

**Exercise:** Show that the term $\sigma$ works for $\exists^{\mathbb{R}}A$. Note that we need $\delta_1$ is Woodin to verify $\sigma$ captures $\exists^{\mathbb{R}}A$; this is because for any real $x \in \exists^{\mathbb{R}}A$, let $y$ be such that $(y, x) \in A$; we need to iterate $M$ by $\Sigma$ to make $y$ generic at $\delta_1$, hence the Woodinness of $\delta_1$ is needed here. □

**Example 4.6.** *$\mathcal{M}_n$ term captures $\Sigma^1_{n+1}, \Pi^1_{n+1}$ sets at its bottom Woodin cardinal (Why?). By the above lemmas, if $\forall x$ $\mathcal{M}^\sharp_n(x)$ exists, then all projective sets are determined.*

**Exercise 4.7** (Hard)**.** *Suppose $\delta$ is Woodin in $M$ and $(M, \Sigma)$ Suslin captures $A \subseteq \mathbb{R}\times\mathbb{R}$ at $\delta$. Then for all $\eta < \delta$, $(M, \Sigma)$ Suslin understands $\forall^{\mathbb{R}}A, \exists^{\mathbb{R}}A$ at $\eta$, i.e. there is a tree $T$ for $\forall^{\mathbb{R}}A$ (similarly for $\exists^{\mathbb{R}}A$) in $M$ such that $p[T] \cap M[g] = \forall^{\mathbb{R}}A \cap M[g]$ for $g \subseteq Col(\omega, \eta)$ $M$-generic.*

The following lemmata will be used substantially in the core model induction arguments of the last section of these notes.

**Lemma 4.8** (Term condensation for sjs)**.** *Let $\mathcal{A}$ be a sjs and $N \in M$ be transitive models of a large fragment of* ZFC. *Suppose $\{\tau_A : A \in \mathcal{A}\} \in M$ is such that for a comeager set $\mathcal{C}$ of $Col(\omega, N)$-generic $g$ over $M$, for each $A \in \mathcal{A}$, $\tau^g_A = A \cap M[g]$. Then for any $\pi : M^* \to M$ $\Sigma_1$ elementary that contains all relevant objects, letting $(N^*, \sigma_A) = \pi^{-1}(N, \tau_A)$, then whenever $g \subseteq Col(\omega, N^*)$ is $M^*$-generic, for all $A \in \mathcal{A}$,*

$$\sigma^g_A = A \cap M^*[g].$$

*Proof.* Fix $A \in \mathcal{A}$ and $(\psi_n : n < \omega)$ be the scale on $A$ such that the corresponding prewellorderings $(\leq_n : n < \omega)$ are in $\mathcal{A}$. Let $\tau_n \in M$ be the term for the prewellorderings $\leq_n$ (i.e. for any $G \in \mathcal{C}$, $\tau^G_n = \leq_n \cap M[g]$). Let $\phi_n \in M$ be the term for the norm $\psi_n$. Let $U_n \in M$ be the term for the n-th level of the tree associated with these norms, i.e. for all $G \in \mathcal{C}$,

$$U^G_n = \{(x \restriction n, \phi^G_0(x), \dots, \phi^G_n(x)) : n < \omega \wedge x \in M[G] \cap A\}$$

It is easy to see that $U_n \in M$ and is independent of $M$-generics $G$ (i.e. for any two $M$-generics $G_0, G_1$, for any $\vec{a}$, $\vec{a} \in U^{G_0}_n$ iff $\vec{a} \in U^{G_1}_n$). **Exercise:** Verify this.

Let $U$ be the term for the tree whose n-th level is $U_n$.

**Claim 4.9.** *For any $M$-generic $G$, $A \cap M[G] \subseteq p[U^G] \subseteq A$.*

*Proof.* $A \cap M[G] \subseteq p[U^G]$ is obvious. Suppose $(x, f) \in [U^G]$. Let $n < \omega$ and $x_n \in A \cap M[G]$ such that $x \restriction n = x_n \restriction n$ and $(x_n, \phi^G_0(x_n) = f(0), \dots, \phi_{n-1^G}(x) = f(n-1)) \in U^G_n$. For any $i$, $\phi^G_i(x_n)$ is eventually constant as $n \to \omega$. Since $(\psi_i : i < \omega)$ is a scale on $A$ so $x \in A$. So $p[U^G] \subseteq A$. □

By the claim and elementarity, over $M^*$, the following holds:

$$\emptyset \Vdash_{Col(\omega, N^*)} \forall x (x \in \sigma_A \to \forall n (x \restriction n, \pi^{-1}(\phi_0(x), \dots, \pi^{-1}(\phi_{n-1}(x)))) \in \pi^{-1}(U_n)).$$

Let $U^*$ be the term for a tree whose levels are $\pi^{-1}(U_n)$. It's easy to see that for any $M^*$-generic $G \subset Col(\omega, N^*)$, $\sigma_A^G \subseteq p[U^G] \subseteq A$. **Exercise:** Verify this.

Run the argument above for $\neg A \in \mathcal{A}$. As a result, we get $\sigma_{\neg A}^G \subseteq \neg A$. This gives us the desired conclusion.

□

**Lemma 4.10** (Term capturing for sets in gaps)**.** *Let $\alpha, \beta$ be such that $[\alpha, \beta]$ is a gap ($\alpha$ could be $\beta$). Suppose $J_\beta(\mathbb{R}) \vDash \mathsf{AD}$. Let $z \in N \cap \mathbb{R}$ and $N$ is a $\mathsf{ZFC}^-$ model. Let $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$. Let $\kappa \in N$ be an $N$-cardinal such that $C_\Gamma((H_{\kappa^+})^N) \subset N$. Let $A \subseteq \mathbb{R}$ be $OD_z^{<\beta}$. Then there is a term $\tau \in N^{Col(\omega,\kappa)}$ such that for comeager many $h \subseteq Col(\omega, \kappa)$ which are $N$-generic,*

$$\tau^h = A \cap N[h].$$

**Exercise:** Read the proof of this (cf. [11, Lemma 3.7.5]).

## 4.2. Capturing hypotheses

Fix a cardinal $\mu$ and $g \subseteq Col(\omega, \mu)$ (or $g \subseteq Col(\omega, <\mu)$) (in (i), $\mu$ is such that $cof(\kappa) < \mu < \kappa$ and $\mu^\omega \subseteq \mu$; in (ii), $\mu = \kappa$; in (iii), $\mu = \kappa$ and $g \subseteq Col(\omega, <\mu)$).

The following definitions occur in $V[g]$.

**Definition 4.11** (($k, A$)-Woodin mouse)**.** *Let $A \subseteq \mathbb{R}$. $(M, \Sigma)$ is a $(k, A)$-Woodin mouse if*

*(a) $M \vDash \mathsf{ZFC}$ and there are $\delta_0 < \cdots < \delta_k$ such that $M \vDash$ "$\delta_i$ is Woodin" for all $i \leq k$.*

*(b) There are trees $(T, U) \in M$ such that $M \vDash T, U$ are $\delta_k^+$-absolutely complemented.*

*(c) For any $i : M \to N$ according to $\Sigma$, $p[i(T)] \subseteq A$ and $p[i(U)] \subseteq \mathbb{R} \backslash A$.*

**Exercise 4.12.** *Show that $(M, \Sigma)$ above Suslin captures $A$ at $\delta_k$.*

**Definition 4.13** ($W_\alpha^*$)**.** *Let $A \subseteq \mathbb{R}$ and suppose there are scales $\vec{\varphi}, \vec{\psi}$ on $A, \neg A$ such that the sequences of prewellorders $\varphi^*, \psi^*$ are in $J_\alpha(\mathbb{R})$. Then for all $k < \omega$, any $x \in \mathbb{R}$, there is a pair $(M, \Sigma)$ such that*

1. *$(M, \Sigma)$ is a $(k, A)$ Woodin mouse.*

2. *$\Sigma \restriction HC \in J_\alpha(\mathbb{R})$.*

**Theorem 4.14.** *$W_\alpha^*$ implies $J_\alpha(\mathbb{R}) \vDash \mathsf{AD}$. Furthermore, $\mathsf{MC}$ holds in $J_\alpha(\mathbb{R})$.*

*Proof.* Suppose not. Let $\gamma + 1 \leq \alpha$ be least such that $J_{\gamma+1}(\mathbb{R}) \vDash \neg\mathsf{AD}$. So $\gamma + 1$ begins a $\Sigma_1$-gap as the statement "there is a non-determined set" is $\Sigma_1$.

Suppose $\gamma$ is not crtical, then $[\alpha, \gamma]$ is a strong gap (as in case (C)(b)), where $\alpha = sup\{\beta : \beta < \gamma$ is critical$\}$. Then by Kechris-Woodin theorem, $J_{\gamma+1}(\mathbb{R}) \vDash \mathsf{AD}$. Contradiction.

So $\gamma$ is critical. Let $U \in J_{\gamma+1}(\mathbb{R}) \backslash J_\gamma(\mathbb{R})$ and let $(N, \Sigma)$ be a $(1, U)$-coarse Woodin mouse. So $(N, \Sigma)$ Suslin captures $U$ at $\delta_1$ (the second Woodin of $N$). This means $U$ is determined. □

To any $\Sigma_1$ formula $\theta(v)$ in $\mathsf{LST}^+$, we associate formulae $\theta^k(v)$ for $k \in \omega$, such that $\theta^k(v)$ is $\Sigma_k$, and for any $\gamma$ and any real $x$,

$$J_{\gamma+1}(\mathbb{R}) \models \theta[x] \Leftrightarrow \exists k < \omega \; J_\gamma(\mathbb{R}) \models \theta^{k[x]}.$$

**Definition 4.15.** *Suppose $\theta(v)$ is a $\Sigma_1$ formula in $\mathsf{LST}^+$ and $z$ is a real. A $\langle \theta, z\rangle$-witness is a sound $(\omega, \omega_1, \omega_1 + 1)$-iterable $z$-mouse $\mathcal{M}$ in which there are $\delta_0 < \cdots < \delta_9$, trees $T, U$ such that $\mathcal{M}$ satisfies*

*(a)* $\mathsf{ZFC}$.

*(b)* $\delta_0, \ldots, \delta_9$ *are Woodin.*

*(c)* $T, U$ *are* $\delta_9^+$*-absolutely complemented.*

*(d) For some $k$, $p[T]$ is the $\Sigma_{k+3}$-theory of $J_\gamma(\mathbb{R})$ (in the language with names for each real), where $\gamma$ is least such that $J_\gamma(\mathbb{R}) \models \theta^k[z]$.*

**Remark 4.16.** *(d) states that $p[T]$ is a maximal consistent $\Sigma_{k+3}$-theory of a well-founded model of $V = L(\mathbb{R}) + \theta^k[z] + \forall \beta J_\beta(\mathbb{R}) \models \neg\theta[z]$. This can be expressed by a formula of the form*

$$\forall^{\mathbb{R}} x_0 \exists^{\mathbb{R}} x_1 \psi(x_0, x_1, (p[T], p[U])), \tag{4.1}$$

*where $\psi$ only involves number quantifiers.*

**Lemma 4.17.** *If there is a $\langle \theta, z\rangle$-witness, then $L(\mathbb{R}) \models \theta[z]$.*

*Proof.* We may assume we have a $\langle \theta, z\rangle$-witness $(\mathcal{M}, \Sigma)$ such that $\Sigma$ has the Dodd-Jensen property for compositions of normal trees.

**Claim 4.18.** *For any $x \in \mathbb{R}$, let $i_0 : \mathcal{M} \to \mathcal{N}_0$, $i_1 : \mathcal{M} \to \mathcal{N}_1$ be iterations according to $\Sigma$ such that $x$ is generic over both $\mathcal{M}_0, \mathcal{M}_1$ at $\delta_l$ for some $l \leq 9$. Then $x \in p[i_0(T)] \Leftrightarrow x \in p[i_1(T)]$.*

*Proof.* Suppose $x \in p[i_0(T)]$ but $x \in p[i_1(U)]$. Execute a simultaneous comparision between $\mathcal{N}_0, \mathcal{N}_1$ and at the same time an $x$-genericity iteration at the images of $\delta_l$. This results in a pair of maps $j_0 : \mathcal{N}_0 \to \mathcal{P}_0$, $j_1 : \mathcal{N}_1 \to \mathcal{P}_1$ such that

1. $j_0 \circ i_0 = j_1 \circ i_1$ and $\mathcal{P}_0 = \mathcal{P}_1$.
2. $x \in \mathcal{P}_0[h] = \mathcal{P}_1[h]$ for $h \subseteq Col(\omega, j_0 \circ i_0(\delta_l))$ $\mathcal{P}_0$-generic.
3. $x \in p[j_0 \circ i_0(T)] \cap p[j_1 \circ i_1(U)]$.

(3) clearly is a contradiction. □

Let $A = \bigcup\{p[i(T)]^{N[g]} : i : \mathcal{M} \to \mathcal{N}$ according to $\Sigma$ and $g \subseteq Col(\omega, i(\delta_0))$ in $V$ is $\mathcal{N}-generic\}$. Now use genericity iterations and the formula in 4.1 to show $A$ is $\Sigma_{k+3}$-theory of a level of $L(\mathbb{R})$ that satisfies $\theta[z]$, so indeed $L(\mathbb{R}) \models \theta[z]$. □

In a core model induction, we need to show the converse of Lemma 4.17 holds (for $\alpha$ limit, ordinal). The following is the more precise statement.

**Definition 4.19** ($W_\alpha$)**.** *Suppose $\theta$ is $\Sigma_1$, $z \in \mathbb{R}$ and $J_\alpha(\mathbb{R}) \vDash \theta[z]$, then there is a $\langle \theta, z\rangle$ witness $\mathcal{M}$ with a strategy $\Sigma$ such that $\Sigma \restriction HC \in J_\alpha(\mathbb{R})$.*

The following uses the notions of gaps in $L(\mathbb{R})$.

**Lemma 4.20.** *Suppose $\alpha$ is a limit ordinal. $W^*_\alpha$ implies $W_\alpha$.*

*Proof.* Let $\beta < \alpha$ (using $\alpha$ limit) be least such that $J_{\beta+1}(\mathbb{R}) \vDash \theta[z]$. So $\beta$ ends a $\Sigma_1$-gap (and $\beta+1$ begins a new gap). Using $W^*_\alpha$ applied to the set $A = \Sigma_3^{J_{\beta+2}(\mathbb{R})}$. Let $M, \Sigma$ be a $(9, A)$-Woodin mouse and $z \in M$ and the trees $(T, U)$ as part of the witness. Note that $A, \neg A$ have scales in $J_\alpha(\mathbb{R})$ by the scales analysis.

Note that every $OD^{\beta+1}(V^M_{\delta_9})$ subset of $V^M_{\delta_9}$ is in $M$. **Exercise:** Verify this. Note the following two facts: (i) the relation $y \in OD^{\beta+1}(x)$ is $\Sigma_1^{J_{\beta+2}(\mathbb{R})}$ because this is equivalent to saying $\exists \mathcal{M}(y \in \mathcal{M} \wedge J_{\beta+2}(\mathbb{R}) \vDash$ "$\mathcal{M}$ is an $\omega_1$-iterable $x$-mouse"; (ii) being $OD^{\beta+1}$-full is $\Pi_1^{J_{\beta+2}(\mathbb{R})}$).

The above fact is recorded in $p[T]$. And this implies for club many $\pi : H \to V^M_\lambda$ for $\lambda > \delta_9$, where $crt(\pi) = \eta < \delta_9$, for every $b \in OD^{\beta+1}(V^M_\eta) \cap \wp(V^M_\eta), b \in H$.

Let $N = L[E, z]$ be the result of the full-background construction in $M$ and let $\eta$ be a cardinal of $N$ in the club above and $N|\eta$ is definable over $M|\eta$. So $\eta$ is not Woodin in $N$. Let $\mathcal{Q} \lhd N$ be the first level of $N$ such that $N|\eta \lhd \mathcal{Q} \wedge \mathcal{Q} \notin OD^{\beta+1}(N|\eta)$. $\mathcal{Q}$ exists because $\eta$ is Woodin in $N$ with respect to all functions in $OD^{\beta+1}(N|\eta)$.[1] $\eta$ is a cutpoint Woodin of $\mathcal{Q}$.

Let $g \subseteq Col(\omega, \eta)$ be $\mathcal{Q}$-generic and $x$ codes $g$ (we may assume $z \leq_T x$). We can rearrange $\mathcal{Q}[g]$ into an $x$-mouse $\mathcal{R}$. Let $\mathbb{R}^* = OD^{\beta+1}(x) \cap \mathbb{R}$. Note that $\mathbb{R}^{\mathcal{R}} = \mathbb{R}^*$; this is because $\mathcal{R}$ is the first $x$-mouse that does not have iteration strategy in $J_{\beta+2}(\mathbb{R})$. There is a unique $\beta^* \in \mathcal{R}$ and a $\Sigma_1$ map

$$j : J_{\beta^*+2}(\mathbb{R}^*) \to J_{\beta+2}(\mathbb{R}).$$

This comes from the fact that every set in $J_{\beta+2}(\mathbb{R})$ has a scale in $J_{\beta+2}(\mathbb{R})$ and every $OD^{\beta+1}(x)$ relation in $J_{\beta+2}(\mathbb{R})$ has a uniformization in $J_{\beta+2}(\mathbb{R})$. If $\Gamma = OD^{\beta+1}$, then $(\mathbb{R}^*, B \cap C_\Gamma(x)) \prec_{\Sigma_1} (\mathbb{R}, B)$ for any $B \in OD^{\beta+1}(x)$; the point is every $OD^{\beta+1}(x)$ set of reals has a member $u$ such that $\{u\} \in C_\Gamma(x)$. Applying this for any $B \in OD^{\beta+1}(x)$ coding a $k$-th reduct, giving a $\Sigma_1$-embedding into the $k$-th reduct, which in turns lifts to a $\Sigma_k$-embedding from $J_{\beta^*+1}(\mathbb{R}^*) \to J_{\beta+1}(\mathbb{R})$.

Let $k$ be such that $J_\beta(\mathbb{R}) \vDash \theta^k[z]$ and $T^*$ be the $\Sigma_{k+3}$-theory of $J_\beta(\mathbb{R})$. $T^*, \neg T^*$ are $\Sigma_1^{J_{\beta+1}(\mathbb{R})}(z)$ and have scales which are $\Sigma_1^{J_{\beta+1}(\mathbb{R})}(z)$. So there are $OD^{\beta+1}(z)$ trees $U_0, U_1$ associated with these scales definable over $\mathcal{R}$ from $z$. They are in $\mathcal{Q}$ by homogeneity. Let $j(U^*_0, U^*_1) = (U_0, U_1)$. Then $U^*0, U^*_1$ are $OD(z)$ in $J_{\beta^*+1}(\mathbb{R}^*)$. So $U^*_0, U^*_1 \in \mathcal{Q}$ by homogeneity. So $\mathcal{Q}$ easily gives a $\langle \theta, z\rangle$-witness (as witnessed by $U^*_0, U^*_1$). $\square$

[1] If $f \in OD^{\beta+1}(M|\eta)$ witnessing $\eta$ is not Woodin in $M$, then we can obtain a function $g \in OD^{\beta+1}(V^N_\eta)$ witnessing $\eta$ is not Woodin in $N$; we use the fact that $N|\eta$ is the $\eta$-th model of the $L[E, z]$ construction and $N|\eta$ is definable over $V^M_\eta$. This contradicts the definition of $\eta$.

**Corollary 4.21.** *Assume $W_\alpha$ holds. If $x, y \in \mathbb{R}$ and $y$ is $OD(x)$ in $J_\gamma(\mathbb{R})$ for some $\gamma < \alpha$, then there is a $x$-mouse $\mathcal{M}$ such that $y \in \mathcal{M}$ and $J_\alpha(\mathbb{R}) \vDash \mathcal{M}$ is iterable.*

*Proof.* Exercise. □

## 5. Mouse operators, strategy mice

This section summarizes basic definitions and facts about mouse operators and strategy mice. The notions are developed in details in [16], in which we carefully explain why we define mouse operators and strategy mice in a certain way and explain errors and short-comings of similar notions defined in various literature. The reader may ignore the content of this section on the first read. For those interested in core model inductions beyond $L(\mathbb{R})$, the content of this section may prove useful. For those just interested in $L(\mathbb{R})$, it is still useful to read this section to see the difficulty with defining the correct notions of mouse operators and strategy mice and the related condensation properties. These operators $\mathcal{F}$ and their condensation properties are defined in certain ways to ensure that various backgrounded constructions relative to $\mathcal{F}$ can be shown to converge.

### 5.0.1. $\mathcal{F}$-PREMICE

**Definition 5.1.** *Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}, \dot{B}, \dot{S}$, and constant symbols $\dot{a}$, $\dot{\mathfrak{P}}$. Let $\mathcal{L}_0^- = \mathcal{L}_0 \backslash \{\dot{E}, \dot{B}\}$.*

*Let $a$ be transitive. Let $\varrho : a \to \mathrm{rank}(a)$ be the rank function. We write $\hat{a} = \mathrm{trancl}(\{(a, \varrho)\})$. Let $\mathfrak{P} \in \mathcal{J}_1(\hat{a})$.*

*A **$\mathcal{J}$-structure over** $a$ **(with parameter** $\mathfrak{P}$**) (for** $\mathcal{L}_0$**)** is a structure $\mathcal{M}$ for $\mathcal{L}_0$ such that $a^{\mathcal{M}} = a$, ($\mathfrak{P}^{\mathcal{M}} = \mathfrak{P}$), and there is $\lambda \in [1, Ord)$ such that $|\mathcal{M}| = \mathcal{J}_\lambda^{S^{\mathcal{M}}}(\hat{a})$.*

*Here we also let $l(\mathcal{M})$ denote $\lambda$, the **length** of $\mathcal{M}$, and let $\hat{a}^{\mathcal{M}}$ denote $\hat{a}$.*

*For $\alpha \in [1, \lambda]$ let $\mathcal{M}_\alpha = \mathcal{J}_\alpha^{S^{\mathcal{M}}}(\hat{a})$. We say that $\mathcal{M}$ is **acceptable** iff for each $\alpha < \lambda$ and $\tau < o(\mathcal{M}_\alpha)$, if*

$$\mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_\alpha \neq \mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_{\alpha+1},$$

*then there is a surjection $\tau^{<\omega} \times \hat{a}^{<\omega} \to \mathcal{M}_\alpha$ in $\mathcal{M}_{\alpha+1}$.*

*A **$\mathcal{J}$-structure (for** $\mathcal{L}_0$**)** is a $\mathcal{J}$-structure over $a$, for some $a$.*

As all $\mathcal{J}$-structures we consider will be for $\mathcal{L}_0$, we will omit the phrase "for $\mathcal{L}_0$". We also often omit the phrase "with parameter $\mathfrak{P}$". Note that if $\mathcal{M}$ is a $\mathcal{J}$-structure over $a$ then $|\mathcal{M}|$ is transtive and rud-closed, $\hat{a} \in M$ and $o \cap M = \mathrm{rank}(M)$. This last point is because we construct from $\hat{a}$ instead of $a$.

