DETERMINACY AND SCALES

MY NOTES ON TREVOR WILSON'S LECTURES

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CAUTION: these notes were typed up during the lectures, and so are probably full of typos along with misheard or misread parts (and perhaps genuine misunderstandings).

Section 1. *Suslin Sets and Scales*

Recall that $\mathbb{R} = \mathcal{N} = \omega^\omega$ is baire space. Everything we're doing will be within $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$. Recall the definition of suslin sets. Moreover, when writing Σ_n or Π_n, the language is $\{\in, \dot{\mathbb{R}}\}$, where $\dot{\mathbb{R}}$ is a constant symbol for the reals.

1 • 1. Definition 1

A set $A \subseteq \mathbb{R}$ is κ-*suslin* iff $A = p[T]$ for some tree T on $\omega \times \kappa$, i.e. $T \subseteq \bigcup_{n\in\omega} \omega^n \times \kappa^n$. Note that $[T]$ is the set of branches of T, i.e. $[T] \subseteq \omega^\omega \times \kappa^\omega$, and $p[T]$ is the projection into $\mathbb{R}$.

A set $A \subseteq \mathbb{R}$ is *suslin* iff it is κ-suslin for some κ.

For example, $\undertilde{\mathbf{\Sigma}}^1_1$ is equivalent to ω-suslin, practically by definition. $\undertilde{\mathbf{\Pi}}^1_1$ are ω_1-suslin.

Now let's say what scales are, a definition due to Moschovakis. The main point is to get uniformization results. The notion is useful because it can be propogated using second periodicity.

1 • 2. Definition 2

A *scale* on A is a sequence $\langle \varphi_n : n \in \omega \rangle$ where

(a) $\varphi_n : A \to \mathsf{Ord}$ for each n; and

(b) for all $x_0, x_1, \cdots \in A$ if $\lim_{i\to\omega} x_i = x \in \mathbb{R}$ and $\lim_{i\to\omega} \varphi_n(x_i) = \lambda_n \in \mathsf{Ord}$ (i.e. is eventually constant),

1. $x \in A$; and
2. $\varphi_n(x) \leq \lambda_n$ for all n (lower semi-continuity).

If just (1) holds, then $\vec{\varphi}$ is called a *semi-scale*.

Exercise 1

Suppose $A = p[T]$ with T on $\omega \times \kappa$ and $\varphi_n(x) = \ell_x(n)$, where for $x \in A$, $\ell_x(n)$ is the $<_{\text{lex}}$-least $f \in \kappa^\omega$ such that $\langle x, f \rangle \in [T]$. Therefore $\langle \varphi_n : n \in \omega \rangle$ is a semi-scale on A.

If $\psi_n(x)$ is defined to be equal to the (lexiographic rank of) $\langle \ell_x(0) \cdots \ell_x(n) \rangle$, then $\langle \psi_n : n \in \omega \rangle$ is a scale on A.

You can also go in the other direction.

Exercise 2

If $\langle \varphi_n : n < \omega \rangle$ is a semi-scale on A and

$$T = \left\{ \langle x \restriction n, \langle \varphi_0(x), \cdots, \varphi_{n-1}(x) \rangle \rangle : n < \omega \wedge x \in A \right\}.$$

Therefore $A = p[T]$.

So we have the equivalence of being suslin, having a scale, and having a semi-scale. So it's important the kind of complexity of the scales. And this is the next point: how do you measure the complexity of a scale?

[The following definition takes place in lecture 3.] Now let's properly introduce the product space (mostly we just think of this as the reals, but sometimes we must work in greater generality).

1 • 3. Definition 3

A *product space* is $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_n$ where each X_i is ω^ω $(= \mathbb{R})$ or ω.
A *pointset* is a subset A of a product space X.
A *pointclass* is a set of pointsets, e.g. Π^1_1 or $\Sigma_n^{J_\beta(\mathbb{R})}$)

Now if you have a product space X where each $\mathrm{X}_i = \omega$—and thus X is countable—then $A \subseteq \mathrm{X}$ is just called a "real", somewhat confusingly. [Now we return to lecture 1.]

Don't worry too much about this next definition, since it'll apply to everything we'll be working with.

1 • 4. Definition 4

A point-class Γ is *adequate* iff it contains all recursive point sets and it's closed under $\wedge$, $\vee$, and bounded number quantification.

For example, this'll include Σ^1_n, Π^1_n, and their bold-face counterparts.

1 • 5. Definition 5

An adequate Γ has the *scale property* ($\mathsf{Scale}(\Gamma)$) iff every $A \in \Gamma$ has a Γ-scale:

$$\{\langle n, x, y\rangle : x \in A \wedge y \in A \to \varphi_n(x) \leq \varphi_n(y)\} \in \Gamma$$
$$\{\langle n, x, y\rangle : x \in A \wedge y \in A \to \varphi_n(x) < \varphi_n(y)\} \in \Gamma$$

In essence, the complexity of the scale is at most that of Γ. And the scale property is a kind of closure property.

1 • 6. Theorem 1 (Scale(Π^1_1))

Scale(Π^1_1) holds.

Proof .:.

Enough to show that WO (the set of reals coding well-orderings of ω) has a Π^1_1-scale. For this, you can define for each $x \in \mathrm{WO}$, $\varphi_n(x)$ to be the rank of n in the well-ordering coded by x. It's an exercise to show that $\vec{\varphi}$ defined this way is a Π^1_1-scale. ⊣

So a corollary of this (due to Addison), which came before the notion of scale was isolated, is uniformization.

1 • 7. Corollary 1

$\mathsf{Unif}(\Pi^1_1)$: Every Π^1_1 binary relation contains a function whose graph is Π^1_1, and who has the same domain.
A further corollary is that this relativizes to any particular real parameter, and hence $\mathsf{Unif}(\boldsymbol{\Pi}^1_1)$.

§ 1.A. Propogation of scales

The following due to Moschovakis is useful.

1.A • 1. Theorem 2

Assume the following
1. Γ is adequate and $\Sigma^0_1 \subseteq \Gamma$; and
2. $\forall^{\mathbb{R}}\Gamma \subseteq \Gamma$, i.e. Γ is closed under universal quantification over reals.

Therefore $\mathsf{Scale}(\Gamma)$ implies $\mathsf{Scale}(\exists^{\mathbb{R}}\Gamma)$.

For example, if $\Gamma = \Pi^1_1$, then the theorem tells us that $\mathsf{Scale}(\Sigma^1_2)$.

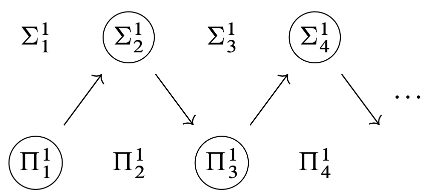
1.A • 2. Theorem 3 (2nd Periodicity)

Assume
1. Γ is adequate and $\Sigma^0_1 \subseteq \Gamma$; and
2. $\exists^{\mathbb{R}}\Gamma \subseteq \Gamma$, i.e. Γ is closed under existential quantification over reals.
3. $\mathrm{Det}(\underset{\sim}{\mathbf{\Gamma}} \cap \check{\underset{\sim}{\mathbf{\Gamma}}})$, where $\check{\underset{\sim}{\mathbf{\Gamma}}}$ is the dual pointclass: the complements of sets in $\underset{\sim}{\mathbf{\Gamma}}$.

