

THE AXIOM OF DETERMINACY IMPLIES DEPENDENT CHOICES IN MICE

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ABSTRACT. We show that the Axiom of Dependent Choices, DC, holds in countably iterable, passive premice \mathcal{M} constructed over their reals which satisfy the Axiom of Determinacy, AD, in a $\text{ZF} + \text{DC}_{\mathbb{R}, \mathcal{M}}$ background universe. This generalizes an argument of Kechris for $L(\mathbb{R})$ using Steel's analysis of scales in mice. In particular, we show that for any $n \leq \omega$ and any countable set of reals A so that $M_n(A) \cap \mathbb{R} = A$ and $M_n(A) \models \text{AD}$, we have that $M_n(A) \models \text{DC}$.

1. INTRODUCTION

We prove that in passive, countably iterable mice \mathcal{M} constructed over their reals, AD, the *Axiom of Determinacy*, implies DC, the *Axiom of Dependent Choices*, working in a background universe which satisfies $\text{ZF} + \text{DC}_{\mathbb{R}, \mathcal{M}}$. Here we write $\mathbb{R}^{\mathcal{M}} = \mathbb{R} \cap \mathcal{M}$ for the set of reals in \mathcal{M} .

Recall that DC is the following statement: For every nonempty set X and every binary relation P on X ,

$$\forall a \in X \exists b \in X P(a, b) \Rightarrow \exists f: \omega \rightarrow X \forall n P(f(n), f(n+1)).$$

Moreover, $\text{DC}_{\mathbb{R}}$ denotes DC restricted to the case where $X = \mathbb{R}$ and more generally, for some nonempty set Y , DC_Y denotes DC restricted to the case where $X = Y$.

Gödel's constructible universe over the reals $L(\mathbb{R})$ is the closure of \mathbb{R} under the definable power set operation. Kechris showed in [Ke84] that in $L(\mathbb{R})$, the Axiom of Determinacy implies the Axiom of Dependent Choices. His proof is based on the analysis of scales in $L(\mathbb{R})$ which was developed by Martin, Moschovakis, and Steel (see [MMS82], [Mo08], [MaSt08], and [St08a]). A generalization of [Ke84] and the analysis of scales to the Dodd-Jensen core model over \mathbb{R} was shown by Cunningham in [Cu95]. We prove the following more general result for arbitrary mice building on the analysis of scales in mice from [St08b]. Note that, in contrast to Kechris's result for $L(\mathbb{R})$, our result requires $\text{DC}_{\mathbb{R}, \mathcal{M}}$ to hold in V in order to consider countable elementary substructures of \mathcal{M} . We will make it clear in the proof where the countability of the model in question is used.

Theorem 1.1 (ZF). *Let \mathcal{M} be a passive, countably iterable $\mathbb{R}^{\mathcal{M}}$ -premouse such that $\mathcal{M} \models \text{AD}$ and suppose that $\text{DC}_{\mathbb{R}, \mathcal{M}}$ holds in V . Then $\mathcal{M} \models \text{DC}$.*

For countable mice it is not necessary to assume $\text{DC}_{\mathbb{R}, \mathcal{M}}$, see Theorem 2.1. In particular, Theorem 1.1 holds for mice of the form $M_n(A)$ for some $n \leq \omega$ and some countable set of reals A such that $M_n(A) \cap \mathbb{R} = A$. This result is for example used in [AgMu], where the authors derive a model with $\omega + n$ Woodin cardinals from a model of the form $M_n(A)$ with $M_n(A) \cap \mathbb{R} = A$ which satisfies the Axiom of Determinacy.

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2. COUNTABLE MICE IN A ZF BACKGROUND UNIVERSE

For simplicity, we first show the following version of Theorem 1.1 for countable mice and argue in the next section that this implies Theorem 1.1. As mentioned above, we do not require any form of choice in the background universe for this result.

Theorem 2.1 (ZF). *Let \mathcal{M} be a countable, passive, $(\omega_1 + 1)$ -iterable $\mathbb{R}^{\mathcal{M}}$ -premouse such that $\mathcal{M} \models \text{AD}$. Then $\mathcal{M} \models \text{DC}$.*

For the definition of premeice and $(\omega_1 + 1)$ -iterability we refer the reader to [St10], and to [MS94] and [SchStZe02] for more background. Moreover, we refer to [St08b] for the notion of X -premeice for arbitrary sets X . First, we recall the notion of iterability we use in the statement of Theorem 1.1.

