HOD COMPUTATIONS

NOTES ON SANDRA MÜLLER'S LECTURES

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Section 1. *Some Like it* HOD

One of the purposes of this talk is to answer the question, why are we interested in HOD? Partly the answer is that HOD computations seem useful or necessary if we're going to push the core model induction forward. But the answer will depend on who "we" is. For those interested in core model induction this is a good enough reason, but there are others without this kind of priority. There are some other reasons to be interested in HOD, however.

With HOD there is always two competing or opposed lines of thought. In general, if we want to talk about HOD, we want to understand it as a canonical model. But in general HOD isn't canonical. For example, $\mathrm{HOD}^{\mathrm{HOD}}$ can be strictly contained in HOD, which is not what you would expect from a supposedly canonical model—this is not the case for L, for example. But on the other hand, what else do we ask of canonical models?

One thing we always want to know is that canonical models are close to V. The whole goal of the inner model problem in some sense is to construct canonical inner models that have close properties to V, like having all the large cardinals of V. And HOD has this kind of property in the sense that V can be $\mathrm{HOD}^{\mathrm{V}[G]}$ where $\mathrm{V}[G]$ is a class forcing generic extension[i]. And this is nice, because if we start, say, in a universe with a supercompact cardinal, then there's consistently a supercompact in HOD just by arranging this situation. With this result, we can basically make HOD look like anything we want, in some sense.

But because of this, there's good and bad news. The good news is that every known large cardinal is compatible with HOD, something not true for the canonical inner models researched. And so if we understand HOD very well and have this become a canonical inner model, this could lead to a solution of the inner model problem: we can have all the large cardinals we want in it. For the bad news, HOD can also be as bad as we want, since it can inherent any badness from V. For example, GCH can fail at every regular cardinal in HOD.

So these results don't really tell us about the deeper structure of HOD. The only thing they tell us is that asking to prove something about HOD in general—without any additional hypotheses—is not a good question. And so far all of these results essentially follow from the fact that HOD can be very close to V. But it get's even worse and we

[i] This result is due to Roguski using McAloon coding, and I believe it's from the late 1960s to early 1970s.

understand much less, because HOD can be very far from V. For example, we have the following fairly recent results. In particular, HOD can be *very* wrong about the large cardinals of V. Of the several results they proved, we have the following.

1.1. Result (Cheng–Friedman–Hamkins)

It is consistent that every measurable cardinal in V is not even weakly compact in HOD.
It is consistent that there is a supercompact cardinal that is not even weakly compact in HOD.

So downward absoluteness of large cardinals fails as badly as you can imagine. So can be seen as really bad news. This is beecause when we construct our inner models, we usually expect that we've constructed them in such a way that we inherit the large cardinals from V. If we look at $\mathrm{L}[\mu]$, for example, the point is to ensure that we have a measurable cardinal.

And there are more results which demonstrate the oddity of HOD. One property we usually desire in our inner models is weak covering: we want to compute the successors of singulars correctly. And usually you prove this under some large cardinal hypotheses, but this is a very canonical property of our core models. But this can fail for HOD about as badly as we could expect it to.

1.2. Result (Cummings–Friedman–Golshani)

It is consistent from GCH with large cardinals that $(\alpha^+)^{\mathrm{HOD}} < \alpha^+$ for every infinite cardinal α.

The general picture I wanted to give is that if we don't assume anything, we cannot expect to have HOD as a canonical inner model.

Now most of these results, I believe, are forcing results. So returning to inner model theory, we can return to a time around or before there were any Mitchell–Steel inner models. Basically, we have $L[\mu]$, and now the question is what is a canonical inner model? One direction is the Mitchell–Steel direction: we start putting large extenders on our sequence and start constructing M_1 or M_2. Another direction follows some results of Woodin at the same time, which showed that HOD can serve as a canonical inner model if we assume some determinacy. So this is the setting under which we want to study HOD. There are several theorems in this direction, showing that under determinacy, HOD of an inner model has large cardinals.

- (Solovay) Under $\mathsf{ZF} + \mathsf{AD}$, $\mathrm{HOD} \vDash$ "ω_1^{V} is measurable".
- (Woodin) Under $\mathsf{ZF} + \mathsf{DC} + \Delta^1_2$-determinacy, for a Turing cone of x, $\mathrm{HOD}^{\mathrm{L}[x]} \vDash$ "$\omega_2^{\mathrm{L}[x]}$ is Woodin".
- (Woodin) Under $\mathsf{ZF} + \mathsf{AD}$, $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})} \vDash$ "$\Theta^{\mathrm{L}(\mathbb{R})}$ is Woodin", where Θ is the supremum of ordinals α with a surjection $\pi : \omega^\omega \to \alpha$.
- (Woodin) Under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$, $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})} \vDash$ "$(\underset{\sim}{\delta}^2_1)^{\mathrm{L}(\mathbb{R})}$ is $< \Theta^{\mathrm{L}(\mathbb{R})}$-strong".

Under these hypotheses, we want to investigate these models of $\mathrm{HOD}^{L[x]}$, $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$. This is because if we want to investigate inner models for a Woodin cardinal, these two would be good candidates. Of couse, this requires an additional determinacy hypothesis, but in order to construct the Mitchell–Steel models with a Woodin cardinal we also need some additional hypothesis, and determinacy is a very natural one.

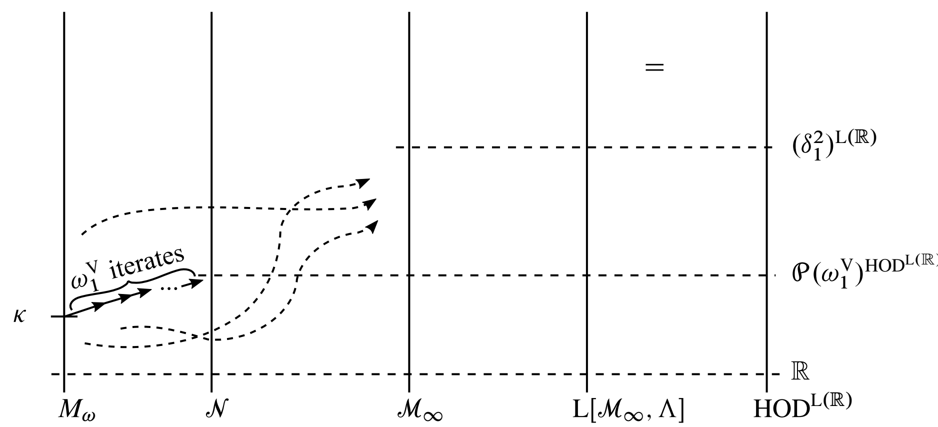
Now if we want to use these models, the first question we should ask is whether these satisfy GCH. More generally, what do these models look like? Looking away from just $\mathrm{HOD}^{L[x]}$ and $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$, we can ask even more generally what HOD^M is for some "natural" inner model $M \vDash \mathsf{AD}^+$. These are the driving questions behind this area of research. But it turns out that as stated, we cannot even answer the first of these about GCH.

The main question we will investigate this week will be the model $\mathrm{HOD}^{\mathrm{L}[x]} \vDash$ "$\omega_1^{\mathrm{L}[x]}$ is Woodin". Showing that the other model thinks $\Theta^{\mathrm{L}(\mathbb{R})}$ is Woodin is essentially the same argument. But the arguments are nice, since they use a lot of the ideas we have in inner model theory, and they can be good examples to figure out how these things fit together.

Let's give a bit of a history of looking at $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ assuming $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$. What can we say about $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$? So we might as well assume V = L(ℝ). The first result is purely descriptive set theoretic. And independently, Steel and Woodin showed what the reals of this model are, see Steel's "Inner models with many Woodin cardinals".

1. (Becker, 1980) Under $\mathsf{AD}^{L(\mathbb{R})}$ (and possibly $\mathsf{DC}_{\mathbb{R}}$), $\mathrm{HOD}^{L(\mathbb{R})} \models \mathsf{GCH}_\alpha$ for all $\alpha \leq \omega_1^V$, where GCH_α is the statement $2^\alpha = \alpha^+$.
2. (Steel and Woodin, 1993) Under the same assumptions, $\mathrm{HOD}^{L(\mathbb{R})} \cap \mathbb{R} = M_\omega \cap \mathbb{R}$, where M_ω is just the minimal model with ω many Woodin models; and moreover
3. (Steel and Woodin, 1993) $\mathrm{HOD}^{L(\mathbb{R})} \cap \mathcal{P}(\omega_1^V) = \mathcal{N} \cap \mathcal{P}(\omega_1^V)$, where $\mathcal{N}$ is the ω_1^V iterate of M_ω by its least measure.
4. (Steel, 1995) Under the same assumptions, let $\mathcal{M}_\infty$ be the direct limit of (some particular) iterates of M_ω. Thus $\mathrm{HOD}^{L(\mathbb{R})} \cap V_{(\delta_1^2)^{L(\mathbb{R})}} = \mathcal{M}_\infty \cap V_{(\delta_1^2)^{L(\mathbb{R})}}$, where $(\delta_1^2)^{L(\mathbb{R})}$ is the supremum of all α with a $\Delta_1^{L(\mathbb{R})}$ surjection $f : \mathbb{R} \to \alpha$.
5. (Woodin, ~1996) Under the same assumptions, $\mathrm{HOD}^{L(\mathbb{R})} = L[\mathcal{M}_\infty, \Lambda]$ where Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

And of course, equality here for these models is equality in the sense of their universes, not any additional structure (like extender sequences). Their fine structure will be quite different. The following picture gives a general idea.



We also have a corollary of (4) above.

1.3. Corollary

Under the same assumptions, $\mathrm{HOD}^{L(\mathbb{R})} \models \mathsf{GCH}$.

This uses that $\underset{\sim}{\delta}_1^2$ is $<\Theta$-strong so that GCH holds up to Θ, and $\mathrm{HOD}^{L(\mathbb{R})}$ is $L(\mathscr{P})$ for some $\mathscr{P} \leq \Theta$, in fact $\mathscr{P}$ is a version of the Vopěnka algebra which adds ℝ to HOD.

What this tells us is that if we want to understand HOD completely, then the models we currently look at are not enough. We have to add this partial iteration strategy for example. But we know that $\mathrm{HOD}^{L(\mathbb{R})}$ is, which tells us that we should be looking instead of $\mathrm{HOD}^{L[x]}$, because we know very little about it. That said, a similar analysis can be carried out for $\mathrm{HOD}^{L[x,G]}$, where G is $\mathrm{Col}(\omega, <\kappa)$-generic over $L[x]$ and κ is the least inaccessible in $L[x]$.

1.4. Theorem (The Main Topic)

(Woodin) Assume Δ_2^1-determinacy for a Turing cone of x—boldface is easer to see, but there is an argument from lightface determinacy as well—and that $M_1^\sharp$ exists. Therefore $\mathrm{HOD}^{L[x,G]} = L[\mathcal{M}_\infty, \Lambda]$, where
1. G is generic over $L[x]$ for $\mathrm{Col}(\omega, <\kappa_x)$ where κ_x is the least inaccesible of $L[x]$;
2. $\mathcal{M}_\infty$ is the direct limit of iterations of M_1; and
3. Λ a restriction of the canonical iteration strategy Σ_0 of M_1 (in particular, to finite full stacks $\vec{\mathcal{T}}$ on $\mathcal{M}_\infty$ such that $\vec{\mathcal{T}} \in \mathcal{M}_\infty \mid \kappa_\infty$, where κ_∞ is the least inaccessible cardinal of $\mathcal{M}_\infty$ strictly above δ_∞).

This result due to Woodin is the main goal of this tutorial. The boldface $\utilde{\mathbf{\Delta}}^1_2$ version is easier to see, but there is an argument from lightface determinacy as well.

§ 1A. Summary of some known computations of HOD

Beyond the result above, we also have the following. As a reference, see Steel and Woodin's "HOD as a core model".