Therefore $\mathsf{Scale}(\Gamma)$ implies $\mathsf{Scale}(\forall^{\mathbb{R}}\Gamma)$.

To get $\underset{\sim}{\mathbf{\Gamma}}$ from Γ, you allow parameters in whatever definitions are allowed in Γ. To truly define $\underset{\sim}{\mathbf{\Gamma}}$ in general, you'd need to take all the pre-images of continuous functions [or something, I didn't quite catch what was said here, but he says that he'll just take it to mean allowing parameters when Γ is a notion of definability without parameters].

For example, if $\Gamma = \Sigma^1_2$, then $\mathrm{Det}(\underset{\sim}{\mathbf{\Delta}}^1_2)$ yields the scale property on Π^1_3. And to continue this pattern, if you assume all of projective determinacy, PD, then you get the scale property for the following (circled) point classes.



1.A • 3. Definition 6

A set A is *suslin co-suslin* (ScS) iff A and its complement are both suslin.

With this terminology, we can rephrase the previous example. Previously the hypothesis was determinacy, and the consequence was scales. But we can (roughly) have it the other way around: $\mathsf{Det}(\mathrm{ScS})$ implies PD.

§ 1.B. Determinacy

The next goal is to outline the proof of the following theorem of Kechris and Woodin.

1.B • 1. Theorem 4

Assume $\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$. Therefore $\mathsf{Det}(\mathrm{ScS})$ implies $\mathsf{AD}^{L(\mathbb{R})}$.

Proof Outline .:.

Note $\mathsf{ZF} + \mathsf{DC} +$ cofinally many $\kappa < \Theta$[i] with the strong partition property implies $\mathsf{Det}(\mathrm{Suslin})$, which is due to Kechris–Kleinberg–Moschovakis–Woodin.

How do we show this? You need to work globally, not locally. And rather than Gödel's hierarchy of $L_\alpha(\mathbb{R})$s, we

[i] Θ is the least ordinal without a surjection from $\mathbb{R}$ to Θ.

should use Jensen's hierarchy of $J_\alpha(\mathbb{R})$s: $J_1(\mathbb{R}) = V_{\omega+1}$, with $J_{\alpha+1}(\mathbb{R})$ as the rud-closure of $J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\}$ with unions at limits.

Recall: a Σ_1-gap (in $L(\mathbb{R})$) is a maximal interval $[\alpha, \beta] \subseteq \mathsf{Ord}$ such that $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$, i.e. Σ_1-elementary with parameters in $\mathbb{R}$ (possibly $\alpha = \beta$). Note the Σ_1-gaps partition Ord. α begins a Σ_1-gap iff new Σ_1-facts about reals are witnessed cofinally often in the hierarchy below $J_\alpha(\mathbb{R})$.

Because of this, and using a bit of fine structure, we have the following fact. If α begins a gap, then there is a $\Sigma_1^{J_\alpha(\mathbb{R})}$ (light-face) partial surjection from $\mathbb{R}$ onto $J_\alpha(\mathbb{R})$. And the converse also holds, quite easily.

Assume towards a contradiction that $\mathsf{Det}(\text{ScS})$ holds, but $\neg\mathsf{AD}^{L(\mathbb{R})}$. Let β be the least such that $J_{\beta+1}(\mathbb{R}) \models \neg\mathsf{AD}$ (the first counter example happens at a successor stage). Note that $\beta+1$ begins a Σ_1-gap, since the failure of AD is a Σ_1 statement, and $\neg\mathsf{AD}$ doesn't hold below $J_{\beta+1}(\mathbb{R})$. Let $[\alpha, \beta]$ be the previous Σ_1-gap. So $J_\alpha(\mathbb{R}) \models \mathsf{AD}$ (and again, we might have $\alpha = \beta$).

Note: because PD holds, (and we started counting the $J(\mathbb{R})$ hierarchy at 1 with $J_1(\mathbb{R}) = V_{\omega+1}$) $J_2(\mathbb{R}) \models \mathsf{AD}$, so $\alpha > 1$. This is just because if $\alpha = 1$ then $\beta = 1$, so $J_{\beta+1}(\mathbb{R}) = J_2(\mathbb{R}) \models \mathsf{AD}$. Now the first step is to apply the following theorem of Steel.

1.B • 2. Theorem 5

If $\alpha > 1$ and $J_\alpha(\mathbb{R}) \models \mathsf{AD}$, then $\Sigma_1^{J_\alpha(\mathbb{R})}(\mathbb{R})$ (i.e. Σ_1 over $J_\alpha(\mathbb{R})$ with real parameters) has the scale property

Proof .:.

The proof uses closed game representations $z \mapsto G_z$ for A in $\Sigma_1^{J_\alpha(\mathbb{R})}(\mathbb{R})$. Here G_z is a closed game continuously associated to z:

$$\begin{array}{llll} \text{I}: x_0 \in \mathbb{R}, \beta_0 \in \mathsf{Ord} & & x_1, \beta_1 & \cdots \\ \text{II}: & y_0 \in \mathbb{R} & & y_1 \cdots \end{array}$$

where $z \in A$ iff I has a WQS [winning quasi-strategy(?)] in G_z.

We also have the following useful fact. If α beings a Σ_1-gap, then the two notions $\utilde{\boldsymbol{\Sigma}}_1^{J_\alpha(\mathbb{R})}$ and $\Sigma_1^{J_\alpha(\mathbb{R})}(\mathbb{R})$ are the same. The proof of this uses the Σ_1 partial surjection from $\mathbb{R}$ to $J_\alpha(\mathbb{R})$.

So $\utilde{\boldsymbol{\Sigma}}_1^{J_\alpha(\mathbb{R})}$ has the scale property. To show the ultimate goal of $\mathsf{Det}(\text{ScS}) \Rightarrow J_{\beta+1}(\mathbb{R}) \models \mathsf{AD}$, we need to construct more scales. This may or may not be possible by periodicity. The case where we can do this is the inadmissible case.

The inadmissible case is when $J_\alpha(\mathbb{R}) \not\models \Sigma_0$-collection. Furthermore, if $J_\alpha(\mathbb{R})$ is inadmissible, then $\alpha = \beta$ (a trivial gap). Also, $\forall^{\mathbb{R}} \utilde{\boldsymbol{\Sigma}}_1^{J_\alpha(\mathbb{R})} = \utilde{\boldsymbol{\Pi}}_2^{J_\alpha(\mathbb{R})}$ (do this as an exercise, using a total Σ_1 function from some X to $J_\alpha(\mathbb{R})$ which is cofinal). And similarly, we can show $\exists^{\mathbb{R}} \utilde{\boldsymbol{\Pi}}_2^{J_\alpha(\mathbb{R})} = \utilde{\boldsymbol{\Sigma}}_3^{J_\alpha(\mathbb{R})}$, and so forth. Then by 2nd Periodicity (1.A • 2), we get two different things: $\mathsf{Scale}(\utilde{\boldsymbol{\Pi}}_2^{J_\alpha(\mathbb{R})})$, $\mathsf{Scale}(\utilde{\boldsymbol{\Sigma}}_3^{J_\alpha(\mathbb{R})})$, etc.; and $\mathsf{Det}\big(\bigcup_{n\in\omega} \utilde{\boldsymbol{\Sigma}}_n^{J_\alpha(\mathbb{R})}\big)$, which is just the same as $J_{\alpha+1}(\mathbb{R}) \models \mathsf{AD}$. This is because we're taking definable subsets to get to the next level. Since $\alpha = \beta$, we get our contradiction in this case.