Definition 2.2. Let A be a set of reals and suppose \mathcal{M} is an A -premeice. We say that \mathcal{M} is *countably iterable* iff whenever $\bar{\mathcal{M}}$ is a countable \bar{A} -premeice for a set of reals \bar{A} and there is an elementary embedding $\pi: \bar{\mathcal{M}} \rightarrow \mathcal{M}$, then $\bar{\mathcal{M}}$ is $(\omega_1 + 1)$ -iterable.

To prove Theorem 2.1, we will show that the argument in the proof of [St08b, Theorem 4.1] which yields the existence of scales in \mathcal{M} using $\mathcal{M} \models \text{DC}$, can be used to show the existence of quasi-scales without using DC in \mathcal{M} . Moreover, we sketch how we can adapt the argument from [Ke84] for $V = L(\mathbb{R})$ to obtain $\mathcal{M} \models \text{DC}$ from these quasi-scales.

Following the notation in [St08b], we write $K(\mathbb{R})$ for the model-theoretic union of all ω -sound, countably iterable premeice over \mathbb{R} which project to \mathbb{R} . Using $\text{DC}_{\mathbb{R}}$, it is easy to show that any two such premeice \mathcal{M} and \mathcal{N} line up, i.e. satisfy $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$. Therefore $K(\mathbb{R})$ is well-defined.

Remark. If we consider premeice $\mathcal{M}(\mathbb{R})$ constructed over all reals $\mathbb{R} = \mathbb{R}^V$, e.g. $\mathcal{M}(\mathbb{R}) = K(\mathbb{R})$ or $\mathcal{M}(\mathbb{R}) = M_1(\mathbb{R})$, it is easy to see that DC in V (and in fact, using the argument at the beginning of the proof of Theorem 2.1, even $\text{DC}_{\mathbb{R}}$ in V) already implies DC in $\mathcal{M}(\mathbb{R})$ as every function $f: \omega \rightarrow \mathbb{R}$ witnessing DC in V can be coded by a single real and is therefore already contained in $\mathcal{M}(\mathbb{R})$. But the same does not hold in general for models \mathcal{M} as in Theorem 1.1 with $\mathbb{R}^{\mathcal{M}} \subsetneq \mathbb{R}$ since if $f: \omega \rightarrow \mathbb{R}^{\mathcal{M}}$ is a function witnessing DC in V for reals in \mathcal{M} for some relation P , it can be coded by a single real in V , but this real need not be in $\mathbb{R}^{\mathcal{M}}$.

For the reader's convenience, we repeat parts of the arguments from [Ke84] and [St08b] to point out the modifications we need to make. We start by recalling the notions of quasi-norm and quasi-scale which go back to [Ke84].

Definition 2.3. Let $B \subseteq \mathbb{R}$. A relation \leq on B is a *quasi-norm* iff

- (1) \leq is a linear preordering on B , i.e. \leq is reflexive, transitive, and for all $x, y \in B$, $x \leq y$ or $y \leq x$, and
- (2) there is no infinite descending chain in $<$, where for $x, y \in B$, we write $x < y$ iff $x \leq y$ and $\neg(y \leq x)$.

Definition 2.4. Let $B \subseteq \mathbb{R}$. A *quasi-scale* on B is a sequence of quasi-norms $(\leq_i)_{i < \omega}$ on B such that if $x_i \in B$ for $i < \omega$ with $x_i \rightarrow x$ as $i \rightarrow \infty$ and if for each i there is some $n_i \in \omega$ such that $x_k \equiv_i x_{n_i}$ for all $k \geq n_i$,¹ then

- (1) $x \in B$ (*limit property*), and

¹We write $x \equiv_i y$ iff $x \leq_i y$ and $y \leq_i x$.

(2) for all $i < \omega$, $x \leq_i x_{n_i}$ (*lower semi-continuity*).

If we replace (2) in Definition 2.3 by “every nonempty subset of B has a \leq -least element”, we obtain the usual definitions of norm and scale. Hence, under $\text{DC}_{\mathbb{R}}$ every quasi-scale is a scale.