- (Müller–Sargsyan): under $\utilde{\mathbf{\Pi}}^1_{n+2}$-determinacy for a Turing cone of x, $\mathrm{HOD}^{M_n(x,g)} = M_n(\mathcal{M}_\infty \mid \delta_\infty, \Lambda)$, where
  1. g is $\mathrm{Col}(\omega, < \kappa)$-generic over $M_n(x)$;
  2. $\mathcal{M}_\infty$ is a direct limit of iterates of M_{n+1} with bottom Woodin cardinal δ_∞; and
  3. Λ is a partial iteration strategy for $\mathcal{M}_\infty \mid \delta_\infty$.
- (Woodin, late 1990s) Assume AD^+ and that there is no boldface pointclass $\Gamma \subsetneq \mathcal{P}(\mathbb{R})$ such that $\mathrm{L}(\Gamma, \mathbb{R}) \models \mathsf{AD}_\mathbb{R} +$ "Θ has uncountable cofinality". Therefore $\mathrm{HOD}^{\mathrm{L}(\Gamma,\mathbb{R})}$ is a hod mouse.
- (Sargsyan, 2009) Assume AD^+ and that there is no boldface pointclass $\Gamma \subsetneq \mathcal{P}(\mathbb{R})$ such that $\mathrm{L}(\Gamma, \mathbb{R}) \models \mathsf{AD}_\mathbb{R} +$ "Θ is regular". Therefore $\mathrm{HOD}^{\mathrm{L}(\Gamma,\mathbb{R})}$ is a hod mouse.
- (Sargsyan–Steel) The same as the above holds for the minimal model of $\mathsf{AD}^+ +$"the largest Suslin cardinal is a member of the Solovay sequence".
- (Sargsyan, 2018; combined with Steel, 2016) The same as the above for the minimal model of $\mathsf{AD}^+ +$"there is a Woodin limit of Woodins".

Here the "minimal model" of something just means that there is no inner model with the property.

Section 2. The Direct Limit System

§ 2A. The mice we want to iterate

Firstly, we will follow the reference stated above: "HOD as a core model" by Steel and Woodin. To begin, we need to talk about the mice we want to iterate, because what we're going to be looking at iterates of M_1, in some sense. For the sake of completeness, we have the following definition.

2A.1. Definition

Let $M_1^\sharp$ be the least active ($(\omega, \omega_1, \omega_1)$-iterable) premouse which is not 1-small, if it exists.
A premouse M is *active* iff it has a top extender.
A premouse M is 1-small iff there is no extender E_α on the extender sequence of M such that

$$M \mid \mathrm{crit}(E_\alpha) \models \text{"there is a Woodin cardinal"}.$$

So not being 1-small means that we *do* have an extender on the sequence that sees a Woodin cardinal below it. Now if $M_1^\sharp$ exists, we can also talk about M_1. So what is M_1? Informally, when we take our $M_1^\sharp$, we have this top extender which we can hit without moving the Woodin cardinal, effectively stretching the model a little bit. And if we keep doing this Ord-many times, we basically iterate the extender out of the universe. The result of this process is denoted M_1.

We have this structure, which we've assumed is ω_1-iterable. And this structure is very simple: there is a unique ω_1-iteration strategy for $M_1^\sharp$. This unique iteration will be denoted Σ_0. For those with the background, it is the one given by Q-structures. For now, let's just assume we have this strategy. And this comprises our basic setup. In addition to this, we will assume the following from now on:

- $M_1^\sharp$ exists;
- $x \in \mathbb{R}$ is a fixed real such that $M_1^\sharp \in \mathrm{L}[x]$;
- κ_x is the least inaccessible cardinal of $\mathrm{L}[x]$; and
- G is $\mathrm{Col}(\omega, < \kappa_x)$-generic over $\mathrm{L}[x]$.

Note that it makes sense for $M_1^\sharp$ to be coded in this way using fine structure[ii]. And just to remind you, our goal is to show the following result. We're not going to be able to compute $\mathrm{HOD}^{\mathrm{L}[x]}$, but instead $\mathrm{HOD}^{\mathrm{L}[x,G]}$.

2A.2. Theorem (Our Goal)

Under our assumptions, $\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[\mathcal{M}_\infty, \Lambda]$ where $\mathcal{M}_\infty$ is the direct limit of all Σ_0-iterates of initial segments of M_1 that are countable in $\mathrm{L}[x, G]$, and where Λ is a fragment of Σ_0[iii].

This is our overall goal, but what we will show along the way is that the Woodin cardinal of $\mathcal{M}_\infty$, δ_∞, is precisely $\omega_2^{L[x,G]} = (\kappa_x^+)^{L[x]}$. Moreover, up to this level, $\mathrm{HOD}^{L[x,G]} \cap \mathrm{V}_{\delta_\infty} = \mathcal{M}_\infty \cap \mathrm{V}_{\delta_\infty}$. One issue in trying to prove this is that $\mathcal{M}_\infty$ is an iterate of M_1, but HOD has no idea how to define the iterates of M_1. Since we're dealing with HOD rather than HOD_x[iv], this model has no chance to figure out what M_1 is. So we need some cleverness to get around this.

Instead, we look at things *similar* to M_1, and primarily work with these. The first thing we want to show is that $\mathcal{M}_\infty$ can be defined. Firstly, we need to define our direct limit system.

§ 2B. The direct limit system externally

2B.1. Definition

A premouse N is *M_1-like* iff there is some ordinal δ such that
1. $N = \mathrm{L}[N \mid \delta]$;
2. $N \models$ "δ is Woodin (as witnessed by its extender sequence)";
3. for every $\eta < \delta$, $\mathrm{L}[N \mid \eta] \models$ "η is not Woodin"; and
4. $N \models \forall \eta < \delta$ "I am (η, η)-iterable".

Write δ^N for the unique such δ.

Here (α, β)-iterable has α refer to the height of the (linear) stack, and β refer to the length of the normal trees. Remember for us that δ is going to be countable in V. So this amount of iterability actually can be seen by M_1, using that $M_1 \mid \delta$ is closed under $\sharp$s. Note also that $N \mid \delta$ is a kind of fine-structural notion, cutting up to the height of δ in its J-hierarchy. A nice exercise is to show that M_1 itself is M_1-like. This isn't too bad with the fourth condition being the most difficult.

Now we have two facts with the first being more conceptual, showing that these things exist.

2B.2. Result (Mitchell, Steel)

If there is a Woodin cardinal, then there is an M_1-like model.

The next fact, also left unproven, says that we have a form of condensation for these premice. If we look at L, if you take an elementary substructure of L, it collapses to an initial segment. But instead of looking at *all* elementary substructures, we will look at Σ_1-hulls without parameters.

[ii] This is a minimal structure, and because of the sharp it projects to ω and can be coded in a countable object, or a real. Alternatively, you can avoid fine structure by noting that the defining property of $M_1^\sharp$ is preserved by elementary substructures.

[iii] In particular, Λ is Σ_0 restricted to finite stacks $\vec{\mathcal{T}}$ of normal trees on $\mathcal{M}_\infty$ such that $\vec{\mathcal{T}} \in \mathcal{M}_\infty \mid \kappa_\infty$, where κ_∞ is the least inaccessible in $\mathcal{M}_\infty$ above δ_∞.

[iv] Note that $\mathrm{HOD}^{\mathrm{L}[x]}$ isn't the same as $\mathrm{HOD}_x^{\mathrm{L}[x]} = \mathrm{L}[x]$.

2B.3. Result (Weak Condensation)

Let N be an M_1-like pre-mouse and H the transitive collapse of $\mathrm{Hull}_1^{N|\gamma}$ for some sufficiently large γ. Therefore $H \trianglelefteq N$, i.e. $H = N \mid \alpha$ for some α.

Something that makes this result harder is that we're only dealing with M_1-structures, and so are limited in the amount of iterability we have. These two facts will be used later.

Now the next thing we want to look at is a kind of countable version of M_1-like. Right now we have proper class models that are M_1-like, but sometimes it's easier to work with a countable model, and so we want to define *suitable* initial segments which already have all the necessary information.

2B.4. Definition

A premouse $\mathcal{P}$ is *suitable* iff $\mathcal{P} = N \mid \nu$ for some M_1-like N such that $N \models$ "$\nu = (\delta^N)^+$". In this case, write $\nu^N = \nu$ and $N^- = N \mid \nu^N$.

So we have the models we want to iterate, but we can't quite iterate freely. We have to do it in a slightly more clever way. The problem is that if we just take an iteration strategy for M_1 and we iterate it, this isn't something L[x] can see. So now we want to define weaker versions of iterability for those premice which L[x] *can* see. And the way we will do this is by looking at simpler trees. The iteration strategy for $M_1^\#$ is the one given by Q-structures. And the goal is to have—assuming we have a tree which is nice (i.e. given by Q-structures)—that we can find a branch. And this amounts to writing down all the nice properties we can in a first-order way that allow us to find a branch.

2B.5. Definition

Let $\mathcal{T}$ be a normal iteration tree on an M_1-like (suitable) N. Therefore

1. $\mathcal{T}$ is *short* iff either
   a. $\mathcal{T}$ has a last model $\mathcal{M}_\alpha$ (such that $\mathcal{M}_\alpha$ is suitable or $[0,\alpha]_\mathcal{T}$ drops), or
   b. $\mathcal{T}$ has limit length, $Q(\mathcal{T})$ exists, and $Q(\mathcal{T}) \trianglelefteq \mathrm{L}[\mathcal{M}(\mathcal{T})]$;
2. $\mathcal{T}$ is *maximal* iff $\mathcal{T}$ is not short.

If the tree $\mathcal{T}$ is short, then we know how to iterate, but what do we do with maximal trees? We want to find a natural way to extend maximal trees, but we cannot really find branches; we have no Q-structure. One way to think about this is with the search of M_1-like models. And if $\mathcal{T}$ is maximal, there is a very canonical one: $\mathrm{L}[\mathcal{M}(\mathcal{T})]$. Proving this serves as a nice exercise. This leads to the following definition, essentially saying that we can iterate as long as we see short trees.

So the plan is to iterate using short trees as long as we can, and if we encounter a maximal tree, we use $\mathrm{L}[\mathcal{M}(\mathcal{T})]$.

2B.6. Definition

Let N be M_1-like (suitable). Then N is *short tree iterable* iff whenever $\mathcal{T}$ is a short tree on N, then

1. If $\mathcal{T}$ has a last model, then it can be freely extended by one more ultrapower. More precisely, every putative—meaning it's last model need not well-founded—normal iteration tree $\mathcal{U}$ extending $\mathcal{T}$ which has length $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T}) + 1$ has a well-founded last model .
2. If $\mathcal{T}$ has limit length, then $\mathcal{T}$ has a cofinal, well-founded branch b such that $Q(b, \mathcal{T}) = Q(\mathcal{T})$.
(3. for all $\alpha < \mathrm{lh}(\mathcal{T})$,
   i. if $[0,\alpha]_T$ does not drop then $\mathcal{M}_\alpha^\mathcal{T}$ is suitable, and
   ii. if $[0,\alpha]_T$ drops then no $\mathcal{R} \trianglelefteq \mathcal{M}_\alpha^\mathcal{T}$ is suitable.)

We demand that our trees here are "correctly guided" by Q-structures in the sense that for limit $\lambda < \mathrm{lh}(\mathcal{T})$ with branch choice b for $\mathcal{T} \restriction \lambda$, $Q(b, \mathcal{T} \restriction \lambda)$ exists and $Q(b, \mathcal{T} \restriction \lambda) \trianglelefteq \mathrm{L}[\mathcal{M}(\mathcal{T} \restriction \lambda)]$, and this is the branch we should pick. Note that short tree iterability is very absolute: it's a Π_1 notion. And this is the main reason we're defining this notion: usual iterability is not so absolute.

2B.7. Lemma (Absoluteness Lemma)

a. For any $\mathscr{P}$ the following are equivalent:
   1. $\mathscr{P}$ is suitable and short tree iterable;
   2. whenever W is a transitive model of ZFC^- (which is $\mathsf{ZFC} - \mathsf{Powerset}$) such that $\mathscr{P} \in W$, then $W \models$ "$\mathscr{P}$ is suitable and short tree iterable".

b. For any fixed, transitive $S \models \mathsf{ZFC}^-$ containing ω_1 and any $\mathscr{P}$ countable in S, the following are equivalent:
   1. $\mathscr{P}$ is suitable and short tree iterable;
   2. $S \models$ "$\mathscr{P}$ is suitable and short tree iterable".

Proof .:.

We will only show (a), as (b) is an easy consequence. That (2) implies (1) is easy, so we will focus on (1) implying (2). Suppose that $\mathscr{P}$ is suitable and short tree iterable. Let W be a transitive model of ZFC^- with $\mathscr{P} \in W$. Clearly $\mathscr{P}$ is then suitable in W. Now suppose $W \models$ "$\mathcal{T}$ is a short tree of limit length on $\mathscr{P}$". Therefore $Q(\mathcal{T}) \in W$ witnesses that $\mathcal{T}$ is short in V as well.