The admissible case is when $J_\alpha(\mathbb{R}) \models \Sigma_0$ collection. Here we can have $\alpha = \beta$ or $\alpha < \beta$, but what we do will work for both cases equally. Now if we try to go through the levels in the same way, already at the first stage we get stuck: $\forall^{\mathbb{R}} \utilde{\boldsymbol{\Sigma}}_1^{J_\alpha(\mathbb{R})} = \utilde{\boldsymbol{\Sigma}}_1^{J_\alpha(\mathbb{R})}$. So how do we generate new scale point clases? 2nd Periodicity (1.A • 2) is not immediately useful. [end of lecture 1]

So far, we have in the projective case, $\alpha = 1$. In the inadmissible case, $\alpha > 1$ and $J_\alpha(\mathbb{R}) \not\models \Sigma_0$-collection. In both of these cases, we have $\alpha = \beta$ and we used 2nd Periodicity (1.A • 2) to show that every set of reals in

$J_{\beta+1}(\mathbb{R})$ is suslin and therefore also co-suslin, and so determined. What's left is the admissible case.

So we know $J_\alpha(\mathbb{R}) \vDash \Sigma_0$-collection, and again, $\utilde{\Sigma}_1^{J_\alpha(\mathbb{R})}$ has the scale property. The difficulty here is that $\utilde{\Sigma}_1^{J_\alpha(\mathbb{R})}$ is closed under both real quantifiers. However, we can define a projective-like hierarchy at the end of the gap $[\alpha, \beta]$.

Define n to be the least such that $\utilde{\Sigma}_n^{J_\beta(\mathbb{R})} \cap \wp(\mathbb{R}) \not\subseteq J_\beta(\mathbb{R})$. Such an n exists, since some counter example to AD is newly added over $J_\beta(\mathbb{R})$. There is a $\utilde{\Sigma}_n^{J_\beta(\mathbb{R})}$ partial surjection from $\mathbb{R}$ to $J_\beta(\mathbb{R})$. As an exercise, show

$$\utilde{\Pi}_{n+1}^{J_\beta(\mathbb{R})} = \forall^{\mathbb{R}} \left(\utilde{\Sigma}_n^{J_{\beta+1}(\mathbb{R})}(\mathbb{R}) \vee \utilde{\Pi}_n^{J_{\beta+1}(\mathbb{R})}(\mathbb{R}) \right).$$

Usually, we just have $\utilde{\Sigma}_{n+1}^{J_\beta(\mathbb{R})} = \exists^{\mathbb{R}} \utilde{\Pi}_{n+1}^{J_\beta(\mathbb{R})}$ and so forth. These pointclasses, however, may or may not have the scale property (depending on propertes of the gap, and in particular on properties of $\utilde{\Sigma}_n^{J_\beta(\mathbb{R})}$). Now we have the following definition due to Steel.

1.B • 3. Definition 7

> Assume $[\alpha, \beta]$ is a Σ_1-gap with $J_\alpha(\mathbb{R}) \vDash \Sigma_0$-collection, and n is the least where $\utilde{\Sigma}_n^{J_\beta(\mathbb{R})} \cap \wp(\mathbb{R}) \not\subseteq J_\beta(\mathbb{R})$.
> $[\alpha, \beta]$ is a *strong gap* iff every Σ_n-type (i.e. set of Σ_n-formulas) realized in $J_\beta(\mathbb{R})$ is realized in $J_\gamma(\mathbb{R})$ for some $\gamma < \beta$ (and therefore for some $\gamma < \alpha$ by definition of a gap: $J_\alpha(\mathbb{R}) \prec_1 J_\beta(\mathbb{R})$.
> $[\alpha, \beta]$ is a *weak gap* iff it is not a strong gap.

Note that the language here is the language of set theory with a constant symbol for $\mathbb{R}$: $\{\in, R\}$. So strong and weak gaps are sub-cases of the admissible case. As an exercise, assume $\alpha = \beta$ with $[\alpha, \alpha]$ a Σ_1-gap, and $J_\alpha(\mathbb{R}) \vDash \Sigma_0$-collection, and show that $[\alpha, \alpha]$ is a strong gap (with $n = 1$).

Case 1: $[\alpha, \beta]$ is a weak gap. The following is a theorem of Steel.

1.B • 4. Theorem 6

> Let α, β, and n be as in Definition 7 (1.B • 3). Assume $J_\alpha(\mathbb{R}) \vDash$ AD. Therefore
> 1. $\utilde{\Sigma}_n^{J_\beta(\mathbb{R})}$ has the scale property; and
> 2. $\utilde{\Sigma}_n^{J_\beta(\mathbb{R})} \cap \wp(\mathbb{R}) = \bigcup^\omega (J_\beta(\mathbb{R}) \cap \wp(\mathbb{R}))$, where $\bigcup^\omega$ represents countable unions.

The proof of this theorem supposes that $J_\beta(\mathbb{R}) = \bigcup_{i<\omega} H_i$, where $H_i \prec_{n-1} J_\beta(\mathbb{R})$. And then we have closed game representations for all these $\utilde{\Sigma}_n^{H_i}$, $i < \omega$ simultaneously.

And as another exercise, using Theorem 6 (1.B • 4) (2), show that $\utilde{\Pi}_{n+1}^{J_\beta(\mathbb{R})} = \forall^{\mathbb{R}} \utilde{\Sigma}_n^{J_\beta(\mathbb{R})}$ (by simplifying $\forall^{\mathbb{R}} \left(\utilde{\Sigma}_n^{J_\beta(\mathbb{R})}(\mathbb{R}) \vee \utilde{\Pi}_n^{J_\beta(\mathbb{R})}(\mathbb{R}) \right)$), and $\utilde{\Sigma}_{n+2}^{J_\beta(\mathbb{R})} = \exists^{\mathbb{R}} \utilde{\Pi}_{n+1}^{J_\beta(\mathbb{R})}$. Note that we get further results. So these propogate the scale property by Theorem 6 (1.B • 4) (1), 2nd Periodicity (1.A • 2), and Det(ScS). So these point classes all have the scale property, and so are determined. Hence $J_{\beta+1} \vDash$ AD, and this completes the weak-gap case.

Case 2: $[\alpha, \beta]$ is a strong gap. We need something entirely different here. In this case, there is no appropriate scale construction. You don't show determinacy using scales as in the previous case. Instead, consider the following theorem of Kechris and Woodin, which gives exactly what we want, and completes the proof of $\mathsf{AD}^{L(\mathbb{R})}$.

1.B • 5. Theorem 7 (Determinacy Transfer)

> Let $[\alpha, \beta]$ be a strong gap. Therefore $J_\alpha(\mathbb{R}) \vDash$ AD implies $J_{\beta+1}(\mathbb{R}) \vDash$ AD.