We shall need the following lemma from [Ke84] which is motivated by the proof of the Third Periodicity Theorem (see [Mo09, Theorem 6E.1]). Recall that $\text{AC}_{\omega, \mathbb{R}}$ denotes countable choice for reals, i.e., for all relations P on $\omega \times \mathbb{R}$,

$$\forall n \in \omega \exists r \in \mathbb{R} P(n, r) \Rightarrow \exists f: \omega \rightarrow \mathbb{R} \forall n \in \omega P(n, f(n)).$$

Lemma 2.5 ($\text{AC}_{\omega, \mathbb{R}}$). *Suppose B is a nonempty set of reals and $(\leq_i)_i$ is a quasi-scale on B . Let Γ be a pointclass containing B such that the relation*

$$R(i, x, y) \Leftrightarrow (x, y \in B \wedge x \leq_i y)$$

is in Γ . Moreover, suppose that Γ is closed under recursive substitutions, \neg , \wedge , \vee , and existential and universal quantification over \mathbb{R} . Then B contains a real x such that $\{(n, m) \in \omega \times \omega: x(n) = m\}$ is in Γ .

Now we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Work in the countable mouse \mathcal{M} and note that it suffices to prove $\text{DC}_{\mathbb{R}^{\mathcal{M}}}$ since there is a definable surjection $F: \text{Ord}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}} \rightarrow \mathcal{M}$ (see Proposition 2.4 in [St08b]). Suppose $\text{DC}_{\mathbb{R}^{\mathcal{M}}}$ fails, i.e. there is a relation $P \subseteq \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$ such that $\forall x \in \mathbb{R}^{\mathcal{M}} \exists y \in \mathbb{R}^{\mathcal{M}} P(x, y)$, but there is no $f: \omega \rightarrow \mathbb{R}^{\mathcal{M}}$ with $P(f(n), f(n+1))$ for all $n \in \omega$.

Let $\xi < \text{Ord}^{\mathcal{M}}$ be a large enough limit ordinal such that $P \in \mathcal{M}|\xi$ and $\mathcal{M}|\xi$ is passive. We may assume that such a limit ordinal exists because the general case when $\text{Ord}^{\mathcal{M}}$ need not be a limit of limit ordinals can be shown similarly using the S -hierarchy (see the end of the proof of Theorem 2.1 in [St08a]). Let α be the least ordinal below ξ such that $\mathcal{M}|\alpha \prec_1 \mathcal{M}|\xi$ (in the sense of Definition 4.4 in [St08b]) and note that α is a limit ordinal. The statement

$$\begin{aligned} & \exists P \subseteq \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}} (\forall x \in \mathbb{R}^{\mathcal{M}} \exists y \in \mathbb{R}^{\mathcal{M}} P(x, y)) \wedge \\ & \neg \exists f: \omega \rightarrow \mathbb{R}^{\mathcal{M}} \forall n P(f(n), f(n+1)) \end{aligned}$$

is Σ_1 in the parameter $\mathbb{R}^{\mathcal{M}}$ as any $f: \omega \rightarrow \mathbb{R}^{\mathcal{M}}$ can be coded by a real. Therefore it follows that there is a counterexample to $\text{DC}_{\mathbb{R}^{\mathcal{M}}}$ (in \mathcal{M}) inside $\mathcal{M}|\alpha$. To finish the proof, we use the following lemma.

Lemma 2.6. *Every relation $P \subseteq \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$ in $\mathcal{M}|\alpha$ can be uniformized in \mathcal{M} , i.e. there is a function $F: \mathbb{R}^{\mathcal{M}} \rightarrow \mathbb{R}^{\mathcal{M}}$ in \mathcal{M} such that for all $x \in \mathbb{R}^{\mathcal{M}}$,*

$$\exists y P(x, y) \Rightarrow P(x, F(x)).$$

Applying Lemma 2.6 to the counterexample P above, we can define a function $f: \omega \rightarrow \mathbb{R}^{\mathcal{M}}$ by letting $f(0) = a \in \mathbb{R}^{\mathcal{M}}$ be arbitrary and $f(n+1) = F(f(n))$. Then $P(f(n), f(n+1))$ holds for all n , contradicting the choice of P . So it suffices to prove Lemma 2.6.

