Orientations of graphs with uncountable chromatic number

Dániel T. Soukup

http://renyi.hu/~dsoukup/
Introduction

Goal: present results on **chromatic number of directed uncountable graphs**.

- first organized effort (undirected case): P. Erdős and A. Hajnal in the 1960s;
- significant contributions: P. Komjáth, S. Todorcevic, S. Shelah, C. Thomassen...
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What is the chromatic number?

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The **chromatic number** of a graph $G$, denoted by $\text{Chr}(G)$, is the least cardinal $\kappa$ such that the vertices of $G$ can be covered by $\kappa$ many independent sets.

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The first results

- **Tutte, 1954:** There are $\Delta$-free graphs of arbitrary large finite chromatic number.

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Two giants of combinatorics share a passion: Erdős and William T. Tutte play “Go” at Tutte’s home in Westmountrose, Ontario, 1985. Another favorite game of Erdős’s was Ping-Pong.
Obligatory subgraphs

What graphs must occur as subgraphs of uncountably chromatic graphs?

- Erdős-Rado, 1959: There are $\Delta$-free graphs with size and chromatic number $\kappa$ for each infinite $\kappa$.

- Erdős-Hajnal, 1966: If $\text{Chr}(G) > \omega$ then $K_{n,\omega_1}$ embeds into $G$ for each $n \in \omega$.

In particular, any even cycle embeds into $G$. 
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- **Erdős et al, Thomassen 1983:** If $Chr(G) > \omega$ then there is an $n \in \omega$ such that any odd cycle of length bigger than $n$ embeds into $G$.

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If $\kappa$ is large enough then the chromatic number of $\text{Sh}_n(\kappa)$ is large.

No odd cycles of length $\leq 2n - 1$. 
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The **dichromatic number** of a digraph $D$, denoted by $\overrightarrow{\chi}(D)$, is the least cardinal $\kappa$ such that the vertices of $D$ can be covered by $\kappa$ many acyclic sets.

What are the implications of large dichromatic number? How is $\overrightarrow{\chi}(D)$ related to the chromatic number of the underlying graph?

If $\overrightarrow{\chi}(D) \geq \kappa$ then the underlying directed graph must have chromatic number $\geq \kappa$. 

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The girth and dichromatic number

[Bokal et al, 2004] There are digraphs with arbitrary large digirth and arbitrary large finite dichromatic number.

How about uncountable dichromatic number?

[DS, 2016] Let $\lambda = \exp_n(\kappa)$ for some $2 \leq n < \omega$ and infinite $\kappa$. Then there is an orientation $D$ of $\text{Sh}_n(\lambda)$ so that whenever $G : [\lambda]^n \rightarrow \kappa$ then there is a monochromatic directed 4-cycle in $D$.

In particular, short odd cycles can be avoided while the dichromatic number is as large as we wish.
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Recall: $C_4 \leftrightarrow G$ if $\chi(G) > \omega$.

[DS, 2016] Consistently, for each $n \in \omega$ there is a digraph $D = D_n$ on vertex set $\omega_1$ so that

1. $D$ has no directed cycles of length $\leq n$, and
2. $C_{n+1} \rightarrow D[X]$ for every uncountable $X \subseteq \omega_1$.

Consistently, there are graphs with uncountable dichromatic number and arbitrarily large digirth. Compactess arguments give the [Bokal et al, 2004] result as a corollary.
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Recall: large dichromatic # implies large chromatic # for the underlying graph.

\[ \text{[Erdős, Neumann-Lara, 1979]} \text{ Is there a function } f : \mathbb{N} \to \mathbb{N} \text{ so that } \chi(G) \geq f(n) \text{ implies } \overrightarrow{\chi}(D) \geq n \text{ for some orientation } D \text{ of } G? \]

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Yes and no.
A few open problems

Is there an elementary way of defining a digraph $D$ with $\chi^-(D) > \omega$?

Does $\chi^-(D) > \omega$ imply that cycles of all but finitely many length embed into $D$?

Does $\chi^-(D) > \omega$ imply that there is a strongly 2-connected subgraph of $D$?

Suppose that $G$ has orientations $D_\xi$ so that $\sup \chi^-(D_\xi) = \kappa$. Is there a single orientation $D$ with $\chi^-(D) = \kappa$?
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