Again, the proof of this does not use scales [the theorem is proven in lecture 3]. So we finally have all the cases confirmed. ⊣

We also have a related theorem of Martin, which must be first introduced with a definition.

1.B • 6. Definition 8

For $x \in \mathbb{R}$, we say $x \in \mathrm{OD}^{<\gamma}$ iff for some $\gamma_0 < \gamma$, x is first-order definable over $J_{\gamma_0}(\mathbb{R})$ with ordinal parameters.

1.B • 7. Theorem 8 (OD Reflection)

Let $[\alpha, \beta]$ be a strong gap with n the least such that $\utilde{\boldsymbol{\Sigma}}_n^{J_\beta(\mathbb{R})} \cap \wp(\mathbb{R}) \not\subseteq J\beta(\mathbb{R})$. Assume $J_\alpha(\mathbb{R}) \models \mathsf{AD}$. Therefore for every $x \in \mathbb{R}$, $x \in \mathrm{OD}^{<\beta+1}$ implies $x \in \mathrm{OD}^{<\alpha}$.

If you can characterize the $\mathrm{OD}^{<\alpha}$ with mouse sets, then the ostensably more complicated reals in $\mathrm{OD}^{<\beta+1}$ are already in the mice. [end of lecture 2]

For this lecture, assume the following for today. All of this takes place within Case 2 of Theorem 4 (1.B • 1)

- $\mathsf{ZF} + \mathsf{DC}_\mathbb{R}$ (as usual)
- $[\alpha, \beta]$ is a Σ_1-gap in $L(\mathbb{R})$
- $J_\alpha(\mathbb{R}) \models \mathsf{AD} + \Sigma_0$-collection
- $[\alpha, \beta]$ is a strong gap. As a reminder, this means that every Σ_n-type realized in $J_\beta(\mathbb{R})$ is realized in some lower $J_\gamma(\mathbb{R})$, where $\gamma < \beta$ (equivalently $\gamma < \alpha$), and where n is the least number such that $\utilde{\boldsymbol{\Sigma}}_n^{J_\beta(\mathbb{R})} \not\subseteq J_\beta(\mathbb{R})$.

For now we want to show Determinacy Transfer (1.B • 5), i.e. $J_{\beta+1} \models \mathsf{AD}$, as well as OD Reflection (1.B • 7), i.e. every $\mathrm{OD}^{<\beta+1}$ real is $\mathrm{OD}^{<\alpha}$. One thing to point out is that if you consider β instead of $\beta+1$, i.e. that $\mathrm{OD}^{<\beta} \cap \mathbb{R} \subseteq \mathrm{OD}^{<\alpha}$, and $J_\beta \models \mathsf{AD}$, then these are automatic, following from the facts that $J_\alpha(\mathbb{R}) \prec_1 J_\beta(\mathbb{R})$, that $\neg\mathsf{AD}$ is Σ_1, and that membership in $\mathrm{OD}^{<\gamma}$ is $\Sigma_1^{J_\gamma(\mathbb{R})}$. Σ_n-type reflection is needed to get from β to $\beta+1$.

One thing we will need is something called the *envelope of* $\Sigma_1^{J_\alpha(\mathbb{R})}$, with a reference being Wilson's *The envelope of a pointclass under a local determinacy hypothesis*. Firstly, recall Definition 3 (1 • 3):

1.B • 8. Definition 9

A *product space* is $\mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_n$ where each X_i is ω^ω ($= \mathbb{R}$) or ω.
A *pointset* is a subset A of a product space X.
A *pointclass* is a set of pointsets, e.g. Π_1^1 or $\Sigma_n^{J_\beta(\mathbb{R})}$)

A characterization due to Martin will allow the definition of an envelope. Originally it was used in the context of full determinacy, and so it wasn't used to show OD Reflection (1.B • 7), but you can use it in this way.

1.B • 9. Definition 10

For a pointclass Γ and X a product space, $\overline{\Gamma}$ is defined as follows. For all $A \subseteq \mathrm{X}$, $A \in \overline{\Gamma}$ iff for every countable $\sigma \subseteq \mathrm{X}$, there is an $A' \subseteq \mathrm{X}$ in Γ with $A \cap \sigma = A' \cap \sigma$. Informally, A is countably approximated by pointsets in Γ in this sense.

As the notation might sugggest, $\Gamma \mapsto \overline{\Gamma}$ is a kind of closure operation. In other words, $\Gamma_1 \subseteq \Gamma_2 \to \overline{\Gamma}_1 \subseteq \overline{\Gamma}_2$, $\Gamma \subseteq \overline{\Gamma}$, and $\overline{\overline{\Gamma}} = \overline{\Gamma}$. Typically this is applied to Γ which is some local version of OD. Note the definition of $\mathrm{OD}^{<\gamma}$ also makes sense as the definition of a pointclass (applying to sets of reals as well).

Now we can define an envelope, although this terminology won't be used. We call $\overline{\mathrm{OD}^{<\alpha}}$ the *envelope* of $\Sigma_1^{J_\alpha(\mathbb{R})}$. Really we'll just work with $\overline{\mathrm{OD}^{<\alpha}}$ rather than this terminology. The *relativized* envelope is just $\overline{\mathrm{OD}^{<\alpha}(z)}$ where $z \in \mathbb{R}$. We also have the *boldface* envelope $\bigcup_{z\in\mathbb{R}} \overline{\mathrm{OD}^{<\alpha}(z)}$, which is sometimes called the envelope of $\utilde{\boldsymbol{\Sigma}}_1^{J_\alpha(\mathbb{R})}$, although our definition doesn't involve this pointclass.

Note the difference between $\bigcup_{z\in\mathbb{R}} \overline{\mathrm{OD}^{<\alpha}(z)}$ and $\overline{\bigcup_{z\in\mathbb{R}} \mathrm{OD}^{<\alpha}(z)}$, with the latter giving all pointsets, i.e. the full powerset. Note also that $\mathrm{OD}^{<\alpha}$ is closed under boolean combinations, which yields that $\overline{\mathrm{OD}^{<\alpha}}$ is too, mostly just because intersection with σ as in Definition 10 (1.B • 9) is closed under these. Under the additional assumptions we have, we can actually show that it's closed under real quantification (albeit harder to do). Furthermore, these properties hold for the relativized and boldface variants.

Now we get on with the proof of both OD Reflection (1.B • 7) and Determinacy Transfer (1.B • 5).

1.B • 10. Lemma 1

Assume the following for every $z \in \mathbb{R}$,

1. For every $s \in [\omega\beta]^{<\omega}$; $\Sigma_n^{J_\beta(\mathbb{R})}(s, z) \subseteq \overline{\mathrm{OD}^{<\alpha}(z)}$, using Σ_n-type reflection.
2. $\overline{\mathrm{OD}^{<\alpha}(z)}$ is closed under real quanitification $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ (or just one), using an idea of Martin;
3. $\mathsf{Det}(\overline{\mathrm{OD}^{<\alpha}(z)})$ (we already know $\mathsf{Det}(\mathrm{OD}^{<\alpha}(z))$), using an idea of Kechris and Woodin.

Therefore both OD Reflection (1.B • 7) and Determinacy Transfer (1.B • 5) hold.