Proof of Lemma 2.6. The proof divides into three claims. The first claim uses fine structural arguments to obtain definability for the sets of reals in $\mathcal{M}|\alpha$. The key part of the argument is Claim 2, where we show the existence of quasi-scales. Finally, in Claim 3 we piece Claim 2 and Lemma 2.5 together to obtain a basis result which will imply the existence of a uniformizing function, as desired.

Claim 1. *Every set of reals in $\mathcal{M}|\alpha$ is Σ_1 -definable in $\mathcal{M}|\alpha$ from parameters in $\mathbb{R}^{\mathcal{M}} \cup \{\mathbb{R}^{\mathcal{M}}\}$.*

Proof. Standard fine structural arguments show that $\mathcal{M}|\alpha$ has a Σ_1 Skolem function which is Σ_1 definable in $\mathcal{M}|\alpha$ (without parameters). As in the proof of Lemma 1.11 in [St08a] for $L(\mathbb{R})$, this together with the fact that we chose α minimal with the property that $\mathcal{M}|\alpha \prec_1 \mathcal{M}|\xi$ yields that there is a partial surjection $h: \mathbb{R}^{\mathcal{M}} \rightarrow \mathcal{M}|\alpha$ such that the graph of h is Σ_1 definable in $\mathcal{M}|\alpha$ from parameter $\mathbb{R}^{\mathcal{M}}$. Hence, every set of reals in $\mathcal{M}|\alpha$ is Σ_1 definable in $\mathcal{M}|\alpha$ from parameters in $\mathbb{R}^{\mathcal{M}} \cup \{\mathbb{R}^{\mathcal{M}}\}$, as desired. \square

Claim 2. *Let $B \subseteq \mathbb{R}^{\mathcal{M}}$ be a set of reals which is Σ_1 -definable in $\mathcal{M}|\alpha$ from some real parameter r and the parameter $\mathbb{R}^{\mathcal{M}}$. Then there is a quasi-scale $(\leq_i)_{i < \omega}$ on B which is also Σ_1 -definable in $\mathcal{M}|\alpha$ from the parameters r and $\mathbb{R}^{\mathcal{M}}$.*

Proof. Here we use Steel's analysis of scales in mice (see [St08b]) under $\text{DC}_{\mathbb{R}}$ and observe that it can be used to obtain a quasi-scale without any use of $\text{DC}_{\mathbb{R}}$. In order to show how to do this, we sketch parts of his argument below.

So let $B \subseteq \mathbb{R}^{\mathcal{M}}$ be a set of reals which is Σ_1 -definable over $\mathcal{M}|\alpha$ with some real parameter r and parameter $\mathbb{R}^{\mathcal{M}}$. Hence for some Σ_1 formula φ ,

$$x \in B \text{ iff } \mathcal{M}|\alpha \models \varphi(x, r, \mathbb{R}^{\mathcal{M}})$$

for all $x \in \mathbb{R}^{\mathcal{M}}$. Recall that α is a limit ordinal. If $\mathcal{M}|\alpha$ satisfies “ Θ exists”, let $\alpha^* = \Theta^{\mathcal{M}|\alpha}$, otherwise let $\alpha^* = \alpha$. Now write for each $\beta < \alpha^*$ and $x \in \mathbb{R}^{\mathcal{M}}$,

$$x \in B^\beta \text{ iff } \mathcal{M}|\beta \models \varphi(x, r, \mathbb{R}^{\mathcal{M}}).$$

By [St08b, Lemma 3.2], applied inside $\text{HOD}_{x, \Sigma}$, where x is a real coding $\mathcal{M}|\alpha$ and Σ is an iteration strategy for \mathcal{M} , we obtain $B = \bigcup_{\beta < \alpha^*} B^\beta$. Note that Σ is amenable to $\text{HOD}_{x, \Sigma}$, so the (canonically well-ordered) fragment $\Sigma \cap \text{HOD}_{x, \Sigma}$ is available within the model $\text{HOD}_{x, \Sigma}$ and witnesses iterability there. Moreover, $\text{HOD}_{x, \Sigma}$ is a model of the Axiom of Choice. Steel constructs in the proof of Theorem 4.1 in [St08b] a closed game representation $x \mapsto G_x^\beta$ of B^β for each $\beta < \alpha^*$. We briefly sketch the argument here to show that it can be done in our situation as well. First, recall the definition of a closed game representation, which was essentially introduced in [Mo08].