$\mathcal{F}$-premice will be $\mathcal{J}$-structures of the following form.

**Definition 5.2.** *A **$\mathcal{J}$-model over** $a$ **(with parameter** $\mathfrak{P}$**)** is an acceptable $\mathcal{J}$-structure over $a$ (with parameter $\mathfrak{P}$), of the form*

$$\mathcal{M} = (M; E, B, S, a, \mathfrak{P})$$

*where $\dot{E}^{\mathcal{M}} = E$, etc, and letting $\lambda = l(\mathcal{M})$, the following hold.*

1. *$\mathcal{M}$ is amenable.*

2. *$S = \langle S_\xi \mid \xi \in [1, \lambda) \rangle$ is a sequence of $\mathcal{J}$-models over $a$ (with parameter $\mathfrak{P}$).*

3. *For each $\xi \in [1, \lambda)$, $\dot{S}^{S_\xi} = S \upharpoonright \xi$ and $\mathcal{M}_\xi = |S_\xi|$.*

4. *Suppose $E \neq \emptyset$. Then $B = \emptyset$ and there is an extender $F$ over $\mathcal{M}$ which is $\hat{a} \times \gamma$-complete for all $\gamma < \text{crit}(F)$ and such that the premouse axioms [17, Definition 2.2.1] hold for $(\mathcal{M}, F)$, and $E$ codes $\tilde{F} \cup \{G\}$ where: (i) $\tilde{F} \subseteq M$ is the amenable code for $F$ (as in [15]); and (ii) if $F$ is not type 2 then $G = \emptyset$, and otherwise $G$ is the "longest" non-type Z proper segment of $F$ in $\mathcal{M}$.*[2]

Our notion of a "$\mathcal{J}$-model over $a$" is a bit different from the notion of "model with parameter $a$" in [11] or [17, Definition 2.1.1] in that we build into our notion some fine structure and we do not have the predicate $l$ used in [17, Definition 2.1.1]. Note that with notation as above, if $\lambda$ is a successor ordinal then $M = J(S^{\mathcal{M}}_{\lambda-1})$, and otherwise, $M = \bigcup_{\alpha<\lambda} |S_\alpha|$. The predicate $\dot{B}$ will be used to code extra information (like a (partial) branch of a tree in $M$).

**Definition 5.3.** *Let $\mathcal{M}$ be a $\mathcal{J}$-model over $a$ (with parameter $\mathfrak{P}$). Let $E^{\mathcal{M}}$ denote $\dot{E}^{\mathcal{M}}$, etc. Let $\lambda = l(\mathcal{M})$, $S^{\mathcal{M}}_0 = a$, $S^{\mathcal{M}}_\lambda = \mathcal{M}$, and $\mathcal{M}|\xi = S^{\mathcal{M}}_\xi$ for all $\xi \leq \lambda$. An **(initial) segment** of $\mathcal{M}$ is just a structure of the form $\mathcal{M}|\xi$ for some $\xi \in [1, \lambda]$. We write $\mathcal{P} \trianglelefteq \mathcal{M}$ iff $\mathcal{P}$ is a segment of $\mathcal{M}$, and $\mathcal{P} \triangleleft \mathcal{M}$ iff $\mathcal{P} \trianglelefteq \mathcal{M}$ and $\mathcal{P} \neq \mathcal{M}$. Let $\mathcal{M}||\xi$ be the structure having the same universe and predicates as $\mathcal{M}|\xi$, except that $E^{\mathcal{M}||\xi} = \emptyset$. We say that $\mathcal{M}$ is $E$-**active** iff $E^{\mathcal{M}} \neq \emptyset$, and $B$-**active** iff $B^{\mathcal{M}} \neq \emptyset$. **Active** means either $E$-active or $B$-active; $E$-**passive** means not $E$-active; $B$-**passive** means not $B$-active; and **passive** means not active.*

*Given a $\mathcal{J}$-model $\mathcal{M}_1$ over $b$ and a $\mathcal{J}$-model $\mathcal{M}_2$ over $\mathcal{M}_1$, we write $\mathcal{M}_2 \downarrow b$ for the $\mathcal{J}$-model $\mathcal{M}$ over $b$, such that $\mathcal{M}$ is "$\mathcal{M}_1 \frown \mathcal{M}_2$". That is, $|\mathcal{M}| = |\mathcal{M}_2|$, $a^{\mathcal{M}} = b$, $E^{\mathcal{M}} = E^{\mathcal{M}_2}$, $B^{\mathcal{M}} = B^{\mathcal{M}_2}$, and $\mathcal{P} \triangleleft \mathcal{M}$ iff $\mathcal{P} \trianglelefteq \mathcal{M}_1$ or there is $\mathcal{Q} \triangleleft \mathcal{M}_2$ such that $\mathcal{P} = \mathcal{Q} \downarrow b$, when such an $\mathcal{M}$ exists. Existence depends on whether the $\mathcal{J}$-structure $\mathcal{M}$ is acceptable.*

In the following, the variable $i$ should be interpreted as follows. When $i = 0$, we ignore history, and so $\mathcal{P}$ is treated as a coarse object when determining $\mathcal{F}(0, \mathcal{P})$. When $i = 1$ we respect the history (given it exists).

**Definition 5.4.** *An **operator** $\mathcal{F}$ **with domain** $D$ is a function with domain $D$, such that for some cone $C = C_{\mathcal{F}}$, possibly self-wellordered (sword)*[3]*, $D$ is the set of pairs $(i, X)$ such that either:*

- *$i = 0$ and $X \in C$, or*

[2] We use $G$ explicitly, instead of the code $\gamma^{\mathcal{M}}$ used for $G$ in [3, Section 2], because $G$ does not depend on which (if there is any) wellorder of $\mathcal{M}$ we use. This ensures that certain pure mouse operators are forgetful.

[3] $C$ is a cone if there are a cardinal $\kappa$ and a transitive set $a \in H_\kappa$ such that $C$ is the set of $b \in H_\kappa$ such that $a \in L_1(b)$; $a$ is called the base of the cone. A set $a$ is self-wellordered if there is a well-ordering of $a$ in $L_1(a)$. A set $C$ is a self-wellordered cone if $C$ is the restriction of a cone $C'$ to its own self-wellordered elements

- $i = 1$ *and* $X$ *is a* $\mathcal{J}$*-model over* $X_1 \in C$,

*and for each* $(i, X) \in D$, $\mathcal{F}(i, X)$ *is a* $\mathcal{J}$*-model over* $X$ *such that for each* $\mathcal{P} \trianglelefteq \mathcal{F}(i, X)$, $\mathcal{P}$ *is fully sound. (Note that* $\mathcal{P}$ *is a* $\mathcal{J}$*-model over* $X$*, so soundness is in this sense.)*

*Let* $\mathcal{F}, D$ *be as above. We say* $\mathcal{F}$ *is* ***forgetful*** *iff* $\mathcal{F}(0, X) = \mathcal{F}(1, X)$ *whenever* $(0, X), (1, X) \in D$*, and whenever* $X$ *is a* $\mathcal{J}$*-model over* $X_1$*, and* $X_1$ *is a* $\mathcal{J}$*-model over* $X_2 \in C$*, we have* $\mathcal{F}(1, X) = \mathcal{F}(1, X \downarrow X_2)$*. Otherwise we say* $\mathcal{F}$ *is* ***historical****. Even when* $\mathcal{F}$ *is historical, we often just write* $\mathcal{F}(X)$ *instead of* $\mathcal{F}(i, X)$ *when the nature of* $\mathcal{F}$ *is clear from the context. We say* $\mathcal{F}$ *is* ***basic*** *iff for all* $(i, X) \in D$ *and* $\mathcal{P} \trianglelefteq \mathcal{F}(i, X)$*, we have* $E^{\mathcal{P}} = \emptyset$*. We say* $\mathcal{F}$ *is* ***projecting*** *iff for all* $(i, X) \in D$*, we have* $\rho_\omega^{\mathcal{F}(i,X)} = X$*.*

Here are some illustrations. Strategy operators (to be explained in more detail later) are basic, and as usually defined, projecting and historical. Suppose we have an iteration strategy $\Sigma$ and we want to build a $\mathcal{J}$-model $\mathcal{N}$ (over some $a$) that codes a fragment of $\Sigma$ via its predicate $\dot{B}$. We feed $\Sigma$ into $\mathcal{N}$ by always providing $b = \Sigma(\mathcal{T})$, for the $<$-$\mathcal{N}$-least tree $\mathcal{T}$ for which this information is required. So given a reasonably closed level $\mathcal{P} \triangleleft \mathcal{N}$, the choice of which tree $\mathcal{T}$ should be processed next will usually depend on the information regarding $\Sigma$ already encoded in $\mathcal{P}$ (its history). Using an operator $\mathcal{F}$ to build $\mathcal{N}$, then $\mathcal{F}(i, \mathcal{P})$ will be a structure extending $\mathcal{P}$ and over which $b = \Sigma(\mathcal{T})$ is encoded. The variable $i$ should be interpreted as follows. When $i = 1$, we respect the history of $\mathcal{P}$ when selecting $\mathcal{T}$. When $i = 0$ we ignore history when selecting $\mathcal{T}$ . The operator $\mathcal{F}(X) = X^\#$ is forgetful and projecting, and not basic; here $\mathcal{F}(X) = \mathcal{F}(0, X)$.

**Definition 5.5.** *For any* $P$ *and any ordinal* $\alpha \geq 1$*, the operator* $\mathcal{J}_\alpha^{\mathrm{m}}(\,\cdot\,; P)$ *is defined as follows.*[4] *For* $X$ *such that* $P \in \mathcal{J}_1(\hat{X})$*, let* $\mathcal{J}_\alpha^{\mathrm{m}}(X; P)$ *be the* $\mathcal{J}$*-model* $\mathcal{M}$ *over* $X$*, with parameter* $P$*, such that* $|\mathcal{M}| = \mathcal{J}_\alpha(\hat{X})$ *and for each* $\beta \in [1, \alpha]$*,* $\mathcal{M}|\beta$ *is passive. Clearly* $\mathcal{J}_\alpha^{\mathrm{m}}(\,\cdot\,; P)$ *is basic and forgetful. If* $P = \emptyset$ *or we wish to supress* $P$*, we just write* $\mathcal{J}_\alpha^{\mathrm{m}}(\,\cdot\,)$*.*

**Definition 5.6** (Potential $\mathcal{F}$-premouse, $C_{\mathcal{F}}$)**.** *Let* $\mathcal{F}$ *be an operator with domain* $D$ *of self-wellordered sets. Let* $b \in C_{\mathcal{F}}$*, so there is a well-ordering of* $b$ *in* $L_1[b]$*. A* ***potential*** $\mathcal{F}$***-premouse over*** $b$ *is an acceptable* $\mathcal{J}$*-model* $\mathcal{M}$ *over* $b$ *such that there is an ordinal* $\iota > 0$ *and an increasing, closed sequence* $\langle \zeta_\alpha \rangle_{\alpha \leq \iota}$ *of ordinals such that for each* $\alpha \leq \iota$*, we have:*

1. $0 = \zeta_0 \leq \zeta_\alpha \leq \zeta_\iota = l(\mathcal{M})$ *(so* $\mathcal{M}|\zeta_0 = b$ *and* $\mathcal{M}|\zeta_\iota = \mathcal{M}$*).*
2. *If* $1 < \iota$ *then* $\mathcal{M}|\zeta_1 = \mathcal{F}(0, b)$*.*
3. *If* $1 = \iota$ *then* $\mathcal{M} \trianglelefteq \mathcal{F}(0, b)$*.*
4. *If* $1 < \alpha + 1 < \iota$ *then* $\mathcal{M}|\zeta_{\alpha+1} = \mathcal{F}(1, \mathcal{M}|\zeta_\alpha) \downarrow b$*.*
5. *If* $1 < \alpha + 1 = \iota$*, then* $\mathcal{M} \trianglelefteq \mathcal{F}(1, \mathcal{M}|\zeta_\alpha) \downarrow b$*.*
6. *Suppose* $\alpha$ *is a limit. Then* $\mathcal{M}|\zeta_\alpha$ *is* $B$*-passive, and if* $E$*-active, then* $\mathrm{crit}(E^{\mathcal{M}|\zeta_\alpha}) > \mathrm{rank}(b)$*.*

[4] The "m" is for "model".

*We say that $\mathcal{M}$ is **($\mathcal{F}$-)whole** iff $\iota$ is a limit or else, $\iota = \alpha + 1$ and $\mathcal{M} = \mathcal{F}(\mathcal{M}|\zeta_\alpha) \downarrow b$.*

*A **(potential) $\mathcal{F}$-premouse** is a (potential) $\mathcal{F}$-premouse over $b$, for some $b$.*

**Definition 5.7.** *Let $\mathcal{F}$ be an operator and $b \in C_{\mathcal{F}}$. Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse over $b$. A **potential continuing $\mathcal{F}$-premouse over** $\mathcal{N}$ is a $\mathcal{J}$-model $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M} \downarrow b$ is a potential $\mathcal{F}$-premouse over $b$. (Therefore $\mathcal{N}$ is a whole strong cutpoint of $\mathcal{M}$.)*

*We say that $\mathcal{M}$ (as above) is **whole** iff $\mathcal{M} \downarrow b$ is whole.*

*A **(potential) continuing $\mathcal{F}$-premouse** is a (potential) continuing $\mathcal{F}$-premouse over $b$, for some $b$.*

**Definition 5.8.** $\mathrm{Lp}^{\mathcal{F}}(a)$ *denotes the stack of all countably $\mathcal{F}$-iterable $\mathcal{F}$-premice $\mathcal{M}$ over $a$ such that $\mathcal{M}$ is fully sound and projects to $a$.*[5]

*Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse over $b$, for $b \in C_{\mathcal{F}}$. Then $\mathrm{Lp}_+^{\mathcal{F}}(\mathcal{N})$ denotes the stack of all countably $\mathcal{F}$-iterable (above $o(\mathcal{N})$) continuing $\mathcal{F}$-premice $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M} \downarrow b$ is fully sound and projects to $\mathcal{N}$.*

*We say that $\mathcal{F}$ is **uniformly** $\Sigma_1$ iff there are $\Sigma_1$ formulas $\varphi_1$ and $\varphi_2$ in $\mathcal{L}_0^-$ such that whenever $\mathcal{M}$ is a (continuing) $\mathcal{F}$-premouse, then the set of whole proper segments of $\mathcal{M}$ is defined over $\mathcal{M}$ by $\varphi_1$ ($\varphi_2$). For such an operator $\mathcal{F}$, let $\varphi_{\mathrm{wh}}^{\mathcal{F}}$ denote the least such $\varphi_1$.*

**Definition 5.9** (Mouse operator)**.** *Let $Y$ be a projecting, uniformly $\Sigma_1$ operator. A **$Y$-mouse operator $\mathcal{F}$ with domain** $D$ is an operator with domain $D$ such for each $(0, X) \in D$, $\mathcal{F}(0, X) \lhd \mathrm{Lp}^Y(X)$, and for each $(1, X) \in D$, $\mathcal{F}(1, X) \lhd \mathrm{Lp}_+^Y(X)$.*[6] *(So any $Y$-mouse operator is an operator.) A $Y$-mouse operator $\mathcal{F}$ is called **first order** if there are formulas $\varphi_1$ and $\varphi_2$ in the language of $Y$-premice such that $\mathcal{F}(0, X)$ ($\mathcal{F}(1, X)$) is the first $\mathcal{M} \lhd Lp^Y(X)$ ($Lp_+^Y(X)$) satisfying $\varphi_1$ ($\varphi_2$).*

*A **mouse operator** is a $\mathcal{J}_1^{\mathrm{m}}$-mouse operator.*

We can then define $\mathcal{F}$-solidity, the $L^{\mathcal{F}}[\mathbb{E}]$-construction etc. as usual (see [16] for more details). We now define the kind of condensation that mouse operators need to satisfy to ensure the $L^{\mathcal{F}}[\mathbb{E}]$ converges.

**Definition 5.10.** *Let $\mathcal{M}_1, \mathcal{M}_2$ be $k$-sound $\mathcal{J}$-models over $a_1, a_2$ and let $\pi : \mathcal{M}_1 \to \mathcal{M}_2$. Then $\pi$ is **(weakly, nearly) $k$-good** iff $\pi \upharpoonright a_1 = \mathrm{id}$, $\pi(a_1) = a_2$, and $\pi$ is a (weak, near) $k$-embedding (as in [3]).*

**Definition 5.11.** *Given a $\mathcal{J}$-model $\mathcal{N}$ over $a$, and $\mathcal{M} \lhd \mathcal{N}$ such that $\mathcal{M}$ is fully sound, the **$\mathcal{M}$-drop-down sequence** of $\mathcal{N}$ is the sequence of pairs $\langle(\mathcal{Q}_n, m_n)\rangle_{n<k}$ of maximal length such that $\mathcal{Q}_0 = \mathcal{M}$ and $m_0 = \omega$ and for each $n + 1 < k$:*

1. *$\mathcal{M} \lhd \mathcal{Q}_{n+1} \trianglelefteq \mathcal{N}$ and $\mathcal{Q}_n \trianglelefteq \mathcal{Q}_{n+1}$,*
2. *every proper segment of $\mathcal{Q}_{n+1}$ is fully sound,*

[5] Countable substructures of $\mathcal{M}$ are $(\omega, \omega_1 + 1)$-$\mathcal{F}$-iterable, i.e. all iterates are $\mathcal{F}$-premice. See [16, Section 2] for more details on $\mathcal{F}$-iterability.

[6] This restricts the usual notion defined in [11].

3. *$\rho_{m_n}(\mathcal{Q}_n)$ is an a-cardinal of $\mathcal{Q}_{n+1}$,*

4. *$0 < m_{n+1} < \omega$,*

5. *$\mathcal{Q}_{n+1}$ is $(m_{n+1}-1)$-sound,*

6. *$\rho_{m_{n+1}}(\mathcal{Q}_{n+1}) < \rho_{m_n}(\mathcal{Q}_n) \leq \rho_{m_{n+1}-1}(\mathcal{Q}_{n+1})$.*

**Definition 5.12.** *Let $\mathcal{F}$ be an operator and let $C$ be some class of $E$-active $\mathcal{F}$-premice. Let $b$ be transitive. A **(C-certified) $L^{\mathcal{F}}[\mathbb{E}, b]$-construction** is a sequence $\langle \mathcal{N}_\alpha \rangle_{\alpha \leq \lambda}$ with the following properties. We omit the phrase "over $b$".*

*We have $\mathcal{N}_0 = b$ and $\mathcal{N}_1 = \mathcal{F}(0, b)$.*

*Let $\alpha \in (0, \lambda]$. Then $\mathcal{N}_\alpha$ is an $\mathcal{F}$-premouse, and if $\alpha$ is a limit then $\mathcal{N}_\alpha$ is the lim inf of the $\mathcal{N}_\beta$ for $\beta < \alpha$. Now suppose that $\alpha < \lambda$. Then either:*

- *$\mathcal{N}_\alpha$ is passive and is a limit of whole proper segments and $\mathcal{N}_{\alpha+1} = (\mathcal{N}_\alpha, G)$ for some extender $G$ (with $\mathcal{N}_{\alpha+1} \in C$); or*

- *$\mathcal{N}_\alpha$ is $\omega$-$\mathcal{F}$-solid. Let $\mathcal{M}_\alpha = \mathcal{C}_\omega(\mathcal{N}_\alpha)$. Let $\mathcal{M}$ be the largest whole segment of $\mathcal{M}_\alpha$. So either $\mathcal{M}_\alpha = \mathcal{M}$ or $\mathcal{M}_\alpha \downarrow \mathcal{M} \trianglelefteq \mathcal{F}_1(\mathcal{M})$. Let $\mathcal{N} \trianglelefteq \mathcal{F}_1(\mathcal{M})$ be least such that either $\mathcal{N} = \mathcal{F}_1(\mathcal{M})$ or for some $k+1 < \omega$, $(\mathcal{N} \downarrow b, k+1)$ is on the $\mathcal{M}_\alpha$-drop-down sequence of $\mathcal{N} \downarrow b$. Then $\mathcal{N}_{\alpha+1} = \mathcal{N} \downarrow b$. (Note $\mathcal{M}_\alpha \triangleleft \mathcal{N}_{\alpha+1}$.)*

**Definition 5.13.** *Let $Y$ be an operator. We say that $Y$ **condenses coarsely** iff for all $i \in \{0, 1\}$ and $(i, \bar{X}), (i, X) \in \mathrm{dom}(Y)$, and all $\mathcal{J}$-models $\mathcal{M}^+$ over $\bar{X}$, if $\pi : \mathcal{M}^+ \to Y_i(X)$ is fully elementary, fixes the parameters in the definition of $Y$, then*

1. *if $i = 0$ then $\mathcal{M}^+ \trianglelefteq Y_0(\bar{X})$; and*

2. *if $i = 1$ and $X$ is a sound whole $Y$-premouse, then $\mathcal{M}^+ \trianglelefteq Y_1(\bar{X})$.*

**Definition 5.14.** *Let $Y$ be a projecting, uniformly $\Sigma_1$ operator. We say that $Y$ **condenses finely** iff $Y$ condenses coarsely and we have the following. Let $k < \omega$. Let $\mathcal{M}^*$ be a $Y$-premouse over $a$, with a largest whole proper segment $\mathcal{M}$, such that $\mathcal{M}^+ = \mathcal{M}^* \downarrow \mathcal{M}$ is sound and $\rho_{k+1}(\mathcal{M}^+) = \mathcal{M}$. Let $\mathcal{P}^*, \bar{a}, \mathcal{P}, \mathcal{P}^+$ be likewise. Let $\mathcal{N}$ be a sound whole $Y$-premouse over $\bar{a}$. Let $G \subseteq \mathrm{Col}(\omega, \mathcal{P} \cup \mathcal{N})$ be $V$-generic. Let $\mathcal{N}^+, \pi, \sigma \in V[G]$, with $\mathcal{N}^+$ a sound $\mathcal{J}$-model over $\mathcal{N}$ such that $\mathcal{N}^* = \mathcal{N}^+ \downarrow \bar{a}$ is defined (i.e. acceptable). Suppose $\pi : \mathcal{N}^* \to \mathcal{M}^*$ is such that $\pi(\mathcal{N}) = \mathcal{M}$ and either:*

1. *$\mathcal{M}^*$ is $k$-sound and $\mathcal{N}^* = \mathcal{C}_{k+1}(\mathcal{M}^*)$; or*

2. *$(\mathcal{N}^*, k+1)$ is in the $\mathcal{N}$-dropdown sequence of $\mathcal{N}^*$, and likewise $(\mathcal{P}^*, k+1), \mathcal{P}$, and either:*
   - *(a) $\pi$ is $k$-good, or*
   - *(b) $\pi$ is fully elementary, or*
   - *(c) $\pi$ is a weak $k$-embedding, $\sigma : \mathcal{P}^* \to \mathcal{N}^*$ is $k$-good, $\sigma(\mathcal{P}) = \mathcal{N}$ and $\pi \circ \sigma \in V$ is a near $k$-embedding.*

*Then $\mathcal{N}^+ \trianglelefteq Y_1(\mathcal{N})$.*

*We say that $Y$ **almost condenses finely** iff $\mathcal{N}^+ \trianglelefteq Y_1(\mathcal{N})$ whenever the hypotheses above hold with $\mathcal{N}^+, \pi, \sigma \in V$.*

In fact, the two notions above are equivalent.

