Strongly surjective linear orders

Dániel T. Soukup

http://www.logic.univie.ac.at/~soukupd73/
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- various consistency results;
- is there an example in ZFC?
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Consider the class of linear orders with order preserving embeddings.

Among countable linear orders:
- $\omega$ and $-\omega$ are the only **minimal** linear orders;
- $\mathbb{Q}$ is the **unique dense** l.o. without endpoints.

How about uncountable linear orders?
- $\omega_1$ and $-\omega_1$ are minimal,
- $L$ is short if $\omega_1, -\omega_1 \not\rightarrow L$
- suborders of $\mathbb{R}$, or
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Select $\ell_k \in f^{-1}(k)$ and note that $K \simeq \{\ell_k : k \in K\} \hookrightarrow L$.

[CCM 2015] When is this implication reversible?

$L$ is strongly surjective if $K \leftrightarrow L$ implies $L \rightarrow K$.

- $\omega, -\omega$ and $\mathbb{Q}$ are strongly surjective,
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If $L \subseteq \mathbb{R}$ is Borel and strongly surjective then $|L| \leq \omega$. 
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Uncountable suborders of $\mathbb{R}$

[CCM 2016] If $A$ is the unique $\kappa$-dense suborder of $\mathbb{R}$ (i.e. $\text{BA}_\kappa$ holds) then $A$ is strongly surjective.

[Baumgartner 1970] $\text{PFA} \rightarrow \text{BA}_{\aleph_1}$ [Neeman ?] $\text{Con} (\text{BA}_{\aleph_2})$

Note: these examples are all minimal and homogeneous under MA.

Consistently, there is an $\aleph_1$-dense, strongly surjective $L \subseteq \mathbb{R}$ which is not minimal and not homogeneous.

[Abraham, Rubin, Shelah 1985] Consistently, $\text{MA}_{\aleph_1} + \text{OCA} + \text{ISA}$. 
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Some questions

Does every uncountable, strongly surjective l.o. contain a minimal suborder?

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\text{MA}_{\aleph_1} \not\rightarrow \text{there is an uncountable, strongly surjective } L \subseteq \mathbb{R}
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There is no 2-entangled, strongly surjective linear order.

- 2-entangled implies no minimal suborder, and
- \(\text{Con}(\text{MA}_{\aleph_1} + \text{every uncountable } L \subseteq \mathbb{R} \text{ has a 2-entangled suborder})\).

Suppose \(L \subseteq \mathbb{R}\) is strongly surjective and rigid. Is \(L\) countable?
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The effects of CH and relatives

CH implies $\neg \text{BA}_{\aleph_1}$ and there are no minimal suborders of $\mathbb{R}$.

[CCM 2016] $2^{\aleph_0} < 2^{\kappa} \Rightarrow$ no strongly surjective $L \subseteq \mathbb{R}$ of size $\kappa$.

Every uncountable, strongly surjective linear order is Aronszajn if

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(◊⁺) There is a strongly surjective, lex. ordered Suslin-tree $T$.

Key property [CCM 2016]:

$T$ is Suslin $+$ doubly isomorphic to all large subtrees.

1. [Baumgartner 1982] the proof is oversimplified (false lemma);
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(\(\text{CH} + \text{axiom (A)}\)) Every strongly surjective linear order is countable.

From [Moore 2007]:

- \((A) \equiv \) any ladder system colouring can be uniformized on an arbitrary Aronszajn tree;
- \((\text{CH} + (A)) \) \(\omega_1\) and \(-\omega_1\) are the only minimal uncountable l. orders;
- \((A)\) is forced from \(\text{CH}\) using a CSI of proper posets with NNR.
Open problems

Consistently, are there strongly surjective linear orders of size $> \aleph_2$?

[ARS 1985] Is it consistent that $\neg \text{BA}_{\aleph_1}$ but $A \leftrightarrow B$ or $B \leftrightarrow A$ for any two $\aleph_1$-dense $A, B \subseteq \mathbb{R}$?

Suppose that $L$ is strongly surjective and $x \in L$. Is $L \setminus \{x\}$ strongly surjective?
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Can Countryman lines be strongly surjective?
They are minimal under $\text{MA}_{\aleph_1}$.

Is the universal A-line $\eta_C$ strongly surjective under PFA?

[AS 1985] Is it consistent that there is a unique Suslin tree?
I.e. there is a Suslin tree, but any two Suslin trees are isomorphic on a club.
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Thank you for your attention!