Definition. Let x be a real and G_x a closed game where Player I plays elements of ${}^\omega\omega \times \gamma$ for some ordinal γ and Player II plays elements of ${}^\omega\omega$, and there is some relation $\mathcal{Q} \subseteq ({}^\omega\omega)^{<\omega} \times \gamma^{<\omega}$ such that Player I wins the run $((x_0, \gamma_0), x_1, (x_2, \gamma_1), x_3, \dots)$ of G_x iff

$$\forall n \mathcal{Q}((x \upharpoonright n, x_0 \upharpoonright n, \dots, x_n \upharpoonright n), (\gamma_0, \dots, \gamma_n)).$$

In particular, G_x is continuously associated to x . We say $x \mapsto G_x$ is a *closed game representation of B* iff B is the set of all x such that Player I has a winning quasi-strategy in G_x .

We now define a closed game representation $x \mapsto G_x^\beta$ of B^β for each $\beta < \alpha^*$. Fix $\beta < \alpha^*$ and x . Let G_x^β be the following game:

$$\begin{array}{c|cccc} \text{I} & i_0, x_0, \gamma_0 & i_1, x_2, \gamma_1 & \dots & \\ \hline \text{II} & & x_1 & x_3 & \dots \end{array}$$

The rules of the game ask Player I to play $i_0, i_1, \dots \in \{0, 1\}$ in order to code a theory T in the language $\mathcal{L}_{\text{pm}}(\{\dot{x}_i : i \in \omega\})$ of premisses with additional constant symbols $\{\dot{x}_i : i \in \omega\}$ such that every model \mathcal{N}^* of T is well-founded. Furthermore, the players alternate playing reals $x_i, i \in \omega$, and Player I plays additional ordinals $\gamma_i, i \in \omega$. The theory ensures that for every model \mathcal{N}^* of T , for all $i \in \omega$, $(\dot{x}_i)^{\mathcal{N}^*} = x_i$ and the definable closure of $\{x_i : i \in \omega\}$ in $\mathcal{N}^* \upharpoonright \mathcal{L}_{\text{pm}}$ is an elementary submodel \mathcal{N} of $\mathcal{N}^* \upharpoonright \mathcal{L}_{\text{pm}}$. By considering its transitive collapse we can assume that \mathcal{N} is

transitive. The winning conditions for Player I require that he plays the theory T such that

$$\mathcal{N} \models \text{“}V = K(\mathbb{R}) + \varphi(x, r, \mathbb{R}) + \text{all of my proper initial segments} \\ \text{do not satisfy } \varphi(x, r, \mathbb{R})\text{”}.$$

In addition, he is using the ordinals γ_i to not only verify well-foundedness of \mathcal{N} by embedding the ordinals into $\omega\beta$, but also to verify iterability of \mathcal{N} by embedding the local HOD's of \mathcal{N} into the local HOD's of $\mathcal{M}|\beta$. This latter embedding corresponds to the embedding of the ordinals. This amount of details suffices for our sketch of the argument, the formal definition of G_x^β can be found in [St08b, Section 4].

Let

$$B_k^\beta(x, u) \Leftrightarrow u \text{ is a position of length } k \text{ from which}$$

Player I has a winning quasi-strategy in G_x^β .

We aim to show that each B_k^β is in $\mathcal{M}|\alpha$ and that the map $(\beta, k) \mapsto B_k^\beta$ is Σ_1 definable over $\mathcal{M}|\alpha$ with parameters r and $\mathbb{R}^{\mathcal{M}}$. In order to do that, we consider *honest positions* in the game G_x^β , which are positions where Player I played the theory T up to this point according to the theory of an initial segment $\mathcal{M}|\xi$ of the true model $\mathcal{M}|\beta$ and the embeddings induced by the ordinals γ_i according to an elementary embedding between the local HOD's of $\mathcal{M}|\xi$ and the local HOD's of the true model $\mathcal{M}|\beta$.