Proof .:.

To need to understand the projective hierarchy at the end of the gap a bit better. We need a light-face version of one of the exercises from last time.

1.B • 11. Exercise 1

For every $z \in \mathbb{R}$ and sufficiently large (with respect to $\subseteq$) finite set $s \in [\omega\beta]^{<\omega}$, we have that

$$\Pi_{n+1}^{J_\beta(\mathbb{R})}(s, z) = \forall^{\mathbb{R}} \left(\Sigma_n^{J_\beta(\mathbb{R})}(s, z) \vee \Pi_n^{J_\beta(\mathbb{R})}(s, z)\right)$$

$$\Sigma_{n+2}^{J_\beta(\mathbb{R})}(s, z) = \exists^{\mathbb{R}} \Pi_{n+1}^{J_\beta(\mathbb{R})}(s, z), \text{ and so on.}$$

In essense, we want a complicated s where a partial surjection $\mathbb{R} \to J_\beta(\mathbb{R})$ is $\Sigma_n^{J_\beta(\mathbb{R})}(s, w)$ for some $w \in \mathbb{R}$.

Claim 1 (OD Reflection)

For every $x \in \mathbb{R}$, $x \in \mathrm{OD}^{<\beta+1}$ implies $x \in \mathrm{OD}^{<\alpha}$.

Proof .:.

By (1), the exercise, and (2) with $z = \emptyset$, we get that $\mathrm{OD}^{<\beta+} \subseteq \overline{\mathrm{OD}^{<\alpha}}$ [cf. roughly 54 minutes in the recording for an explanation]. Now for A a "real"—meaning here a subset of a countable product space X—we have $A \in \overline{\mathrm{OD}^{<\alpha}}$ iff $A \in \mathrm{OD}^{<\alpha}$. This is just because for countable A, a countable approximation tells you the whole thing. So each $x \in \mathrm{OD}^{<\beta+1} \cap \mathbb{R}$ is in $\mathrm{OD}^{<\alpha}$. ⊣

Note that this proof doesn't really that depend that much on what we call a "real", allowing us to go between the various senses. Now let's work towards Determinacy Transfer (1.B • 5).

Claim 2 (Determinacy Transfer)

$J_\alpha(\mathbb{R}) \vDash \mathsf{AD}$ implies $J_{\beta+1}(\mathbb{R}) \vDash \mathsf{AD}$.

Proof .:.

Suppose $J_\alpha(\mathbb{R}) \vDash \mathsf{AD}$. Remember as a Σ_1-gap, $J_{\beta+1}(\mathbb{R}) \vDash \mathsf{AD}$. By (1), the exercise, and (2) for all $z \in \mathbb{R}$, we get that $J_{\beta+1}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \bigcup_{k<\omega} \utilde{\Sigma}_k^{J_\beta(\mathbb{R})} \cap \mathcal{P}(\mathbb{R})$. Moreover, (1) and (2) tell us that this is contained in the bold-face envelope $\bigcup_{z\in\mathbb{R}} \overline{\mathrm{OD}^{<\alpha}(z)}$. Given (3), all of this union is determined. ⊣

Hence we have proven both. ⊣

But we haven't really shown any determinacy transfer, just a more abstract for of the result. So we really need to prove the assumptions of Lemma 1 (1.B • 10).

1.B • 12. Lemma 2

The following hold for every $z \in \mathbb{R}$:

1. For every $s \in [\omega\beta]^{<\omega}$, $\Sigma_n^{J_\beta(\mathbb{R})}(s,z) \subseteq \overline{\mathrm{OD}^{<\alpha}(z)}$;
2. $\overline{\mathrm{OD}^{<\alpha}(z)}$ is closed under real quanitification $\exists^{\mathbb{R}}$ and $\forall^{\mathbb{R}}$ (or just one); and
3. $\mathsf{Det}(\overline{\mathrm{OD}^{<\alpha}(z)})$.

Proof .:.

1. Let $s \in [\omega\beta]^{<\omega}$. For simplicity, assume $z = \emptyset$. We want to show $\Sigma_n^{J_\beta(\mathbb{R})}(s) \subseteq \overline{\mathrm{OD}^{<\alpha}}$. Let X be a product space and φ a Σ_n-formula. Consider the subset $A \subseteq \mathrm{X}$ defined by $x \in A$ iff $J_\beta(\mathbb{R}) \vDash \varphi[x,s]$.

   Let $\sigma \subseteq \mathrm{X}$ be a countable pointset, say $\sigma = \{y_k : k < \omega\}$. Now we have all the things we need to apply Σ_n-type reflection. Since $[\alpha,\beta]$ is a strong gap, the Σ_n-type of $\langle s, \langle y_k : k < \omega\rangle\rangle$ in $J_\beta(\mathbb{R})$ is realized by some $\langle \bar{s}, \langle y_k : k < \omega\rangle\rangle$ in $J_\gamma(\mathbb{R})$ for some $\gamma < \alpha$. The reason for this is because the jth digit of each y_k is part of the type, while s might be smaller (although $\bar{s}$ is still a parameter).

   Now we can define a simpler set $A' \subseteq \mathrm{X}$ by saying $x \in A'$ iff $J_\gamma(\mathbb{R}) \vDash \varphi[x,\bar{s}]$, where $\gamma < \alpha$. Thus $A' \in \mathrm{OD}^{<\alpha}$, and furthermore $A \cap \sigma = A' \cap \sigma$, since $y_k \in \sigma$ is reflected in the Σ_n-type. Hence by Definition 10 (1.B • 9), $A \in \overline{\mathrm{OD}^{<\alpha}}$.

2. Again, let's assume $z = \emptyset$. We want to show closure of $\overline{\mathrm{OD}^{<\alpha}}$ under $\exists^{\mathbb{R}}$. For this, we'll use the following lemma in essence due to Martin (although not stated in this terminology), giving an equivalent condition for being in this envelope.

   1.B • 13. Lemma 3

   For every product space X, and $A \subseteq \mathrm{X}$, the following are equivalent:

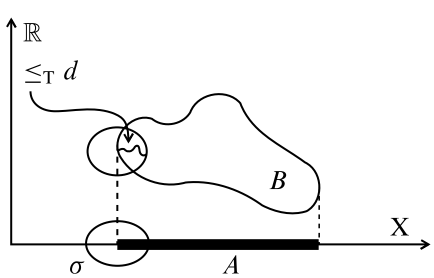
   I. $A \in \overline{\mathrm{OD}^{<\alpha}}$, i.e. for all countable $\sigma \subseteq \mathrm{X}$, there is an $A' \subseteq \mathrm{X}$ in $\mathrm{OD}^{<\alpha}$ such that $A \cap \sigma = A' \cap \sigma$;

   II. For all countable $\sigma \subseteq \mathrm{X}$, for a Turing-cone of Turing-degrees d, there is an $A' \subseteq \mathrm{X}$ in $\mathrm{OD}^{<\alpha}(d)$ with $A \cap \sigma = A' \cap \sigma$.

   Proof Idea .:.

   Clearly, (I) implies (II). To show (II) implies (I), the idea is that we can smooth out the dependence on d using Martin's measure on the Turing degrees. Since we don't have determinacy, we use the fact that $J_\alpha(\mathbb{R}) \vDash$ Turing determinacy.