**Lemma 5.15.** *Let $Y$ be an operator on a cone with base in* HC*. Suppose that $Y$ almost condenses finely. Then $Y$ condenses finely.*

We end this section with the following lemma (proved in Section 2 of [16]), which states that the $L^{\mathcal{F}}[\mathbb{E}]$-construction (relative to some class of background extenders) runs smoothly for a certain class of operators. In the following, if $(\mathcal{N}, G) \in C$, then $G$ is backgrounded as in [3] or as in [13] (we additionally demand that the structure $N$ in [13, Definition 1.1] is closed under $\mathcal{F}$).

**Lemma 5.16.** *Let $\mathcal{F}$ be a projecting, uniformly $\Sigma_1$ operator which condenses finely. Suppose $\mathcal{F}$ is defined on a cone with bases in* HC*. Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha \leq \lambda}$ be the ($C$-certified) $L^{\mathcal{F}}[\mathbb{E}, b]$-construction for $b \in C_{\mathcal{F}}$. Then (a) $\mathcal{N}_\lambda$ is $0$-$\mathcal{F}$-solid (i.e., is an $\mathcal{F}$-premouse).*

*Now suppose that $\mathcal{N}_\lambda$ is $k$-$\mathcal{F}$-solid.*

*Suppose that for a club of countable elementary $\pi : \mathcal{M} \to \mathcal{C}_k(\mathcal{N}_\lambda)$, there is an $\mathcal{F}$-putative, $(k, \omega_1, \omega_1 + 1)$-iteration strategy $\Sigma$ for $\mathcal{M}$, such that every tree $\mathcal{T}$ via $\Sigma$ is $(\pi, \mathbb{C})$-realizable.*[7]

*Then (b) $\mathcal{N}_\lambda$ is $(k+1)$-$\mathcal{F}$-solid.*

**Lemma 5.17.** *Let $Y, \mathcal{F}$ be uniformly $\Sigma_1$ operators defined on a cone over some $\mathcal{H}_\kappa$, with bases in* HC*.*[8] *Suppose that $Y$ condenses finely. Suppose that $\mathcal{F}$ is a whole continuing $Y$-mouse operator. Then $\mathcal{F}$ condenses finely.*

The following lemma gives a stronger condensation property than fine condensation in certain circumstances. So if $\mathcal{F}$ satisfies the hypothesis of Lemma 5.18 (particularly, if $\mathcal{F}$ is one of the operators constructed in our core model induction) then the $L^{\mathcal{F}}[\mathbb{E}]$-construction converges by Lemma 5.16.

**Lemma 5.18.** *Let $Y, \mathcal{F}$ be uniformly $\Sigma_1$ operators with bases in* HC*. Suppose that $Y$ condenses finely. Suppose that $\mathcal{F}$ is a whole continuing $Y$-mouse operator. Then (a) $\mathcal{F}$ condenses finely. Moreover, (b) let $\mathcal{M}$ be an $\mathcal{F}$-whole $\mathcal{F}$-premouse. Let $\pi : \mathcal{N} \to \mathcal{M}$ be fully elementary with $a^{\mathcal{N}} \in C_{\mathcal{F}}$. Then $\mathcal{N}$ is an $\mathcal{F}$-whole $\mathcal{F}$-premouse. So regarding $\mathcal{F}$, the conclusion of 5.13 may be modified by replacing "$\trianglelefteq$" with "$=$".*

**Remark 5.19.** *In the context of the core model induction of this paper (and elsewhere), we often construct mouse operators $\mathcal{F}$ defined over some $\mathcal{H}_\kappa$ with base $a \notin$ HC. So given an $\mathcal{F}$-premouse $\mathcal{N}$, $\pi : \mathcal{N}^* \to \mathcal{N}$ elementary, and $\mathcal{N}^*$ countable, $\mathcal{N}^*$ may not be an $\mathcal{F}$ premouse. We have to make some changes for the theory above to work for these $\mathcal{F}$. For instance, in Lemma 5.16, with the notation as there, we can modify the hypothesis of the lemma in one of two ways:*

[7] See [16, Section 2] for a precise definition of $(\pi, \mathbb{C})$-realizability. Roughly speaking this means that models along the tree $\mathcal{T}$ are embedded into the $\mathcal{N}_\alpha$'s.

[8] We also say "operator over $\mathcal{H}_\kappa$ with bases in HC" for short.

1. *We can either require that $a \in \mathcal{M}$, $|\mathcal{M}| = |a|$, and the $(\pi, \mathbb{C})$-realizable strategy $\Sigma$ is $(k, |a|^+, |a|^+ + 1)$-iterable.*

2. *We can still require $\mathcal{M}$ is countable but the strategy $\Sigma$ is a $(k, \omega_1, \omega_1 + 1)$-$\mathcal{F}^\pi$-strategy, where $\mathcal{F}^\pi$ is the $\pi$-pullback operator of $\mathcal{F}$.*[9]

### 5.0.2. STRATEGY PREMICE

We now proceed to defining $\Sigma$-premice, for an iteration strategy $\Sigma$. We first define the operator to be used to feed in $\Sigma$.

**Definition 5.20** ($\mathfrak{B}(a, \mathcal{T}, b)$, $b^{\mathcal{N}}$)**.** *Let $a, \mathcal{P}$ be transitive, with $\mathcal{P} \in \mathcal{J}_1(\hat{a})$. Let $\lambda > 0$ and let $\mathcal{T}$ be an iteration tree*[10] *on $\mathcal{P}$, of length $\omega\lambda$, with $\mathcal{T} \restriction \beta \in a$ for all $\beta \leq \omega\lambda$. Let $b \subseteq \omega\lambda$. We define $\mathcal{N} = \mathfrak{B}(a, \mathcal{T}, b)$ recursively on $\mathrm{lh}(\mathcal{T})$, as the $\mathcal{J}$-model $\mathcal{N}$ over $a$, with parameter $\mathcal{P}$,*[11] *such that:*

1. *$l(\mathcal{N}) = \lambda$,*

2. *for each $\gamma \in (0, \lambda)$, $\mathcal{N}|\gamma = \mathfrak{B}(a, \mathcal{T} \restriction \omega\gamma, [0, \omega\gamma]_{\mathcal{T}})$,*

3. *$B^{\mathcal{N}}$ is the set of ordinals $o(a) + \gamma$ such that $\gamma \in b$,*

4. *$E^{\mathcal{N}} = \emptyset$.*

*We also write $b^{\mathcal{N}} = b$.*

It is easy to see that every initial segment of $\mathcal{N}$ is sound, so $\mathcal{N}$ is acceptable and is indeed a $\mathcal{J}$-model (not just a $\mathcal{J}$-structure).

In the context of a $\Sigma$-premouse $\mathcal{M}$ for an iteration strategy $\Sigma$, if $\mathcal{T}$ is the $<_{\mathcal{M}}$-least tree for which $\mathcal{M}$ lacks instruction regarding $\Sigma(\mathcal{T})$, then $\mathcal{M}$ will already have been instructed regarding $\Sigma(\mathcal{T} \restriction \alpha)$ for all $\alpha < \mathrm{lh}(\mathcal{T})$. Therefore if $\mathrm{lh}(\mathcal{T}) > \omega$ then $\mathfrak{B}(\mathcal{M}, \mathcal{T}, \Sigma(\mathcal{T}))$ codes redundant information (the branches already in $\mathcal{T}$) before coding $\Sigma(\mathcal{T})$. This redundancy seems to allow one to prove slightly stronger condensation properties, given that $\Sigma$ has nice condensation properties (see Lemma 5.27). It also simplifies the definition.

**Definition 5.21.** *Let $\Sigma$ be a partial iteration strategy. Let $C$ be a class of iteration trees, closed under initial segment. We say that $(\Sigma, C)$ is* ***suitably condensing*** *iff for every $\mathcal{T} \in C$ such that $\mathcal{T}$ is via $\Sigma$ and $\mathrm{lh}(\mathcal{T}) = \lambda + 1$ for some limit $\lambda$, either (i) $\Sigma$ has hull condensation with respect to $\mathcal{T}$, or (ii) $b^{\mathcal{T}}$ does not drop and $\Sigma$ has branch condensation with respect to $\mathcal{T}$, that is, any hull $\mathcal{U}^\frown c$ of $\mathcal{T}^\frown b$ is according to $\Sigma$.*

When $C$ is the class of all iteration trees according to $\Sigma$, we simply omit it from our notation.

[9] For instance, if $\mathcal{F}$ corresponds to a strategy $\Sigma$, then $\mathcal{F}^\pi$ corresponds to $\Sigma^\pi$, the $\pi$-pullback of $\Sigma$. If $\mathcal{F}$ is a first order mouse operator defined by $(\varphi, a)$, then $\mathcal{F}^\pi$ is defined by $(\varphi, \pi^{-1}(a))$.

[10] We formally take an *iteration tree* to include the entire sequence $\langle M_\alpha^{\mathcal{T}} \rangle_{\alpha < \mathrm{lh}(\mathcal{T})}$ of models. So it is $\Sigma_0(\mathcal{T}, \mathfrak{P})$ to assert that "$\mathcal{T}$ is an iteration tree on $\mathfrak{P}$".

[11] $\mathcal{P} = M_0^{\mathcal{T}}$ is determined by $\mathcal{T}$.

**Definition 5.22.** *Let $\varphi$ be an $\mathcal{L}_0$-formula. Let $\mathcal{P}$ be transitive. Let $\mathcal{M}$ be a $\mathcal{J}$-model (over some $a$), with parameter $\mathcal{P}$. Let $\mathcal{T} \in \mathcal{M}$. We say that $\varphi$* ***selects** $\mathcal{T}$ **for*** *$\mathcal{M}$, and write $\mathcal{T} = \mathcal{T}^{\mathcal{M}}_{\varphi}$, iff*

*(a) $\mathcal{T}$ is the unique $x \in \mathcal{M}$ such that $\mathcal{M} \models \varphi(x)$,*

*(b) $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ of limit length,*

*(c) for every $\mathcal{N} \triangleleft \mathcal{M}$, we have $\mathcal{N} \not\models \varphi(\mathcal{T})$, and*

*(d) for every limit $\lambda < \mathrm{lh}(\mathcal{T})$, there is $\mathcal{N} \triangleleft \mathcal{M}$ such that $\mathcal{N} \models \varphi(\mathcal{T} \upharpoonright \lambda)$.*

One instance of $\phi(\mathcal{P}, \mathcal{T})$ is, in the case $a$ is self-wellordered, the formula "$\mathcal{T}$ is the least tree on $\mathcal{P}$ that doesn't have a cofinal branch", where least is computed with respect to the canonical well-order of the model.

**Definition 5.23** (Potential $\mathcal{P}$-strategy-premouse, $\Sigma^{\mathcal{M}}$)**.** *Let $\varphi \in \mathcal{L}_0$. Let $\mathcal{P}, a$ be transitive with $\mathcal{P} \in \mathcal{J}_1(\hat{a})$. A* ***potential** $\mathcal{P}$**-strategy-premouse (over** $a$**, of type** $\varphi$**)*** *is a $\mathcal{J}$-model $\mathcal{M}$ over $a$, with parameter $\mathcal{P}$, such that the $\mathfrak{B}$ operator is used to feed in an iteration strategy for trees on $\mathcal{P}$, using the sequence of trees naturally determined by $S^{\mathcal{M}}$ and selection by $\varphi$. We let $\Sigma^{\mathcal{M}}$ denote the partial strategy coded by the predicates $B^{\mathcal{M}|\eta}$, for $\eta \leq l(\mathcal{M})$.*

*In more detail, there is an increasing, closed sequence of ordinals $\langle \eta_\alpha \rangle_{\alpha \leq \iota}$ with the following properties. We will also define $\Sigma^{\mathcal{M}|\eta}$ for all $\eta \in [1, l(\mathcal{M})]$ and $\mathcal{T}_\eta = \mathcal{T}^{\mathcal{M}}_{\eta}$ for all $\eta \in [1, l(\mathcal{M}))$.*

1. *$1 = \eta_0$ and $\mathcal{M}|1 = \mathcal{J}^{\mathrm{m}}_1(a; \mathcal{P})$ and $\Sigma^{\mathcal{M}|1} = \emptyset$.*
2. *$l(\mathcal{M}) = \eta_\iota$, so $\mathcal{M}|\eta_\iota = \mathcal{M}$.*
3. *Given $\eta \leq l(\mathcal{M})$ such that $B^{\mathcal{M}|\eta} = \emptyset$, we set $\Sigma^{\mathcal{M}|\eta} = \bigcup_{\eta' < \eta} \Sigma^{\mathcal{M}|\eta'}$.*

*Let $\eta \in [1, l(\mathcal{M})]$. Suppose there is $\gamma \in [1, \eta]$ and $\mathcal{T} \in \mathcal{M}|\gamma$ such that $\mathcal{T} = \mathcal{T}^{\mathcal{M}|\gamma}_{\varphi}$, and $\mathcal{T}$ is via $\Sigma^{\mathcal{M}|\eta}$, but no proper extension of $\mathcal{T}$ is via $\Sigma^{\mathcal{M}|\eta}$. Taking $\gamma$ minimal such, let $\mathcal{T}_\eta = \mathcal{T}^{\mathcal{M}|\gamma}_{\varphi}$. Otherwise let $\mathcal{T}_\eta = \emptyset$.*

4. *Let $\alpha + 1 \leq \iota$. Suppose $\mathcal{T}_{\eta_\alpha} = \emptyset$. Then $\eta_{\alpha+1} = \eta_\alpha + 1$ and $\mathcal{M}|\eta_{\alpha+1} = \mathcal{J}^{\mathrm{m}}_1(\mathcal{M}|\eta_\alpha; \mathcal{P}) \downarrow a$.*
5. *Let $\alpha + 1 \leq \iota$. Suppose $\mathcal{T} = \mathcal{T}_{\eta_\alpha} \neq \emptyset$. Let $\omega\lambda = \mathrm{lh}(\mathcal{T})$. Then for some $b \subseteq \omega\lambda$, and $\mathcal{S} = \mathfrak{B}(\mathcal{M}|\eta_\alpha, \mathcal{T}, b)$, we have:*
   *(a) $\mathcal{M}|\eta_{\alpha+1} \trianglelefteq \mathcal{S}$.*
   *(b) If $\alpha + 1 < \iota$ then $\mathcal{M}|\eta_{\alpha+1} = \mathcal{S}$.*
   *(c) If $\mathcal{S} \trianglelefteq \mathcal{M}$ then $b$ is a $\mathcal{T}$-cofinal branch.*[12]
   *(d) For $\eta \in [\eta_\alpha, l(\mathcal{M})]$ such that $\eta < l(\mathcal{S})$, $\Sigma^{\mathcal{M}|\eta} = \Sigma^{\mathcal{M}|\eta_\alpha}$.*
   *(e) If $\mathcal{S} \trianglelefteq \mathcal{M}$ then then $\Sigma^{\mathcal{S}} = \Sigma^{\mathcal{M}|\eta_\alpha} \cup \{(\mathcal{T}, b^{\mathcal{S}})\}$.*

[12] We allow $\mathcal{M}^{\mathcal{T}}_b$ to be illfounded, but then $\mathcal{T}^\frown b$ is not an iteration tree, so is not continued by $\Sigma^{\mathcal{M}}$.

6. *For each limit $\alpha \leq \iota$, $B^{\mathcal{M}|\eta_\alpha} = \emptyset$.*

**Definition 5.24** (Whole)**.** *Let $\mathcal{M}$ be a potential $\mathcal{P}$-strategy-premouse of type $\varphi$. We say $\mathcal{P}$ is **$\varphi$-whole** (or just **whole** if $\varphi$ is fixed) iff for every $\eta < l(\mathcal{M})$, if $\mathcal{T}_\eta \neq \emptyset$ and $\mathcal{T}_\eta \neq \mathcal{T}_{\eta'}$ for all $\eta' < \eta$, then for some $b$, $\mathfrak{B}(\mathcal{M}|\eta, \mathcal{T}_\eta, b) \trianglelefteq \mathcal{M}$.*[13]

**Definition 5.25** (Potential $\Sigma$-premouse)**.** *Let $\Sigma$ be a (partial) iteration strategy for a transitive structure $\mathcal{P}$. A **potential $\Sigma$-premouse (over $a$, of type $\varphi$)** is a potential $\mathcal{P}$-strategy premouse $\mathcal{M}$ (over $a$, of type $\varphi$) such that $\Sigma^{\mathcal{M}} \subseteq \Sigma$.*[14]

**Definition 5.26.** *Let $\mathcal{R}, \mathcal{M}$ be $\mathcal{J}$-structures for $\mathcal{L}_0$, $a = a^{\mathcal{R}}$ and $b = a^{\mathcal{M}}$. Suppose that $a, b$ code $\mathcal{P}, \mathcal{Q}$ respectively. Let $\pi : \mathcal{R} \to \mathcal{M}$ (or*

$$\pi : o(\mathcal{R}) \cup \mathcal{P} \cup \{\mathcal{P}\} \to o(\mathcal{M}) \cup \mathcal{Q} \cup \{\mathcal{Q}\}$$

*respectively). Then $\pi$ is a $(\mathcal{P}, \mathcal{Q})$**-weak** $0$**-embedding** (resp., $(\mathcal{P}, \mathcal{Q})$**-very weak** $0$**-embedding**) iff $\pi(\mathcal{P}) = (\mathcal{Q})$ and with respect to the language $\mathcal{L}_0$, $\pi$ is $\Sigma_0$-elementary, and there is an $X \subseteq \mathcal{R}$ (resp., $X \subseteq o(\mathcal{R})$) such that $X$ is cofinal in $\in^{\mathcal{R}}$ and $\pi$ is $\Sigma_1$-elementary on parameters in $X \cup \mathcal{P} \cup \{\mathcal{P}\}$. If also $\mathcal{P} = \mathcal{Q}$ and $\pi \restriction \mathcal{P} \cup \{\mathcal{P}\} = \mathrm{id}$, then we just say that $\pi$ is a $\mathcal{P}$**-weak** $0$**-embedding** (resp., $\mathcal{P}$**-very weak** $0$**-embedding**).*

Note that, for $(\mathcal{P}, \mathcal{Q})$-weak 0-embeddings, we can in fact take $X \subseteq o(\mathcal{R})$. The following lemma is again proved in [16, Section 3].

**Lemma 5.27.** *Let $\mathcal{M}$ be a $\mathcal{P}$-strategy premouse over $a$, of type $\varphi$, where $\varphi$ is $\Sigma_1$. Let $\mathcal{R}$ be a $\mathcal{J}$-structure for $\mathcal{L}_0$ and $a' = a^{\mathcal{R}}$, and let $\mathcal{P}'$ be a transitive structure coded by $a'$.*

1. *Suppose $\pi : \mathcal{R} \to \mathcal{M}$ is a partial map such that $\pi(\mathcal{P}') = \mathcal{P}$ and either:*

   *(a) $\pi$ is a $(\mathcal{P}', \mathcal{P})$-weak $0$-embedding, or*

   *(b) $\pi$ is a $(\mathcal{P}', \mathcal{P})$-very weak $0$-embedding, and if $E^{\mathcal{R}} \neq \emptyset$ then item 4 of 5.2 holds for $E^{\mathcal{R}}$.*

   *Then $\mathcal{R}$ is a $\mathcal{P}'$-strategy premouse of type $\varphi$. Moreover, if $\pi \restriction \{\mathcal{P}'\} \cup \mathcal{P}' = \mathrm{id}$ and if $\mathcal{M}$ is a $\Sigma$-premouse, where $(\Sigma, \mathrm{dom}(\Sigma^{\mathcal{M}}))$ is suitably condensing, then $\mathcal{R}$ is also a $\Sigma$-premouse.*

2. *Suppose there is $\pi : \mathcal{M} \to \mathcal{R}$ is such that $\pi(a, \mathcal{P}) = (a', \mathcal{P}')$ and either*

   *(a) $\pi$ is $\Sigma_2$-elementary; or*

   *(b) $\pi$ is cofinal and $\Sigma_1$-elementary, and $B^{\mathcal{M}} = \emptyset$.*

   *Then $\mathcal{R}$ is a $\mathcal{P}'$-strategy premouse of type $\varphi$, and $\mathcal{R}$ is whole iff $\mathcal{M}$ is whole.*

[13] *$\varphi$-whole* depends on $\varphi$ as the definition of $\mathcal{T}_\eta$ does.

[14] If $\mathcal{M}$ is a model all of whose proper segments are potential $\Sigma$-premice, and the rules for potential $\mathcal{P}$-strategy premice require that $B^{\mathcal{M}}$ code a $\mathcal{T}$-cofinal branch, but $\Sigma(\mathcal{T})$ is not defined, then $\mathcal{M}$ is not a potential $\Sigma$-premouse, whatever its predicates are.