Now let b be a branch through $\mathcal{T}$ such that $Q(b, \mathcal{T}) = Q(\mathcal{T})$ in B. Let η be sufficiently large so that if g is $\mathrm{Col}(\omega, \eta)$-generic over W, then Q, $\mathcal{T}$, and $Q(\mathcal{T})$ are all countable in $W[g]$.

In V$[g]$, by Σ^1_1 absoluteness, $W[g]$ correctly satisfies the Σ^1_1-statement "there is an x that codes a branch b_x through $\mathcal{T}$ where $Q(b_x, \mathcal{T}) = Q(\mathcal{T})$" (parameters are codes for $Q(\mathcal{T})$ and $\mathcal{T}$). Hence there is some branch c through $\mathcal{T}$ with $Q(c, \mathcal{T}) = Q(\mathcal{T})$ in $W[g]$ as well. By uniqueness of Q-structures, $b = c$ and hence $b \in W[g]$. Since g was arbitrary, by homogeneity we get $b \in W$ as desired. ⊣

As a summary, if we want to define the direct limit system of these suitable models, we need to find out what we can see about these iterations. Moreover, we want to do these in a definable way. We've thus far split up the iteration trees into two kinds: the good kindsm (short trees) and the bad kinds (maximal trees). What was good about short trees is that they come with a description of what to do at limit stages. This is a result of having Q-structures, which are simple enough that we can see them. So this raises the question, what do we do if we have a maximal tree?

Well, firstly, we have this notion of short tree iterability, but we should still check that the structures we're interested in satisfy this. Note that we built some iterability into the definition of begin M_1-like.

2B.8. Theorem

Let N be M_1-like or suitable. Therefore N is short tree iterable.

Proof .:.

Let's consider the case when N is M_1-like. By Absoluteness Lemma (2B.7) (b), it suffices to see that $N[g] \models$ "N is short tree iterable", where g is $\mathrm{Col}(\omega, (\delta^N)^+)$-generic over N.

So suppose not. In N, let γ be a sufficiently large, regular cardinal. By Weak Condensation (2B.3), let $\pi : H \to N \mid \gamma$ be Σ_m-elementary for some large enough m, where $H \trianglelefteq N \mid \omega_1^N$.

Let $h \in N$ be $\mathrm{Col}(\omega, (\delta^H)^+)$-generic over H. Then there is a counterexample to H's short tree iterability in $H[h]$. In other words, there is a tree $\mathcal{T}$ on H such that $\mathcal{T}$ is short in $H[h]$, and $\mathcal{T}$ witnesses the failure of short tree iterability of H in $H[h]$.

$\mathcal{T}$ is short in N, according to the unique ω_1^N iteration strategy for $N \mid \omega_1^N$—recall that $N \models$"I am (η, η)-iterable" for every $\eta < \delta$, so in particular, for $\eta = \omega_1^N$. So $\mathcal{T}$ cannot have a last ill-founded model in N. Hence $\mathcal{T}$ has limit length, and $H[h] \models$ "$Q(\mathcal{T})$ exists", noting that $Q(\mathcal{T})$ is the same in N and $H[h]$.

Since H and $\mathcal{T}$ are countable in N, N has a unique branch b through $\mathcal{T}$ with $Q(\mathcal{T}) = Q(b, \mathcal{T})$. This unique b

is in $H[h]$ as well by Σ^1_1-absoluteness again. But this contradicts the choice of $\mathcal{T}$ as a counterexample to short tree iterability in $H[h]$. ⊣

There are a couple other facts and ideas about iterability that must be introduced in order to deal with maximal trees. For example, from the concept of a normal iterate, we add "pseudo" to essentially mean that whenever we hit a maximal tree, we don't even try. For the sake of simplicity, whenever N is written, it is M_1-like.

2B.9. Definition

$\mathcal{R}$ is a *pseudo-normal-iterate* of N iff $\mathcal{R}$ is also M_1-like (suitable) and there is a normal tree $\mathcal{T}$ on N such that

1. $\mathcal{R}$ is the last model of $\mathcal{T}$; or
2. $\mathcal{T}$ is maximal and $\mathcal{R} = \mathrm{L}[\mathcal{M}(\mathcal{T})]$ ($\mathcal{R}$ is the suitable initial segment of $\mathrm{L}[\mathcal{M}(\mathcal{T})]$).

Note that this makes sense because if $\mathcal{T}$ is maximal, then $\mathrm{L}[\mathcal{M}(\mathcal{T})]$ is M_1-like. So this is a very good example of an iteration where we can see the last model, but we don't know the branch we took to get there. But the first thing to notice is that this weak notion of iterability is enough to give a version of comparison, and a notion of genericity iteration.

2B.10. Theorem (Pseudo-Comparison)

Let $\mathcal{P}$ and $\mathcal{Q}$ be suitable (or M_1-like). Therefore they have a common pseudo-normal iterate $\mathcal{R}$ such that $\mathcal{R} \in \mathrm{L}[\mathcal{P}, \mathcal{Q}]$ and moreover, $\delta^{\mathcal{R}} \leq (\max(\delta^{\mathcal{P}}, \delta^{\mathcal{Q}})^+)^{\mathrm{L}[\mathcal{P},\mathcal{Q}]}$.

Proof .:.

Work in $\mathrm{L}[\mathcal{P}, \mathcal{Q}]$ for suitable $\mathcal{P}$ and $\mathcal{Q}$, and let $(\mathcal{T}, \mathcal{U})$ on $(\mathcal{P}, \mathcal{Q})$ be the result of the standard comparison process with choosing according to the short tree strategies at limit stages. (By Absoluteness Lemma (2B.7), the restrictions of the short tree strategies to $\mathrm{L}[\mathcal{P}, \mathcal{Q}]$ are definable over $\mathrm{L}[\mathcal{P}, \mathcal{Q}]$.) If we reach a successor step with no disagreements, then we are done.

So write $\mu = (\max(\delta^{\mathcal{P}}, \delta^{\mathcal{Q}})^+)^{\mathrm{L}[\mathcal{P},\mathcal{Q}]}$. Then the process cannot last $\mu + 1$ steps, because in this case we have branches of length μ in $\mathrm{L}[\mathcal{P}, \mathcal{Q}]$—the trees have been short up to this point—and μ is a regular cardinal of $\mathrm{L}[\mathcal{P}, \mathcal{Q}]$. So the standard termination argument applies.

Finally, we might produce trees $\mathcal{T}$ and $\mathcal{U}$ of length $\leq \mu$ with one being maximal. But then $\mathcal{M}(\mathcal{T}) = \mathcal{M}(\mathcal{U})$, so they must both be maximal, and $\mathcal{R} = \mathrm{L}[\mathcal{M}(\mathcal{T})] = \mathrm{L}[\mathcal{M}(\mathcal{U})]$ is the desired common pseudo-iterate of $\mathrm{L}[\mathcal{P}]$ and $\mathrm{L}[\mathcal{Q}]$. ⊣

So we do the whole comparison argument inside $\mathrm{L}[\mathcal{P}, \mathcal{Q}]$. Again, all of this comes down to the fact that we have Q-structures as extremely simple. Thing are so absolute because we just look at initial segments of L.

It's also notable that the arguments for $\mathrm{L}[x]$ and $\mathrm{L}[x, G]$ break down here, and not so much at the very complicated arguments later. The reason for this is that we're trying to define $\mathcal{M}_\infty$, which is some direct limit of some countable mice. And these mice are pseudo-iterates of these M_1-like or suitable models. But we want to do it in such a way that everything is countable and we've defined a nice system. And if we compare two such countable models, we want their common co-iterate to be countable. But if we compare these two models $\mathcal{P}$ and $\mathcal{Q}$ that are suitable and countable in $\mathrm{L}[x]$, then we only have that their common pseudo-iterate $\mathcal{R}$ has size $\leq \omega_1^{\mathrm{L}[x]}$. Since $\mathcal{R}$ is countable in $\mathrm{L}[x, G]$, this is why we consider $\mathrm{HOD}^{\mathrm{L}[x,G]}$ instead of $\mathrm{HOD}^{\mathrm{L}[x]}$.

But this raises the question, can we improve Pseudo-Comparison (2B.10) to show that $\mathcal{R}$ is countable in $\mathrm{L}[x]$? In a result due to Farmer Schlutzenberg in 2015[v], under sufficiently large cardinals, the answer is no. Explicitly, assume Turing determinacy and that $M_1^\sharp$ exists and is fully iterable. Therefore for a cone of reals x, there is some suitable $\mathcal{P}$ such that the pseudo-comparison of M_1 with $\mathrm{L}[\mathcal{P}]$ has length $\omega_1^{\mathrm{L}[x]}$.

[v] published in 2018

2B.11. Theorem (Pseudo-Genericity Iteration)

Let $\mathcal{P}$ be suitable (or M_1-like), and z a real. Therefore in $\mathrm{L}[\mathcal{P}, z]$, there is a pseudo-normal iterate $\mathcal{R}$ of $\mathcal{P}$ such that z is $\mathbb{W}^{\mathcal{R}}$-generic over $\mathcal{R}$—here $\mathbb{W}^{\mathcal{R}}$ is Woodin's extender algebra in $\mathcal{R}$—and $\delta^{\mathcal{R}} \leq ((\delta^{\mathcal{P}})^+)^{\mathrm{L}[\mathcal{P},z]}$.

The proof of this is similar to the argument in Pseudo-Comparison (2B.10), using the proof of the standard genericity iteration result.

2B.12. Definition

Let N be M_1-like and κ the least inaccessible of $N > \delta^N$. Let H be $\mathrm{Col}(\omega, <\kappa)$-generic over N. Then we call $N[H]$ a *derived model* of N. Here $\mathrm{Col}(\alpha, <\beta)$ is the finite support product of $\mathrm{Col}(\alpha, \gamma)$ for all $\gamma < \beta$.

Usually, $\mathrm{L}(\mathbb{R})^{N[H]}$ or $\mathrm{L}(\mathbb{R}^*)$ for $\mathbb{R}^* = \bigcup_{\eta<\kappa} \mathbb{R} \cap N[H \cap \mathrm{Col}(\omega, <\eta)]$ would be called the derived model. This depends on H of course, so "*a* derived model" would be more accurate. But the first order theory is independent of H because the forcing is sufficiently homogeneous. So we can say "the" with no ambiguity.

Looking forward, we want to compute the theory of $\mathrm{L}[x, G]$ inside a generic extension of $\mathcal{M}_\infty$. And this will be called the derived model resemblence. The following corollary is a step towards that. It tells us that not only find a real generic, but we can also iterate to make the derived model $\mathrm{L}[x, G]$.

2B.13. Corollary

Let N be M_1-like and suppose $N \mid \delta^N$ is countable in $\mathrm{L}[x, G]$. Therefore there is a pseudo-normal iterate $\mathcal{P}$ of N such that $\mathcal{P} \mid \delta^{\mathcal{P}}$ is countable in $\mathrm{L}[x, G]$, and $\mathrm{L}[x, G]$ is a derived model of $\mathcal{P}$.

Proof .:.

Let $y \in \mathbb{R} \cap \mathrm{L}[x, G]$ code $N \mid \delta^N$. Apply Pseudo-Genericity Iteration (2B.11) *inside* $\mathrm{L}[x, G]$ to obtain a pseudo-normal iterate $\mathcal{P}$ of N such that x is $\mathbb{W}^{\mathcal{P}}$-generic over $\mathcal{P}$ and $\delta^{\mathcal{P}} \leq ((\delta^N)^+)^{L[N|\nu^N,x]}$. Because y codes $N \mid \delta^N$, in particular δ^N is countable in $\mathrm{L}[x, y]$. Hence we actually get the inequalities

$$\delta^{\mathcal{P}} \leq ((\delta^N)^+)^{L[N|\nu^N,x]} \leq \omega_1^{\mathrm{L}[x,y]} < \kappa_x.$$

Here κ_x is the least inaccessible in $\mathrm{L}[X]$, which then satisfies $\kappa_x = \omega_1^{\mathrm{L}[x,G]}$, and so $\omega_1^{\mathrm{L}[x,y]} < \omega_1^{\mathrm{L}[x,G]}$.

In fact, κ_x is the least inaccessible of $\mathcal{P}$ above $\delta^{\mathcal{P}}$. To see this, if there were some inaccessible in between, it would be in $\mathcal{P}[x]$ since $\mathbb{W}^{\mathcal{P}}$ has the $\delta^{\mathcal{P}}$-cc. As $\mathrm{L}[x] \subseteq \mathcal{P}[x]$, by downward absoluteness, it would be in $\mathrm{L}[x]$, contradicting that κ_x is the least inaccessible there.

