Consistency strength lower bounds for the proper forcing axiom via the core model induction

J. Steel. PFA *implies* $\mathrm{AD}^{L(\mathbb{R})}$. ***The Journal of Symbolic Logic***, vol. 70(4) (2005), pp. 1255–1296.

G. Sargsyan. *Nontame mouse from the failure of square at a singular strong limit cardinal.* ***Journal of Mathematical Logic***, vol. 14(1) (2014), 1450003 (47 pages).

G. Sargsyan. *Covering with universally Baire operators.* ***Advances in Mathematics***, vol. 268 (2015), pp. 603–665.

N. Trang. PFA *and guessing models.* ***Israel Journal of Mathematics***, vol. 215 (2016), pp. 607–667.

The proper forcing axiom, PFA, plays a crucial role in modern set theory and in addition has several applications outside of set theory (cf. J. T. Moore, *The Proper Forcing Axiom*, ***Proceedings of the International Congress of Mathematicians***, Hyderabad, India, 2010, for an expository article including the definition of PFA and some of these applications). It is a natural strengthening of Martin's Axiom and was formulated and shown to be consistent relative to the existence of a supercompact cardinal by Baumgartner in the early 1980's. Since then, a major open question in set theory is whether this is optimal, i.e., whether the consistency of a supercompact cardinal follows from PFA. More generally, the relation between forcing axioms and large cardinals is a central theme in modern set theory. Work on this has led to many new ideas and also driven the advancement of powerful techniques used to get closer to an answer. The aim of this review is to outline some recent development in the search for an optimal large cardinal strength lower bound for PFA and give an idea of the relevant techniques.

There are different ways to tackle the question about the strength of PFA. One approach is to study the methods which are used to obtain models of PFA. Following this theme, Viale and Weiß showed in *On the consistency strength of the proper forcing axiom*, ***Advances in Mathematics***, vol. 228 (2011), pp. 2672–2687, that to force PFA with a proper forcing a supercompact cardinal is indeed necessary. Here, we aim to derive consistency strength from PFA by proving that under PFA inner models with large cardinals exist. In the arguments we want to consider, strength is not derived from PFA directly but from consequences involving combinatorial principles called *square sequences.* Todorčević proved in *A note on the proper forcing axiom*, ***Contemporary Mathematics***, vol. 31 (1984), pp. 209—218, that if PFA holds, $\square(\kappa)$ fails for all regular cardinals $\kappa > \omega_1$. Here $\square(\kappa)$ denotes a combinatorial principle introduced by Todorčević stating the existence of a coherent sequence of clubs of length κ which cannot be threaded. This easily implies the failure of $\square_\kappa$ for all uncountable cardinals κ, where $\square_\kappa$ is a similar but stronger combinatorial principle introduced by Jensen stating the existence of a coherent sequence of length κ^+ consisting of clubs with small ordertypes at limits. All lower bound results for PFA described below crucially use the failure of a version of these combinatorial principles.

As the arguments require a lot of inner model theoretic techniques, we start by outlining the general pattern underlying these. The general method used is the *core model induction* which allows us to construct mice with stronger and stronger large cardinals. While we focus on the consistency strength of PFA here, it should be mentioned that the core model induction technique is a very general tool that can be used to obtain lower bounds in terms of large cardinal strength in a variety of settings, even with a limited amount of choice. Examples include statements about ideals on ω_1 (cf. Ketchersid's PhD thesis *Toward* $\mathrm{AD}_{\mathbb{R}}$ *from the Continuum Hypothesis and an* ω_1*-dense ideal*, Berkeley, 2000), the failure of the unique branch hypothesis, UBH, for tame trees (cf. Sargsyan, Trang, *Non-tame mice from tame failures of the unique branch hypothesis*,

Canadian Journal of Mathematics, vol. 66(4) (2014), pp. 903–923 and *Tame failures of the unique branch hypothesis and models of* $\mathrm{AD}_{\mathbb{R}} + \Theta$ *is regular*, ***Journal of Mathematical Logic***, vol. 16(2) (2016), 1650007), ω_2-guessing models (cf. the paper by Trang under review), and the hypothesis that all uncountable cardinals are singular under ZF (cf. Busche, Schindler, *The strength of choiceless patterns of singular and weakly compact cardinals*, ***Annals of Pure and Applied Logic***, vol. 159 (2009), pp. 198–248 and recent so far unpublished results of Adolf).