3. *Suppose* $B^{\mathcal{M}} \neq \emptyset$. *Let* $\mathcal{T} = \mathcal{T}_\eta^{\mathcal{M}}$ *where* $\eta < l(\mathcal{M})$ *is largest such that* $\mathcal{M}|\eta$ *is whole. Let* $b = b^{\mathcal{M}}$ *and* $\omega\gamma = \bigcup b$. *So* $\mathcal{M} \trianglelefteq \mathfrak{B}(\mathcal{M}|\eta, \mathcal{T}, b)$. *Suppose there is* $\pi : \mathcal{M} \to \mathcal{R}$ *such that* $\pi(\mathcal{P}) = \mathcal{P}'$ *and* $\pi$ *is cofinal and* $\Sigma_1$*-elementary. Let* $\omega\gamma' = \sup \pi``\omega\gamma$.

   (a) $\mathcal{R}$ *is a* $\mathcal{P}'$*-strategy premouse of type* $\varphi$ *iff we have either (i)* $\omega\gamma' = \mathrm{lh}(\pi(\mathcal{T}))$, *or (ii)* $\omega\gamma' < \mathrm{lh}(\pi(\mathcal{T}))$ *and* $b^{\mathcal{R}} = [0, \omega\gamma']_{\pi(\mathcal{T})}$.

   (b) *If either* $b^{\mathcal{M}} \in \mathcal{M}$ *or* $\pi$ *is continuous at* $\mathrm{lh}(\mathcal{T})$ *then* $\mathcal{R}$ *is a* $\mathcal{P}'$*-strategy premouse of type* $\varphi$.

**Remark 5.28.** *The preceding lemma left open the possibility that* $\mathcal{R}$ *fails to be a* $\mathcal{P}$*-strategy premouse under certain circumstances (because* $B^{\mathcal{R}}$ *should be coding a branch that has in fact already been coded at some proper segment of* $\mathcal{R}$*, but codes some other branch instead). In the main circumstance we are interested in, this does not arise, for a couple of reasons. Suppose that* $\Sigma$ *is an iteration strategy for* $\mathcal{P}$ *with hull condensation,* $\mathcal{M}$ *is a* $\Sigma$*-premouse, and* $\Lambda$ *is a* $\Sigma$*-strategy for* $\mathcal{M}$*. Suppose* $\pi : \mathcal{M} \to \mathcal{R}$ *is a degree* $0$ *iteration embedding and* $B^{\mathcal{M}} \neq \emptyset$ *and* $\pi$ *is discontinuous at* $\mathrm{lh}(\mathcal{T})$*. Then [16, Section 3] shows that* $b^{\mathcal{M}} \in \mathcal{M}$*. (It's not relevant whether* $\pi$ *itself is via* $\Lambda$*.) It then follows from 3b of Lemma 5.27 that* $\mathcal{R}$ *is a* $\Sigma$*-mouse.*

*The other reason is that, supposing* $\pi : \mathcal{M} \to \mathcal{R}$ *is via* $\Lambda$ *(so* $\pi \upharpoonright \mathcal{P} \cup \{\mathcal{P}\} = \mathrm{id}$*), then trivially,* $B^{\mathcal{R}}$ *must code branches according to* $\Sigma$*. We can obtain such a* $\Lambda$ *given that we can realize iterates of* $\mathcal{M}$ *back into a fixed* $\Sigma$*-premouse (with* $\mathcal{P}$*-weak* $0$*-embeddings as realization maps).*

**Definition 5.29.** *Let* $\mathcal{P}$ *be transitive and* $\Sigma$ *a partial iteration strategy for* $\mathcal{P}$*. Let* $\varphi \in \mathcal{L}_0$*. Let* $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$ *be the operator such that:*

1. $\mathcal{F}_0(a) = \mathcal{J}_1^{\mathrm{m}}(a; \mathcal{P})$*, for all transitive* $a$ *such that* $\mathcal{P} \in \mathcal{J}_1(\hat{a})$*;*

2. *Let* $\mathcal{M}$ *be a sound branch-whole* $\Sigma$*-premouse of type* $\varphi$*. Let* $\lambda = l(\mathcal{M})$ *and with notation as in 5.23, let* $\mathcal{T} = \mathcal{T}_\lambda$*. If* $\mathcal{T} = \emptyset$ *then* $\mathcal{F}_1(\mathcal{M}) = \mathcal{J}_1^{\mathrm{m}}(\mathcal{M}; \mathcal{P})$*. If* $\mathcal{T} \neq \emptyset$ *then* $\mathcal{F}_1(\mathcal{M}) = \mathfrak{B}(\mathcal{M}, \mathcal{T}, b)$ *where* $b = \Sigma(\mathcal{T})$*.*

*We say that* $\mathcal{F}$ *is a* ***strategy operator****.*

**Lemma 5.30.** *Let* $\mathcal{P}$ *be countable and transitive. Let* $\varphi$ *be a formula of* $\mathcal{L}_0$*. Let* $\Sigma$ *be a partial strategy for* $\mathcal{P}$*. Let* $D_\varphi$ *be the class of iteration trees* $\mathcal{T}$ *on* $\mathcal{P}$ *such that for some* $\mathcal{J}$*-model* $\mathcal{M}$*, with parameter* $\mathcal{P}$*, we have* $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}}$*. Suppose that* $(\Sigma, D_\varphi)$ *is suitably condensing. Then* $\mathcal{F}_{\Sigma,\varphi}$ *is uniformly* $\Sigma_1$*, projecting, and condenses finely.*

**Definition 5.31.** *Let* $a$ *be transitive and let* $\mathcal{F}$ *be an operator. We say that* $\mathcal{M}_1^{\mathcal{F},\#}(a)$ ***exists*** *iff there is a* $(0, |a|, |a|+1)$*-*$\mathcal{F}$*-iterable, non-1-small* $\mathcal{F}$*-premouse over* $a$*. We write* $\mathcal{M}_1^{\mathcal{F},\#}(a)$ *for the least such sound structure. For* $\Sigma, \mathcal{P}, a, \varphi$ *as in 5.29, we write* $\mathcal{M}_1^{\Sigma,\varphi,\#}(a)$ *for* $\mathcal{M}_1^{\mathcal{F}_{\Sigma,\varphi},\#}(a)$*.*

*Let* $\mathcal{L}_0^+$ *be the language* $\mathcal{L}_0 \cup \{\dot{\prec}, \dot{\Sigma}\}$*, where* $\dot{\prec}$ *is the binary relation defined by "*$\dot{a}$ *is self-wellordered, with ordering* $\prec_{\dot{a}}$*, and* $\dot{\prec}$ *is the canonical wellorder of the universe extending* $\prec_{\dot{a}}$*", and* $\dot{\Sigma}$ *is the partial function defined "*$\dot{\mathfrak{P}}$ *is a transitive structure and the universe is a potential*

*$\dot{\mathfrak{P}}$-strategy premouse over $\dot{a}$ and $\dot{\Sigma}$ is the associated partial putative iteration strategy for $\dot{\mathfrak{P}}$". Let $\varphi_{all}(\mathcal{T})$ be the $\mathcal{L}_0$-formula "$\mathcal{T}$ is the $\dot{\prec}$-least limit length iteration tree $\mathcal{U}$ on $\dot{\mathfrak{P}}$ such that $\mathcal{U}$ is via $\dot{\Sigma}$, but no proper extension of $\mathcal{U}$ is via $\dot{\Sigma}$". Then for $\Sigma, \mathcal{P}, a$ as in 5.29, we sometimes write $\mathcal{M}_1^{\Sigma,\#}(a)$ for $\mathcal{M}_1^{\mathcal{F}_{\Sigma,\varphi_{all}},\#}(a)$.*

*Let $\kappa$ be a cardinal and suppose that $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}(a)$ exists and is $(0, \kappa^+ + 1)$-iterable. We write $\Lambda_{\mathfrak{M}}$ for the unique $(0, \kappa^+ + 1)$-iteration strategy for $\mathfrak{M}$ (given that $\kappa$ is fixed).*

**Definition 5.32.** *We say that $(\mathcal{F}, \Sigma, \varphi, D, a, \mathfrak{P})$ is **suitable** iff $a$ is transitive and $\mathcal{M}_1^{\mathcal{F},\#}(a)$ exists, where either*

1. *$\mathcal{F}$ is a projecting, uniformly $\Sigma_1$ operator, $C_{\mathcal{F}}$ is the (possibly swo'd) cone above $a$, $D$ is the set of pairs $(i, X) \in \mathrm{dom}(\mathcal{F})$ such that either $i = 0$ or $X$ is a sound whole $\mathcal{F}$-premouse, and $\Sigma = \varphi = 0$, or*

2. *$\mathcal{P}, \Sigma, \varphi, D_\varphi$ are as in 5.30, $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$, $D_\varphi \subseteq D$, $D$ is a class of limit length iteration trees on $\mathcal{P}$, via $\Sigma$, $\Sigma(\mathcal{T})$ is defined for all $\mathcal{T} \in D$, $(\Sigma, D)$ is suitably condensing and $\mathcal{P} \in \mathcal{J}_1(\hat{a})$.*

*We write $\mathcal{G}_{\mathcal{F}}$ for the function with domain $C_{\mathcal{F}}$, such that for all $x \in C_{\mathcal{F}}$, $\mathcal{G}_{\mathcal{F}}(x) = \Sigma(x)$ in case (ii), and in case (i), $\mathcal{G}_{\mathcal{F}}(0, x) = \mathcal{F}(0, x)$ and $\mathcal{G}_{\mathcal{F}}(1, x)$ is the least $\mathcal{R} \trianglelefteq \mathcal{F}_1(x) \downarrow a^x$ such that either $\mathcal{R} = \mathcal{F}_1(X) \downarrow a^X$ or $\mathcal{R}$ is unsound.*

**Lemma 5.33.** *Let $\mathcal{F}$ be as in 5.32 and $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}$. Then $\Lambda_{\mathfrak{M}}$ has branch condensation and hull condensation.*

### 5.0.3. G-ORGANIZED $\mathcal{F}$-PREMICE

Now we give an outline of the general treatment of [16] on $\mathcal{F}$-premice over an arbitrary set; following the terminology of [16], we will call these $g$-organized $\mathcal{F}$-premice and $\Theta$-$g$-organized $\mathcal{F}$-premice. For $(\Theta)$-$g$-organized $\mathcal{F}$-premice to be useful, we need to assume that the following absoluteness property holds of the operator $\mathcal{F}$. We then show that if $\mathcal{F}$ is the operator for a nice enough iteration strategy, then it does hold. We write $\mathfrak{M}$ for $\mathfrak{M}_{\mathcal{F}}$ and fix $a, \mathfrak{P}, \mathcal{F}, \mathcal{P}, C$ as in the previous subsection. In the following, $\delta^{\mathfrak{M}}$ denotes the Woodin cardinal of $\mathfrak{M}$. Again, the reader should see [16] for proofs of lemmas stated here.

We need to work with the $g$-organized hierarchies to ensure various $S$-constructions succeed. This in turn is important in extending operators to generic extensions (cf. Lemma 6.5). We need to work with the $\Theta$-$g$-organized hierarchies (a slight variation of the $g$-organized hierarchies), in particular, we need to work in the hierarchy of $\Theta$-$g$-organized mice over $\mathbb{R}$ to ensure the scales analysis goes through as in $Lp(\mathbb{R})$.

**Definition 5.34.** *Let $(\mathcal{F}, \Sigma, \varphi, C, a, \mathfrak{P})$ be suitable. We say that $\mathcal{M}_1^{\mathcal{F},\sharp}(a)$ **generically interprets** $\mathcal{F}$[15] iff, writing $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}(a)$, there are formulas $\Phi, \Psi$ in $\mathcal{L}_0$ such that there is some $\gamma > \delta^{\mathfrak{M}}$ such*

[15] In [16], this notion is called $\mathcal{F}$ determines itself on generic extensions. In this paper, "determines itself on generic extensions" will have a different meaning, as defined later.

*that $\mathfrak{M}|\gamma \vDash \Phi$ and for any non-dropping $\Sigma_{\mathfrak{M}}$-iterate $\mathcal{N}$ of $\mathfrak{M}$, via a countable iteration tree $\mathcal{T}$, any $\mathcal{N}$-cardinal $\delta$, any $\gamma \in Ord$ such that $\mathcal{N}|\gamma \vDash \Phi$ & "$\delta$ is Woodin", and any $g$ which is set-generic over $\mathcal{N}|\gamma$ (with $g \in V$), then $(\mathcal{N}|\gamma)[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$, and $\mathcal{G}_{\mathcal{F}} \upharpoonright (\mathcal{N}|\gamma)[g]$ is defined over $(\mathcal{N}|\gamma)[g]$ by $\Psi$. We say such a pair $(\Phi, \Psi)$* ***generically determines*** *$(\mathcal{F}, \Sigma, \varphi, C, a)$ (or just $\mathcal{F}$).*

*We say an operator $\mathcal{F}$ is* ***nice*** *iff for some $\Sigma, \varphi, C, a, \mathfrak{P}$, $(\mathcal{F}, \Sigma, \varphi, C, a, \mathfrak{P})$ is suitable and $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$.*

*Let $\mathcal{P} \in \mathrm{HC}$, let $\Sigma$ be an iteration strategy for $\mathcal{P}$ and let $C$ be the class of all limit length trees via $\Sigma$. Suppose $\mathcal{M}_1^{\Sigma,\#}(\mathcal{P})$ exists, $(\Sigma, C)$ is suitably condensing. We say that $\mathcal{M}_1^{\Sigma,\#}(\mathcal{P})$* ***generically interprets*** *$\Sigma$ iff some $(\Phi, \Psi)$ generically determines $(\mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}, \Sigma, \varphi_{\mathrm{all}}, C, \mathcal{P})$. (Note then that the latter is suitable.)*

**Lemma 5.35.** *Let $\mathcal{N}, \delta$, etc, be as in 5.34, except that we allow $\mathcal{T}$ to have uncountable length, and allow $g$ to be in a set-generic extension of $V$. Then $(\mathcal{N}|\gamma)[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$ and letting $\mathcal{G}'$ be the interpretation of $\Psi$ over $(\mathcal{N}|\gamma)[g]$, $\mathcal{G}' \upharpoonright C = \mathcal{G}_{\mathcal{F}} \upharpoonright (\mathcal{N}|\gamma)[g]$.*

We fix a nice $\mathcal{F}$, $\mathfrak{M}$, $\Lambda_{\mathfrak{M}} = \Lambda$, $(\Phi, \Psi)$ for the rest of the section. We define $\mathcal{M}_1^{\Sigma}$ from $\mathfrak{M}$ in the standard way.

See [16, Section 4] for a proof that if $\Sigma$ is a strategy (of a hod mouse, a suitable mouse) with branch condensation and is fullness preserving with respect to mice in some sufficiently closed, determined pointclass $\Gamma$ or if $\Sigma$ is the unique strategy of a sound $(Y)$-mouse for some mouse operator $Y$ that is projecting, uniformly $\Sigma_1$, $\mathcal{M}_1^{Y,\sharp}$ generically interprets $Y$, and condenses finely then $M_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$.

Now we are ready to define $g$-organized $\mathcal{F}$-premice.

**Definition 5.36** (Sargsyan, [6])**.** *Let $M$ be a transitive structure. Let $\dot{G}$ be the name for the generic $G \subseteq \mathrm{Col}(\omega, M)$ and let $\dot{x}_{\dot{G}}$ be the canonical name for the real coding $\{(n, m) \mid G(n) \in G(m)\}$, where we identify $G$ with $\bigcup G$. The* ***tree $\mathcal{T}_M$ for making $M$ generically generic****, is the iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of maximal length such that:*

1. *$\mathcal{T}$ is via $\Lambda$ and is everywhere non-dropping.*
2. *$\mathcal{T} \upharpoonright o(M)+1$ is the tree given by linearly iterating the first total measure of $\mathfrak{M}$ and its images.*
3. *Suppose $\mathrm{lh}(\mathcal{T}) \geq o(M)+2$ and let $\alpha+1 \in (o(M), \mathrm{lh}(\mathcal{T}))$. Let $\delta = \delta(\mathcal{M}_\alpha^{\mathcal{T}})$ and let $\mathbb{B} = \mathbb{B}(M_\alpha^{\mathcal{T}})$ be the extender algebra of $M_\alpha^{\mathcal{T}}$ at $\delta$. Then $E_\alpha^{\mathcal{T}}$ is the extender $E$ with least index in $M_\alpha^{\mathcal{T}}$ such that for some condition $p \in \mathrm{Col}(\omega, M)$, $p \Vdash$ "There is a $\mathbb{B}$-axiom induced by $E$ which fails for $\dot{x}_{\dot{G}}$".*

*Assuming that $\mathfrak{M}$ is sufficiently iterable, then $\mathcal{T}_M$ exists and has successor length.*

Sargsyan noticed that one can feed in $\mathcal{F}$ into a structure $\mathcal{N}$ indirectly, by feeding in the branches for $\mathcal{T}_{\mathcal{M}}$, for various $\mathcal{M} \trianglelefteq \mathcal{N}$. The operator ${}^{\mathrm{g}}\mathcal{F}$, defined below, and used in building g-organized $\mathcal{F}$-premice, feeds in branches for such $\mathcal{T}_{\mathcal{M}}$. We will also ensure that being such a structure is first-order - other than wellfoundedness and the correctness of the branches - by allowing sufficient spacing between these branches.

In the following, we let $\mathcal{N}^{\mathcal{T}}$ denote the last model of the tree $\mathcal{T}$.

**Definition 5.37.** *Given a formula $\Phi$. Given a successor length, nowhere dropping tree $\mathcal{T}$ on $\mathfrak{M}$, let $P^\Phi(\mathcal{T})$ be the least $P \trianglelefteq \mathcal{N}^{\mathcal{T}}$ such that for some cardinal $\delta'$ of $\mathcal{N}^{\mathcal{T}}$, we have $\delta' < o(P)$ and $P \models \Phi+$ "$\delta'$ is Woodin". Let $\lambda = \lambda^\Phi(\mathcal{T})$ be least such that $P^\Phi(\mathcal{T}) \trianglelefteq M^{\mathcal{T}}_\lambda$. Then $\delta'$ is a cardinal of $M^{\mathcal{T}}_\lambda$. Let $I^\Phi = I^\Phi(\mathcal{T})$ be the set of limit ordinals $\leq \lambda$.*

We can now define the operator used for g-organization:

**Definition 5.38** ($^{\mathrm{g}}\mathcal{F}$)**.** *We define the forgetful operator $^{\mathrm{g}}\mathcal{F}$, for $\mathcal{F}$ such that $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$ as witnessed by a pair $(\Phi, \Psi)$. Let $b$ be a transitive structure with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$.*

*We define $\mathcal{M} = {}^{\mathrm{g}}\mathcal{F}(b)$, a $\mathcal{J}$-model over $b$, with parameter $\mathfrak{M}$, as follows.*

*For each $\alpha \leq l(\mathcal{M})$, $E^{\mathcal{M}|\alpha} = \emptyset$.*

*Let $\alpha_0$ be the least $\alpha$ such that $\mathcal{J}_\alpha(b) \models \mathsf{ZF}$. Then $\mathcal{M}|\alpha_0 = \mathcal{J}^{\mathrm{m}}_{\alpha_0}(b; \mathfrak{M})$.*

*Let $\mathcal{T} = \mathcal{T}_{\mathcal{M}|\alpha_0}$. We use the notation $P^\Phi = P^\Phi(\mathcal{T})$, $\lambda = \lambda^\Phi(\mathcal{T})$, etc, as in 5.37. The predicates $B^{\mathcal{M}|\gamma}$ for $\alpha_0 < \gamma \leq l(\mathcal{M})$ will be used to feed in branches for $\mathcal{T} \restriction \lambda + 1$, and therefore $P^\Phi$ itself, into $\mathcal{M}$. Let $\langle \xi_\alpha \rangle_{\alpha < \iota}$ enumerate $I^\Phi \cup \{0\}$.*

*There is a closed, increasing sequence of ordinals $\langle \eta_\alpha \rangle_{\alpha \leq \iota}$ and an increasing sequence of ordinals $\langle \gamma_\alpha \rangle_{\alpha \leq \iota}$ such that:*

1. *$\eta_1 = \gamma_0 = \eta_0 = \alpha_0$.*
2. *For each $\alpha < \iota$, $\eta_\alpha \leq \gamma_\alpha \leq \eta_{\alpha+1}$, and if $\alpha > 0$ then $\gamma_\alpha < \eta_{\alpha+1}$.*
3. *$\gamma_\iota = l(\mathcal{M})$, so $\mathcal{M} = \mathcal{M}|\gamma_\iota$.*
4. *Let $\alpha \in (0, \iota)$. Then $\gamma_\alpha$ is the least ordinal of the form $\eta_\alpha + \tau$ such that $\mathcal{T} \restriction \xi_\alpha \in \mathcal{J}_\tau(\mathcal{M}|\eta_\alpha)$ and if $\alpha > \alpha_0$ then $\delta(\mathcal{T} \restriction \xi_\alpha) < \tau$. (We explain below why such $\tau$ exists.) And $\mathcal{M}|\gamma_\alpha = \mathcal{J}^{\mathrm{m}}_\tau(\mathcal{M}|\eta_\alpha; \mathfrak{M}) \downarrow b$.*
5. *Let $\alpha \in (0, \iota)$. Then $\mathcal{M}|\eta_{\alpha+1} = \mathfrak{B}(\mathcal{M}|\gamma_\alpha, \mathcal{T} \restriction \xi_\alpha, \Lambda(\mathcal{T} \restriction \xi_\alpha)) \downarrow b$.*
6. *Let $\alpha < \iota$ be a limit. Then $\mathcal{M}|\eta_\alpha$ is passive.*
7. *$\gamma_\iota$ is the least ordinal of the form $\eta_\iota + \tau$ such that $\mathcal{T} \restriction \lambda + 1 \in \mathcal{J}_{\eta_\iota + \tau}(\mathcal{M}|\eta_\iota)$ and $\tau > o(M^{\mathcal{T}}_\lambda)$; with this $\tau$, $\mathcal{M} = \mathcal{J}^{\mathrm{m}}_\tau(\mathcal{M}|\eta_\iota; \mathfrak{M}) \downarrow b$ and furthermore, ${}^g\mathcal{F}(b)$ is acceptable and every strict segment of ${}^g\mathcal{F}(b)$ is sound.*

**Remark 5.39.** *It's not hard to see (cf. [16]) $\bar{\mathcal{M}} \trianglelefteq \mathcal{M} = {}^{\mathrm{g}}\mathcal{F}(b)$, the sequences $\langle \mathcal{M}|\eta_\alpha \rangle_{\alpha \leq \iota} \cap \bar{\mathcal{M}}$ and $\langle \mathcal{M}|\gamma_\alpha \rangle_{\alpha \leq \iota} \cap \bar{\mathcal{M}}$ and $\langle \mathcal{T} \restriction \alpha \rangle_{\alpha \leq \lambda+1} \cap \bar{\mathcal{M}}$ are $\Sigma_1^{\bar{\mathcal{M}}}$ in $\mathcal{L}_0^-$, uniformly in $b$ and $\bar{\mathcal{M}}$.*

**Definition 5.40.** *Let $b$ be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$. A* ***potential g-organized $\mathcal{F}$-premouse over*** *$b$ is a potential $^{\mathrm{g}}\mathcal{F}$-premouse over $b$, with parameter $\mathfrak{M}$.*

**Lemma 5.41.** *There is a formula $\varphi_{\mathrm{g}}$ in $\mathcal{L}_0$, such that for any transitive $b$ with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$, and any $\mathcal{J}$-structure $\mathcal{M}$ over $b$, $\mathcal{M}$ is a potential g-organized $\mathcal{F}$-premouse over $b$ iff $\mathcal{M}$ is a potential $\Lambda_{\mathfrak{M}}$-premouse over $b$, of type $\varphi_{\mathrm{g}}$.*

**Lemma 5.42.** *${}^{\mathrm{g}}\mathcal{F}$ is projecting, uniformly $\Sigma_1$, basic, and condenses finely.*

**Definition 5.43.** *Let $\mathcal{M}$ be a g-organized $\mathcal{F}$-premouse over $b$. We say $\mathcal{M}$ is **$\mathcal{F}$-closed** iff $\mathcal{M}$ is a limit of ${}^{\mathrm{g}}\mathcal{F}$-whole proper segments.*

Because $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$, $\mathcal{F}$-closure ensures closure under $\mathcal{G}_{\mathcal{F}}$:

**Lemma 5.44.** *Let $\mathcal{M}$ be an $\mathcal{F}$-closed g-organized $\mathcal{F}$-premouse over $b$. Then $\mathcal{M}$ is closed under $\mathcal{G}_{\mathcal{F}}$. In fact, for any set generic extension $\mathcal{M}[g]$ of $\mathcal{M}$, with $g \in V$, $\mathcal{M}[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}_{\mathcal{F}} \restriction \mathcal{M}[g]$ is definable over $\mathcal{M}[g]$, via a formula in $\mathcal{L}_0^-$, uniformly in $\mathcal{M}, g$.*

The analysis of scales in $\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}}(\mathbb{R})$ runs into a problem (see [16, Remark 6.8] for an explanation). Therefore we will analyze scales in a slightly different hierarchy.