Definition. We say a position $u = ((i_n, x_{2n}, \gamma_n, x_{2n+1}) : n < k)$ in the game G_x^β is (β, x) -*honest* iff $\mathcal{M}|\beta \models \varphi(x, r, \mathbb{R})$ and if $\xi \leq \beta$ is least such that

$$\mathcal{M}|\xi \models \varphi(x, r, \mathbb{R}),$$

then $x_0 = x$ if $k > 0$ and if $\mathcal{M}^+|\xi$ denotes the canonical expansion of $\mathcal{M}|\xi$ to the language $\mathcal{L}_{\text{pm}}(\{x_i : i < 2k\})$ by letting $(\dot{x}_i)^{\mathcal{M}^+} = x_i$ for $i < 2k$,

- (1) $\mathcal{M}^+|\xi$ satisfies all sentences in T determined up to the position u ,
- (2) the embedding given by $\mathcal{M}^+|\xi$ and the ordinals γ_i for $i < k$ is well-defined and can be extended to an order-preserving map

$$\pi : \omega\xi \rightarrow \omega\beta,$$

and

- (3) this embedding can be extended to an elementary embedding between the relevant local HOD's.

The formal definition of honest positions can be found in [St08b, Section 4]. We let $H_k^\beta(x, u)$ iff u is a (β, x) -honest position of length k . The following subclaim concerning the definability of honest positions is the analogue of [St08b, Claim 4.2].

Subclaim 2.1. *Each H_k^β is in $\mathcal{M}|\alpha$ and the map $(\beta, k) \mapsto H_k^\beta$ is Σ_1 -definable in $\mathcal{M}|\alpha$ from parameters r and $\mathbb{R}^{\mathcal{M}}$.*

Moreover, we also get an analogue of [St08b, Claim 4.3], stating that the positions u from which Player I has a winning quasi-strategy in G_x^β are precisely the (β, x) -honest positions.

Subclaim 2.2. *For all positions u in G_x^β and all natural numbers k , $B_k^\beta(x, u)$ if, and only if, $H_k^\beta(x, u)$.*

Proof. It is easy to see that $H_k^\beta(x, u)$ implies $B_k^\beta(x, u)$ as Player I can win from an (β, x) -honest position u by continuing to play according to the true model $\mathcal{M}|\beta$. For the other implication, let σ be a winning quasi-strategy for Player I from a position u in G_x^β . Recall that $\mathbb{R}^{\mathcal{M}}$ is countable in V and consider a complete run

$((i_n, x_{2n}, \gamma_n, x_{2n+1}) : n < \omega)$ of G_x^β according to σ such that $\{x_i : i \in \omega\} = \mathbb{R}^{\mathcal{M}}$. Moreover, consider the canonical model \mathcal{N} associated to this run of G_x^β as above. We need to show that \mathcal{N} is an initial segment of $\mathcal{M}|\beta$.

This part of the proof uses a comparison argument. Recall that the standard proof of the comparison lemma (see for example Theorem 3.11 in [St10]) uses a reflection argument to a small elementary substructure and hence DC. But $\mathcal{M}|\beta$ and hence \mathcal{N} is $(\omega_1 + 1)$ -iterable in V , so we can perform the comparison in $\text{HOD}_{x, \Sigma, \Sigma'}$, where x is a real coding $\mathcal{M}|\beta$ and \mathcal{N} , and Σ and Σ' are iteration strategies for $\mathcal{M}|\beta$ and \mathcal{N} respectively. Similar as before, Σ and Σ' are amenable to $\text{HOD}_{x, \Sigma, \Sigma'}$ and their (canonically well-ordered) fragments $\Sigma \cap \text{HOD}_{x, \Sigma, \Sigma'}$ and $\Sigma' \cap \text{HOD}_{x, \Sigma, \Sigma'}$ witness iterability in $\text{HOD}_{x, \Sigma, \Sigma'}$, which is a model of the Axiom of Choice. So there is no further assumption on \mathcal{M} needed and we obtain that \mathcal{N} is an initial segment of $\mathcal{M}|\beta$, in fact that $\mathcal{N} = \mathcal{M}|\xi$, where ξ is least such that $\mathcal{M}|\xi \models \varphi(x, r, \mathbb{R})$, as in the proof of [St08b, Claim 4.3]. \square

Now let $(\leq_i^\beta)_i$ be the quasi-scale on B^β constructed from the closed game representation as in 2.6 in [Ke84] using the *fake sup*, *min*, and *fake inf* method. Then $(\beta, i) \mapsto \leq_i^\beta$ is Σ_1 definable over $\mathcal{M}|\alpha$ with parameters r and $\mathbb{R}^{\mathcal{M}}$ as well, as desired. \square

Using Claim 2 together with Lemma 2.5 we can now show the following claim.