   Assuming this lemma, let $B \subseteq \mathrm{X} \times \mathbb{R}$, such that $B \in \overline{\mathrm{OD}^{<\alpha}}$. Define $A = \exists^{\mathbb{R}} B \subseteq \mathrm{X}$.

   We want to show $A \in \overline{\mathrm{OD}^{<\alpha}}$. We will show II holds, and thus conclude I, completing the proof of (2). Let $\sigma \subseteq \mathrm{X}$ be countable. Now for a Turing cone of a degree d,

   $$\forall x \in \sigma[x \in A \leftrightarrow \exists y \leq_{\mathrm{T}} d(\langle x,y\rangle \in B)].$$

   If we just want the σ part of A, we don't have to project all of B, we can just use a portion of B of Turing degree $\leq_{\mathrm{T}} d$, and project that down. Now define $\sigma^* = \sigma \times \{y \in \mathbb{R} : y \leq_{\mathrm{T}} d\}$. In the above form of the definition of A, we don't use all of B, we only use $B \cap \sigma^*$. Note that σ^* is countable and $B \in \overline{\mathrm{OD}^{<\alpha}}$. There is some $B' \in \mathrm{OD}^{<\alpha}$ with $B \cap \sigma^* = B' \cap \sigma^*$ and thus

$$\forall x \in \sigma[x \in A \leftrightarrow \exists y \leq_T d(\langle x, y\rangle \in B')].$$

The condition on the right-hand-side is $\mathrm{OD}^{<\alpha}(d)$. So by Lemma 3 (1.B • 13), $A \in \overline{\mathrm{OD}^{<\alpha}}$. Hence (2) is proven. [End of lecture 3]

3. Once more, without loss of generality, $z = 0$.

1.B • 14. Definition 11

Turing ideal is a set $\mathcal{I} \subseteq \mathbb{R}$ closed under recursive joins $\oplus$, where $x \oplus y = \langle x(0), y(0), x(1), y(1), \cdots\rangle$; and which is closed under $\leq_T$.
For a set $A \subseteq \mathbb{R}$ and Turing ideal $\mathcal{I}$, we say A is *determined on* $\mathcal{I}$ iff there is a strategy "in $\mathcal{I}$" (coded by a real in $\mathcal{I}$) for the game G_A for either player that defeats every play in $\mathcal{I}$.

For example, "determined on $\mathbb{R}$" is equivalent to being determined. With these definitions, we can state the following lemma due to Kechris and Woodin.

1.B • 15. Lemma 4

Let $J_\alpha(\mathbb{R}) \models \mathsf{AD} + \Sigma_0$-collection. Then there is a real t such that for every countable Turing ideal $\mathcal{I}$ such that $t \in \mathcal{I}$, and A is determined on $\mathcal{I}$ whenever A is $\mathrm{OD}^{<\alpha}(z)$.

The proof of the lemma is similar to the proof for the following theorem of Kechris and Solovay. Note that OD here is "light-face", which is important for getting these results.

1.B • 16. Theorem 9

Δ^1_2 determinacy implies that for a Turing code of x, $L[x] \models$ OD determinacy.

The lemma implies the determinacy we need, but it's not immediately obvious. We need to see how it interacts with the $\Gamma \mapsto \overline{\Gamma}$ operation. Assuming the lemma, we will work towards proving (3). Supposing not, there is a non-determined $A \in \overline{\mathrm{OD}^{<\alpha}}$. By a Skolem hull argument, We can use $\mathsf{DC}_\mathbb{R}$ to get a countable Turing ideal $\mathcal{I}$ with $t \in \mathcal{I}$ as in Lemma 4 (1.B • 15) such that A is not determined on $\mathcal{I}$.

Using the defining property of the envelop, since $A \in \overline{\mathrm{OD}^{<*}}$, there is an $A' \in \mathrm{OD}^{<\alpha}$ such that $A \cap \mathcal{I} = A' \cap \mathcal{I}$. Thus A' is not determined on $\mathcal{I}$. But this contradicts the choice of t as in Lemma 4 (1.B • 15). ⊣

Note that (1) just uses that $[\alpha, \beta]$ is a strong gap. (2) and (3) just use the facts $J_\alpha(\mathbb{R}) \models \mathsf{AD}$, that α begins an admissible gap, and that $\mathsf{DC}_\mathbb{R}$ holds.

Section 2. Beyond $L(\mathbb{R})$

First of all, what is it about $L(\mathbb{R})$ that we want to talk about generalizations of? We have the following theorem due to Martin and Steel.

2 • 1. Theorem 10

Assume $\mathsf{AD} + V = L(\mathbb{R})$. Therefore Σ_1 has the scale property. (The local version for $\Sigma_1^{J_\alpha(\mathbb{R})}$ is due to steel.)

Again, Σ_1 is in the context of the language $\{\in, \dot{\mathbb{R}}\}$, where $\dot{\mathbb{R}}$ is a constant symbol for $\mathbb{R}$. So Σ_1 in this context means

$\Sigma_1(\{\mathbb{R}\})$ under normal circumstances.

The theorem can be generalized beyond $L(\mathbb{R})$ using a strengthening AD^+ of AD, defined by Woodin.

2 • 2. Definition 12

> AD^+ is the following:
> 1. $\mathsf{DC}_{\mathbb{R}}$;
> 2. every set of reals is ∞-Borel; and
> 3. ordinal determinacy in the sense that certain games on ordinals are determined.

The particular games that are determined can be explained as follows. For every $A \subseteq \mathbb{R}$, every $\lambda < \Theta$—i.e. for all λ which admits a surjection from $\mathbb{R}$ to λ—and every continuous $f : \lambda^\omega \to \mathbb{R} = \omega^\omega$, the game on λ with payoffset in $f^{-1}"A$ is determined. So ordinal determinacy doesn't say all ordinal games are determined (which would be inconsistent). Note that ordinal determinacy implies AD, since one could take f as the identity.

Note that whether ADAD^+ is open. However, we have $\mathsf{AD} + V = L(\mathbb{R}) \Rightarrow \mathsf{AD}^+$, and that AD^+ holds in derived models[ii]. The reason for bringing up AD^+ is for stating the following theorem due to Woodin.

2 • 3. Theorem 11

> Assume $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$. Then Σ_1 has the scale property. If instead we don't assume $V = L(\wp(\mathbb{R}))$, we could say Σ_1^2 has the scale property.

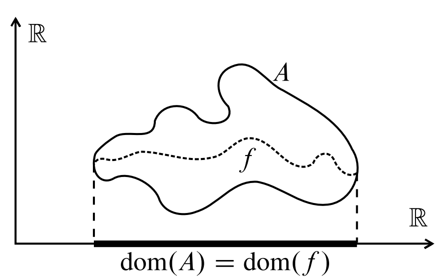
One of the reasons for assuming $V = L(\wp(\mathbb{R}))$ is that it's a natural generalization in the form of Theorem 10 (2 • 1). Again, note the language of $\{\in, \dot{\mathbb{R}}\}$.

§ 2.A. Uniformization

2.A • 1. Definition 13

> Let $A \subseteq \mathbb{R} \times \mathbb{R}$. A *uniformization* of A is a function $f \subseteq A$ such that $\mathrm{dom}(f) = \mathrm{dom}(A)$.