**Definition 5.45.** *Let $X \subseteq \mathbb{R}$. We say that $X$ is **self-scaled** iff there are scales on $X$ and $\mathbb{R}\backslash X$ which are analytical (i.e., $\Sigma^1_n$ for some $n < \omega$) in $X$.*

**Definition 5.46.** *Let $b$ be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$.*

*Then ${}^{\mathsf{G}}\mathcal{F}(b)$ denotes the least $\mathcal{N} \trianglelefteq {}^{\mathrm{g}}\mathcal{F}(b)$ such that either $\mathcal{N} = {}^{\mathrm{g}}\mathcal{F}(b)$ or $\mathcal{J}_1(\mathcal{N}) \models$"$\Theta$ does not exist". (Therefore $\mathcal{J}_1^{\mathrm{m}}(b;\mathfrak{M}) \trianglelefteq {}^{\mathsf{G}}\mathcal{F}(b)$.)*

*We say that $\mathcal{M}$ is a **potential $\Theta$-g-organized $\mathcal{F}$-premouse over** $X$ iff $\mathfrak{M} \in \mathrm{HC}^{\mathcal{M}}$ and for some $X \subseteq \mathrm{HC}^{\mathcal{M}}$, $\mathcal{M}$ is a potential ${}^{\mathsf{G}}\mathcal{F}$-premouse over $(\mathrm{HC}^{\mathcal{M}}, X)$ with parameter $\mathfrak{M}$ and $\mathcal{M} \models$"$X$ is self-scaled". We write $X^{\mathcal{M}} = X$.*

In our application to core model induction, we will be most interested in the cases that either $X = \emptyset$ or $X = \mathcal{F} \restriction \mathrm{HC}^{\mathcal{M}}$. Clearly $\Theta$-g-organized $\mathcal{F}$-premousehood is not first order. Certain aspects of the definition, however, are:

**Definition 5.47.** *Let "I am a $\Theta$-g-organized premouse over $X$" be the $\mathcal{L}_0$ formula $\psi$ such that for all $\mathcal{J}$-structures $\mathcal{M}$ and $X \in \mathcal{M}$ we have $\mathcal{M} \models \psi(X)$ iff (i) $X \subseteq \mathrm{HC}^{\mathcal{M}}$; (ii) $\mathcal{M}$ is a $\mathcal{J}$-model over $(\mathrm{HC}^{\mathcal{M}}, X)$; (iii) $\mathcal{M}|1 \models$"$X$ is self-scaled"; (iv) every proper segment of $\mathcal{M}$ is sound; and (v) for every $\mathcal{N} \trianglelefteq \mathcal{M}$:*

1. *if $\mathcal{N} \models$"$\Theta$ exists" then $\mathcal{N} \downarrow (\mathcal{N}|\Theta^{\mathcal{N}})$ is a $\mathfrak{P}^{\mathcal{N}}$-strategy premouse of type $\varphi_{\mathrm{g}}$;*
2. *if $\mathcal{N} \models$"$\Theta$ does not exist" then $\mathcal{N}$ is passive.*

**Lemma 5.48.** *Let $\mathcal{M}$ be a $\mathcal{J}$-structure and $X \in \mathcal{M}$. Then the following are equivalent: (i) $\mathcal{M}$ is a $\Theta$-g-organized $\mathcal{F}$-premouse over $X$; (ii) $\mathcal{M} \models$"I am a $\Theta$-g-organized premouse over $X$" and $\mathfrak{P}^{\mathcal{M}} = \mathfrak{M}$ and $\Sigma^{\mathcal{M}} \subseteq \Lambda_{\mathfrak{M}}$; (iii) $\mathcal{M}|1$ is a $\Theta$-g-organized premouse over $X$ and every proper segment of $\mathcal{M}$ is sound and for every $\mathcal{N} \trianglelefteq \mathcal{M}$,*

1. *if $\mathcal{N} \vDash$ "$\Theta$ exists" then $\mathcal{N} \downarrow (\mathcal{N}|\Theta^{\mathcal{N}})$ is a g-organized $\mathcal{F}$-premouse;*

2. *if $\mathcal{N} \vDash$ "$\Theta$ does not exist" then $\mathcal{N}$ is passive.*

**Lemma 5.49.** *${}^{\mathsf{G}}\mathcal{F}$ is basic and condenses finely.*

**Definition 5.50.** *Suppose $\mathcal{F}$ is a nice operator and is an iteration strategy and $X \subseteq \mathbb{R}$ is self-scaled. We define $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, X)$ as the stack of all $\Theta$-g-organized $\mathcal{F}$-mice $\mathcal{N}$ over $(H_{\omega_1}, X)$ (with parameter $\mathfrak{M}$). We also say ($\Theta$-g-organized) $\mathcal{F}$-premouse **over** $\mathbb{R}$ to in fact mean over $H_{\omega_1}$.*

**Remark 5.51.** *It's not hard to see that for any such $X$ as in Definition 5.50, $\wp(\mathbb{R}) \cap Lp^{{}^{\mathsf{g}}\mathcal{F}}(\mathbb{R}, X) = \wp(\mathbb{R}) \cap Lp^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, X)$. Suppose $\mathcal{M}$ is an initial segment of the first hierarchy and $\mathcal{M}$ is $E$-active. Note that $\mathcal{M} \vDash$ "$\Theta$ exists" and $\mathcal{M}|\Theta$ is $\mathcal{F}$-closed. By induction below $\mathcal{M}|\Theta^{\mathcal{M}}$, $\mathcal{M}|\Theta^{\mathcal{M}}$ can be rearranged into an initial segment $\mathcal{N}'$ of the second hierarchy. Above $\Theta^{\mathcal{M}}$, we simply copy the $E$ and $B$-sequence from $\mathcal{M}$ over to obtain an $\mathcal{N} \lhd Lp^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, X)$ extending $\mathcal{N}'$.*

In core model induction applications, we often have a pair $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is a hod premouse and $\Sigma$ is $\mathcal{P}$'s strategy with branch condensation and is fullness preserving (relative to mice in some pointclass) or $\mathcal{P}$ is a sound (hybrid) premouse projecting to some countable set $a$ and $\Sigma$ is the unique (normal) $\omega_1 + 1$-strategy for $\mathcal{P}$. Let $\mathcal{F}$ be the operator corresponding to $\Sigma$ (using the formula $\varphi_{\text{all}}$) and suppose $\mathcal{M}_1^{\mathcal{F},\sharp}$ exists. [16, Lemma 4.8] shows that $\mathcal{F}$ condenses finely and $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$. Also, the core model induction will give us that $\mathcal{F} \restriction \mathbb{R}$ is self-scaled. [16]Thus, we can define $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ as above (assuming sufficient iterability of $\mathcal{M}_1^{\mathcal{F},\sharp}$). A core model induction is then used to prove that $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \mathsf{AD}^+$. What's needed to prove this is the scales analysis of $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$, from the optimal hypothesis (similar to those used by Steel; see [14] and [10]).[17] This is carried out in [16]; we will not go into details here, though we simply note that for the scales analysis to go through under optimal hypotheses, we need to work with the $\Theta$-g-organized hierarchy, instead of the g-organized hieararchy.

### 5.0.4. CORE MODEL INDUCTION OPERATORS

Suppose $\mathcal{F}$ is a nice operator and $\Gamma$ is an inductive-like pointclass that is determined. Let $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\sharp}$. $\mathrm{Lp}^{{}^g\mathcal{F}}(x)$ is defined as in the previous section. We write $\mathrm{Lp}^{{}^g\mathcal{F},\Gamma}(x)$ for the stack of ${}^g\mathcal{F}$-premice $\mathcal{M}$ over $x$ such that every countable, transitive $\mathcal{M}^*$ embeddable into $\mathcal{M}$ has an $\omega_1$-${}^g\mathcal{F}$-iteration strategy in $\Gamma$.

**Definition 5.52.** *Let $t \in \mathrm{HC}$ with $\mathfrak{M} \in \mathcal{J}_1(t)$. Let $1 \leq k < \omega$. A premouse $\mathcal{N}$ over $t$ is $\mathcal{F}$-$\Gamma$-$k$-*suitable *(or just $k$-*suitable *if $\Gamma$ and $\mathcal{F}$ are clear from the context) iff there is a strictly increasing sequence $\langle \delta_i \rangle_{i<k}$ such that*

[16]We abuse notation here, and will continue to do so in the future. Technically, we should write $\mathcal{F} \restriction \mathrm{HC}$.

[17]Suppose $\mathcal{P} = \mathcal{M}_1^\sharp$ and $\Sigma$ is $\mathcal{P}$'s unique iteration strategy. Let $\mathcal{F}$ be the operator corresponding to $\Sigma$. Suppose $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{MC}$. Then in fact $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}) \cap \wp(\mathbb{R}) = \mathrm{Lp}(\mathbb{R}) \cap \wp(\mathbb{R})$. This is because in $L(\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}))$, $L(\wp(\mathbb{R})) \vDash \mathsf{AD}^+ + \Theta = \theta_0 + \mathsf{MC}$ and hence by [7], in $L(\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}))$, $\wp(\mathbb{R}) \subseteq \mathrm{Lp}(\mathbb{R})$. Therefore, even though the hierarchies $\mathrm{Lp}(\mathbb{R})$ and $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ are different, as far as sets of reals are concerned, we don't lose any information by analyzing the scales pattern in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ instead of that in $\mathrm{Lp}(\mathbb{R})$.

1. $\forall \delta \in \mathcal{N}$, $\mathcal{N} \models$ "$\delta$ *is Woodin" if and only if* $\exists i < k(\delta = \delta_i)$.

2. $o(\mathcal{N}) = \sup_{i<\omega}(\delta_{k-1}^{+i})^{\mathcal{N}}$.

3. *If* $\mathcal{N}|\eta$ *is a* ${}^{\mathsf{g}}\mathcal{F}$*-whole strong cutpoint of* $\mathcal{N}$ *then* $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathrm{Lp}^{{}^g\mathcal{F},\Gamma}(\mathcal{N}|\eta)$.[18]

4. *Let* $\xi < o(\mathcal{N})$*, where* $\mathcal{N} \models$ "$\xi$ *is not Woodin". Then* $C_\Gamma(\mathcal{N}|\xi) \models$ "$\xi$ *is not Woodin".*

*We write* $\delta_i^{\mathcal{N}} = \delta_i$*; also let* $\delta_{-1}^{\mathcal{N}} = 0$ *and* $\delta_k^{\mathcal{N}} = o(\mathcal{N})$.

**Definition 5.53** (relativizes well)**.** *Let* $\mathcal{F}$ *be a* $Y$*-mouse operator for some operator* $Y$*. We say that* $\mathcal{F}$ ***relativizes well*** *if there is a formula* $\phi(x, y, z)$ *such that for any* $a, b \in dom(\mathcal{F})$ *such that* $a \in L_1(b)$ *and have the same cardinality, whenever* $N$ *is a transitive model of* $\mathsf{ZFC}^-$ *such that* $N$ *is closed under* $Y$*,* $\mathcal{F}(b) \in N$ *then* $\mathcal{F}(a) \in N$ *and is the unique* $x \in N$ *such that* $N \models \phi[x, a, \mathcal{F}(b)]$.

**Definition 5.54** (determines itself on generic extensions)**.** *Suppose* $\mathcal{F}$ *is a* $Y$*-mouse operator for some operator* $Y$*. We say that* $\mathcal{F}$ ***determines itself on generic extensions*** *if there is a formula* $\phi(x, y, z)$*, a parameter* $a$ *such that for almost all transitive structures* $N$ *of* $\mathsf{ZFC}^-$ *such that* $\omega_1 \subset N$*,* $N$ *contains* $a$ *and is closed under* $\mathcal{F}$*, for any generic extension* $N[g]$ *of* $N$ *in* $V$*,* $\mathcal{F} \cap N[g] \in N[g]$ *and is definable over* $N[g]$ *via* $(\phi, a)$*, i.e. for any* $x \in N[g] \cap \mathrm{dom}(\mathcal{F})$*,* $\mathcal{F}(a) = b$ *if and only if* $b$ *is the unique* $c \in N[g]$ *such that* $N[g] \models \phi[x, c, a]$.[19]

The following definition gives examples of "nice model operators". This is not a standard definition and is given here for convenience more than anything. These are the kind of model operators that the core model induction in this paper deals with. We by no means claim that these operators are all the useful model operators that one might consider. Recall we fixed a $V$-generic $G \subseteq Col(\omega, \kappa)$. These operators are obtained from the $W_\alpha^*, W_\alpha$ hypotheses formulated for $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\alpha$; these are straightforward generalizations of the $W_\alpha^*, W_\alpha$ hypotheses for $J_\alpha(\mathbb{R})$.

**Definition 5.55** (Core model induction operators)**.** *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair below* $\kappa$*; assume furthermore that* $\Sigma$ *is a* $(\lambda^+, \lambda^+)$*-strategy. Let* $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}$ *(note that* $\mathcal{F}$*,* ${}^g\mathcal{F}$ *are basic, projecting, uniformly* $\Sigma_1$*, and condenses finely). Assume* $\mathcal{F} \restriction \mathbb{R}$ *is self-scaled. We say* $J$ *is a* $\Sigma$ *core model induction operator or just a* $\Sigma$*-cmi operator if in* $V[G]$*, one of the following holds:*

1. *$J$ is a projecting, uniformly* $\Sigma_1$*, first order* $\mathcal{F}$*-mouse operator (or* ${}^{\mathsf{g}}\mathcal{F}$*-mouse operator) defined on a cone of* $(H_{\omega_1})^{V[G]}$ *above some* $a \in (H_{\omega_1})^{V[G]}$*. Furthermore,* $J$ *relativizes well.*

2. *For some* $\alpha \in \mathrm{OR}$ *such that* $\alpha$ *ends either a weak or a strong gap in the sense of [14] and [16], letting* $M = Lp^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})||\alpha$ *and* $\Gamma = (\Sigma_1)^M$*,* $M \models \mathsf{AD}^+ + \mathsf{MC}(\Sigma)$[20]*. For some transitive* $b \in H_{\omega_1}^{V[G]}$ *and some g-organized* $\mathcal{F}$*-premouse* $\mathcal{Q}$ *over* $b$*,* $J = \mathcal{F}_\Lambda$*, where* $\Lambda$

[18]Literally we should write "$\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathrm{Lp}^\Gamma(\mathcal{N}|\eta) \downarrow t$", but we will be lax about this from now on.

[19]By "almost all", we mean for all such $N$ with the properties listed above and $N$ satisfies some additional property. In practice, this additional property is: $N$ is closed under $\mathcal{M}_1^{\mathcal{F},\sharp}$.

[20]$\mathsf{MC}(\Sigma)$ stands for the Mouse Capturing relative to $\Sigma$ which says that for $x, y \in \mathbb{R}$, $x$ is $\mathrm{OD}(\Sigma, y)$ (or equivalently $x$ is $\mathrm{OD}(\mathcal{F}, y)$) iff $x$ is in some ${}^{\mathsf{g}}\mathcal{F}$-mouse over $y$. $\mathsf{SMC}$ is the statement that for every hod pair $(\mathcal{P}, \Sigma)$ such that $\Sigma$ is fullness preserving and has branch condensation, then $\mathsf{MC}(\Sigma)$ holds.

*is an $(\omega_1,\omega_1)$-iteration strategy for a 1-suitable (or more fully $\mathcal{F}$-$\Gamma$-1-suitable) $\mathcal{Q}$ which is $\Gamma$-fullness preserving, has branch condensation and is guided by some self-justifying-system (sjs) $\vec{A}=(A_i : i<\omega)$ such that $\vec{A}\in OD^M_{b,\Sigma,x}$ for some real $x$ and $\vec{A}$ seals the gap that ends at $\alpha$*[21].

## 6. Lifting Operators

We assume the hypothesis of (i). Suppose $(\mathcal{P}^*,\Sigma)$ is a hod pair below $\kappa$ such that $\Sigma$ is an $(\kappa^+,\kappa^+)$-strategy in $V[G]$ and $\Sigma\restriction V\in V$ (or $(\mathcal{P}^*,\Sigma)=(\emptyset,\emptyset)$). We fix a $V$-generic $G\subseteq Col(\omega,\gamma)$, where $\gamma<\kappa$ is such that $\gamma^\omega$ and $\gamma< cof(Lp^\Sigma(A))$ for some $A\subseteq\kappa$ coding $H_\kappa$. Suppose $J$ is a $\Sigma$-cmi-operator. We assume $J$ is defined on a cone in $H^{V[G]}_{\kappa^+}$ above some $x\in H^V_{\kappa^+}$ and $J\restriction V\in V$. [22]

Let $\mathcal{F}=\mathcal{F}_{\Sigma,\varphi_{\rm all}}$. Again, we fix $A\subseteq\kappa$ coding $V_\kappa$ and $\mathrm{cof}(o(Lp^{\mathcal{F}}(A)))<\gamma<\kappa$, where $\gamma$ is countably closed and a $V$-generic $G\subseteq Col(\omega,\gamma)$.

We may assume $A$ code $\mathcal{P}^*$. We set

$$\mathrm{Lp}^\Sigma_1(A)=\mathrm{Lp}^{\mathcal{F}}(A).$$

Suppose $\mathrm{Lp}^\Sigma_\alpha(A)$ has been defined for $\alpha<\kappa^+$,

$$\mathrm{Lp}^\Sigma_{\alpha+1}(A)=\mathrm{Lp}^{\mathcal{F}}_+(\mathrm{Lp}^\Sigma_\alpha(A)),$$ [23] and

for $\xi<\kappa^+$ limit,

$$\mathrm{Lp}^\Sigma_\xi(A)=\textstyle\bigcup_{\alpha<\xi}\mathrm{Lp}^\Sigma_\alpha(A).$$

**Lemma 6.1.** *Let $A$ be as above. Then $Lp^\Sigma_{\kappa^+}(A)\vDash\lambda^+$ exists. Furthermore, $cof(\lambda^+)^{Lp^\Sigma_{\kappa^+}(A)}<\kappa$.*

*Proof.* Suppose not. This easily implies that we can construct over $\mathrm{Lp}^\Sigma_{\kappa^+}(A)$ a $\square_\kappa$-sequence[24]. This contradicts $\neg\square_\lambda$ in $V$. Now the cofinality clause follows from the fact that $\kappa$ is singular. $\square$

Let $S$ be the set of $X\prec H_{\kappa^{++}}$ such that $\gamma\subseteq X$, $|X|=\gamma$, $X^\omega\subseteq X$, and $X$ is cofinal in the ordinal height of $\mathrm{Lp}^\Sigma(A)$ and $J,(\mathcal{P}^*\cup\{\mathcal{P}^*\},\Sigma)\in X$.[25] So $S$ is stationary. We let $\pi_X: M_X\to H_{\kappa^{++}}$ be the uncollapsed map and $\lambda_X$ be the critical point of $\pi_X$; note that $\lambda_X>\gamma$. We now prove some lemmas about "lifting" operators.

**Lemma 6.2** (Full hulls)**.** *Suppose $B^*\subseteq\kappa$. Suppose $X\in S$ such that $A^*\in X$ and $X$ is cofinal in $Lp^\Sigma(B^*)$. Let $\pi_X(B)=B^*$. Then $Lp^\Sigma(B)\subseteq M_X$.*

[21] This implies that $\vec{A}$ is Wadge cofinal in $\mathbf{Env}(\mathbf{\Gamma})$, where $\Gamma=\Sigma^M_1$. Note that $\mathbf{Env}(\mathbf{\Gamma})=\wp(\mathbb{R})^{\mathbf{M}}$ if $\alpha$ ends a weak gap and $\mathbf{Env}(\mathbf{\Gamma})=\wp(\mathbb{R})^{\mathrm{Lp}^{\mathbf{\Sigma}}(\mathbb{R})|(\alpha+\mathbf{1})}$ if $\alpha$ ends a strong gap.

[22] We note the specific requirement that the cone over which $J$ is defined is above some $x\in V$. These are the $\Sigma$-cmi-operators that we will propagate in our core model induction. We will not deal with all $\Sigma$-cmi-operators.