Hence by Solovay's factoring lemma, $\mathrm{L}[x, G] = \mathcal{P}[H] = \mathcal{P}[x][H']$—as $\mathbb{W}^{\mathcal{P}}$ is absorbed into $\mathrm{Col}(\omega, <\kappa_x)$ as $\delta^{\mathcal{P}} < \kappa_x$—for some $\mathrm{Col}(\omega, <\kappa_x$-generic H (note that $\mathcal{P}[x] \in \mathrm{L}[x, G_\eta]$ where $G_\eta = G \cap \mathrm{Col}(\omega, \eta)$ for $\eta < \kappa_x$, yielding that $\mathrm{L}[x, G] = \mathrm{L}[x][G_\eta][G]$). ⊣

Next we analyze how much $\mathrm{L}[x, G]$ can see in terms of Σ_0 on maximal trees. We have already seen that $\mathrm{L}[x]$ sees the short tree fragments of Σ_0. The following lemma uses the minimality of M_1-like models.

Basically, we want to define pieces of the maps in $\mathrm{L}[x, G]$. The way we are going to do this is to try to define them on certain hulls of the models. What we want to do is to define pieces of the models which look like hulls, and these give pieces of the embedding. And we want to do it in a way where $\mathrm{L}[x, G]$ can see that. To that end, we have one more definition following the lemma.

2B.14. Lemma

Let M be M_1-like. Therefore

1. if Γ is a proper class, then $\mathrm{Hull}^M(\Gamma)$—the uncollapsed hull—is cofinal in δ^M, and
2. if Γ is an infinite set of indiscernibles for $M = \mathrm{L}[M \mid \delta^M]$, then $\mathrm{Hull}^M(\Gamma)$ is cofinal in δ^M.

Proof .:.

We will prove (11) as the the proof of (2) is similar. Write $X = \mathrm{Hull}^M(\Gamma)$, and suppose $\gamma = \sup(X \cap \delta^M) \neq \delta^M$. Set $Y = \mathrm{Hull}^M(\Gamma \cup \gamma)$ and note that as δ^M is regular, $Y \cap \delta^M$ is cofinal in δ^M iff $X \cap \delta^M$ is cofinal in δ^M.

Let H be the transitive collapse of Y. By elementarity, γ is Woodin in H as δ^M collapses to γ: $\delta^M \in Y$ as δ^M is definable in M as the unique Woodin cardinal. Moreover, H is M_1-like (Γ is a proper class, telling us that Y and H are proper classes) with $\delta^H = \gamma$. In fact, $H \mid \gamma = M \mid \gamma$ by definition of Y.

But this contradicts the minimality condition of Definition 2B.1 for M because $\mathrm{L}[M \mid \gamma] = \mathrm{L}[H \mid \gamma] \vDash$ "$\delta^H = \gamma$ is Woodin". ⊣

Now we should confirm that normal iteration trees of limit length have at most one cofinal, well-founded branch on M_1-like models. This need not happen in general, as maximal trees on a suitable model might have more than one cofinal, well-founded branch. The existence of branches is more absolute for trees on suitable models.

The proof of this fact uses Lemma 2B.14 and the following lemma. The proof of the lemma below uses a kind of "zipper argument" result, so we omit its proof.

2B.15. Lemma

Suppose $\mathcal{T}$ is a normal iteration tree on M of finite length and s is a cofinal subset of $\delta(\mathcal{T})$. Therefore there is at most one cofinal branch b with an $\alpha \in b$ such that $i^{\mathcal{T}}_{\alpha,b}$ exists and $s \subseteq \mathrm{ran}(i^{\mathcal{T}}_{\alpha,b})$.

2B.16. Theorem

Let $\mathcal{T}$ be a normal iteration tree of limit length on an M_1-like model. Therefore $\mathcal{T}$ has at most one cofinal, well-founded branch.

Proof .:.

Let $\mathcal{T}$ be a normal iteration tree on M, and b, c cofinal, well-founded branches through $\mathcal{T}$. We have two cases.

Case 1: $\mathcal{T}$ is short. Therefore $b = c$ follows just from the uniqueness of Q-structures: $Q(b, \mathcal{T}) = Q(c, \mathcal{T})$. By Lemma 2B.15, $b = c$ because Q-structures determine a canonical, cofinal subset of $\mathrm{ran}(i^{\mathcal{T}}_{\alpha,b}) \cap \delta(\mathcal{T})$ for some $\alpha \in b$.

Case 2: $\mathcal{T}$ is maximal. Therefore b and c cannot drop, as otherwise we would have Q-structures. (Here we don't require fullness preservation, this is why the branches might drop.) Moreover, the lack of Q-structures also tells us that $i_b(\delta^M) = i_c(\delta^M) = \delta(\mathcal{T})$.

Now let $\Gamma = \{\alpha : i_b(\alpha) = \alpha = i_c(\alpha)\}$ with $X = \mathrm{Hull}^{L[\mathcal{M}(\mathcal{T})]}(\Gamma)$. Thus Γ is a proper class, so by Lemma 2B.14, X is cofinal in $\delta(\mathcal{T})$. Moreover, $X \subseteq \mathrm{ran}(i_b) \cap \mathrm{ran}(i_c)$. So Lemma 2B.15 implies $b = c$. ⊣

So far we've only looked at individual trees. But in general what we need to do is look at stacks of them. So we look at a pseudo-normal iterate. But then maybe we hit a maximal tree, get a new model. And now we start taking a new tree on the new model. We will allow finite stacks of these.

2B.17. Definition

Let $k < \omega$ and $\mathcal{P}$ be suitable or M_1-like. We say that $(\vec{\mathcal{T}}, \vec{\mathcal{P}}) = (\langle \mathcal{T}_i : i < k\rangle, \langle \mathcal{P}_i : i \leq k\rangle)$ for $\vec{\mathcal{T}}$ is a *finite full stack* of length k on $\mathcal{P}$ iff

1. $\mathcal{P}_0 = \mathcal{P}$; and
2. for all $i < k$, $\mathcal{P}_{i+1}$ is a pseudo-normal iterate of $\mathcal{P}_i$ as witnessed by $\mathcal{T}_i$.

In this case, we call $\mathcal{P}_k$ the *last model* of $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$, and say that $\mathcal{P}_k$ is a *pseudo-iterate* of $\mathcal{P}$.

Analogously, we can define infinite full stacks $(\langle \mathcal{T}_i : i < \omega\rangle, \langle \mathcal{P}_i : i < \omega\rangle)$. Note that the $\mathcal{P}_i$ are all suitable or all

M_1-like. This is part of Definition 2B.9: we want to make sure that they are always fullness preserving. Note that maximal trees on a suitable model might have more than one cofinal, well-founded branch as the existence of branches is more absolute for trees on suitable models.

And this explains the notion of finite full stacks. We want to make sure that whenever we build these iterations, we don't lose this information. So all of these mice are still suitable or M_1-like. And one thing this also ensures is that we always have maps. What we want to see in $\mathrm{L}[x, G]$ is part of the iteration maps

Moreover, there are no drops. Full stacks are like plays of the weak iteration game with the difference that we omitted the branches and demand fullness for the iterates.

Now we want to look at the pieces of the maps we're interested in. To introduce some notation, for a finite, non-empty set of ordinals s, write s^- for $s \setminus \{\max(s)\}$. You should think of these ordinals as being really large. Let N be M_1-like and $\max(s) > \delta^N$. We also write

$$\begin{aligned}\gamma_s^N &= \sup(\mathrm{Hull}^{N|\max(s)}(s^-) \cap \delta^N),\\ \mathrm{Th}_s^N &= \{\langle \varphi, t\rangle : t \in (\gamma_s^N \cup s^-)^{<\omega} \text{ and } N \mid \max(s) \vDash \varphi[t]\}, \text{ and}\\ H_s^N &= \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-).\end{aligned}$$

This tells you to take an infintie set of indiscernibles, and γ_s^N will just be equal to δ^N. So this H_s^N is going to be cofinal. But in this case, if we just take a finite segment, we might get something bounded. So in some sense, for every finite s, this gives us some bounded piece of information. Our goal is to understand this for many finite s. Combining this with the Lemma 2B.14, the motivation is that if we take an infinite set of indiscernibles, we will generate the whole structure back. For example, in the direct limit system, you want to argue that one model is stronger than another if it preserves more of the theory.

With these new concepts, consider the following exercise to get used to these ideas.

Exercise 1

Show that $\gamma_s^N = \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) \cap \delta^N$, noting that δ^N is regular in N.

Solution .:.

Write $X = \mathrm{Hull}^{N|\max(s)}(s^-)$, i.e. $\gamma_s^N = \sup(X \cap \delta^N)$. Without loss of generality, $\gamma_s^N < \delta^N$ because otherwise $\gamma_x^N = \delta^N = \mathrm{Hull}^{N|\max(s)}(\delta^N \cup s^-) \cap \delta^N$. Now suppose there is some $\xi \in \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) \cap \delta^N$ with $\xi \notin X$. We are done if we can show that there is some $\xi' \in X$ with $\xi < \xi' < \delta^N$ because, giving " $\geq$ ",

$$\gamma_s^N \leq \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-) \cap \delta^N.$$

Say ξ is defined in $N|\max(s)$ by $\varphi(\vec{\alpha}, s^-, x)$ with $\vec{\alpha} \in (\gamma_s^N)^{<\omega}$. In other words, $\xi = \{x : N \mid \max(s) \vDash \varphi(\vec{\alpha}, s^-, x)\}$. Let $\vec{\beta} \in X$ be a finite set of ordinals which are pointwise above $\vec{\alpha}$ (say $|\vec{\alpha}| = |\vec{\beta}|$). Let

$$\xi' = \sup\left\{\eta < \delta^N : \begin{array}{c}\eta = \{x : N \mid \max(s) \vDash \varphi(\vec{\alpha}', s', x)\} \text{ for some } \vec{\alpha}' \\ \text{that is pointwise below } \vec{\beta} \text{ with the same length}\end{array}\right\}.$$

Therefore $\xi' \in X$ since $s', \vec{\beta} \in X$. Moreover, $\xi' \geq \xi$ because ξ shows up in the sup as one of those ηs from the point of view of V. ⊣

2B.18. Definition

Let $\mathcal{P}$ be suitable and let $\mathcal{Q}$ be a pseudo-normal iterate of $\mathcal{P}$ witnessed by the normal tree $\mathcal{T}$. Let s be a finite set of ordinals with $\max(s) > \delta^{\mathcal{P}}$. Then

1. b is a *branch choice* for $\mathcal{T}$ iff
   a. $\mathrm{lh}(\mathcal{T}) = \alpha + 1$ and $b = [0, \alpha]_T$ (write $i_b^{\mathcal{T}}$ for $i_{0,\alpha}^{\mathcal{T}}$; or
   b. $\mathcal{T}$ is maximal and b is a cofinal branch of $\mathcal{T}$ such that $Q = \mathcal{M}_b^{\mathcal{T}}$.
2. We say the branch choice b *respects* s iff $i_b^{\mathcal{T}}(\mathrm{Th}_s^{\mathcal{P}}) = \mathrm{Th}_s^{\mathcal{Q}}$, which follows from $i_b^{\mathcal{P}}(s) = s$.

Note that in this case, b does not drop. And we get a similar definition for finite full stacks: for $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ a finite full stack, $\vec{b}$ is a branch choice for $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ iff each b_i is a branch choice for $\mathcal{T}_i$; and $\vec{b}$ respects s iff each b_i respects s: each $i^{\mathcal{T}_n}_{b_n} : \mathcal{P}_n \to \mathcal{P}_{n+1}$ has $i^{\mathcal{T}_n}_{b_n}(\mathrm{Th}^{\mathcal{P}_n}_s) = \mathrm{Th}^{\mathcal{P}_{n+1}}_s$ for appropriate n.

2B.19. Definition

Let $\mathcal{P}$ be suitable, and $\delta^{\mathcal{P}} < \omega_1$. Let s be a finite set of ordinals such that $\max(s) > \delta^{\mathcal{P}}$. Then we say that $\mathcal{P}$ is *s-iterable* iff whenever $(\vec{\mathcal{T}}, \vec{\mathcal{P}}) \in \mathrm{HC}$—i.e. is hereditarily countable—is a finite full stack on $\mathcal{P}$, then there is a branch choice for $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ that respects s.

So we're demanding something strong here. We're not just asking for some branch for some tree somewhere, but for branches which carry along a lot of information. One thing to note is that s-iterability says that if we have a model then for every pseudo-normal iterate, and every finite full stack, we have a branch choice for s. In particular, any pseudo-iterate of an s-iterable model is s-iterable as well, following directly from the definition. And this is good news, because we are trying to define some kind of directed system, and we want it to be upwards closed in some sense.

There's also some new notation for finite full stacks and branch choices. Write $\pi_{\vec{\mathcal{T}},\vec{b},n} = i^{\mathcal{T}_n}_{b_n}$ for all appropriate n, say $n < k$, and in this case,

$$\pi_{\vec{\mathcal{T}},\vec{b}} = \pi_{\vec{\mathcal{T}},\vec{b},k-1} \circ \pi_{\vec{\mathcal{T}},\vec{b},k-2} \circ \cdots \circ \pi_{\vec{\mathcal{T}},\vec{b},0}.$$

Moving forward, we have a downward absoluteness for proper classes, just as with the case of short tree iterability.

2B.20. Lemma

Let $\mathcal{P}$ be suitable and s-iterable, and suppose W is a transitive proper class ZFC model such that $\mathcal{P} \in \mathrm{HC}^W$. Therefore $W \models$ "$\mathcal{P}$ is s-iterable".

Proof .:.

Let $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ be a finite full stack on $\mathcal{P}$ in HC^W. Then it is a finite full stack on $\mathcal{P}$ in HC^{V}, so first of all its last models $\mathcal{P}_i$ are short tree iterable in V as they are suitable by Theorem 2B.8, and hence in W by absoluteness of short tree iterability as in Absoluteness Lemma (2B.7).

Let $z \in W$ be a real coding $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ and all the $\mathrm{Th}^{\mathcal{P}_i}_s$s. Then the existence of a branch choice $\vec{b}$ respecting s is a $\Sigma^1_1(z)$ fact which holds in V, and thus by Σ^1_1-absoluteness, in W as well. ⊣

One thing to confirm is that these s-iterability embeddings are unique. Even though a given tree may have more than one branch choice respecting s, we will see now that the embeddings associated with these branches agree up to γ_s.

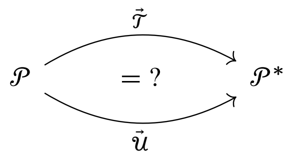
2B.21. Lemma (Uniqueness of s-iterability embedings)

Suppose $\mathcal{P}$ is suitable, $s \in [\mathrm{Ord}]^{<\omega}$ with $\max(s) > \delta^{\mathcal{P}}$, and $\mathcal{T}$ is a finite full stack on $\mathcal{P}$. Suppose $\vec{b}$ and $\vec{c}$ are branch choices for $\vec{\mathcal{T}}$ that respect s. Therefore $\pi_{\vec{\mathcal{T}},\vec{b}} \restriction \gamma^{\mathcal{P}}_s = \pi_{\vec{\mathcal{T}},\vec{c}} \restriction \gamma^{\mathcal{P}}_s$.

Moreover, if $\vec{\mathcal{T}}$ consists of just one normal tree $\mathcal{T}$ with last model $\mathcal{Q}$ and b and c are branch choices for $\mathcal{T}$ respecting s, then for the least $\xi \in b$ such that $\mathrm{crit}(E^{\mathcal{T}}_\xi) > \gamma^{\mathcal{Q}}_s$, we have $b \cap \xi = c \cap \xi$.

The proof of this follows from a careful analysis of the zipper argument—see also Lemma 2B.14.

But note that $\pi_{\vec{\mathcal{T}},\vec{b}} \restriction \gamma^{\mathcal{P}}_s$ might still depend on the choice of $\vec{\mathcal{T}}$. Remember we want to piece together bits of the embeddings. The first thing we want is that it doesn't depend on the branch choice. But it might still depend on the tree. We might have the following situation: we have our $\mathcal{P}$ and iterate $\mathcal{P}^*$, but we have two different ways of getting there: $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$. So we need something stronger: not only do we want it do be independent of the branch, we also want it to be independent of the tree on the stack. This is can be understood with the following picture.



So in essence, we just define that this will be true. And then it makes sense to talk about unique embeddings, because in the direct limit system, there are going to be different ways to get to the same point. But if we want to piece together the embeddings, if they don't agree on each other, we're out of luck. Ideally, we'd have full equality or agreement, but this is too much, so we only demand agreement on a sufficient part.

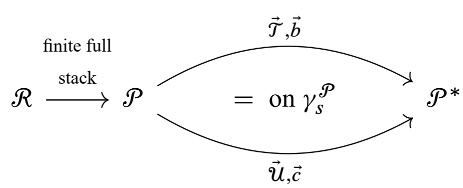
2B.22. Definition

Let $\mathcal{R}$ be suitable with $\delta^{\mathcal{R}} < \omega_1$ and let $s \in [\mathrm{Ord}]^{<\omega}$ with $\max(s) > \delta^{\mathcal{R}}$. We say that $\mathcal{R}$ is strongly s-iterable iff whenever

- $\mathcal{P} \in \mathrm{HC}$ is a pseudo-iterate of $\mathcal{R}$;
- $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ and $(\vec{\mathcal{U}}, \vec{\mathcal{Q}})$ are finite full stacks on $\mathcal{P}$ in HC with a common last model; and
- $\vec{b}$ and $\vec{c}$ are (non-dropping) branch choices for $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ respecting s;

then $\pi_{\vec{\mathcal{T}},\vec{b}} \restriction \gamma_s^{\mathcal{P}} = \pi_{\vec{\mathcal{U}},\vec{c}} \restriction \gamma_s^{\mathcal{P}}$.

This is represented in the following picture, a modified version from before.



Note again that any pseudo-iterate in HC of a strongly s-iterable premouse is itself strongly s-iterable. Again, we get an absoluteness result: strong s-iterability is downward absolute in the sense of Lemma 2B.20 with a similar proof.

2B.23. Lemma

Let $\mathcal{P}$ be suitable and strongly s-iterable, and suppose W is a transitive proper class ZFC model such that $\mathcal{P} \in \mathrm{HC}^W$. Therefore $W \models$ "$\mathcal{P}$ is strongly s-iterable".

So this tells us this strong s-iterability is a nice notion. But of course, we need to know that such things exist. This is nice argument, which relates to the core model induction, if you've heard of the A-iterability proof. In some sense, this is a baby version of it in that the underlying ideas are the same, but the A-iterability proof is ten times more complicated.

2B.24. Theorem

Let N br M_1-like and Σ an $(\omega, \omega, \omega_1)$-iteration strategy for N which has the Dodd-Jensen property. Let $\mathcal{R} = N^-$—the suitable initial segment of N—and let W be a transitive, proper class ZFC model such that $\mathcal{R} \in \mathrm{HC}^W$. Therefore

1. $W \models$ "$\mathcal{R}$ is strongly s-iterable" whenever s is a finite set of uncountable V-cardinals; and
2. if s is a finite set of ordinals, then there is a Σ-iterate $\mathcal{Q}$ of $\mathcal{R}$ such that $W \models$ "$\mathcal{Q}$ is countable and strongly s-iterable".

In particular, this can be applied for $W = \mathrm{V}$.

Proof .:.

(1) uses the Dodd-Jensen property while (2) uses a bad sequence argument. Recall that the Dodd-Jensen property tells us that if we take two iteration embeddings into the same last model, then they agree.

1. Note that the s-iterable part is completely trivial: $\mathcal{R}$ is s-iterable in V through branch choices by Σ, because every iteration embedding according to some $\vec{\mathcal{T}}$ will fix s: $i^{\mathcal{T}_n}(s) = s$ for all n. This is because we're dealing with countable things while s is a finite set of uncountable V-carindals, meaning that they are high above $\mathcal{R}$.

   Now we argue the interesting part: that $\mathcal{R}$ is in fact strongly s-iterable. We argue that this is the case for V, and hence in W by Lemma 2B.23. Let $\mathcal{P}$ be a countable pseudo-iterate of $\mathcal{R}$, i.e. a Σ-iterate of $\mathcal{R}$ (if we consider the full M_1-like model N, there is a unique branch). Let $(\vec{\mathcal{T}}, \vec{\mathcal{P}})$ and $(\vec{\mathcal{U}}, \vec{\mathcal{Q}})$ both be finite full stacks on $\mathcal{P}$ with a common last model $\mathcal{P}_k = \mathcal{Q}_j$ (the lengths can be different). Here, we only care about the last model, not the branch, and they all agree up to $\mathcal{M}(\vec{\mathcal{T}})$, which is what we need if $\mathcal{T}$ is maximal. If $\mathcal{T}$ is short, we pick the branch given by the Q-structure.

   Now let $\vec{b}$ and $\vec{c}$ be branch choices for $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ respecting s. Let $\vec{a}$ and $\vec{d}$ be the branch choices for $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ made by Σ. Therefore by the uniqueness of s-iterability embeddings, we have
$$\pi_{\vec{\mathcal{T}},\vec{b}} \restriction \gamma_s^{\mathcal{P}} = \pi_{\vec{\mathcal{T}},\vec{a}} \restriction \gamma_s^{\mathcal{P}}, \quad \text{and} \quad \pi_{\mathcal{U},\vec{d}} \restriction \gamma_s^{\mathcal{P}} = \pi_{\vec{\mathcal{U}},\vec{c}} \restriction \gamma_s^{\mathcal{P}}.$$
   But by the Dodd-Jensen lemma, $\pi_{\vec{\mathcal{T}},\vec{a}} \restriction \gamma_s^{\mathcal{P}} = \pi_{\vec{\mathcal{U}},\vec{d}} \restriction \gamma_s^{\mathcal{P}}$, and hence all four are equal, as desired.

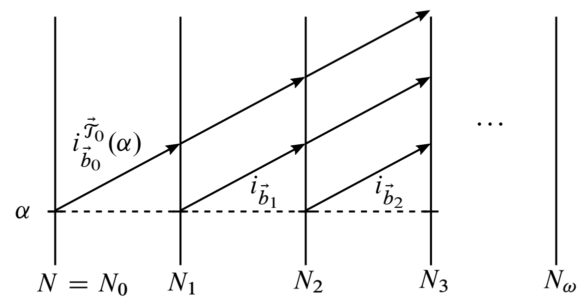
2. Work in W and suppose that there is no s-iterable $\mathcal{Q}$ (we will take care of the "strong" part later). Construct pseudo-iterates of $\mathcal{R}$ as follows:

$$\begin{array}{ccccccc} N = N_0 & \xrightarrow{\vec{\mathcal{T}}_0} & N_1 & \xrightarrow{\vec{\mathcal{T}}_1} & N_2 & \xrightarrow{\vec{\mathcal{T}}_2} & \cdots \\ \mathcal{R} = N^- = \mathcal{R}_0 & \longrightarrow & \mathcal{R}_1 & \longrightarrow & \mathcal{R}_2 & \longrightarrow & \cdots \end{array}$$

   2B.25. Figure 1: Bad Sequence Argument

   Let $\vec{b}_i = \Sigma(\vec{\mathcal{T}}_i)$ be given by Σ. For each of these finite stacks of trees $\vec{\mathcal{T}}_i$, no branch choice $\vec{b}$ has the property that $i^{\vec{\mathcal{T}}_i}_{\vec{b}}(\mathrm{Th}^{\mathcal{R}_i}_s) = \mathrm{Th}^{\mathcal{R}_{i+1}}_s$, as otherwise we would have found our $\mathcal{Q}$. There is a little argument here that is a bit more complicated, since we want to show this in W: $\vec{b}_i$ might not be in W since Σ is in V. But by absoluteness, the argument goes through. We also have $i^{\vec{\mathcal{T}}_i}_{\vec{b}_i}(\mathrm{Th}^{\mathcal{R}_i}_s) \neq \mathrm{Th}^{\mathcal{R}_{i+1}}_s$, because otherwise there would be a branch with $i^{\vec{T}_i}_{\vec{b}}(\mathrm{Th}^{\mathcal{R}_i}_s) = \mathrm{Th}^{\mathcal{R}_{i+1}}_s$ as well by absoluteness. Hence, $i^{\vec{\mathcal{T}}_i}_{\vec{b}_i}(s) \neq s$ for all $i < \omega$. Because if it were equal to s, then it would move the theory correctly.

   Note that the direct limit along these branches, N_ω, is well-founded, choosing the branches according tot he iteration strategy Σ (we consider the proper class model here: we want to use s which is above $\mathcal{R} \cap \mathrm{Ord}$). But this is a contradiction since the branch embedding of each $\vec{\mathcal{T}}_i$ moves some ordinal in s strictly upward. In other words, N_ω has an infinite decreasing sequence of ordinals. To better understand this, consider the following simple case where $s = \{\alpha\}$ (since s is always finite, there is some ordinal $\alpha \in s$ moved infinitely often).



$$i_{(\vec{b}_0,\vec{b}_1,\cdots)}(\alpha) > i_{(\vec{b}_1,\vec{b}_2,\cdots)}(\alpha) > i_{(\vec{b}_2,\cdots)}(\alpha) > \cdots$$

So there is an s-iterable $\mathcal{Q}$ (in W) which is Σ-iterate of $\mathcal{R}$ in V. The argument for (1) can now be used to show that $\mathcal{Q}$ is in fact strongly s-iterable in W (both parts of the argument apply to $W = \mathrm{V}$ as well). ⊣

So we proved that we do have strongly s-iterable premice, and in fact if we start with any suitable premouse, we can iterate it a little bit, and make it s-iterable for any given finite s we want. Now we are finally ready to define the direct limit system.

§ 2C. The internal system converging to $\mathrm{HOD}^{\mathrm{L}[x,G]} \restriction \omega_2^{\mathrm{L}[x,G]}$

We will for the most part work in $\mathrm{L}[x,G]$. We will have to step outside of this model for a bit to show well-foundedness, but for now we will work internally in $\mathrm{L}[x,G]$. We will now define a direct limit system that converges to $\mathrm{HOD}^{\mathrm{L}[x,G]} \restriction \omega_2^{\mathrm{L}[x,G]}$. Let

$$\mathcal{I} = \{\langle \mathcal{P}, s\rangle : \mathcal{P} \text{ is suitable, countable, and strongly } s\text{-iterable}\},$$

and for $\langle \mathcal{P}, s\rangle, \langle \mathcal{Q}, t\rangle \in \mathcal{I}$, let

$$\langle \mathcal{P}, s\rangle \leqslant^* \langle \mathcal{Q}, t\rangle \quad \text{iff} \quad \mathcal{Q} \text{ is a pseudo-iterate of } \mathcal{P} \text{ and } s \subseteq t.$$

So as we go further and further along, we get more and more agreement, as we allow more and more parameters, and hence we define more and more of the model. So we want to use the $\mathcal{P}$s as the building blocks, and we get embeddings between them from the s-iterability property.

2C.1. Lemma

$\leqslant^*$ is a directed partial order.

Proof .:.

Reflexivity, anti-symmetry, and transitivity are clear. The interesting property is directedness: let $\langle \mathcal{P}, s\rangle, \langle \mathcal{Q}, t\rangle \in \mathcal{I}$: By Theorem 2B.24, there is a Σ_0-iterate $\mathcal{R}$ of M_1^-, i.e. $M_1 \mid \nu^{M_1}$, that is strongly $(s \cup t)$-iterable in $\mathrm{L}[x,G]$. Simultaneously comparing $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ yields that the comparison terminates in countably many steps, because $\omega_1^{\mathrm{L}[\mathcal{P},\mathcal{Q},\mathcal{R}]}$ is countable in $\mathrm{L}[x,G]$.

As a pseudo-iterate of $\mathcal{R}$, the common pseudo-iterate $\mathcal{S}$ is strongly $(s \cup t)$-iterable, and is the desired $\leqslant^*$-upper bound for $\langle \mathcal{P}, s\rangle, \langle \mathcal{Q}, t\rangle$. ⊣

$\mathcal{I}$ can be viewed as the set of indices in our direct limit system, where the structure indexed by some $\langle \mathcal{P}, s\rangle \in \mathcal{I}$ is $H_s^{\mathcal{P}}$. Now for $\langle \mathcal{P}, s\rangle \leqslant^* \langle \mathcal{Q}, t\rangle$, there is a natural embedding $\pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},t\rangle} : H_s^{\mathcal{P}} \to H_t^{\mathcal{Q}}$ given by $\pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},t\rangle}(s) = s$ and $\pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},t\rangle} \restriction \gamma_s^{\mathcal{P}} = \pi_{\vec{\mathcal{T}},\vec{b}} \restriction \gamma_s^{\mathcal{P}}$ for any finite full stacks $\vec{\mathcal{T}}$ on $\mathcal{P}$ with last model $\mathcal{Q}$ and branch choices $\vec{b}$ respecting s. This is where strong s-iterability comes in: we want things to be independent of the choice of $\vec{\mathcal{T}}$ and $\vec{b}$. This generates a unique embedding on $H_s^{\mathcal{P}}$ since $H_s^{\mathcal{P}}$ is generated by $s \cup \gamma_s^{\mathcal{P}}$, and $\pi_{\vec{\mathcal{T}},\vec{b}}(\mathrm{Th}_s^{\mathcal{P}}) = \mathrm{Th}_s^{\mathcal{Q}}$.

This embedding is Σ_0-elementary, and the embeddings commute: if $\langle \mathcal{P}, s\rangle \leqslant^* \langle \mathcal{Q}, t\rangle \leqslant^* \langle \mathcal{R}, u\rangle$, then $\pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{R},u\rangle} =$

$\pi_{\langle \mathcal{Q},t\rangle,\langle \mathcal{R},u\rangle} \circ \pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},t\rangle}$ by strong s-iterability again. Hence $(\mathcal{I}, \leqslant^*)$ together with the $H_s^{\mathcal{P}}$s and the $\pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},t\rangle}$s indeed forms a directed system $\mathcal{F}$.

Let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ and $\pi_{\langle \mathcal{P},s\rangle,\infty} : H_s^{\mathcal{P}} \to \mathcal{M}_\infty$ be the canonical direct limit embedding. Moreover, let $\delta_\infty = \pi_{\langle \mathcal{P},s\rangle,\infty}(\delta^{\mathcal{P}})$ be the common value for any (and every) $\langle \mathcal{P}, s\rangle \in \mathcal{I}$. Note that $\mathcal{M}_\infty$ is a proper class model (the s can be arbitrarily high) and δ_∞ is the unique Woodin cardinal of $\mathcal{M}_\infty$.

Our next goal is to show that $\mathcal{M}_\infty$ is well-founded. Without loss of generality, assume that the well-founded part of $\mathcal{M}_\infty$ is transitive, just replacing it with the transitive collapse). So far, everything we did was definable in $\mathrm{L}[x, G]$. Now we need to step outside $\mathrm{L}[x, G]$ again. We will argue from the outside that $\mathcal{M}_\infty$ is well-founded. How do we do this? We consider a different system that we can trivially see that it is well-founded.

Consider the directed system $\mathcal{F}^\Sigma \cap \mathrm{L}[x, G]$ of all Σ-iterates N of M_1 such that $N^- \in \mathrm{HC}^{\mathrm{L}[x,G]}$ with the iteration maps given by Σ. This is a directed system, because for any iterates of M_1, we can compare them and get a third iterate of M_1. Moreover, if we do this in $\mathrm{L}[x, G]$, we can ensure that the result of this comparison is countable. So all of our preparation tells us that this is a directed system. Let $\mathcal{M}^+_\infty$ be its direct limit.

Now it's easy to see that $\mathcal{M}^+_\infty$ is well-founded. This is because Σ has the Dodd-Jensen property, so the iteration maps commute and since Σ is an $(\omega, \omega_1, \omega_1)$-iteration strategy, $\mathcal{M}^+_\infty$ is well-founded. Without loss of generality, we can also assume $\mathcal{M}^+_\infty$ is transitive, yielding that it is M_1-like. From here, the following lemma gives the well-foundedness of $\mathcal{M}_\infty$. Now it suffices to prove the following lemma. In essence, we pull back from σ, and push forward again.

2C.2. Lemma

There is an elementary $\sigma : \mathcal{M}_\infty \to \mathcal{M}^+_\infty$ such that $\sigma \restriction (\delta_\infty + 1) = \mathrm{id}$. In particular, $\mathcal{M}_\infty = \mathcal{M}^+_\infty$.

Proof .:.

First we will show the following claim.

Claim 1

For every $\langle \mathcal{P}, s\rangle \in \mathcal{I}$, there is a $\langle \mathcal{Q}, s\rangle$ such that
1. $\langle \mathcal{P}, s\rangle \leqslant^* \langle \mathcal{Q}, s\rangle$;
2. $\mathrm{L}[\mathcal{Q}] \in \mathcal{F}^\Sigma \cap \mathrm{L}[x, G]$; and
3. if $\vec{\mathcal{T}}$ is a finite full stack on $\mathrm{L}[\mathcal{Q}]$ such that $\vec{\mathcal{T}} \in \mathrm{HC}^{\mathrm{L}[x,G]}$ and $\vec{b}$ is the branch choice for $\vec{\mathcal{T}}$ made by Σ, then $i_{b_n}(s) = s$ for all $n < \mathrm{lh}(\vec{\mathcal{T}})$.

Proof .:.

This is another bad sequence argument (see Bad Sequence Argument (2B.25)). First, compare $\mathrm{L}[\mathcal{P}]$ with M_1 and obtain $\mathrm{L}[\mathcal{Q}_0] \in \mathcal{F}^\Sigma \cap \mathrm{L}[x, G]$ with $\langle \mathcal{P}, s\rangle \leqslant^* \langle \mathcal{Q}_0, s\rangle$. If $\mathcal{Q}_0$ doesn't work, let $\mathcal{Q}_1$ be the last model of an s-bad stack on $\mathcal{Q}_0$, and so forth. As before, the direct limit of the $\mathrm{L}[\mathcal{Q}_i]$s is ill-founded because some ordinal in s is moved infinitely many times. But the iteration maps are given by Σ, so the direct limit has to be well-founded, a contradiction. ⊣

Now let $x \in \mathcal{M}_\infty$. Say $x = \pi_{\langle \mathcal{Q},s\rangle,\infty}(\bar{x})$. By Claim 1, we may assume that $\mathrm{L}[\mathcal{Q}] \in \mathcal{F}^\Sigma \cap \mathrm{L}[x, G]$ and (3) holds.

Let $\pi^\Sigma_{\mathcal{Q},\infty} : \mathrm{L}[\mathcal{Q}] \to \mathcal{M}^+_\infty$ be the iteration map given by Σ and set $\sigma(x) = \pi^\Sigma_{\mathcal{Q},\infty}(\bar{x})$. Then it is easy to check that σ is well-defined and elementary. The key idea is that for $\langle \mathcal{Q}, s\rangle$ as above, the maps of the system $\mathcal{F}^\Sigma \cap \mathrm{L}[x, G]$ all come from branches respecting s and hence agree with the maps from $\mathcal{F}$ on $\gamma_s^{\mathcal{Q}}$. This also shows that $\sigma \restriction \delta_\infty = \mathrm{id}$. By an earlier lemma, $\mathrm{ran}(\sigma)$ is cofinal in $\delta^{\mathcal{M}^+_\infty}$ and hence $\sigma(\delta_\infty) = \delta^{\mathcal{M}^+_\infty}$. This is because $\mathcal{M}^+_\infty$ is M_1-like and $\mathrm{ran}(\sigma)$ is the hull in $\mathcal{M}^+_\infty$ of δ_∞ together with all $\pi^\Sigma_{\mathcal{Q},\infty}(\alpha)$ such that α is fixed from $\mathcal{Q}$ onward by all the maps $\pi_{\mathcal{Q},\mathcal{R}}$ of $\mathcal{F}^\Sigma \cap \mathrm{L}[x, G]$. ⊣

So we showed that $\mathcal{M}_\infty$ is well-founded, M_1-like, and—viewed outside of $\mathrm{L}[x, G]$—a Σ-iterate of M_1 via a stack of normal trees. Now we have two more properties, which are should be somewhat natural and unsurprising if you know the abstract theorems of Woodin about HOD.

2C.3. Lemma

1. The least measurable cardinal of $\mathcal{M}_\infty$ is $\omega_1^{\mathrm{L}[x,G]}$.
2. $\delta_\infty = \omega_2^{\mathrm{L}[x,G]}$.

Proof .:.

1. For a non-dropping Σ_0-iterate N of M_1, let $j_{N,\infty} : N \to \mathcal{M}_\infty$ be the map determined by the direct limit system. Write κ_N for the least measurable cardinal of N.

Claim 1

$\kappa_N = \mathrm{crit}(j_{N,\infty})$.

Proof .:.

First, note that $j_{N,\infty}(\kappa_N) \neq \kappa_N$ since if $\mathcal{U}$ is a measure on κ_N, taking $N_1 = \mathrm{Ult}(N, \mathcal{U})$ has N_1 show up in the direct limit system (if N is strongly s-iterable then so is N_1). But $j_{N,N_1}(\kappa_N) = \kappa_N$.

For $\alpha < \kappa_N$, $j_{N,\infty}(\alpha) = \alpha$, because κ_N is the least cardinal with a total measure of N and the iteration is non-dropping, so no partial measures can be applied. ⊣

To see that $\kappa_{\mathcal{M}_\infty} \geq \omega_1^{\mathrm{L}[x,G]}$, let $\alpha < \omega_1^{\mathrm{L}[x,G]}$. For some N, we can then iterate N to N^* such that $j_{N,N^*}(\kappa_N) > \alpha$. To see that $\kappa_{\mathcal{M}_\infty} \leq \omega_1^{\mathrm{L}[x,G]}$, suppose not: $\kappa_{\mathcal{M}_\infty} > \omega_1^{\mathrm{L}[x,G]}$. This cannot happen because otherwise for some N, because N is countable in $\mathrm{L}[x,G]$, $\mathrm{crit}(j_{N,\infty}) < \kappa_N$, which contradicts Claim 1. Hence (a) holds.

2a: To see that $\delta_\infty \leq \omega_2^{\mathrm{L}[x,G]}$, let $\alpha < \delta_\infty$. Then we can fix some $\langle \mathcal{P}, s\rangle \in \mathcal{I}$ and $\beta < \gamma_s^{\mathcal{P}}$ such that $\pi_{\langle \mathcal{P},s\rangle,\infty}(\beta) = \alpha$ as elements in the direct limit have to come from somewhere. Let

$$A = \{\langle \mathcal{Q}, \gamma\rangle : \langle \mathcal{P}, s\rangle \leqslant^* \langle \mathcal{Q}, s\rangle \text{ and } \gamma < \pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},s\rangle}(\beta)\},$$

and let $f : A \to \mathrm{Ord}$ be given by $f(\mathcal{Q}, \gamma) = \pi_{\langle \mathcal{Q},s\rangle,\infty}(\gamma)$.

Therefore $\alpha \subseteq \mathrm{ran}(f)$. To see this, for every $\alpha' < \alpha$, there is a $\beta' < \beta$ such that $\pi_{\langle \mathcal{P},s\rangle,\infty}(\beta') = \alpha'$. So there is a $\langle \mathcal{Q}, s\rangle$ and $\gamma = \pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},s\rangle}(\beta') < \pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},s\rangle}(\beta)$ such that $\pi_{\langle \mathcal{Q},s\rangle,\infty}(\gamma) = \pi_{\langle \mathcal{P},s\rangle,\infty}(\beta') = \alpha'$, and hence $\alpha' = f(\mathcal{Q}, \gamma) \in \mathrm{ran}(f)$.

But $f, A \in \mathrm{L}[x,G]$ and A is coded by a set of reals in $\mathrm{L}[x,G]$ as the mice are countable. By CH in $\mathrm{L}[x,G]$, $\alpha < \omega_2^{\mathrm{L}[x,G]}$. Since $\alpha < \delta_\infty$ was arbitrary, we get that $\delta_\infty \leq \omega_2^{\mathrm{L}[x,G]}$.

2b: To see that $\omega_2^{\mathrm{L}[x,G]} \leq \delta_\infty$, let $\alpha < \omega_2^{\mathrm{L}[x,G]}$. Our goal here will be to produce an order preserving map $\eta \mapsto \mu$ of α into δ_∞, which would complete the proof since $\alpha < \delta_\infty$.

Recall that $x^\#$ exists and $\omega_2^{\mathrm{L}[x,G]} = (\kappa_x^+)^{\mathrm{L}[X]}$ where κ_x is the least inaccessible of $\mathrm{L}[x]$ (κ_x is not an x-indiscernible). So there is a term τ and a finite set of uncountable V-cardinals such that

$$\forall \eta < \alpha \exists \beta < \kappa_x (\tau^{\mathrm{L}_{\max(s)}[x]}[\beta, s^-] = \eta).$$

Fix $\eta < \beta$ and β like this. Using genericity iterations, let N be a Σ-iterate of M_1 such that N^- is countable in $\mathrm{L}[x,G]$, $\langle N^-, s\rangle \in \mathcal{I}$, and x is $\mathbb{W}^N$ generic over N. By first iterating the least measure, we can arrange that $\beta < \kappa^N$, where κ^N is the least measurable cardinal of N. Furthermore, we can ensure the following:

($\star$) Whenever $\pi : N \to \mathcal{R}$ is a Σ_N-iteration map such that $\mathcal{R}^-$ is countable in $\mathrm{L}[x,G]$, then $\pi(\eta) = \eta$.

Now let

$$\mathcal{P}^N = \left\{\xi : \exists\beta < \kappa^N \exists p \in \mathbb{W}^N \left(p \Vdash \tau^{\mathrm{L}_{\max(s)}[x]}[\beta, s^-] = \xi\right)\right\}.$$

Since $\mathbb{W}^N$ has the δ^N-cc in N, $\mathcal{P}$ has order type $< \delta^N$. Let μ^N_η be the unique γ such that η is the γth element of $\mathcal{P}^N$ (note that $\eta \in \mathcal{P}^N$ by choice of τ). And let $\mu_\eta = \pi_{\langle N^-,s\rangle,\infty}(\mu^N_\eta)$.

Claim 2

μ_η is well-defined, i.e. independent of N, and if $\eta \leq \xi < \alpha$, then $\mu_\eta \leq \mu_\xi$.

Proof .:.

Let $\eta < \xi < \alpha$, let $\eta = \tau^{\mathrm{L}_{\max(s)}[x]}[\beta, s^-]$, and let $\xi = \tau^{\mathrm{L}_{\max(s)}[x]}[\gamma, s^-]$. Moreover, let N and M be Σ-iterates of M_1 such that μ^N_η and μ^M_ξ are defined.

Let $\mathcal{R}$ be a common Σ-iterate of N and M such that $\mathcal{R}^-$ is countable in $\mathrm{L}[x, G]$ and let $i : N \to \mathcal{R}$ and $j : M \to \mathcal{R}$ be the iteration maps. Thus $i(\beta) = \beta$ and $j(\gamma) = \gamma$ as $\beta < \kappa^N$, $\gamma < \gamma^N$, and the iterations are non-dropping. Moreover, by our choice of N and M with ($\star$), $i(\eta) = \eta$ and $j(\xi) = \xi$—remember $\langle N^-, s\rangle, \langle M^-, s\rangle \in \mathcal{I}$—and therefore $i(\mu^N_\eta) = \mu^{\mathcal{R}}_\eta$ and $j(\mu^M_\xi) = \mu^{\mathcal{R}}_\xi$. But then

$$\eta \leq \xi \leftrightarrow \mu^{\mathcal{R}}_\eta \leq \mu^{\mathcal{R}}_\xi \leftrightarrow \mu_\eta \leq \mu_\xi,$$

as desired. ⊣

Claim 2 implies $\alpha < \delta_\infty$, as desired. ⊣

So this completes the first step of the proof of Our Goal (2A.2): we have our $\mathcal{M}_\infty$. The next goal is to go the other way around, and to show that this $\mathcal{M}_\infty$ is strong enough to compute (as much as possible of) $\mathrm{HOD}^{\mathrm{L}[x,G]}$.

Section 3. $\mathrm{L}[x, G]$ *versus* $\mathcal{M}_\infty[H]$

Since $\mathcal{F}$ is definable in $\mathrm{L}[x, G]$, we have that $\mathcal{M}_\infty \subseteq \mathrm{L}[x, G]$. For "$\supseteq$", we want to "compute" $\mathrm{HOD}^{\mathrm{L}[x,G]}$ inside $\mathcal{M}_\infty$ using "its derived model as a surogate for $\mathrm{L}[x, G]$". What we first do is collapse the first inaccesible above δ_∞ in $\mathcal{M}_\infty$, and see what we can do from there. This is where the genericity iterations come in. Using these genericity iterations, we can take points in our direct limit system, and we can iterate them a little to make $\mathrm{L}[x, G]$ the derived model. What we basically need for the next step is to make this more formal.

We will make this more precise now. Set κ_∞ as the least inaccessible of $\mathcal{M}_\infty$ strictly above δ_∞, and let H be $\mathrm{Col}(\omega_1, <\kappa_\infty)$-generic over $\mathcal{M}_\infty$ (note the homogeneity). What we want to do with this is compute the theory of $\mathrm{L}[x, G]$, because what we're actually interested in is $\mathrm{HOD}^{\mathrm{L}[x,G]}$, we don't need to compute everything, just what's definable. And by the homogeneity, it doesn't matter which generic we take.

Because we want to check where the ordinals in our direct system get mapped to in $\mathcal{M}_\infty$, let's introduce some notation. For $\alpha \in \mathrm{Ord}$, pick $\langle \mathcal{Q}, s\rangle \in \mathcal{I}^{\mathrm{L}[x,G]}$ such that $\alpha \in s$, which means that from this point on, we will not move the theory where we use α any more. Set $\alpha^* = \pi_{\langle \mathcal{Q},s\rangle,\infty}(\alpha)$. Note that α^* does not depend on the choice of $\langle \mathcal{Q}, s\rangle$, because we can look at a common iterate, which must move α to the same place. Let $s^* = \{\alpha^* : \alpha \in s\}$.

This * embedding will play a very important roll for the following lemma. We want to to compute $\mathrm{HOD}^{\mathrm{L}[x,G]}$, but we will not be showing that we can get $\mathrm{HOD}^{\mathrm{L}[x,G]}$ from an iteration strategy. Instead we will get $\mathrm{HOD}^{\mathrm{L}[x,G]}$ from this * embedding. And from there it's a small step to get to the iteration strategy. But to get there, we need to the following lemma.

3.1. Lemma (Derived Model Resemblance)

Let $\langle \mathcal{P}, s\rangle \in \mathcal{I}^{\mathrm{L}[x,G]}$, $\bar{\eta} < \gamma_s^{\mathcal{P}}$, and $\eta = \pi_{\langle \mathcal{P},s\rangle,\infty}(\bar{\eta})$. Let $t \in [\mathrm{Ord}]^{<\omega}$ and $\varphi(v_0, v_1, v_2)$ be a formula in the language of set theory. Therefore, the following are equivalent:

1. $\mathcal{M}_\infty[H] \vDash \varphi[\mathcal{M}_\infty^-, \eta, t^*]$; and
2. $\mathrm{L}[x, G] \vDash$"There is a countable pseudo-iterate $\mathcal{R}$ of $\mathcal{P}$ such that whenever $\mathcal{Q}$ is a acountable pseudo-iterate of $\mathcal{R}$ and I am a derived model of $\mathrm{L}[\mathcal{Q}]$, then $\varphi(\mathcal{Q}, \pi_{\langle \mathcal{P},s\rangle,\langle \mathcal{Q},s\rangle}(\bar{\eta}), t)$".

Proof .:.

First assume that (2) fails to show that (1) fails. We may assume$\mathcal{P} = N^-$ for some Σ-iterate N of M_1 such that s is respected for all further Σ_N-iterates.

Claim 1

here is a sequence $\langle N_k : k < \omega\rangle$ cofinal in $\mathcal{F}^\Sigma$ such that $N_0 = N$, and for each k, $\mathrm{L}[x, G]$ is a derived model of N_k and $\mathrm{L}[x, G] \vDash \neg\varphi[N_k^-, \bar{\eta}_k, t]$, where $\bar{\eta}_k = \pi_{\langle N_0^-,s\rangle,\langle N_k^-,s\rangle}(\bar{\eta})$ (i.e. the image of $\bar{\eta}$ under the Σ-iteration map).

Proof .:.

In V, let $\langle \mathcal{S}_k : k < \omega\rangle$ iterate all suitable $\mathcal{S} \in \mathcal{F}^\Sigma$, proper class models. Suppose N_k is given. Now coiterate N_k with $\mathcal{S}_k$ to some W. Let $\mathcal{R} = W^-$.