The idea can be represented by the following picture



Note that Suslin sets have uniformizations definable from their associated trees.

2.A • 2. Proposition 1

> If $A \subseteq \mathbb{R} \times \mathbb{R}$ is Suslin, say $A = p[T]$, then it has a uniformization this is definable from T.

Proof .:.

[ii] This is a fact due to Woodin; see Steel DMT.

T is a tree on $\omega \times \omega \times \kappa$ for some cardinal κ. Hence $\langle x, y\rangle \in A$ iff there is some $h \in \kappa^\omega$ such that $\langle x, y, h\rangle \in [T]$ is a branch of T. As a result, $x \in \text{dom}(A)$ iff $\exists y \in \mathbb{R} \exists h \in \kappa^\omega(\langle x, y, h\rangle \in [T])$. For any x, if there is a $y \in \mathbb{R}$ like this, there's a *least* such y "lexiographically", and this is what we'll define this function.

For all $x \in \text{dom}(A)$, define $f(x)$ to be the y such that the ω-sequence $\langle y(0), h(0), y(1), h(1) \cdots\rangle$ is lexiographically-least such that $\langle x, y, h\rangle \in [T]$—so we just discard h. There *is* such a lexiographically-least such element, since $[T]$ is closed. Note that f uniformizes A, and f is definable from T. ⊣

We now have a corollary of Theorem 11 (2 • 3).

2.A • 3. Corollary 2

Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Therefore every Σ_1 subset of $\mathbb{R} \times \mathbb{R}$ has a definable uniformization.

As a contrast to this, if you consider Π_1 instead of Σ_1, the situation is much different, even in the ostensibly weaker AD instead of AD^+. The following theorem of Kechris and Solovay expresses this.

2.A • 4. Theorem 12

Assume $\mathsf{AD} + V = L(\mathcal{P}(\mathbb{R}))$. Therefore there is a Π_1 subset of $\mathbb{R} \times \mathbb{R}$ with no $\text{OD}(\mathbb{R})$ uniformization.

In particular, the relation $\{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} : y \notin \text{OD}_x\}$ has no $\text{OD}(\mathbb{R})$ uniformization. It's not totally obvious that this is Π_1. But as an exercise, you can show that this works. [end of lecture 4]

Recall two previous theorems, the first from Woodin that $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ implies that every Σ_1 pointset is $p[T]$ for some OD T. So every Σ_1 binary relation has an OD uniformization—and in fact a Σ_1-uniformization. And this is the story for Σ_1. For Π_1, the situation is totally different, as seen by Theorem 12 (2.A • 4).

In fact, the relation defined there, $\{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} : y \notin \text{OD}_x\}$ is not $p[T]$ for any $\text{OD}(\mathbb{R})$ tree T. So it's natural to ask what the complexity for this relation's uniformation is, if it has one at all. The answer is that it depends.

2.A • 5. Corollary 3

Assume $\mathsf{AD} + V = L(\mathbb{R})$. Therefore every set is $\text{OD}(\mathbb{R})$. In particular, the Π_1-set $\{\langle x, y\rangle : y \notin \text{OD}_x\}$ is not Suslin, and can't be uniformized.

But can it be uniformized somewhere else? The answer is "yes, possibly so". The goal for the rest of this final lecture is to prove the following theorem, also due to Woodin. Note that another way to state (3) is to say that $\theta_0 < \Theta$. As an exercise, you can show the equivalence. Although this is a shorter way to state it, we will use (3) as stated here.

2.A • 6. Theorem 13

Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Therefore the following are equivalent:

1. every Π_1-pointset is Suslin;
2. every Π_1 binary relation has a uniformization;
3. not every pointset is $\text{OD}(\mathbb{R})$.

Proof .:.

For (1) implying (2), note that every Suslin binary relation has a uniformization. For (2) implying (3), see Kechris–Solovay's Theorem 12 (2.A • 4). (3) implying (1) will take the rest of the lecture.

Firstly, assume that not every pointset is $\text{OD}(\mathbb{R})$. Recall that every Σ_1 set is $p[T]$ for some OD T. Let T be an OD tree (on $\omega \times \kappa$ for some κ for the sake of simplicity). We want to show $\mathbb{R} \setminus p[T]$, a Π_1-set, is Suslin. Woodin showed something a bit stronger in this situation. In particular, the tree T is weakly homogeneous, although this is not how we will proceed. It should also be mentioned that Martin previously showed this assuming $\mathsf{AD}_\mathbb{R}$.

We, however, will show directly that there is a semiscale on our Π_1 set $\mathbb{R} \setminus p[T]$. Recall that having a semiscale is equivalent to being Suslin.

Also recall Definition 2 (1 • 2): a semiscale on $\mathbb{R} \setminus p[T]$ is a sequence $\vec{\varphi}$ such that whenever $x_0, \cdots \in \mathbb{R} \setminus \varphi[T]$ converge to some x and $\langle \varphi_n(x_k) : k < \omega \rangle$ is eventually constant for each n, then $x \in \mathbb{R} \setminus p[T]$. We have the following claim, which is where we use (3).

Claim 1

There is a fine (for any given element, measure 1 many things contain that element) countably complete (CC) measure $\mathcal{U}$ on $\mathcal{P}_{\omega_1}(\mathrm{OD})$, meaning the subsets of size ω_1 of all OD pointsets.

Proof .:.

By the hypothesis, we can say that $\mathrm{OD}(\mathbb{R})$ is a proper initial segment of the Wadge hierarchy (I believe this uses $\mathsf{DC}_{\mathbb{R}}$). So there is a surjection from $\mathbb{R}$ to $\mathrm{OD}(\mathbb{R})$, and in particular, a surjection $f : \mathbb{R} \to \mathrm{OD}$. So there aren't many OD sets. Now we use Martin's measure.

Martin's Measure on the Turing degrees gives a fine, CC measure μ on $\mathcal{P}_{\omega_1}(\mathbb{R})$. The reason why is that this measure will concentrate on countable sets of the form $\{x : x \leq_{\mathrm{T}} d\}$. So this measure is described by the following: $\mu(A) = 1$ iff for a cone of Turing degrees d, these sets $\{x : x \leq_{\mathrm{T}} d\} \in A$.

So we can define a fine, CC measure $\mathcal{U}$ on $\mathcal{P}_{\omega_1}(\mathrm{OD})$ as the push-forward of μ under the surjection f. In particular, we define $B \in \mathcal{U}$ iff for μ-almost every $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, $f"\sigma \in B$.

Exercise 3

Check that this works.

One way to think about this is that pointwise image takes countable subsets to countable subsets. ⊣

We now have another claim, carrying over the μ used in Claim 1.

Claim 2

$\overline{\mathrm{OD}} = \mathrm{OD}$. So using Claim 1, $\mathcal{U}$ is a fine CC measure on $\mathcal{P}_{\omega_1}(\overline{\mathrm{OD}})$.

Proof .:.