[23] $\mathrm{Lp}^{\mathcal{F}}_{*,+}(\mathrm{Lp}^\Sigma_\alpha(A))$ is defined similarly to $\mathrm{Lp}^{\mathcal{F}}_*$ but here we stack continuing, $\mathcal{F}$-sound $\mathcal{F}$-premice.

[24] Squares hold in $\mathrm{Lp}^\Sigma_{\kappa^+}(A)$ because $\Sigma$ has hull and branch condensation.

[25] This means $(\mathcal{P}^*,\Sigma\restriction V)\in X$ and $\Sigma\in X[G]$ but we will abuse notation here.

*Proof.* We just prove the first clause. Suppose not. Then let $\mathcal{M} \lhd \mathrm{Lp}^{\Sigma}(B)$ be the least counterexample. Let $E$ be the $(\lambda_X, \kappa)$-extender derived from $\pi_X$. Let $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, E)$. Then any countable transitive $\mathcal{N}^*$ embeddable into $\mathcal{N}$ (via $\sigma$) is embeddable into $\mathcal{M}$ (via $\tau$) such that $i_E \circ \tau = \sigma$ by countable completeness of $E$. So $\mathcal{N}^*$ is $\omega_1 + 1$ $\Sigma^{\sigma}$-iterable because $\mathcal{M} \lhd \mathrm{Lp}^{\Sigma}(B)$, $\sigma^{-1}(\mathcal{P}^*) = \tau^{-1}(\mathcal{P}^*)$, and $\sigma \restriction \sigma^{-1}(\mathcal{P}^*) = \tau \restriction \sigma^{-1}(\mathcal{P}^*)$. So $\mathcal{N} \lhd \mathrm{Lp}^{\Sigma}(B^*)$. But since $\pi_X$ is cofinal in $\mathrm{Lp}^{\Sigma}(A^*)$, $\mathcal{N} \notin \mathrm{Lp}^{\Sigma}(B^*)$. Contradiction. □

**Lemma 6.3** (Lifting in $V$)**.** *1. If $H$ is defined by $(\psi, a)$ on $H^{V[G]}_{\omega_1}$ (as in clause 1 of 5.55) with $a \in V$ and $H \restriction V \in V$, then $H$ can be extended to an operator $H^+$ defined by $(\psi, a)$ on $H_{\kappa^+}$. Furthermore, $H^+$ relativizes well.*

*2. If $(\mathcal{Q}, F)$ and $\Gamma$ are as in clause 2 of Definition 5.55, where $F$ plays the role of $\Lambda$ there with $(\mathcal{Q}, F \restriction V) \in V$, then $F$ can be extended to a $(\kappa^+, \kappa^+)$-strategy that has branch condensation. Furthermore, there is a unique such extension.*

*Proof.* To prove 1), first let $A^*$ be a bounded subset of $\kappa$ (in the cone above $a$) and let $X \in S$ such that $A, A^* \in X$ and $X$ is cofinal in $\mathrm{Lp}^{\Sigma}(A^*)$. Let $\pi_X(A_X, A^*_X) = (A, A^*)$. We assume $H$ is an $\mathcal{F}$-mouse operator. By Lemma 6.2, $H(A_X) \in M_X$ and since $H$ relativizes well, $H(A^*_X) \in M_X$ (we use the fact that $A^* \in L_1[A]$). Hence we can define $H^+(A^*) = \pi_X(H(A^*_X))$ (as the first level $\mathcal{M} \lhd \mathrm{Lp}^{\Sigma}(A^*)$ that satisfies $\psi[A^*, a]$). This defines $H^+$ on all bounded subsets of $\kappa$.

Now let $A^*$ be a bounded subset of $\kappa^*$. Let $X \prec H_{\kappa^{++}}$ be such that $X$ is countably closed, $|X| < \kappa$, and $X$ is cofinal in $Lp^{\Sigma}(A^*)$. Let $\pi_X : M_X \to X$ be the uncollapse map. By Lemma 6.2, $Lp^{\Sigma}(A^*_X) \subseteq M_X$ where $A^*_X = \pi_X^{-1}(A^*)$. So $H(A^*_X) \in M_X$ and we can define $H^+(A^*)$ to be $\pi_X(H(A^*_X))$. It's easy to see then that $H^+$ also relativizes well (**Exercise:** verify this).

We first prove the "uniqueness" clause of 2). Suppose $F_1$ and $F_2$ are two extensions of $F$ and let $\mathcal{T}$ be according to both $F_1$ and $F_2$. Let $b_1 = F_1(\mathcal{T})$ and $b_2 = F_2(\mathcal{T})$. If $b_1 \neq b_2$ then $\mathrm{cof}(lh(\mathcal{T})) = \omega$. So letting $\mathcal{T}^*$ be a hull of $\mathcal{T}$ such that $|\mathcal{T}^*| \leq \gamma$ and letting $\pi : \mathcal{T}^* \to \mathcal{T}$ be the hull embedding, then $b_1 \cup b_2 \subseteq \mathrm{rng}(\pi)$. Then $\pi^{-1}[b_1] = F(\mathcal{T}^*) \neq \pi^{-1}[b_2] = F(\mathcal{T}^*)$. Contradiction.

To show existence, let $F_{\gamma^+} = F$. Inductively for each $\gamma^+ \leq \xi < \kappa$ such that $\xi$ is a limit ordinal, we define a strategy $F_\xi$ extending $F_\alpha$ for $\alpha < \xi$ and $F_\xi$ acts on trees of length $\xi$. For $X \prec Y \prec H_{\kappa^{++}}$, let $\pi_{X,Y} = \pi_Y^{-1} \circ \pi_X$. Let $\mathcal{T}$ be a tree of length $\xi$ such that for all limit $\xi^* < \xi$, $\mathcal{T} \restriction \xi^*$ is according to $F_{\xi^*}$. We want to define $F_\xi(\mathcal{T})$.

For $X \in S$ such that $X$ is cofinal in $\mathrm{Lp}^{\Sigma}_{+}(A)$, let $(\mathcal{T}_X, \xi_X) = \pi_X^{-1}(\mathcal{T}, \xi)$ and $b_X = F(\mathcal{T}_X)$. Let $c_X$ be the downward closure of $\pi_X[b_X]$ and $c_{X,Y}$ be the downward closure of $\pi_{X,Y}[b_X]$.

**Claim:** $\forall^* X \in S$[26], for any $\gamma < \xi$, $\gamma \in c_X$ or $\gamma \notin c_X$.

*Proof.* The proof is similar to that of Lemma 2.5 in [9] so we only sketch it here. Suppose for contradiction that there is some $\gamma$, there are stationarily many $X \in S$ such that $\gamma \in c_X$ and there are stationarily many $Y \in S$ such that $\gamma \notin c_Y$. Suppose first $\mathrm{cof}(\xi) \in [\omega_1, \gamma]$. Note that $\mathrm{crt}(\pi_X)$,

[26]This means there is a club set $C$

$\mathrm{crt}(\pi_Y) > \gamma$. It's easy then to see that $\pi_X[b_X]$ is cofinal in $\xi$ and $\pi_Y[c_Y]$ is cofinal in $\xi$. Hence $c_X = c_Y$. Contradiction.

Now suppose $\mathrm{cof}(\xi) = \omega$. Fix a surjection $f : \theta \twoheadrightarrow \xi$ (where $\theta = |\xi|$). $\forall^* X \in S$ $(f, \xi) \in X$ so let $(f_X, \xi_X, \theta_X) = \pi_X^{-1}(f, \xi, \theta)$. For each such $X$, let $\alpha_X$ be least such that $f_X[\alpha_X] \cap b_X$ is cofinal in $\xi_X$. By Fodor's lemma,

$$\exists \alpha \exists U \ (U \text{ is stationary} \wedge \forall X \in U \ \alpha_X = \alpha).$$

By symmetry and by thinning out $U$, we may assume

$$X \in U \Rightarrow \pi_X^{-1}(\gamma) \in b_X.$$

Fix $Y \in S$ such that $\gamma \notin c_Y$ and $\alpha < \lambda_Y$. Since $U$ is stationary, there is some $X \in U$ such that $Y \prec X$, which implies

$$\pi_{Y,X}[f_Y[\alpha]] = f_X[\alpha]$$

is cofinal in $b_X$ and hence $\mathcal{T}_Y^\frown \pi_{Y,X}^{-1}[b_X]$ is a hull of $\mathcal{T}_X^\frown b_X$. Since $F$ condenses well, $\pi_{Y,X}^{-1}[b_X] = b_Y$. This contradicts the fact that $\pi_X^{-1}(\gamma) \in b_X$ but $\pi_Y^{-1}(\gamma) \notin b_Y$.

Finally, suppose $\mathrm{cof}(\xi) \geq \gamma^+$. The case $\forall^* X$ $\mathcal{T}_X$ is maximal is proved exactly as in Lemma 1.25 of [9] (**Exercise:** Please read this argument in [9]). The main point is the following: for any $X \prec Y$ in the above club, if $\sup \pi_{X,Y}[\mathcal{T}_X] = \lambda < lh(\mathcal{T}_Y)$, then $\mathcal{T}_Y = \mathcal{T}_Y|\lambda^\frown \mathcal{U}$ where $c_{X,Y} = [0, \lambda]_{\mathcal{T}_Y}$, and $\mathcal{U}$ is a tree on $\mathcal{M}_\lambda^{\mathcal{T}_Y}$. In other words, $b_Y = [0, \lambda]_{\mathcal{T}_Y}^\frown c$, where $c$ is a cofinal branch of $\mathcal{U}$. So $c_{X,Y} \subseteq b_Y$. In the case $\lambda = lh(\mathcal{T}_Y)$, we get $c_{X,Y} = b_Y$.

Suppose $\mathcal{T}_X$ is short and is according to $F$. Note that $\mathrm{lh}(\mathcal{T}_X)$ has uncountable cofinality (in $V$). We claim that $\forall^* X \in S$ $b_X = F(\mathcal{T}_X) \in M_X$. Given the claim we get that for any two such $X \prec Y$ satisfying the claim, $\pi_{X,Y}(b_X)$ is cofinal in $\mathcal{T}_Y$ and hence $\pi_{X,Y}(b_X) = b_Y$. This gives $c_{X,Y}$ is an initial segment of $b_Y$, which is what we want to prove.

Now to see $\forall^* X \in S$ $b_X = F(\mathcal{T}_X) \in M_X$. $\mathcal{Q}(\mathcal{T}_X)$ is the least $\mathcal{Q} \lhd \mathrm{Lp}_+^{\Sigma,\Gamma}(\mathcal{M}(\mathcal{T}_X))$ that defines the failure of Woodinness of $\delta(\mathcal{T}_X)$. Since $\delta(\mathcal{T})$ has uncountable cofinality (in $V$ and in $V[G]$), by a standard interpolation argument, whenever $\mathcal{M}_0, \mathcal{M}_1 \in \mathrm{Lp}_+^{\Sigma,\Gamma}(\mathcal{M}(\mathcal{T}_X))$ then we have either $\mathcal{M}_0 \trianglelefteq \mathcal{M}_1$ or $\mathcal{M}_1 \trianglelefteq \mathcal{M}_0$. So the "leastness" of $\mathcal{Q}(\mathcal{T}_X)$ is justified in this case. By the same proof as that of Lemma 6.2 and the fact that $X$ is cofinal in $\mathrm{Lp}_+^\Sigma(A)$, $\mathcal{M}(T)$ is coded in to $A$, and $\mathrm{Lp}_+^{\Sigma,\Gamma}(\mathcal{M}(\mathcal{T}_X)) \trianglelefteq \mathrm{Lp}_+^\Sigma(\mathcal{M}(\mathcal{T}))$, we get $\mathcal{Q}(\mathcal{T}_X) \in M_X$.

Now $F(\mathcal{T}_X) = b_X$ is the unique branch $b$ such that $\mathcal{Q}(b, \mathcal{T}_X)$ exists and is a $\Sigma$-premouse over $\mathcal{M}(\mathcal{T}_X)$ and hence is $\mathcal{Q}(\mathcal{T}_X)$. The uniqueness of $b_X$ follows from a standard comparison argument. By an absoluteness argument and the fact that $\mathcal{Q}(\mathcal{T}_X) \in M_X$, $b_X \in M_X$. We're done. □

Letting $C$ be the club as in the claim, we can just define

$$\gamma \in F_\xi(\mathcal{T}) \Leftrightarrow \forall X \in C \cap S \ \gamma \in c_X.$$

For $\kappa \leq \xi < \kappa^+$, for $\mathcal{T}$ of length $\xi$ by $F_\xi$, we define $F_{\xi+1}(\mathcal{T})$ in a similar fashion as above, except now we let $S$ be the collection of hulls $X$ such that $|X| < \kappa$, countably closed, and cofinal in $Lp(\mathcal{M}(\mathcal{T}))$.

It's easy to verify that with this definition, the unique extension of $F$ to a $(\kappa^+, \kappa^+)$ strategy has branch condensation (**Exercise:** Verify this). This completes the proof sketch of the lemma. $\square$

**Remark 6.4.** *(i) In the proof of the claim above, we do not use our hypothesis in the cases where $\omega \leq cof(\xi) \leq \gamma$, but we seem to need our hypothesis in the case $cof(\xi) > \gamma$.*

*(ii) If we assume instead full* PFA*, in particular, $\square(\alpha)$ fails for every cardinal $\alpha \geq \omega_3$, then we can show that for any $(\mathcal{N}, \Sigma)$ such that $|\mathcal{N}| \leq \omega_2$ and $\Sigma$ is a $(\omega_3, \omega_3)$-iteration strategy for $\mathcal{N}$ with branch condensation, then we can extend $\Sigma$ uniquely to a strategy $\Sigma^+$ acting on all stacks of normal trees in $V$. Let $\mathcal{T}$ be according to $\Sigma^+$ and $\mathcal{T}$ has limit length. We define $\Sigma^+(\mathcal{T})$ as follows: if cof($lh(\mathcal{T})) < \omega_3$, we define $\Sigma^+(\mathcal{T})$ by the procedure in the proof of the Claim in Lemma 6.3. Otherwise, observe that $\vec{C} = \{[0,\lambda]_{\mathcal{T}} : \lambda < lh(\mathcal{T})\}$ is a coherent sequence and by $\neg\square(lh(\mathcal{T}))$, we get a thread $D$. This in turns gives us a cofinal (necessarily unique and well-founded) branch $b$ of $\mathcal{T}$. Let $\Sigma^+(\mathcal{T})$ be this $b$.*

The following sometimes comes up during CMI beyond $L(\mathbb{R})$, but we do not need it for proving AD holds in $L(\mathbb{R})$.

**Lemma 6.5** (Extending to generic extensions)**.** *1. If $H$ is a defined by $(\psi, a)$ on $H^{V[G]}_{\omega_1}$ as in clause 1 of 5.55 with $a \in H^V_{\gamma^+}$, then $H$ can be extended to a first order mouse operator $H^+$ defined by $(\psi, a)$ on $H^{V[G]}_{\kappa^+}$. Furthermore, $H^+$ relativizes well and if $H$ determines itself on generic extensions then so does $H^+$.*

*2. If $(\mathcal{Q}, \Lambda)$ and $\Gamma$ are as in clause 2 of Definition 5.55, where $F$ plays the role of $J$ there and $(\mathcal{Q}, \Lambda \upharpoonright V) \in V$, then $\Lambda$ can be extended to a unique $(\kappa^+, \kappa^+)$-strategy that has branch condensation in $V[G]$.*

*Proof.* For (1), let $b \in H^{V[G]}_{\kappa^+}$ and let $\tau \in H^V_{\kappa^+}$ be a nice $\mathrm{Col}(\omega, \gamma)$-name for $b$ ($\mathrm{Col}(\omega, \gamma)$ is $\gamma^+$-cc so such a name exists since $\kappa$ is singular strong limit).[27] Assume $H$ is a $\Sigma$-mouse operator (the other case is proved similarly). Let $X \in S$ be such that $\mathcal{P}^* \cup \{\mathcal{P}^*\}, \Sigma, b, \tau, H(\tau) \in X[G]$; here we use Lemma 6.3 to get that $H(\tau)$ is defined. Let $(\bar{b}, \bar{\tau}) = \pi_X^{-1}(b, \tau)$. Then $\pi_X^{-1}(H(\tau)) = H(\bar{\tau}) \in M_X$ by condensation of $H$. Since $H$ relativizes well, $H(\bar{b}) \in M_X[G]$. This means we can define $H^+(b)$ to be $\pi_X(H(\bar{b}))$. We need to see that $H^+(b)$ is countably $\Sigma$-iterable in $V[G]$. So let $\pi : \mathcal{N} \to H^+(b)$ with $\mathcal{N}$ countable transitive in $V[G]$ and $\pi(b^*) = b$. Let $X \subset Y \in S$ be such that $\mathrm{ran}(\pi) \subseteq \mathrm{ran}(\pi_Y)$; then $H(\pi_Y^{-1}(b)) \in M_Y[G]$ and there is an embedding from $\mathcal{N}$ into $H(\pi_Y^{-1}(b))$, so $\mathcal{N}$ has an $(\omega_1, \omega_1+1)$-$\Sigma$-iteration strategy. The definition doesn't depend on the choice of $X$ and it's easy to see that $H^+$ satisfies the conclusion (**Exercise:** Verify this).

For (2), let $M \in H^{V[G]}_{\kappa^+}$ be transitive and $\tau \in H^V_{\kappa^+}$ be a $\mathrm{Col}(\omega, \gamma)$-term for $M$. We define the extension $\Lambda^+$ of $\Lambda$ as follows (it's easy to see that there is at most one such extension). In $N = L^{\Lambda^*}_{\kappa^+}[B, \mathfrak{M}]$, where $B \subseteq \kappa$ codes $tr.cl.(\tau)$ and a well-ordering of $tr.cl.(\tau)$, $\Lambda^*$ is the unique $(\kappa^+, \kappa^+)$-$\Lambda$-strategy for $\mathfrak{M} = \mathcal{M}_1^{\Lambda,\sharp}$ in $V$. $\Lambda^*$ exists by Lemma 6.3.

[27] In particular, a nice $\mathrm{Col}(\omega, \gamma)$-name for a real can be considered a subset of $\gamma$ and hence a nice $\mathrm{Col}(\omega, \gamma)$-name for $\mathbb{R}^{V[G]}$ is an element of $H^V_{\gamma^+}$.

Let $\mathcal{T}_{tr.cl.(\tau)}$ be according to $\Lambda^*$ and be defined as in Definition 5.36. Note that $(\kappa^+)^N < (\kappa^+)^V$ by $\neg\Box_\kappa$ and the fact that $\Box_\kappa$ holds in $N$, so $\mathcal{T}_{tr.cl.(\tau)} \in N$ and has length less than $o(N) = \kappa^+$. Let $\mathcal{R}$ be the last model of $\mathcal{T}_{tr.cl(\tau)}$ and note that by the construction of $\mathcal{T}_{tr.cl.(\tau)}$, $M$ is generic over $\mathcal{R}$. Let $\mathcal{U} \in M$ be a tree according to $\Lambda^+$ of limit length, then set $\Lambda^+(\mathcal{U}) = b$ where $b$ is given by (the proof of) [16, Lemma 4.8] by interpreting $\Lambda$ over generic extensions of $\mathcal{R}$. By a simple reflection argument, it's easy to see that $\Lambda^+(\mathcal{U})$ doesn't depend on $M$. **Exercise:** Verify this.

This completes the construction of $\Lambda^+$. It's easy to see that $F^+$ has branch condensation. □

**Exercise 6.6.** *Formulate and prove analogs of the above lemmata for situations (ii) and (iii).*

## 7. The $J \mapsto \mathcal{M}_1^{J,\sharp}$ step

Let $J$ be a $\Sigma$-cmi-operator for some $\Sigma$ and $J$ is defined on a cone above $x \in H^V_{\gamma^+}$. We now proceed to construct the operator $\mathcal{M}_1^{J,\sharp} : x \mapsto \mathcal{M}_1^{J,\sharp}(x)$. The domain of $\mathcal{M}_1^{J,\sharp}$ will be the same as the domain of $J$. We denote $\mathcal{M}_0^{J,\sharp}(x)$ for the least $E$-active, sound $J$-mouse over $x$.

**Lemma 7.1.** *For every $A$ bounded in $\kappa^+$, $\mathcal{M}_0^{J,\sharp}(A)$ exists.*

*Proof.* By Lemma 6.3, it's enough to show that if $B$ is a bounded subset of $\gamma^+$, then $\mathcal{M}_0^{J,\sharp}(B)$ exists. Suppose not. Let $M = L^J[B]$ (by $L^J[B$, we mean $L^J_{\kappa^+}[B]$, and so $M$ has ordinal height $\kappa^+$). Note that $\kappa^+ > (\kappa^+)^M$ because $\Box_\kappa$ holds in $M$ but fails in $V$.[28] On the other hand, by Jensen covering theorem, $(\kappa^+)^M = \kappa^+$. Contradiction.

□

**Lemma 7.2.** *Suppose $A$ is a bounded subset of $\kappa^+$. Then $\mathcal{M}_1^{J,\sharp}(A)$ exists and is $(\kappa^+,\kappa^+)$-iterable.*

*Proof.* It suffices to show $\mathcal{M}_1^{J,\sharp}(a)$ exists for $a$ a bounded subset of $\gamma^+$ (with $a$ coding $x$). Fix such an $a$ and suppose not. Then the Jensen-Steel core model (cf. [1]) $K^J(a)$ exists[2930]. Let $\xi = (\kappa^+)^{K^J(a)}$. Since $\kappa^+ = \mathrm{o}(K^J(a)) > \gamma^+$ is a limit of cardinals in $K^J(a)$ (by the fact that $\Box_\kappa$ holds in $K^J(a)$ but $\neg\Box_\kappa$ holds in $V$), so $\xi < \kappa^+$. Weak covering (cf. [1, Theorem 1.1 (5)]) gives us,

$$\mathrm{cof}(\xi) \geq |\xi| \geq \kappa. \tag{7.1}$$

7.1 in fact implies $\mathrm{cof}(\xi) = \kappa$. But $\kappa$ is singular so this is impossible.

□

Lemmas 6.3, 6.5, and 7.2 allow us to extend $\mathcal{M}_1^{J,\sharp}$ to $H^{V[G]}_{\kappa^+}$.

**Exercise 7.3.** *Formulate and prove the analogs of the above lemmata for $\kappa$ in (ii) and (iii).*

**Exercise 7.4.** *Give $J$ as above. Let $J_0 = J$, $J^{n+1} = \mathcal{M}_1^{J^n,\sharp}$ for all $n$. Suppose $J^n$ is defined for all $n$. Show that the operators $\mathcal{M}_n^{J,\sharp}$ are defined for all $n$.*

[28] In fact, $\mathrm{cof}(\kappa^+)^M < \kappa$ because $\kappa$ is singular.

[29] By our assumption and the fact that $J$ condenses finely, $K^{c,J}(a)$ (constructed up to $\lambda^+$) converges and is $(\lambda^+,\lambda^+)$-iterable. See Lemma 5.16. We then can use the $K^J$-existence dichotomy [11, Theorem 3.1.9] to conclude $K^J(a)$ exists.

[30] One can also work with the stable core models as done in [11].

## 8. The induction

Again, let $\kappa$ etc be the objects associated with (i). We will prove $W^*_{\alpha+1}$ assuming $W^*_\alpha$, for critical ordinals $\alpha$.

### 8.1. The successor and countable cofinality cases

We assume $W^*_\alpha$ for $\alpha$ as in case (A) ($\alpha = \eta + 1$ for $\eta$ critical), or case (B)(a) ($\mathrm{cof}(\alpha) = \omega$). In particular, we know $J_\alpha(\mathbb{R}) \vDash \mathsf{AD}$ and that for all $\beta \leq \alpha, W^*_\beta$ holds.

We first do the proof below for $\alpha$ satisfying (B)(a). Let $\alpha = sup_n \alpha_n$, where $\alpha_n < \alpha$ begins a gap. Let $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$ and $\Gamma_n = \Sigma_1^{J_{\alpha_n}(\mathbb{R})}$. It is easy to see that:

- $C_\Gamma = \bigcup_n C_{\Gamma_n}$.
- For any real $z$, any set $A \in \Gamma(z)$, there is a sequence $\langle A_n : n < \omega \rangle$ such that $A_n \in \Gamma_n(z)$ and $A = \bigcup_n A_n$.

For each $n$, we have the sequence $\langle J^m_n : m < \omega \rangle$ witnessing $W^*_{\alpha_n+1}$. Furthermore, we take $J = J^0_n$ to be a fine-structural mouse operator and $J^m_n$ is the operator $\mathcal{M}^{J,\sharp}_m$. The domain of these operators is the cone in $H^{V[G]}_{\kappa^+}$ above some set $A_n \in H^V_{\gamma^+}$.

**Definition 8.1.** *For any $A \in H^V_{\gamma^+}$ coding $\oplus A_n$ ($\oplus A_n \in L_1[A]$), let $J^0_\alpha(A)$ be the least $\mathcal{M} \lhd Lp(A)$ such that for some $\gamma \leq o(\mathcal{M})$, $\mathcal{M}|\gamma$ is $\mathsf{ZFC}^-$-model closed under all $J^m_n$ for all $m, n$. We then define $J^n_\alpha(A)$ as $\mathcal{M}^{J^0_\alpha,\sharp}_n(A)$, if this can be done.*

By the previous section, $J^n_\alpha$ is defined on the cone above $A$ over $H^{V[G]}_{\kappa^+}$ and each $J^n_\alpha$ relativizes well and determines itself on generic extensions.

**Lemma 8.2.** *If $J^n_\alpha$ is defined on the cone above $A$ over $H^{V[G]}_{\kappa^+}$ for each $n$, then $W^*_{\alpha+1}$ holds.*

*Proof.* Fix $l < \omega$. In $V[G]$, let $U$ be a set of reals in $J_{\alpha+1}(\mathbb{R})$. Let $\varphi$ be a $\Sigma_k$-formula, and $z \in \mathbb{R}$ be such that $y \in U \Leftrightarrow J_\alpha(\mathbb{R}) \vDash \varphi[y, z]$.

Let $\rho \in H^V_{\gamma^+}$ be such that $\rho^G = z$. Set

$$\mathcal{P} = J^{k+l+2}_\alpha(A, \rho).$$

Let $\Sigma$ be the canonical strategy of $\mathcal{P}$. We may also assume $W$ is a universal $\Gamma(z)$-set and $W = \bigcup_n W_n$, where $W_n$ is a universal $\Gamma_n(z)$-set. Note that $\mathcal{P}[G]$ is closed under $C_{\Gamma_n}$ for each $n$ by the closure of $\mathcal{P}$ (this is why we close $\mathcal{P}$ under the $J^m_n$'s).

Let $\delta_0 < \cdots < \delta_{k+l+1}$ be the Woodin cardinals of $\mathcal{P}$ (and hence of $\mathcal{P}[G]$). We have for each $n$, there is a term $\tau_n \in \mathcal{P}[G]^{Col(\omega,\delta_{k+l+1})}$ for the universal $\Gamma_n$-set $W_n$ witnessing $W_n$ is term captured at $\delta_{k+l+1}$ by $\mathcal{P}[G]$. This follows from Lemma 4.10. We can then define a term $\tau \in \mathcal{P}[G]^{Col(\omega,\delta_{k+l+1})}$, using $\tau_n$'s, for $W$. Using Lemma 4.5, we can define a term $\sigma \in \mathcal{P}[G]^{Col(\omega,\delta_{l+1})}$ for $U$ witnessing $U$ is term captured at $\delta_{l+1}$ by $(\mathcal{P}, \Sigma)$.

**Exercise:** Using $\sigma$, verify that we can build trees $S, T$ witness that $U$ is Suslin captured at $\delta_{l+1}$ by $(\mathcal{P}, \Sigma)$. Basically, $S$ builds $(y, Q, h, \pi)$ such that $y \in \mathbb{R}$, $\pi : Q \to \mathcal{P}[G]|\gamma$ is elementary, where $\gamma$ is a fixed limit ordinal $> \delta_{l+1}$, $\mathcal{P}[G]|\gamma$ is closed under the $C_{\Gamma_n}$'s, $h$ is $Q$-generic for $Col(\omega, \pi^{-1}(\delta_{k+l+1}))$, and $y \in \pi^{-1}(\sigma)_h$. $T$ does a similar thing but for the term that captures the complement of $U$.

□

The case (A) is handled similarly. Let $\Gamma_k = \Sigma_k(J_\eta(\mathbb{R}))$ and let $\Gamma = \Sigma_1(J_\alpha(\mathbb{R}))$. Note that $\Gamma \subseteq \bigcup_k \Gamma_k$. This fact is used to show that if $\mathcal{M}$ is closed under $C_\Gamma$, then for any real $x \in \mathcal{M}$, any $\kappa$ $\mathcal{M}$-cardinal, for any $\Gamma(x)$-set of reals $A$, there is a term $\tau \in \mathcal{M}^{Col(\omega,\kappa)}$ that captures $A$. We can then run the proof of the above lemma.

### 8.2. The uncountable cofinality cases: inadmissible gap

Suppose $\alpha$ is a limit ordinal of uncountable cofinality and $J_\alpha(\mathbb{R})$ is admissible as witnessed by $\phi(v_0, v_1)$ and $x \in \mathbb{R}$, that is, $\phi$ is $\Sigma_1$ and

$$\forall y \exists \beta < \alpha J_\beta(\mathbb{R}) \vDash \phi[x, y],$$

and letting $\beta(x, y)$ be the least such $\beta$ for $y$, then $\alpha = sup\{\beta(x, y) : y \in \mathbb{R}\}$.

Let $G \subseteq Col(\omega, \gamma)$ be $V$-generic as before and $\tau^G = x$, here $\tau \in H^V_{\gamma^+}$. Let $p \in G$ forces all relevant statements. Let $A \in H^V_{\gamma^+}$ be transitive, self-wellordered, and code $\tau$ in some simple way (i.e. $\tau \in L_1[A]$ and a well-ordering of $A$ is in $L_1[A]$). Call such an $A$ *suitable.* For any $A$-premouse $\mathcal{M}$, any $g \times h \subseteq Col(\omega, \gamma) \times Col(\omega, A)$, $\mathcal{M}[g][h]$ can be construed as a $z$-mouse for a real $z = z(g, h)$ obtained from $g, h, A$ and codes $g, h, A$ in some simple fashion.

There is a term $\sigma_A$ for a real defined from $A$ in $\mathcal{M}$ such that whenever $g \times h$ is generic as above, then

$$(\sigma_A^{g\times h})_0 = \tau^g,$$

and

$$\{(\sigma_A^{g\times h})_i : i > 0\} = \{\rho^{g\times h} : \rho \in L_1[A] \wedge \rho^{g\times h} \in \mathbb{R}\}.$$

For $n < \omega$, let

$$\varphi^*_n(v) \equiv \exists\gamma(J_\gamma(\mathbb{R}) \vDash \forall i \in \omega(i > 0 \Rightarrow \phi(v_0, v_1)) \wedge (\gamma + \omega n) \text{ exists}. \tag{8.1}$$

Let $\psi$ be the sentence in the language of $A$-premice such that for any $A$-mouse $\mathcal{M}$: $\mathcal{M} \vDash \psi$ iff whenever $g \times h$ is $\mathcal{M}$-generic as above, and $p \in g$, for any $n$, there is $\gamma < o(\mathcal{M})$ such that $\mathcal{M}[z(g, h)]|\gamma$ is a $\langle\varphi^*_n, \sigma_A^{g\times h}\rangle$-witness.

**Definition 8.3.** *For any suitable $A$, let $J^0(A)$ be the least $\mathcal{M} \lhd Lp(A)$ satisfying $\psi$ if exists. Otherwise, $J^0(A)$ is undefined.*

**Lemma 8.4.** *For any suitable $A$, $J^0(A)$ is defined. Furthermore, $J^0(A)$ is $OD(A)$ in $J_\gamma(\mathbb{R})$ for some $\gamma < \alpha$.*

*Proof.* In $V[G]$, using $W_\alpha$ and $\text{cof}(\alpha) > \omega$, we have a mouse $\mathcal{N}$ over $(A, G)$ such that whenever $H \subseteq Col(\omega, A)$ is $\mathcal{N}$-generic, $\mathcal{N}[H]$ (as a mouse over $z(G \times H)$) is a $\langle \varphi_n^*, \sigma_A^{G\times H}\rangle$ for all $n$ and furthermore, $\mathcal{N}$ ($\mathcal{N}[H]$) has an iteration strategy in $J_\alpha(\mathbb{R})$.**Exercise:** Think through this.

Let $\mathcal{P}$ be the structure constructed over $A$ from the extender sequence of $\mathcal{N}$ (this is the so-called P (S) constructions, cf. [14]). One can show $\mathcal{P}$ is an iterable $A$-mouse such that $\mathcal{P}[G] = \mathcal{N}$. Hence $\mathcal{P} \in V$ and $\mathcal{P} \lhd Lp(A)$.

We need to verify $\mathcal{P} \vDash \psi$. This tells us $J^0(A)$ is defined. **Exercise:** Verify this. □

**Remark 8.5.** *Let $A$ be suitable, $G, H$ be as above, and $\mathcal{M} = J^0(A)$. For each $m$, let $\beta_m$ be the least $\beta$ such that $J_\beta(\mathbb{R}) \vDash \varphi_m^*(\sigma_A^{G\times H})$, then letting $\gamma_m$ be the least $\gamma$ such that $\mathcal{M}|\gamma[z(G, H)]$ is a $(\varphi_m^*, \sigma_A^{G\times H})$-witness, then $\mathcal{M}|\gamma_m$ has iteration strategy in $J_{\beta_m+k}$ for some $k < \omega$. This uses the definition of $\varphi_m^*$.*

**Lemma 8.6.** *$J^0$ relativizes well.*

*Proof.* Let $A, B$ be suitable and $A \in L_1[B]$. Let $N$ be any transitive $\mathsf{ZFC}^-$ model such that $J^0(B) \in N$. We will show how in $N$, one can compute $J^0(A)$ from $J^1(B)$.

Let $G \times H$ be $J^0(A)$-generic for $Col(\omega, \gamma) \times Col(\omega, A)$. Let $\gamma_m$ be the least $\gamma$ such that $J^0(A)|\gamma[z(G, H)]$ is a $\langle \varphi_m^*, \sigma_A^{G\times H}\rangle$-witness. Then $o(J^0(A)) = sup_m \gamma_m$. It suffices to show we can compute $J^0(A)|\gamma_m$ from $J^0(B)$ in $N$ for each $m$. Fix such an $m$.

Let $\beta_m$ be the least $\beta$ such that $J_\beta(\mathbb{R}) \vDash \varphi_m^*(\sigma_A^{G\times H})$. By Remark 8.5, $J^0(A)|\gamma_m$ has an iteration strategy $\Sigma$ in $J_{\beta_m+k}(\mathbb{R})$ for some $k < \omega$. In fact, $\Sigma$ is $OD(A)$ there.

Let $K$ be such that $G \times K$ is $Col(\omega, \gamma) \times Col(\omega, B)$-generic for $J^0(B)$. Assume also $H$ is coded into $K$. If $\beta^*$ is least such that $J_{\beta^*}(\mathbb{R}) \vDash \varphi_m^*(\sigma_B^{G\times K})$, then $\beta_m \leq \beta^*$ (Why?).

We then have that there is an initial segment $\mathcal{N} \lhd J^0(B)$ that can compute the theory of $J_{\beta_m+k}(\mathbb{R})$. This gives $J^0(A)|\gamma_m \in J^0(B)$. This computation is uniform from parameters $A, J^0(B)$. □

Let $J = J^0$. Let $J^n$ for $j > 0$ be the $\mathcal{M}_n^{J,\sharp}$-operator. By Sections 6 and 7, $J^n$ is defined on a cone above $\tau$ in $H_{\kappa^+}^{V[G]}$.

**Lemma 8.7.** *Suppose $J^n$ is defined on a cone above $\tau$ in $H_{\kappa^+}^{V[G]}$. Then $W_{\alpha+1}^*$ holds in $L(\mathbb{R})$ (in $V[G]$).*

*Proof.* Let $U$ be a set of reals in $J_{\alpha+1}(\mathbb{R})$ and $k < \omega$. We want to construct a $(k, U)$-coarse Woodin mouse. Suppose $U$ is $\Sigma_n$-definable over $J_\alpha(\mathbb{R})$ from real parameter $z$. Let $\rho$ be such that $\rho^G = z$. Let $\mathcal{P} = J^{k+n+3}(TC(\tau, \rho))$. We show that $\mathcal{P}[G]$ is the desired witness.

In the following, we assume $n = 1$ (and leave the general case $n > 1$ for the reader to fill in). Let $\psi$ be a $\Sigma_1$ formula defining $U$ (from $z$). Let $\Sigma$ be the canonical strategy for $\mathcal{P}$ (and $\mathcal{P}[G]$). Let $\delta_0 < \cdots < \delta_{k+3}$ be the Woodin cardinals of $\mathcal{P}$.

**Claim 8.8.** *There is a term $\dot{U} \in P[G]^{Col(\omega,\delta_{k+2})}$ that witnesses $(\mathcal{P}[G], \Sigma)$ captures $U$ at $\delta_{k+2}$.*

*Proof.* Let $A = \mathcal{P}|\delta_{k+3}$. Then $\mathcal{P} = J^0(A)$. Whenever $H$ is $Col(\omega, A)$-generic over $\mathcal{P}[G]$, then for each $m$, there is some $\gamma$ such that $\mathcal{P}[z(G,H)]|\gamma$ is a $\langle \varphi^*_m, \varphi_A^{G\times H}\rangle$-witness. So $\mathcal{P}[z(G,H)]|\gamma$ has a tree $T$ such that letting $Th(H,\gamma,k)$ be the $\Sigma_{k+3}$-theory of the least level of $L(\mathbb{R}^{\mathcal{P}[z(G,H)]|\gamma}$ that satisfies $\theta^k[z]$ (here $\varphi^*_m$ plays the role of $\theta$).

We define $\dot{U}$ as follows. For any $h \subseteq Col(\omega, \delta_{k+2})$ be $\mathcal{P}[G]$-generic, for any real $y \in \mathcal{P}[G,h]$: $y \in \dot{U}^h$ iff for any $l \subseteq Col(\omega, \delta_{k+3})$ $\mathcal{P}[G,h]$-generic, there is some $\gamma$ such that

$$\forall \sigma \in L_1[\mathcal{P}[G,h]|\delta_{k+3}]\ \varphi(x, \sigma^l) \in Th(h\times l, \gamma, k)$$

then

$$\psi(y,z) \in Th(h\times l, \gamma, k).$$

**Exercise:** Verify that $\dot{U}^h \subseteq U$. (main point is $Th(h\times l,\gamma,k)$ gives the theory of the first initial segment of $L(\mathbb{R})$ at which $\phi[x,\sigma^l]$ is verified to be true for all $\sigma \in L_1[\mathcal{P}[G][h]|\delta_{k+3}]$).

For the converse, let $y \in U \cap \mathcal{P}[G][h]$. Let $\xi < \alpha$ be least such that $J_\xi(\mathbb{R}) \vDash \psi[y,z]$. Pick $u \in \mathbb{R}$ such that $\beta(u,x) \geq \gamma$. Let $i : \mathcal{P}[G,h] \to \mathcal{Q}[G,h]$ be a $u$-genericity iteration at $\delta_{k+3}$. Let $l \subseteq Col(\omega, i(\delta_{k+3}))$ be $\mathcal{Q}[G,h]$-generic such that $u \in \mathcal{Q}[G,h,l]$. Let $H$ be such that $\mathcal{Q}[G,h,l] = \mathcal{Q}[G,H]$ and let $\mathcal{Q}[z(G,H)]|\gamma^*$ be a $\langle \varphi^*_m, \Sigma_A^{G\times H}\rangle$-witness. As there is a canonical term $\sigma$ for $u$ in $L_1[\mathcal{Q}[G,h]|i(\delta_{k+3})]$, $\varphi[x,u] \in Th(H,\gamma',k)$. By the choice of $\gamma'$, $Th(H,\gamma',k)$ is the $\Sigma_{k+3}$-theory of $J_\xi(\mathbb{R})$ restricted to real parameters in $L_1[\mathcal{Q}[G,H]|i(\delta_{k+3})]$, and hence $\xi \geq \beta(x,u) \geq \gamma$. This gives $y \in i(\dot{U})^l$. This gives $y \in \dot{U}^l$ for any $l \subseteq Col(\omega, \delta_{k+3})$ be $\mathcal{P}[G,h]$-generic. □

**Exercise:** Now complete the proof by building absolutely complementing trees $T, U$ from $\dot{U}$.

□

## 8.3. The uncountable cofinality cases: admissible gap

**Outline:**

Assume $\alpha, \beta$ are as in case (C)(a) or (C)(b). Using Theorem 3.4, we may assume $W^*_\beta$ holds. Let $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$. Let $n$ be the least such that $\rho_n(J_\beta(\mathbb{R})) = \mathbb{R}$ and $U$ be a universal $\Sigma_n^{J_\beta(\mathbb{R})}$. Using Theorem 3.5, there is a sjs $\mathcal{A} = \{A_i : i < \omega\}$ suc $U = \bigcup_i A_{2i}$.

We want to construct a pair $(\mathcal{P}, \Sigma)$ such that

- $\mathcal{P}$ is $\Gamma$-suitable (cf. Definition 5.52). Let $\delta = \delta^{\mathcal{P}}$ be the Woodin cardinal of $\mathcal{P}$.
- For each $i$, there is a term $\tau_i \in \mathcal{P}^{Col(\omega,\delta)}$ witnessing $(\mathcal{P},\Sigma)$ term captures $A_i$ at $\delta$. More precisely, whenever $j : \mathcal{P} \to \mathcal{Q}$ is according to $\Sigma$, whenever $g \subseteq Col(\omega, j(\delta))$, $A_i \cap \mathcal{Q}[g] = j(\tau_i)^g$. We will let $\tau_i^{\mathcal{Q}}$ be the term for $A_i$ whenever $\mathcal{Q}$ is an iterate of $\mathcal{P}$.
- Furthermore, $\Sigma$ is "guided" by $\mathcal{A}$, in other words, whenever $\mathcal{T}$ is according to $\Sigma$, suppose $b = \Sigma(\mathcal{T})$. Then either $\mathcal{T}$ is short and $b$ is the unique branch such that $\mathcal{Q}(b,\mathcal{T}) \lhd Lp^\Gamma(\mathcal{M}(\mathcal{T}))$ or $\mathcal{T}$ is maximal and $b$ is the unique branch $c$ such that $i_c^{\mathcal{T}}(\tau_i^{\mathcal{P}}) = \tau_i^{\mathcal{Q}}$.

We then construct the mouse operators $J^n = \mathcal{M}_n^{\Sigma,\sharp} : x \mapsto \mathcal{M}_n^{\Sigma,\sharp}(x)$. These operators will be defined on a cone above $\mathcal{P}$ in $H_{\kappa^+}^{V[G]}$. Collectively, they witness $W_{\beta+1}^*$.

**Term capturing and strategies with fullness preservation:**

Let $\Gamma = \Sigma_1^{J_\alpha(\mathbb{R})}$. For $A$ a transitive, self-wellordered set in $H_{\gamma^+}^V$, we write $\mathrm{Lp}^\Gamma(A)$ for the collection of $A$-mice $\mathcal{M}$ such that $\rho(\mathcal{M}) = o(A)$, $\mathcal{M}$ is $\omega_1$-iterable in $J_\alpha(\mathbb{R})$. We define $\Gamma$-1-suitable (or just $\Gamma$-suitable) just as in Definition 5.52. Since $\Gamma$ is fixed throughout this section, if $\mathcal{P}$ is $\Gamma$-suitable, we simply say $\mathcal{P}$ is suitable. We say a tree $\mathcal{T}$ on a suitable $\mathcal{P}$ is based on $\delta^{\mathcal{P}}$ if all extenders used in $\mathcal{T}$ are taken from the extender sequence of $\mathcal{P}|\delta^{\mathcal{P}}$ or its images along the tree.

Recall from Lemma 4.10 that if $\mathcal{P}$ is a suitable premouse over some real $z$, and if $B \in OD^{<\beta}(z)$, then for each $\xi \geq \delta^{\mathcal{P}}$, there is (a canonical) term $\tau_{B,\xi}^{\mathcal{P}} \in \mathcal{P}^{Col(\omega,\xi)}$ such that whenever $g \subseteq Col(\omega, \xi)$ is $\mathcal{P}$-generic, $B \cap \mathcal{P}[g] = (\tau_{B,\xi}^{\mathcal{P}})^g$.