Claim 3. *Every nonempty set of reals B in $\mathcal{M}|\alpha$ which is Σ_1 -definable in $\mathcal{M}|\alpha$ from a real parameter r and the parameter $\mathbb{R}^{\mathcal{M}}$, contains an element x which is first-order definable in $\mathcal{M}|\alpha$ from r and $\mathbb{R}^{\mathcal{M}}$.*

Proof. We will use Lemma 2.5 to pick an element out of a set of reals B in a definable way using a quasi-scale on B . Recall that $\text{AC}_{\omega, \mathbb{R}^{\mathcal{M}}}$ holds in \mathcal{M} as a consequence of AD. To obtain Claim 3, apply Lemma 2.5 inside $\mathcal{M}|\alpha$ to a nonempty set $B \subseteq \mathbb{R}^{\mathcal{M}}$ which is Σ_1 -definable in $\mathcal{M}|\alpha$ from some real parameter r and the parameter $\mathbb{R}^{\mathcal{M}}$, the quasi-scale on B obtained in Claim 2, and the pointclass Γ of all sets which are first-order definable in $\mathcal{M}|\alpha$ from the parameters r and $\mathbb{R}^{\mathcal{M}}$. \square

Claim 3 now implies Lemma 2.6. Suppose P is as in Lemma 2.6. By Claim 1 we can in addition assume that P is Σ_1 -definable in $\mathcal{M}|\alpha$ from a parameter $r \in \mathbb{R}^{\mathcal{M}}$ and the parameter $\mathbb{R}^{\mathcal{M}}$. We can define a uniformizing function F as follows. If for a real x , $\neg \exists y P(x, y)$, let $F(x) = x$. Otherwise, let $F(x)$ be the least (with respect to a fixed enumeration of first-order formulae) real z which is first-order definable from x, r , and $\mathbb{R}^{\mathcal{M}}$ in $\mathcal{M}|\alpha$ such that $P(x, z)$. Then $F \in \mathcal{M}$ is the desired uniformization. \square

This finishes the proof of Theorem 2.1. \square

3. UNCOUNTABLE MICE WITH $\text{DC}_{\mathbb{R}^{\mathcal{M}}}$ IN THE BACKGROUND

In this section we argue that instead of working with countable premice \mathcal{M} we can work in a background universe which is a model of $\text{DC}_{\mathbb{R}^{\mathcal{M}}}$, i.e. we derive Theorem 1.1 as a corollary of Theorem 2.1.

Proof of Theorem 1.1. Let \mathcal{M} be a passive, countably iterable $\mathbb{R}^{\mathcal{M}}$ -premouse such that $\mathcal{M} \models \text{AD}$. Using $\text{DC}_{\mathbb{R}^{\mathcal{M}}}$ in V , we can by the standard proof of the Löwenheim-Skolem Theorem consider a countable elementary substructure \mathcal{N} of \mathcal{M} . Then \mathcal{N} is an $(\omega_1 + 1)$ -iterable $\mathbb{R}^{\mathcal{N}}$ -premouse and we can apply Theorem 2.1 to \mathcal{N} . This yields $\mathcal{N} \models \text{DC}$ and hence $\mathcal{M} \models \text{DC}$. \square

Finally, note that the statements in Theorem 1.1 and Theorem 2.1 are in fact equivalent by the following argument. Let \mathcal{M} be a countable, passive, $(\omega_1 + 1)$ -iterable $\mathbb{R}^{\mathcal{M}}$ -premouse such that $\mathcal{M} \models \text{AD}$. Let Σ be an $(\omega_1 + 1)$ -iteration strategy for \mathcal{M} and $x_{\mathcal{M}}$ be a real coding \mathcal{M} . Now apply Theorem 1.1 inside $\text{HOD}_{x_{\mathcal{M}}, \Sigma}$, which is a model of the Axiom of Choice.

Using that for any countable set of reals A the Woodin cardinals in $M_n(A)$ are countable in V , we obtain the following corollary.

Corollary 3.1 (ZF). *Let $n \leq \omega$ and let $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$. Suppose that $M_n^{\sharp}(A)$ exists and is $(\omega_1 + 1)$ -iterable. Moreover, suppose that $M_n(A) \cap \mathbb{R} = A$ and $M_n(A) \models \text{AD}$. Then $M_n(A) \models \text{DC}$.*

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