Defining a well-ordering on $\overline{\mathrm{OD}}$ gives what we want. Define $<$ on $\overline{\mathrm{OD}}$ by $A < B$ iff for μ-almost every $\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$, the $<_{\mathrm{OD}}$-least $A' \in \mathrm{OD}$ with $A \cap \sigma = A' \cap \sigma$ has $A' <_{\mathrm{OD}} B'$, where B' is similarly the $<_{\mathrm{OD}}$-least element of OD with $B \cap \sigma = B' \cap \sigma$.

Exercise 4

Show that $<$ is a well-ordering of $\overline{\mathrm{OD}}$.

This completes the proof of the claim. This is because we have a definition without parameters: everything in $\overline{\mathrm{OD}}$ is definable from its position in $<$, and is thus OD. ⊣

What remains is the following claim. It's equivalent to say for $\mathcal{U}$-almost every C instead of just *some* C.

Claim 3

For some $C \in \mathcal{P}_{\omega_1}(\overline{\mathrm{OD}}) \cap \mathcal{P}(\mathbb{R}^2)$, the pre-well-orderings of $\mathbb{R} \setminus p[T]$ in C form a semiscale on $\mathbb{R} \setminus p[T]$. What this means is that for some (every) enumeration $\langle \leq_n : n < \omega \rangle$ of such pre-well-orderings—letting $\varphi_n(x)$ be the rank of x in $\leqslant_n$—$\vec{\varphi}$ is a semiscale on $\mathbb{R} \setminus p[T]$.

Proof .:.

Suppose towards a contradiction that every C fails to have this property: for every $C \in \mathcal{P}_{\omega_1}(\overline{\mathrm{OD}})$, the pre-well-orderings of $\mathbb{R} \setminus p[T]$ in C do not form a semiscale. Since we don't have much choice, we can't associate these failing so naïvely. But we can in a kind of canonical way.

To introduce some notation, for a sequence of norms $\vec{\sigma}$ on $\mathbb{R} \setminus p[T]$, we say $T_{\vec{\varphi}}$ is the associated tree by collecting initial segments. So $\vec{\varphi}$ being a semiscale is equivalent to $p[T_{\vec{\varphi}}] = \mathbb{R} \setminus p[T]$. We're trying to find a $\vec{\varphi}$ where this happens. Note that the only way $\vec{\varphi}$ can fail at being a semiscale is when $p[T_{\vec{\varphi}}] \cap p[T] \neq \emptyset$.

For each $C \in \mathcal{P}_{\omega_1}(\overline{\mathrm{OD}})$, consider the corresponding game G_C. Intuitively, I is the "bad branch" player trying to show overlap, and II is the "semiscale" player, trying to show that this can't happen. Together x and f are a branch of T.

$$\begin{array}{lllllll} \mathrm{I}: x(0), f(0) & & g(0) & x(1), f(1) & & g(1) & \cdots \\ \mathrm{II}: & \leqslant_0 & & & \leqslant_1 & & \cdots \end{array}$$

The rule for I: for every $n < \omega$, $\langle\langle x(0), \cdots, x(n)\rangle, \langle f(0), \cdots, f(n)\rangle\rangle \in T$. The rule for II: $\leqslant_n \in C$ is a pre-well-ordering of $\mathbb{R} \setminus p[T]$. The other rule for I: $\langle\langle x(0), \cdots, x(n)\rangle, \langle g(0), \cdots, g(n)\rangle\rangle \in T_{\vec{\varphi}}$ where φ_n is the norm corresponding to $\leq_n$.

This game is closed for player I.

Exercise 5

> Player I has a "canonical" winning strategy Σ_C in G_C for every C using the hypothesis that no C forms a semiscale.

We will combine all these Σ_C for $C \in \mathcal{P}_{\omega_1}(\overline{\mathrm{OD}})$ using $\mathcal{U}$ to get $\langle \leqslant_n : n < \omega\rangle$—a play for II—that beats $\mathcal{U}$-almost every Σ_C, a contradiction.

Suppose we (as II) have already played the pre-well-orderings $\leqslant_0$, ..., $\leqslant_{n-1}$. To define $\leqslant_n$, note that each Σ_C gives a play of G_C.

$$\begin{array}{llllll} \mathrm{I}: x_C(0), f_C(0) & & g_C(0) & \cdots & g_C(n-1) & x_C(n), f_C(n) \\ \mathrm{II}: & \leqslant_0 & & \cdots \ \leqslant_{n-1} & & \end{array}$$

Define $\leqslant_n$ on $\mathbb{R} \setminus p[T]$ by $y \leqslant_n z$ iff for every $\mathcal{U}$-almost every C, $\mathrm{rank}_{T_y}(\langle f_C(0), \cdots, f_C(n)\rangle) \leq \mathrm{rank}_{T_z}(\langle f_C(0), \cdots, f_C(n)\rangle)$, where $T_y = \{s \in \kappa^{<\omega} : \langle y \restriction |s|, s\rangle \in T\}$. So $y \in \mathbb{R} \setminus p[T]$ iff T_y is well-founded. (Just define $\mathrm{rank}_{T_y}(s) = 0$ if $s \notin T_y$.)

Exercise 6

> $\leqslant_n$ is in $\overline{\mathrm{OD}}$.

If you look just at the condition on ranks above, the tree is ordinal definable, and the sequence of f_Cs are all ordinals. Now C varies, but using the countable completeness of $\mathcal{U}$ allows this to work.

But as a result of the exercise, we can say for $\mathcal{U}$-almost every $C \in \mathcal{P}_{\omega_1}(\overline{\mathrm{OD}})$, $\{\leqslant_n : n < \omega\} \subseteq C$, so the sequence $\langle \leqslant_n : n \in \omega\rangle$ is a legal play for IIin G_C. Of course, we need to show that these are legal moves, and that I can't survive forever against these moves.

Following Σ_C for I gives a complete run of the game:

$$\begin{array}{l} \mathrm{I}: x_C \in \mathbb{R}, f_C \in \kappa^\omega, g_C \in \mathsf{Ord}^\omega \\ \mathrm{II}: \langle \leqslant_n : n \in \omega\rangle \end{array}$$

We want to show that this is a loss for I for $\mathcal{U}$-almost every C. Note x_C is constant on a $\mathcal{U}$-measure 1 set. The way that you show this uses CC twice: first use CC to fix each digit. Hence we may assume $x_c = x \in \mathbb{R}$ for $\mathcal{U}$-almost every C. So for this x, $x \in p[T] \cap p[T_{\vec{\varphi}}]$, where φ_n is the norm corresponding to $\leqslant_n$.

x witnesses the failure of the semiscale property, so there are $x_0, x_1, \cdots \in \mathbb{R} \setminus p[T]$ converging to $x \in p[T]$ such that for all n, $\langle \varphi_n(x_k) : k < \omega \rangle$ is eventually constant.

Exercise 7

For $\mathcal{U}$-almost every $C \in \mathcal{P}_{\omega_1}(\overline{\text{OD}})$,

1. $\left\langle \text{rank}_{T_{x_k}} \left(\langle f_C(0), \cdots, f_C(n) \rangle \right) : k < \omega \right\rangle$ is eventually constant.
2. letting λ_n be its eventual value, then $\lambda_0 > \lambda_1 > \lambda_2 > \cdots$, a contradiction.

This exercise will prove the claim. ⊣

Proving this final claim proves the theorem. Note that this proof of Claim 3 doesn't use determinacy at all. ⊣