**Theorem 8.9** (Term-relation condensation, Woodin)**.** *Let $z$ be a real and $\mathcal{P}$ be suitable premouse over $z$. Let $\mathcal{A}$ be a self-justifying system containing a universal $\Gamma$-set and for each $B \in \mathcal{A}$, $B \in OD^{<\beta}(z)$. Suppose $\pi : \mathcal{Q} \to \mathcal{P}$ is $\Sigma_1$-elementary such that*

$$\forall B \in \mathcal{A} \forall \xi \geq \delta^{\mathcal{P}} \tau_{B,\xi}^{\mathcal{P}} \in rng(\pi).$$

*Then the following hold.*

*(i) $\mathcal{Q}$ is suitable and forall $B \in \mathcal{A}$, $\pi^{-1}(\tau_{B,\xi}^{\mathcal{P}}) = \tau_{B,\xi^*}^{\mathcal{Q}}$, where $\pi(\xi^*) = \xi$.*

*(ii) $rng(\pi)$ is cofinal in $\delta^{\mathcal{P}}$.*

*(iii) If $\delta^{\mathcal{P}} \subseteq rng(\pi)$, then $\mathcal{P} = \mathcal{Q}$ and $\pi$ is the identity.*

*Proof.* For part (i), the second clause follows from Lemma 4.10 and the fact that for any $\mathcal{P}$-cardinal $\xi \geq \delta^{\mathcal{P}}$, $C_\Gamma(H_{\xi^+}^{\mathcal{P}}) \subseteq \mathcal{P}$. For the first part, note that being $\Gamma$-full is a $\Gamma$-dual fact. **Exercise:** Complete the details here.

For part (ii), let $\lambda = sup(rng(\pi) \cap \delta^{\mathcal{P}}$. Let $\mathcal{R}$ be the transitive collapse of the set of $x$ such that $x$ is the unique $a$ such that $a = \psi^{\mathcal{P}|\xi}(\vec{\eta}, \vec{\tau})$, where $\vec{\eta} \in \lambda^{<\omega}$ and $\vec{\tau}$ is a finite subset of $\{\tau_{B,\xi^*}^{\mathcal{P}|\xi} : B \in \mathcal{A} \wedge \xi^* < \xi\}$. Let $\sigma : \mathcal{R} \to \mathcal{P}$ be the uncollapse map. **Exercise:** Using the regularity of $\delta^{\mathcal{P}}$, verify that $\sigma \restriction \lambda$ is the identity and $\sigma(\lambda) = \delta^{\mathcal{P}}$.

From part (i), $\mathcal{R}$ is suitable. From the exercise, $\mathcal{R}|\lambda = \mathcal{P}|\lambda$ and $\sigma(\lambda) = \delta^{\mathcal{P}}$, so $\mathcal{R} \vDash \lambda$ is Woodin. But $\mathcal{R}$ is suitable, so $Lp^\Gamma(\mathcal{R}|\lambda) = Lp^\Gamma(\mathcal{P}|\lambda) \vDash \lambda$ is Woodin. This is a contradiction.

Part (iii) follows from the proof of part (ii). □

**Definition 8.10.** *Let $\mathcal{P}$ be suitable and $\mathcal{T}$ is a normal tree of length $< \gamma^+$ based on $\delta^{\mathcal{P}}$. $\mathcal{T}$ is short if for all limit $\xi \leq lh(\mathcal{T})$, $Lp^\Gamma(\mathcal{M}(\mathcal{T}|\xi)) \vDash \delta(\mathcal{T}|\xi)$ is not Woodin. Otherwise, we say $\mathcal{T}$ is maximal.*

**Definition 8.11.** *Let $\mathcal{P}$ be suitable and $\Sigma$ is a $\gamma^+$-iteration strategy for $\mathcal{P}$ in $V[G]$. $\Sigma$ is $\Gamma$-fullness preserving (or just fullness preserving) if whenever a normal tree $\mathcal{T}$ is based on $\delta^{\mathcal{P}}$ and is according to $\Sigma$ with last model $\mathcal{Q}$, then*

1. *either $\mathcal{T}$ does not drop in model and $\mathcal{Q}$ is suitable,*

2. *or the $\mathcal{P}$-to-$\mathcal{Q}$ branch drops (in model) and $\mathcal{J}_\alpha(\mathbb{R}) \vDash \mathcal{Q}$ is $\omega_1$-iterable.*

**Remark 8.12.** *$Lp^\Gamma$ can "track" a $\Gamma$-fullness preserving strategy $\Sigma$ in the sense that if $\mathcal{T}$ is according to $\Sigma$ and is short, then $Lp^\Sigma(\mathcal{M}(\mathcal{T}))$ identifies the unique branch $b$ for $\mathcal{T}$ such that $\mathcal{Q}(b,\mathcal{T}) \lhd Lp^\Gamma(\mathcal{M}(\mathcal{T}))$; otherwise, if $\mathcal{T}$ is maximal, even though $Lp^\Gamma$ cannot identify $\Sigma(\mathcal{T})$, it can tell the final model $\mathcal{M}^{\mathcal{T}}_{\Sigma(\mathcal{T})}$ namely $Lp^\Gamma_\omega(\mathcal{M}(\mathcal{T}))$.*

**Definition 8.13.** *A strategy $\Sigma$ condenses well (or has hull condensation) if whenever $\mathcal{T}$ is according to $\Sigma$ and $\mathcal{S}$ is a hull of $\mathcal{T}$, then $\mathcal{S}$ is according to $\Sigma$.*

**Definition 8.14** ($A$-iterability)**.** *Let $\mathcal{G}(\omega, n, \omega_1)$ be the usual iteration game where players play out a linear stack of $n$ normal iteration trees. Let $A \in OD^{<\beta}(z)$. We say a suitable $z$-premouse $\mathcal{P}$ is (weakly) $A$-iterable if for all $n$, there is a fullness preserving $\Sigma$ for II in $\mathcal{G}(\omega, n, \omega_1)$ such that whenever $i : \mathcal{P} \to \mathcal{Q}$ is according $\Sigma$, then $i(\tau^{\mathcal{P}}_{A,\xi}) = \tau^{\mathcal{Q}}_{A,i(\xi)}$ for all $\xi \geq \delta^{\mathcal{P}}$.*

**Exercise 8.15.** *Assume $W^*_\beta$. Show that for any $z \in HC^{V[G]}$, there is a suitable $z$-premouse that is $\omega_1$-iterable with respect to short trees.*

**Hint:** Assume not. Let $z$ be a witness. Use reflection to reflect the failure of the conclusion to some $\xi < \alpha$ and use $W^*_\alpha$ + the proof of Lemma 4.20 to get a contradiction.

**Theorem 8.16** (Woodin)**.** *Suppose $\Gamma$ is an inductive-like pointclass and $\underset{\sim}{\Delta}_\Gamma$ is determined. Suppose mouse capturing holds for $\Gamma$. Let $A \in \underset{\sim}{Env}(\Gamma)$ and $z \in \mathbb{R}$, then there is an $A$-iterable $z$-mouse.*

**Strategies that condense well and are guided by a sjs:**

We say that $\mathcal{P}$ is weakly $\mathcal{A}$-iterable where $\mathcal{A}$ is a countable collection of sets of reals if for any finite $F \subset \mathcal{A}$, $\mathcal{P}$ is weakly $\oplus F$-iterable.

**Theorem 8.17** (Woodin)**.** *Let $\mathcal{A}$ be a countable collection of $OD^{<\beta}(z)$ sets of reals, where $z \in HC^{V[G]}$. Then there is a suitable, weakly $\mathcal{A}$-iterable $z$-premouse.*

*Proof.* For each finite $F \subseteq \mathcal{A}$, let $\Sigma_F$ be the fullness preserving strategy for some $z$-suitable $\mathcal{P}_F$ such that $\Sigma_F$ is weakly $\oplus F$-strategy (cf. Theorem 8.16).

**Exercise 8.18.** *Show that the simultaneous coiteration of the pairs $\{(\mathcal{P}_F, \Sigma_F) : F \in \mathcal{A}^{<\omega}\}$ terminates successfully at some countable ordinal. Furthermore, letting $\mathcal{R}_F$ be the last model of the comparison tree $\mathcal{U}_F$ according to $\Sigma_F$, then $\mathcal{P}_F$-to-$\mathcal{R}_F$ does not drop.*

Let $i_F : \mathcal{P}_F \to \mathcal{R}_F$ be according to $_F$. From the exercise, it is easy to see that for any $F, G$ as above, $\mathcal{R}_F = \mathcal{R}_G$. Let $\mathcal{R} = \mathcal{R}_F$. Then it is clear that $\mathcal{R}$ is weakly $\mathcal{A}$-iterable.

□

**Definition 8.19.** *Let $\mathcal{A}$ be a collection of sets in $OD^{<\beta}(z)$. Let $\mathcal{P}$ be a suitable $z$-premouse. Let $\Sigma$ be an $\omega_1$-strategy for $\mathcal{P}$. $\Sigma$ is said to be guided by $\mathcal{A}$ if $\Sigma$ is fullness preserving and whenever $\mathcal{T}$ is countable of limit length and is according to $\Sigma$, letting $b = \Sigma(\mathcal{T})$, then* *(a) if $\mathcal{T}$ is short, then $\mathcal{Q}(b,\mathcal{T})$ exists and $\mathcal{Q}(b,\mathcal{T}) \lhd Lp^{\Gamma}(\mathcal{M}(\mathcal{T}))$, or*

*(b) if $\mathcal{T}$ is maximal, then $b$ is the unique branch $c$ such that $i_c^{\mathcal{T}}(\tau_{A,\xi}^{\mathcal{P}}) = \tau_{A,i_c^{\mathcal{T}}(\xi)}^{\mathcal{M}_c^{\mathcal{T}}}$, for all $A \in \mathcal{A}$, for all cardinals $\xi \geq \delta^{\mathcal{P}}$.*

**Theorem 8.20** (Woodin)**.** *Let $\mathcal{A}, z$ be as in Definition 8.19. Suppose further that $\mathcal{A}$ is a sjs containing a universal $\Gamma$-set. Then there is a $z$-suitable premouse $\mathcal{P}$ and a unique fullness preserving strategy $\Sigma$ guided by $\mathcal{A}$; furthermore, $\Sigma$ condenses well.*

*Proof.* Let $\mathcal{A} = \{A_i : i < \omega\}$. Let $\mathcal{P}$ be as in the conclusion of Theorem 8.17. Let $\Sigma_n$ be a fullness preserving strategy for $\mathcal{P}$ that witnesses $\mathcal{P}$ is weakly $\oplus_{i\leq n} A_i$-iterable. Let $\mathcal{T}$ be a normal tree of limit length according to all $\Sigma_n$'s. Let $b_n = \Sigma_n(\mathcal{T})$, and $i_n : \mathcal{P} \to \mathcal{M}(\mathcal{T})^+ = \mathcal{M}_{b_n}^{\mathcal{T}}$ be the iteration embedding. For $k < \omega$, let $\nu_k$ be the $k$-th cardinal of $\mathcal{M}(\mathcal{T})^+ \geq \delta(\mathcal{T})$ and set

$$\mathcal{M}_k = \mathcal{M}(\mathcal{T})^+ | \nu_k,$$

$$\tau_{j,k} = \tau_{A_j,\nu_k}^{\mathcal{M}(\mathcal{T})^+},$$

and $\gamma_k$ be the supremum of $\xi$ such that $\xi$ is definable over $\mathcal{M}_k$ from points of the form $\tau_{i,j}$ for $i, j < k$.

**Exercise 8.21.** *Verify that: (i) $\gamma_k$'s are cofinal in $\delta(\mathcal{T})$; (ii) for any $k \leq l$, $E$ is an extender of length $\leq \gamma_k$, then $E$ is used in $b_k$ iff $E$ is used in $b_l$.*

Define now:

$$\xi \in b \Leftrightarrow \exists k \forall l \geq k\ \xi \in b_l. \tag{8.2}$$

**Claim 8.22.** *$b$ is cofinal in $lh(\mathcal{T})$.*

*Proof.* Suppose not and let $\eta = \bigcup b < lh(\mathcal{T})$. Fix $k$ such that $lh(E_\eta^{\mathcal{T}}) < \gamma_k$. All extenders used in $b$ have length $< lh(E_\eta^{\mathcal{T}})$, so by the Exercise,

$$b \subseteq b_l \text{ for all } l \geq k.$$

By closure of the branches, $\eta \in b$ (Why?). Let $F$ be the extender applied to $\mathcal{M}_\eta^{\mathcal{T}}$ along $b_k$. So we have $crt(F) < lh(E_\eta^{\mathcal{T}})$. Furthermore, $lh(F) > \gamma_k$ since $F$ is not used in $b$. But this implies $rng(i_k)$ is not cofinal in $\gamma_k$. Contradiction. □

Let $T_k = Th^{\mathcal{M}_k}(\delta^{\mathcal{M}} \cup \{\tau_{i,j} : i, j < k\})$ and $S_k$ be the corresponding theory defined over $\mathcal{P}$ from the corresponding terms. Then

$$i_{b_k}^{\mathcal{T}}(S_k) = T_k \tag{8.3}$$

as $i_{b_k}^{\mathcal{T}}$ moves the terms $\tau_{i,j}$ correctly for all $i, j < k$.

**Claim 8.23.** *$i_b^{\mathcal{T}}(S_k) = T_k$ for all $k$.*

*Proof.* Fix $k$ and regard $S_k$ as a subset of $\delta^{\mathcal{P}}$. Since $i_b$ is cofinal in $\delta(\mathcal{T})$, it is enough to see that $i_b^{\mathcal{T}}(S_k) \cap lh(E) = T_k \cap lh(E)$ for any extender $E$ used along $b$. Fix such an $E$; let $l \geq k$ be such that $E$ is used in $b_l$ . By 8.3, $i_{b_l}^{\mathcal{T}}(S_k) \cap lh(E) = T_k \cap lh(E)$. Since $E$ is used in both $b, b_l$, we are done. □

By Theorem 8.9(iii), $\mathcal{P}$ is coded by the $S_k$'s and $\mathcal{M}(\mathcal{T})^+$ is coded by the $i_b^{\mathcal{T}}(S_k) = T_k$'s (more precisely, $\mathcal{P}$ is pointwise $\Sigma_0$-definable from ordinals below $\delta^{\mathcal{P}}$ and terms $\tau_{i,j}^{\mathcal{P}}$ and similarly for $\mathcal{M}(\mathcal{T})^+$). $\mathcal{M}_b^{\mathcal{T}}$ is also coded by the $i_b^{\mathcal{T}}(S_k)$'s, so

$$\mathcal{M}_b^{\mathcal{T}} = \mathcal{M}(\mathcal{T})^+.$$

Set $\Sigma(\mathcal{T}) = b$.

Finally, from the above argument we see that:

- $\Sigma(\mathcal{T}) = b$ moves all the terms $\tau_{i,j}^{\mathcal{P}}$'s correctly.
- $\Sigma$ is fullness preserving.
- For any hull $(\mathcal{S}, c)$ of $(\mathcal{T}, b)$, $c = \Sigma(\mathcal{S})$. This follows from Theorem 8.9.

□

**Going back to $V$ - Boolean comparisons:**

Let $\mathcal{A}, z, \mathcal{P}, \Sigma$ as in Theorem 8.20. Let $\tau$ be a term such that $\tau^G = z$; we take $\tau$ to be an element of $H_{\gamma^+}^V$. By considering all possible finite variants of $G$ and comparing the suitable mice associated with them, we will produce a suitable $\tau$-mouse $\mathcal{N}$ that does not depend on $G$. $\mathcal{N}$ will be in $V$ and will have a fullness preserving strategy $\Lambda$ that condenses well and $\Lambda \restriction V \in V$.

We now describe the method of *boolean comparison* that produces $\mathcal{N}$. This is due to Woodin. First, let $p_0 \in G$ be a condition that forces all relevant facts about $\tau$. For each $p \leq p_0$, let $G_p = p \cup G \restriction (\omega \backslash dom(p))$. Then it is easy to see that $G_p$ is $V$-generic and $V[G] = V[G_p]$. Let $\mathcal{A}_p$ be the corresponding sjs of $OD^{<\beta}(\tau^{G_p})$ sets. Let $z_p = (\tau, G_p)$ (so $z = z_{p_0}$). Let $\mathcal{B} = \bigcup_{p \leq p_0} \mathcal{A}_p$. Let $\dot{\mathcal{B}}$ be the symmetric term for $\mathcal{B}$, so that

$$\forall p \leq p_0 \mathcal{B}^{G_p} = \mathcal{B}.$$

Let $\dot{\mathcal{N}}_p, \dot{\Sigma}_p$ be terms such that $p \vDash$ the statement: "$\dot{\Sigma}_p$ is an $\dot{\mathcal{A}}$-guided, fullness preserving strategy for the $(\tau, \dot{G})$ suitable mouse $\dot{\mathcal{N}}_p$ such that $\dot{\Sigma}_p$ condenses well".

Let $\mathcal{N}_p = \dot{\mathcal{N}}_p^{G_p}$ and $\Sigma_p = \dot{\Sigma}_p^{G_p}$. Now we simultaneously compare the $\mathcal{N}_p$'s using strategies $\Sigma_p$'s. This comparison makes sense since $z_p$ and $z_q$ are Turing equivalent so a $z_p$-mouse can be regarded as a $z_q$ mouse. The comparison is successful. Let $i_p : \mathcal{N}_p \to \mathcal{N}_\infty^p$ be the corresponding iteration embedding. $i_p$ exists (Why?). $\mathcal{N}_p^\infty$ and $\mathcal{N}_q^\infty$ are mice over different sets but they have the same extender sequence $E_\infty$. So we abuse notation and write $\mathcal{N}_\infty$. $\mathcal{N}_\infty$ is weakly $\mathcal{A}$-iterable and hence by Theorem 8.20 has a unique $\mathcal{A}$-guided strategy $\Lambda$ that is fullness preserving and condenses well.

The key point is: the comparison above only depends on the set $\{\mathcal{N}_p : p \leq p_0\}$, but not any enumeration of the set. Therefore, there are symmetric terms $\dot{E_\infty}$, $\dot{\Lambda}$ for $E_\infty$ and $\Lambda$. By the S-construction, we can construct a mouse $\mathcal{N}$ over $\tau$ such that $o(\mathcal{N}) = o(\mathcal{N}_\infty)$ and for any $\xi < o(\mathcal{N})$, $\mathcal{N}|\xi[G] = \mathcal{N}_\infty|\xi$. One can check that $\mathcal{N} \in V$, is a suitable $\tau$-mouse and $\Lambda$ induces a $\gamma^+$-strategy $\Psi$ for $\mathcal{N}$ such that $\Psi$ is fullness preserving and condenses well.

**Getting $W^*_{\beta+1}$:** Let $\mathcal{N}, \Psi$ be as in the previous section. Note that $\Psi$, being guided by $\mathcal{A}$, is not in $J_\beta(\mathbb{R})$ but it is in $J_{\beta+1}(\mathbb{R})$. **Exercise:** Verify this.

We then use the results in Sections 5,6,7 to show that the operators $\mathcal{M}_n^{\Sigma,\sharp}$ exist for all $n$ and are defined on a cone over $H_{\kappa^+}^{V[G]}$ over $\tau$.

Given this, we obtain $W^*_{\beta+1}$ as before. Let $U$ be $\Sigma_n^{J_\beta(\mathbb{R})}$-definable with real parameter $z$ and $k < \omega$. Take $z$ that codes $(\mathcal{N}, G)$ and let $P = \mathcal{M}_{k+n}^{\Sigma,\sharp}(z)$. This is the desired $(k, U)$-witness. The proof is more or less as before. We just mention one point: suppose $U = \oplus_{n<\omega, A_n \in \mathcal{A}} A_n$ (**Exercise:** Verify the general case).

**Claim 8.24.** *Let $\Lambda$ be the canonical strategy of $P$, defined all trees in $H_{\kappa^+}^{V[G]}$. For any $\xi \in P$, there is a term $\dot{U} \in P^{Col(\omega,\xi)}$ such that whenever $i : P \to Q$ is according to $\Lambda$ and $l \in V[G]$ is $Q$-generic for $Col(\omega, \xi)$, then $i(\dot{U})^l = U \cap Q[l]$.*

*Proof.* $\dot{U}$ consists of $(p, \tau)$ such that $p$ forces when $j : \mathcal{N} \to \mathcal{M}$ is the $P|\xi$-genericity iteration given by $\Psi \cap P$ (this is definable over $P$), then $\tau$ belongs to $j(\tau^{\mathcal{N}}_{A_i})$ for some $i$. ☐

References


[1] Ronald Jensen and John Steel. K without the measurable. *The Journal of Symbolic Logic*, 78(3):708–734, 2013.

[2] Donald A Martin. The largest countable this, that, and the other. In *Cabal Seminar 79–81*, pages 97–106. Springer, 1983.

[3] William J. Mitchell and John R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1994.

[4] Itay Neeman. Optimal proofs of determinacy. *Bulletin of Symbolic Logic*, 1(3):327–339, 1995.

[5] Itay Neeman. Optimal proofs of determinacy II. *Journal of Mathematical Logic*, 2(02):227–258, 2002.

[6] G. Sargsyan. *Hod mice and the mouse set conjecture*, volume 236 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2014.

[7] G. Sargsyan and J.R. Steel. The mouse set conjecture for sets of reals, available at author's website. 2014.

[8] Ernest Schimmerling. Coherent sequences and threads. *Advances in Mathematics*, 216(1):89–117, 2007.

[9] J. R. Steel. PFA implies $\mathsf{AD}^{L(\mathbb{R})}$. *J. Symbolic Logic*, 70(4):1255–1296, 2005.

[10] J. R. Steel. Scales in $K(\mathbb{R})$ at the end of a weak gap. *J. Symbolic Logic*, 73(2):369–390, 2008.

[11] J. R. Steel and R. D. Schindler. The core model induction; available at https://ivv5hpp.uni-muenster.de/u/rds/.

[12] John R. Steel. Scales in L($\mathbb{R}$). In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 107–156. Springer, Berlin, 1983.

[13] John R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic*. Springer-Verlag, Berlin, 1996.

[14] John R. Steel. Scales in K($\mathbb{R}$). In *Games, scales, and Suslin cardinals. The Cabal Seminar. Vol. I*, volume 31 of *Lect. Notes Log.*, pages 176–208. Assoc. Symbol. Logic, Chicago, IL, 2008.

[15] John R Steel. An outline of inner model theory. *Handbook of set theory*, pages 1595–1684, 2010.

[16] Nam Trang and Farmer Schlutzenberg. Scales in hybrid mice over $\mathbb{R}$. Available at math.unt.edu/∼ntrang/.

[17] Trevor Miles Wilson. *Contributions to Descriptive Inner Model Theory*. PhD thesis, University of California, 2012. Available at author's website.