PROSEMINAR AXIOMATIC SET THEORY I (S2018): 04.05.2018

For all exercises on this sheet, assume ZF.

Exercise 1. Assume ZFC and let κ be an infinite cardinal.

(1) Show that $H(\kappa) \subseteq V_\kappa$.
(2) Suppose that κ is inaccessible. Show that $H(\kappa) = V_\kappa$.

We say a formula is Π_1 if it is of the form $\forall x\, \varphi$ where φ is a Δ_0 formula.
Note that since Δ_0 formulas are absolute, Π_1 formulas are *downward absolute*, i.e. if $\varphi(z)$ is Π_1 and M is a transitive model, then for all $z \in M$, $\varphi(z)$ implies $\varphi^M(z)$.

Exercise 2. Show that the following notions are defined by formulas that ZF proves are equivalent to Π_1 formulas and hence are absolute for every transitive model M of ZF.

(1) x is a cardinal,
(2) x is a regular cardinal,
(3) x is a limit cardinal,
(4) x is a weakly inaccessible cardinal.

Exercise 3. Let κ be a weakly inaccessible cardinal.

(1) Argue that κ is an inaccessible cardinal in L.
(2) Conclude that $L_\kappa \models \mathrm{ZFC} + \text{“}V = L\text{”}$ and hence that the existence of weakly inaccessible cardinals cannot be proven in ZFC.

Hint: Use Exercises 1 and 2 together with results from the lecture.

Recall, that we defined

$$\mathrm{OD} = \bigcup \{\mathrm{OD}_{V_\eta} \mid \eta \in \mathrm{ON}\},$$

where OD_{V_η} is the set of all elements $a \in V_\eta$ that are definable in $(V_\eta, \in)$ with parameters in $\mathrm{ON} \cap V_\eta = \eta$. Moreover, we defined

$$\mathrm{HOD} = \{x \in \mathrm{OD} \mid \mathrm{trcl}(x) \subseteq \mathrm{OD}\},$$

where $\mathrm{trcl}(x)$ denotes the transitive closure of x.

Exercise 4.

(1) Show that all axioms of ZF hold in HOD.
(2) For every $A \in \mathrm{HOD}$, show that there is a well-order of A that is definable from ordinals. Conclude, that the Axiom of Choice holds in HOD.

E-mail address: `mueller.sandra@logic.univie.ac.at`