Note that (2) fails for $\mathcal{R}$ if there is some $\mathcal{Q}$ which is a countable, pseudo-iterate of $\mathcal{R}$ and has $\mathrm{L}[x, G]$ as its derived mode despite $\neg\varphi(\mathcal{Q}, \cdots)$. So let $\mathcal{Q}$ witness the failure of (2) for $\mathcal{R}$, and take $N_{k+1} = \mathrm{L}[\mathcal{Q}]$, a Σ-iterate of W. ⊣

Let κ_{N_k} denote the least inaccessible of N_k above δ^{N_k}. Since $\mathrm{L}[x, G]$ is a derived model of N_k, we have

$$N_k \vDash \left(\mathbb{1} \Vdash_{\mathrm{Col}(\omega,<\kappa_{N_k})} \neg\varphi(N_k^-, \bar{\eta}_k, t)\right).$$

Let $j_{k,\ell} : N_k \to N_\ell$ be the iteration map, and $\mathcal{M}_\infty^+$ the direct limit. Then by elementarity, for all sufficiently large k,

$$\mathcal{M}_\infty^+ \vDash \left(\mathbb{1} \Vdash_{\mathrm{Col}(\omega,<\kappa_{\mathcal{M}_\infty^+})} \neg\varphi(\mathcal{M}_\infty^-, j_{k,\omega}(\bar{\eta}_k), j_{k,\omega}(t))\right).$$

Since Σ_N—the tail strategy after N—respects s, $j_{k,\omega}(\bar{\eta}_k) = \pi_{\langle \mathcal{P},s\rangle,\infty}(\bar{\eta})$ (recall that $\mathcal{P} = N^-$, $\bar{\eta}_k = \pi_{\langle N_0^-,s\rangle,\langle N_k^-,s\rangle}(\bar{\eta})$, and $\bar{\eta} < \gamma_s^{\mathcal{P}}$).

Let k be large enough such that $j_{k^*,k^*+1}(s) = s$ for all $k^* \geq k$ and let $\sigma : \mathcal{M}_\infty \to \mathcal{M}_\infty^+$ be the map defined earlier in the proof of Lemma 2C.2 so that $\sigma \restriction (\delta_\infty) = \mathrm{id}$ and $\sigma(t^*) = j_{k,\omega}(t)$. Thus pulling back under σ, we get

$$\mathcal{M}_\infty^+ \vDash \left(\mathbb{1} \Vdash_{\mathrm{Col}(\omega,<\kappa_\infty)} \neg\varphi(\mathcal{M}_\infty^-, \eta, t^*)\right),$$

in other words, the failure of (1), as desired.

The same argument—now using that in $\mathrm{L}[x, G]$, every countable, suitable $\mathcal{R}$ has a countable, suitable, pseudo-iterate $\mathcal{Q}$ over which x is generic—shows that (2) implies (1). To get $\mathrm{L}[x, G]$ as a derived model, see the earlier Corollary 2B.13, and then we can apply (2) to get $\varphi(\mathcal{Q}, \cdots)$. ⊣.

This is a very useful theorem, because it allows us to compute $\mathrm{HOD}^{\mathrm{L}[x,G]}$. In essence, Derived Model Resemblance (3.1) tells us if we want to compute the theory of $\mathrm{L}[x, G]$, we can just look at what $\mathcal{M}_\infty[H]$ does. But $\mathcal{M}_\infty$, by considering its forcing relation, can already tell us what happens there. This is the key idea behind the lemma.

3.2. Corollary

For any $\alpha_1, \cdots, \alpha_n \in \mathrm{Ord}$, and formula ψ, $\mathrm{L}[x, G] \vDash \psi[\alpha_1, \cdots, \alpha_n]$ iff $\mathcal{M}_\infty[H] \vDash \psi[\alpha_1^*, \cdots \alpha_n^*]$.

Now that we have this, we get the first computation of $\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[\mathcal{M}_\infty, F]$. Giving the * function a proper name, F, we get the following corollary. And this is the first key step. After this, there are some interesting ideas, but mostly it's working to smooth out F to show we can replace it with other parameters that look a bit nicer. What makes this a key step is that $\mathrm{HOD}^{\mathrm{L}[x,G]}$ is given by a small object: $\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[\mathcal{M}_\infty, F \restriction \delta_\infty]$.

If we restrict ourselves to considering all of F, then we're dealing with a huge object: the way F has been defined, F works on ordinals (or finite sequences at least). So the interesting step is that we don't need all of F: if we take a small segment, we get the same result. The argument for the corollary is nice, and other computations of HOD build on it. For instance, if you want to show that it's $\mathrm{L}[\mathcal{M}_\infty, \Sigma]$, what you need to do is show that you can replace $F \restriction \delta_\infty$ by Σ.

3.3. Corollary

Let $F(s) = s^*$. Therefore $\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[\mathcal{M}_\infty, F] = \mathrm{L}\left[\mathcal{M}_\infty, F \restriction \omega_2^{\mathrm{L}[x,G]}\right]$.

Proof .:.

$\mathcal{M}_\infty$ and F are definable (meaning ordinal definable here) over $\mathrm{L}[x,G]$, so we have "$\supseteq$". For the other direction, we show a different characterization of HOD.

Claim 1

$\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[A]$ for some $\mathrm{L}[x,G]$-definable $A \subseteq \omega_2^{\mathrm{L}x,G]}$.

Proof .:.

Let $\mathbb{V}$ be the Vopěnka algebra of $\mathrm{L}[x,G]$ for adding a real to $\mathrm{HOD}^{\mathrm{L}[x,G]}$. The conditions are (ordinal codings of) OD sets of reals, so $|\mathbb{V}| \leq \omega_2^{\mathrm{L}[x,G]}$. Then x is $\mathbb{V}$-generic over $\mathrm{HOD}^{\mathrm{L}[x,G]}$ and $\mathrm{HOD}^{\mathrm{L}[x,G]}[x] = \mathrm{L}[x]$, which is $\mathrm{L}[x]^{\mathrm{L}[x,G]}$.

Let $\mathbb{P} = \mathbb{V} \times \mathrm{Col}(\omega, < \kappa_x)$. Then $\mathbb{P}$ is definable in $\mathrm{L}[x,G]$ and $|\mathbb{P}| \leq \omega_2^{\mathrm{L}[x,G]}$. So it suffices to show $\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[\mathbb{P}]$. Let $a \in \mathrm{HOD}^{\mathrm{L}[x,G]}$ be a set of ordinals (we can code everything into sets of ordinals so this suffices). Note that $\mathrm{L}[\mathbb{P}] \subseteq \mathrm{HOD}^{\mathrm{L}[x,G]} \subseteq \mathrm{L}[x,G]$ and $\langle x, G\rangle$ is $\mathbb{P}$-generic over both $\mathrm{L}[\mathbb{P}]$ and $\mathrm{HOD}^{\mathrm{L}[x,G]}$. So there is some $\tau \in \mathrm{L}[\mathbb{P}]$ such that $\tau_{\langle x,G\rangle} = a \in \mathrm{L}[\mathbb{P}][\langle x,G\rangle]$.

This is forced over $\mathrm{HOD}^{\mathrm{L}[x,G]}$, i.e. there is some $p \in \mathbb{P}$ such that $\mathrm{HOD}^{\mathrm{L}[x,G]} \models (p \Vdash_{\mathbb{P}} \tau = \check{a})$. Note that this only makes sense in $\mathrm{HOD}^{\mathrm{L}[x,G]}$ since $a \in \mathrm{HOD}^{\mathrm{L}[x,G]}$ and we want the standard name for a. Yet $\mathrm{L}[\mathbb{P}]$ can compute a by consulting the forcing relation for $\mathbb{P}$ below p. For this, we don't need a name for a, just check what τ is. ⊣

This implies the corollary as follows. Say $A \subseteq \omega_2^{\mathrm{L}[x,G]}$ is defined by φ, meaning $\xi \in A$ iff $\mathrm{L}[x,G] \models \varphi(\xi)$. By Derived Model Resemblance (3.1) or Corollary 3.2, $\mathcal{M}_\infty \models \left(\mathbb{1} \Vdash_{\mathrm{Col}(\omega,<\kappa_\infty)} \varphi(\xi^*)\right)$ where $F(\xi) = \xi^*$. So $A \in \mathrm{L}\left[\mathcal{M}_\infty, F \restriction \omega_2^{\mathrm{L}[x,G]}\right]$, and $\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[A] \subseteq \mathrm{L}[\mathcal{M}_\infty, F \restriction \omega_2^{\mathrm{L}[x,G]}]$, as desired. ⊣

By Derived Model Resemblance (3.1), $\mathcal{M}_\infty[H]$ is elementarily equivalent to $\mathrm{L}[x,G]$. So it makes sense to consider $\mathcal{F}^* = \mathcal{F}^{\mathcal{M}_\infty[H]}$ and $\mathcal{M}^*_\infty = (\mathcal{M}_\infty)^{\mathcal{M}_\infty[H]}$. As before, $(\mathcal{M}^*_\infty)^-$ is the direct limit of $\mathcal{F}^*$. Derived Model Resemblance (3.1) then yields the following exercise.

Exercise 2

For any $s \in [\mathrm{Ord}]^{<\omega}$, $\mathcal{M}_\infty[H] \models$ "M^-_∞ is strongly s^*-iterable".
Hint: apply it to $\varphi(v_0, v_1, v_2)$ saying "v_0 is v_2-iterable".

Now define the following, giving rise to the next exercise:

$$\pi_\infty = \bigcup \left\{ \pi^{\mathcal{F}^*}_{\langle \mathcal{M}^-_\infty, s^*\rangle, \infty} : s \in [\mathrm{Ord}]^{<\omega} \right\}.$$

Exercise 3

1. $\mathcal{M}_\infty = \bigcup\{H_{s^*}^{\mathcal{M}_\infty^-} : s \in [\mathrm{Ord}]^{<\omega}\}$.
2. $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^*$.
3. For all $\eta < \delta_\infty$, $\pi_\infty(\eta) = \eta^*$.

For (3) apply Derived Model Resemblance (3.1) to $\varphi(v_0, v_1, v_2, v_3)$ saying

$$\text{“}\langle v_0, v_1\rangle \in \mathcal{I} \text{ and } v_2 < \gamma_{v_1}^{v_0}, \text{ and } \pi_{\langle v_0,v_1\rangle,\infty}(v_2) = v_3\text{”}.$$

3.4. Lemma

1. $\mathrm{V}_{\delta_\infty}^{\mathrm{HOD}^{\mathrm{L}[x,G]}} = \mathrm{V}_{\delta_\infty}^{\mathcal{M}_\infty}$; and
2. $\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[\mathcal{M}_\infty, \pi_\infty \restriction \delta_\infty]$.

Proof .:.

(2) follows directly from (3) of Exercise 2 and Corollary 3.3. For (1), note that $\pi_\infty \restriction \alpha \in \mathcal{M}_\infty$ for all $\alpha < \delta_\infty$, because for $\alpha < \delta_\infty$ there is an s such that $\alpha < \gamma_{s^*}^{\mathcal{M}_\infty^-}$ and for such an α,

$$\pi_\infty \restriction (\alpha + 1) = \pi^{\mathcal{F}^*}_{\langle \mathcal{M}_\infty^-, s^*\rangle,\infty} \restriction (\alpha + 1).$$

Let $\alpha < \delta_\infty$ and let $A \subseteq \alpha$ be defined over $\mathrm{L}[x, G]$ by φ and γ. In other words,

$$\begin{aligned} \beta \in A \quad &\text{iff} \quad \mathrm{L}[x, G] \vDash \varphi[\beta, \gamma] \\ &\text{iff} \quad \mathcal{M}_\infty[H] \vDash \varphi[\beta^*, \gamma^*] \quad \text{by Derived Model Resemblance (3.1)} \\ &\text{iff} \quad \mathcal{M}_\infty[H] \vDash \varphi[\pi_\infty(\beta), \gamma^*]. \end{aligned}$$

Since $\pi_\infty \restriction \alpha \in \mathcal{M}_\infty$, we have $A \in \mathcal{M}_\infty$ (γ^* is just an ordinal parameter, and H is generic for a homogeneous forcing). ⊣

We can use this to get another presentation of HOD via a strategy for $\mathcal{M}_\infty$. Let Λ be the restriction of Σ_0 to finite full stacks $\vec{\mathcal{T}}$ on $\mathcal{M}_\infty$ such that $\vec{\mathcal{T}} \in \mathcal{M}_\infty \mid \kappa_\infty$.

3.5. Theorem (Woodin)

$\mathrm{HOD}^{\mathrm{L}[x,G]} = \mathrm{L}[\mathcal{M}_\infty, \pi_\infty \restriction \delta_\infty] = \mathrm{L}[\mathcal{M}_\infty, \Lambda]$.