In our setting, a first baby example in the direction of the core model induction is the argument that under PFA, by appealing to Jensen's Covering Lemma, $0^\sharp$ exists. Developing this idea further, Schimmerling proved in *Combinatorial principles in the core model for one Woodin cardinal*, ***Annals of Pure and Applied Logic***, vol. 74(2) (1995), pp. 153—201, that under PFA, $M_1^\sharp(X)$, the canonical inner model with a Woodin cardinal and a sharp, exists for all sets X. He used a covering result for Steel's core model K, the fact that certain square sequences exist in canonical inner models, and a generalization of Todorčević's result on square sequences from PFA due to Magidor. Building on this, Steel and Woodin independently showed that PFA in fact implies that $M_n^\sharp(X)$, the canonical inner model with n Woodin cardinals and a sharp, exists for all sets X and all $n < \omega$. So in particular, PFA implies projective determinacy. Their argument proceeds inductively on n producing stronger and stronger core models yielding more and more Woodin cardinals. This already explains the terminology *core model induction* but the key ideas only show up beyond the projective hierarchy. At this point the focus starts shifting from inner models with large cardinals towards models of determinacy. To push the argument further and obtain the Axiom of Determinacy in $L(\mathbb{R})$, $\mathrm{AD}^{L(\mathbb{R})}$, descriptive set theoretic methods come into play to organize the induction through the $L(\mathbb{R})$-hierarchy. This technique was first used by Woodin in his proof that PFA together with a strongly inaccessible cardinal implies $\mathrm{AD}^{L(\mathbb{R})}$. An extension of the core model induction technique beyond $L(\mathbb{R})$ was first done by Ketchersid in his PhD thesis.

Before we go into the details of the four papers under review which use and extend the core model induction technique, we should mention that there are also results obtaining strength from PFA using different inner model theoretic tools. First, Andretta, Neeman, and Steel showed in *The domestic levels of* K^c *are iterable*, ***Israel Journal of Mathematics***, vol. 125 (2001), pp. 157–201, that PFA together with a measurable cardinal yields a transitive model of $\mathrm{AD}_{\mathbb{R}}$ containing all reals and ordinals. Then, building on this, Jensen, Schimmerling, Schindler, and Steel proved in *Stacking mice*, ***Journal of Symbolic Logic***, vol. 74(1) (2009), pp. 315–335, that PFA implies the existence of a sharp for a proper class of strong cardinals and a proper class of Woodin cardinals by proving a covering theorem for a certain stack of mice. Moreover, Neeman and Trang showed independently in unpublished work that PFA together with a Woodin cardinal implies the existence of a model with a Woodin cardinal that is a limit of Woodin cardinals. However, in all three results the methods seem hard to generalize to obtain stronger lower bounds or, in the third result, an equally strong lower bound without assuming the existence of large cardinals in addition to PFA.

In the first paper under review, PFA *implies* $\mathrm{AD}^{L(\mathbb{R})}$, Steel proves that AD holds in $L(\mathbb{R})$ from $\neg\Box_\kappa$ for a single singular strong limit cardinal κ, so in particular from PFA without additionally assuming any large cardinals. One key ingredient in the argument is Schimmerling and Zeman's result that $\Box_\kappa$ holds in certain mice as this yields a failure of covering. The proof proceeds inductively through the $L(\mathbb{R})$-hierarchy as given by the Wadge hierarchy of sets of reals in $L(\mathbb{R})$ in the following sense. Given a pointclass Γ such that all sets in Γ are determined, the inductive step moves on to the next scaled pointclass Γ' and he argues that all sets in Γ' are determined. The argument exploits the tight connection between mice and determinacy and during the induction there is an additional inductive hypothesis on the existence of mice which is carried along. This inductive hypothesis is first phrased in a coarse way and it postulates the existence of countable models with finitely many Woodin cardinals together with sufficiently nice definable iteration strategies. But in fact during the argument the constructed mice will actually be fine structural, either in the ordinary sense or in the hybrid sense, meaning that they can be construed from an extender sequence and an iteration strategy instead of just an extender sequence.

Later, Sargsyan in *Nontame mouse from the failure of square at a singular strong limit cardinal* extended Steel's result on the strength of PFA to a core model induction beyond the $L(\mathbb{R})$-hierarchy essentially up to the same level that was reached by Ketchersid. More precisely, he proved the following theorem, where Θ is the supremum of all ordinals α such that there is a surjection $f\colon \mathbb{R} \twoheadrightarrow \alpha$ and θ_0, the first member of the Solovay sequence, is defined similarly but with restricting to functions f which are ordinal definable.

THEOREM 1 (Sargsyan). *Suppose $\neg\Box_\kappa$ holds for some singular strong limit cardinal κ. Then there is a model M containing all reals and ordinals such that*

$$M \vDash \text{“AD}^+ + \theta_0 < \Theta\text{”}.$$

In terms of large cardinal strength, Steel and Woodin proved that the conclusion of this theorem implies the existence of a nontame mouse, i.e., a mouse which has an extender overlapping a Woodin cardinal. The general scheme of proof of Sargsyan's theorem follows the one in Ketchersid's thesis and essentially splits into two parts. We always work under a minimality assumption which is essentially saying that a model M as in the theorem cannot exist in generic extensions of V. Then in what he calls the *internal* step of the core model induction the aim is to prove that a given model satifies AD^+. This is done by the same methods as in Steel's paper PFA *implies* $\mathrm{AD}^{L(\mathbb{R})}$. The more interesting part is the *external* step. Here he builds a model contradicting the minimality assumption by constructing a sufficiently nice iteration strategy Σ in a generic extension of V. This is done using direct limits of hulls and ideas from HOD analyses. The nice properties of Σ will then ensure that when constructing over the reals relative to Σ, the resulting model satisfies $\mathrm{AD}^+ + \theta_0 < \Theta$. Here the fact that it satisfies AD^+ follows from the internal step of the core model induction. The crucial part where Sargsyan's argument is different from Ketchersid's is in the proof that this strategy Σ is sufficiently nice, more precisely that it satisfies a property called branch condensation.

It remained open how to perform a core model induction up to the level of $\mathrm{AD}_\mathbb{R} + \Theta$ is regular. In *Covering with universally Baire operators* Sargsyan provided the first such example in the proof of the following statement. It is beyond the scope of this review to define all concepts involved in the statement of the theorem but we decided to state the theorem anyway and direct the interested reader to Sargsyan's paper for the details.

THEOREM 2 (Sargsyan). *Suppose κ is a measurable limit of strong cardinals that are S-reflecting where $S = \{\eta < \kappa \colon \eta \text{ is strong}\}$. Then one of the following holds:*

1. *Some symmetric hybrid K^c construction below κ converges.*
2. *Covering with lower parts holds at κ.*
3. *There is a model M containing all reals and ordinals such that*

$$M \vDash \text{“AD}^+ + \Theta \textit{ is regular”}.$$

This is closely connected to lower bounds for PFA as under PFA options 1. and 2. in Theorem 2 fail. Thus if PFA holds and there is a measurable limit κ of strong cardinals that are S-reflecting as in the theorem, then there is a model M containing all reals and ordinals such that $M \vDash$ "$\mathrm{AD}^+ + \Theta$ is regular". Sargsyan conjectures that Theorem 2 should still hold when 3. is replaced by "There is a mouse with a superstrong cardinal" and calls this the *UB-Covering Conjecture*. A proof of this conjecture would actually yield the existence of an inner model with a superstrong cardinal from PFA and a measurable limit κ of strong cardinals that are S-reflecting as in the theorem. But it remained open how to obtain a similar result without any large cardinal assumption.

Finally, Trang in PFA *and guessing models* managed to perform a core model induction up to $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular under just PFA. He proved the following theorem.

THEOREM 3 (Trang). *Suppose κ is a cardinal with $\kappa^\omega = \kappa$ and $\neg\square(\alpha)$ holds for every $\alpha \in [\kappa^+, (2^\kappa)^+]$. Then there is a model M containing all reals and ordinals such that*

$$M \vDash \text{“}\mathrm{AD}^+ + \Theta \textit{ is regular”}.$$

Note that PFA implies the hypothesis stated in the theorem for $\kappa = \aleph_2$ as it in addition to the non-existence of square sequences mentioned earlier also implies $2^{\aleph_0} = \aleph_2$ (cf. results of Todorčević and Veličković).

All in all, these results suggest that the core model induction is a powerful technique to obtain lower bounds in consistency strength from forcing axioms such as PFA. In a so far unpublished book Sargsyan and Trang extended the core model induction technique and Trang's results to show that PFA implies the existence of a model M containing all reals and ordinals such that $M \vDash \mathrm{LSA}$. Here LSA is the *Largest Suslin Axiom* stating that AD^+ holds, that there is a largest Suslin cardinal, and that this largest Suslin cardinal is a member of the Solovay sequence. Sargsyan and Trang proved that LSA is consistent relative to a Woodin cardinal that is a limit of Woodin cardinals, so a natural next goal to aim for is to prove that PFA implies the existence of a model with a Woodin cardinal that is a limit of Woodin cardinals. But in their also so far unpublished work on Sealing Sargsyan and Trang showed that this goal seems unreachable with the current understanding of the core model induction technique and therefore requires a major new idea.

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Entry for the Table of Contents: