HOD IN INNER MODELS WITH WOODIN CARDINALS

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Abstract. We analyze the hereditarily ordinal definable sets HOD in $M_n(x)[g]$ for a Turing cone of reals x, where $M_n(x)$ is the canonical inner model with n Woodin cardinals build over x and g is generic over $M_n(x)$ for the Lévy collapse up to its bottom inaccessible cardinal. We prove that assuming $\mathbf{\Pi}^1_{n+2}$-determinacy, for a Turing cone of reals x, $\mathrm{HOD}^{M_n(x)[g]} = M_n(\mathcal{M}_\infty|\kappa_\infty, \Lambda)$, where $\mathcal{M}_\infty$ is a direct limit of iterates of M_{n+1}, δ_∞ is the least Woodin cardinal in $\mathcal{M}_\infty$, κ_∞ is the least inaccessible cardinal in $\mathcal{M}_\infty$ above δ_∞, and Λ is a partial iteration strategy for $\mathcal{M}_\infty$. It will also be shown that under the same hypothesis $\mathrm{HOD}^{M_n(x)[g]}$ satisfies GCH.

1. Introduction

An essential question regarding the theory of inner models is the analysis of the class of all hereditarily ordinal definable sets HOD inside various inner models M of the set theoretic universe V under appropriate determinacy hypotheses. Examples for such inner models M are $L(\mathbb{R})$, $L[x]$, and the canonical proper class x-mouse with n Woodin cardinals $M_n(x)$, but nowadays also larger models of determinacy M are considered.

One motivation for analyzing the internal structure of these models HOD^M is given by Woodin's results in [KW10] that under determinacy hypotheses these models contain large cardinals. He showed in [KW10] for example that assuming Δ^1_2 determinacy there is a Turing cone of reals x such that $\omega_2^{L[x]}$ is a Woodin cardinal in the model $\mathrm{HOD}^{L[x]}$. This result generalizes to higher levels in the projective hierarchy. That means for $n \geq 1$ assuming $\mathbf{\Pi}^1_{n+1}$ determinacy and Π^1_{n+2} determinacy there is a cone of reals x such that $\omega_2^{M_n(x)}$ is a Woodin cardinal in the model $\mathrm{HOD}^{M_n(x)|\delta_x}$, where $M_n(x)$ denotes the canonical proper class x-mouse with n Woodin cardinals and δ_x is the least Woodin cardinal in $M_n(x)$. Moreover, Woodin showed a similar result for $\mathrm{HOD}^{L(\mathbb{R})}$. If we let Θ denote the supremum of all ordinals α

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such that there exists a surjection $\pi : \mathbb{R} \to \alpha$, then assuming $\mathrm{ZF} + \mathrm{AD}$, he showed that $\Theta^{L(\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$ (see [KW10]). The fact that these models of the form HOD^M can have large cardinals as for example Woodin cardinals motivates the question if they are in some sense fine structural as for example the models $L[x], M_n(x)$, and $L(\mathbb{R})$ are. A good test question for this is whether these models HOD^M satisfy the generalized continuum hypothesis GCH. If it turns out that HOD^M is in fact a fine structural model, it would follow that it satisfies the GCH and even stronger combinatorial principles as for example the $\diamondsuit$ principle.

The first model which was analyzed in this sense was $\mathrm{HOD}^{L(\mathbb{R})}$ under the assumption that every set of reals in $L(\mathbb{R})$ is determined (short: $\mathrm{AD}^{L(\mathbb{R})}$). Using purely descriptive set theoretic methods Becker showed in [Be80] under this hypothesis that GCH_α, i.e. $2^\alpha = \alpha^+$, holds in $\mathrm{HOD}^{L(\mathbb{R})}$ for all $\alpha < \omega_1^{L(\mathbb{R})}$. Later J. R. Steel and W. H. Woodin were able to push the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ forward using more recent advances in inner model theory. In 1993 they first showed independently that the reals in $\mathrm{HOD}^{L(\mathbb{R})}$ are the same as the reals in M_ω, the least proper class iterable premouse with ω Woodin cardinals. Then they showed in §4 of [St93] that $\mathrm{HOD}^{L(\mathbb{R})}$ in fact agrees with the inner model N up to $\mathcal{P}(\omega_1^{L(\mathbb{R})})$, where N denotes the $\omega_1^{L(\mathbb{R})}$-th linear iterate of M_ω by its least measure and its images. Building on this, John R. Steel was able to show in [St95] that $\mathrm{HOD}^{L(\mathbb{R})}$ agrees with the inner model $\mathcal{M}_\infty$ up to $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$, where $\mathcal{M}_\infty$ is a direct limit of iterates of M_ω and $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$ is the supremum of all ordinals α such that there exists a surjection $\pi : \mathbb{R} \to \alpha$ which is $\boldsymbol{\Delta}_1^{L(\mathbb{R})}$ definable. Finally, in 1996 W. Hugh Woodin extended this (see [StW16]) and showed that in fact $\mathrm{HOD}^{L(\mathbb{R})} = L[\mathcal{M}_\infty, \Lambda]$, where Λ is a partial iteration strategy for $\mathcal{M}_\infty$. For even larger models of determinacy M the corresponding model HOD^M was first analyzed in [Sa09], where the second author showed that it is fine structural using a layered hierarchy. Models of this form are nowadays called *hod mice*. A different approach for the fine structure of hod mice called the least branch hierarchy is studied in [St16].

The question if $\mathrm{HOD}^{L[x]}$ is a model of GCH or even a fine structural model for a Turing cone of reals x under a suitable determinacy hypothesis remains open until today. What has been done is the analysis of the model $\mathrm{HOD}^{L[x][G]}$, where G is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $\mathrm{HOD}^{L[x]}$ for the least inaccessible cardinal κ_x in $L[x]$. Woodin showed in the 1990's (see [StW16]) that assuming Δ^1_2 determinacy there is a Turing cone of reals x such that $\mathrm{HOD}^{L[x][G]} = L[\mathcal{M}_\infty, \Lambda]$, where $\mathcal{M}_\infty$ is a direct limit of mice (which are iterates of M_1) and Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

In this article, we analyze HOD in the model $M_n(x)[g]$ for any real x of sufficiently high Turing degree under the assumption that every $\boldsymbol{\Pi}^1_{n+2}$ set of reals is determined. Here g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(x)$, where κ denotes the least inaccessible cardinal in $M_n(x)$. We first show that the direct limit model $\mathcal{M}_\infty$, obtained from iterates of suitable premice, agrees up to its bottom Woodin cardinal δ_∞ with $\mathrm{HOD}^{M_n(x)[g]}$. In a second step, we show that the full model $\mathrm{HOD}^{M_n(x)[g]}$ is in fact of the form $M_n(\hat{\mathcal{M}}_\infty|\kappa_\infty, \Lambda)$, where $\hat{\mathcal{M}}_\infty = M_n(\mathcal{M}_\infty|\delta_\infty)$, κ_∞ is the least inaccessible cardinal of $\hat{\mathcal{M}}_\infty$ above δ_∞, and Λ is a partial iteration strategy for $\mathcal{M}_\infty$. Here and below $M_n(\hat{\mathcal{M}}_\infty|\kappa_\infty, \Lambda)$ denotes the canonical fine structural model with n Woodin cardinals build over the coarse objects $\hat{\mathcal{M}}_\infty|\kappa_\infty$ and Λ. Our proof in fact shows that $\mathrm{HOD}^{M_n(x)[g]}$ is a model of GCH, $\Diamond$, and other combinatorial principles which are consequences of fine structure.

In the statement of the following main theorem and in fact everywhere in this article whenever we write HOD^M for some premouse M we mean $\mathrm{HOD}^{\lfloor M \rfloor}$, where $\lfloor M \rfloor$ denotes the universe of the model M. In particular, we do not allow the extender sequence of M as a parameter in the definition of HOD. It will be clear from the context if we consider the model M or the universe $\lfloor M \rfloor$ of M, therefore we decided for the sake of readability to not distinguish the notation for these two objects.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $n < \omega$ and assume $\mathbf{\Pi}^1_{n+2}$-determinacy. Then for a Turing cone of reals x,*

$$\mathrm{HOD}^{M_n(x)[g]} = M_n(\hat{\mathcal{M}}_\infty|\kappa_\infty, \Lambda),$$

where g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(x)$, κ denotes the least inaccessible cardinal in $M_n(x)$, $\hat{\mathcal{M}}_\infty$ is a direct limit of iterates of M_{n+1}, δ_∞ is the least Woodin cardinal in $\hat{\mathcal{M}}_\infty$, κ_∞ is the least inaccessible cardinal of $\hat{\mathcal{M}}_\infty$ above δ_∞, and Λ is a partial iteration strategy for $\mathcal{M}_\infty$.

Our proof in fact shows the following corollary.

Corollary 1.2. *Assume $\mathbf{\Pi}^1_{n+2}$-determinacy. Then for a Turing cone of reals x,*

$$\mathrm{HOD}^{M_n(x)[g]} \models \mathrm{GCH},$$

where g is $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M_n(x)$ and κ denotes the least inaccessible cardinal in $M_n(x)$.

Remark. In fact the full strength of $\mathbf{\Pi}^1_{n+2}$-determinacy is not needed for these results. It suffices to assume that $M_n^\#(x)$ exists and is ω_1-iterable for all reals x (or equivalently $\mathbf{\Pi}^1_{n+1}$-determinacy, see [MSW] and [Ne02]) and that $M_{n+1}^\#$ exists and is ω_1-iterable. This is all we will use in the proof.

Finally, we summarize some open questions related to these results. The following question already appears in [StW16].

Question 1. *Assume $\mathbf{\Delta}^1_2$ determinacy. Is $\mathrm{HOD}^{L[x]}$ for a cone of reals x a fine structural model?*

Question 2. *Assume $\mathbf{\Pi}^1_{n+2}$ determinacy. Is $\mathrm{HOD}^{M_n(x)}$ for a cone of reals x a fine structural model?*

This article is structured as follows. In Section 2 we recall some preliminaries and fix the basic notation. In Section 3 we recall the relevant notions from [Sa13] and define the direct limit system converging to $\mathcal{M}_\infty$, before we compute $\mathrm{HOD}^{M_n(x)[g]}$ up to its Woodin cardinal in Section 4. In Section 5 we then show how this can be used to compute the full model $\mathrm{HOD}^{M_n(x)[g]}$, i.e., we finish the proof of Theorem 1.1. The authors thank Farmer Schlutzenberg for the helpful discussions during the 4th Münster conference on inner model theory in the summer of 2017. Finally, the authors thank the referee for carefully reading the paper and making several helpful comments and suggestions.

2. Preliminaries and notation

Whenever we say *reals* we mean elements of the Baire space ${}^\omega\omega$. We also write $\mathbb{R}$ for ${}^\omega\omega$. HOD denotes the class of all hereditarily ordinal definable sets. Moreover HOD_x for any $x \in {}^\omega\omega$ denotes the class of all sets which are hereditarily ordinal definable over $\{x\}$.[1] That means we let $A \in \mathrm{OD}_x$ iff there is a formula φ such that $A = \{v \mid \varphi(v, \alpha_1, \dots, \alpha_n, x)\}$ for some ordinals $\alpha_1, \dots, \alpha_n$. Then $A \in \mathrm{HOD}_x$ iff $\mathrm{TC}(\{A\}) \subset \mathrm{OD}_x$, where $\mathrm{TC}(\{A\})$ denotes the transitive closure of the set $\{A\}$.

We use the notions of premice and iterability from [St10, §1 − 4] and assume that the reader is familiar with the basic concepts defined there. In most cases we will demand $(\omega, \omega_1, \omega_1)$-iterability in the sense of Definition 4.4 in [St10] for our mice, but in other cases or if it is not clear from the context we will state the precise amount of iterability. We say a *cutpoint* of a premouse $\mathcal{M}$ is an infinite ordinal γ such that there is no extender E on the $\mathcal{M}$-sequence with $\mathrm{crit}(E) \leq \gamma \leq \mathrm{lh}(E)$.[2]

For some ZFC model M and some real $x \in M$ we write $L[E](x)^M$ for the result of a fully backgrounded extender construction above x inside M in the sense of [MS94], with the minimality condition relaxed to ω-small premice. Moreover, we let for a premouse $\mathcal{M}$ with $\mathcal{M} \vDash \mathrm{ZFC}$, a cardinal cutpoint η of $\mathcal{M}$, and a premouse $\mathcal{N}$ of height η such that $\mathcal{N} \in \mathcal{P}(\mathcal{M}|\eta) \cap \mathcal{M}|(\eta + \omega)$, $\mathcal{P}^{\mathcal{M}}(\mathcal{N})$ denote the result of a $\mathcal{P}$-construction over $\mathcal{N}$ inside the model $\mathcal{M}$ in the sense of [SchSt09] or [Sa13, Proposition 2.3 and Definition 2.4].

For $x \in {}^\omega\omega$ and $n \leq \omega$ we let $M_n^\#(x)$, if it exists, denote a countable, sound, ω_1-iterable x-premouse which is not n-small but all of whose proper initial segments are n-small. In fact, ω_1-iterability suffices to show that such an $M_n^\#(x)$ is unique. If $M_n^\#(x)$ exists, we let $M_n(x)$ be the proper

[1] In the literature this is sometimes also called $\mathrm{HOD}_{\{x\}}$.

[2] Such a cutpoint γ is often also called a strong cutpoint.

class premouse obtained by iterating the top extender of $M_n^\#(x)$ out of the universe.

3. The direct limit system

To show that $\mathrm{HOD}^{M_n(x)[g]}$ is a fine structural inner model, we will use an extension of the direct limit system introduced in [Sa13]. For the reader's convenience we will first recall the relevant definitions and results from [Sa13], obtaining a direct limit system which is definable in $M_n(x)$. We use the chance to correct some minor errors in the presentation of that direct limit system in [Sa13]. Then we discuss the changes we need to make to obtain a direct limit system definable in $M_n(x)[g]$. Another application of a similar but slightly different direct limit system as in [Sa13] can be found in [SaSch18].

Fix an arbitrary natural number n. Throughout the rest of this article we will assume that $M_{n+1}^\#$ exists and is $(\omega, \omega_1, \omega_1)$-iterable and fix a real x that codes $M_{n+1}^\#$. This implies $\mathbf{\Pi}_{n+1}^1$ determinacy or equivalently that $M_n^\#(z)$ exists and is $(\omega, \omega_1, \omega_1)$-iterable for all reals z (see [Ne95] and [MSW] for a proof of this equivalence due to Itay Neeman and W. Hugh Woodin). Finally, we fix a $\mathrm{Col}(\omega, {<}\kappa)$-generic g over $M_n(x)$, where κ is the least inaccessible cardinal in $M_n(x)$.

The first direct limit system. We first recall the definition of a lower part premouse.

Definition 3.1. *Let a be a countable, transitive, self-wellordered[3] set. Then we define the lower part model $Lp^n(a)$ as the model theoretic union of all countable a-premice M with $\rho_\omega(M) = a$ which are n-small, sound, and $(\omega, \omega_1, \omega_1)$-iterable.*

If $\mathcal{N}$ is a countable premouse, we also use $Lp^n(\mathcal{N})$ to denote the premouse extending $\mathcal{N}$ which is defined similarly as the model theoretic union of premice $M \trianglerighteq \mathcal{N}$ with $\rho_\omega(M) \leq \mathcal{N} \cap \mathrm{Ord}$ which have $\mathcal{N} \cap \mathrm{Ord}$ as a cutpoint, are n-small above $\mathcal{N} \cap \mathrm{Ord}$, sound above $\mathcal{N} \cap \mathrm{Ord}$, and $(\omega, \omega_1, \omega_1)$-iterable above $\mathcal{N} \cap \mathrm{Ord}$.

Definition 3.2. *A countable premouse $\mathcal{N}$ is n-*suitable *iff there is an ordinal δ such that*

(1) $\mathcal{N} \vDash$ "$\mathrm{ZFC} - $Replacement" and $\mathcal{N} \cap \mathrm{Ord} = \sup_{i<\omega} (\delta^{+i})^{\mathcal{N}}$,
(2) $\mathcal{N} \vDash$ "δ is a Woodin cardinal",
(3) $\mathcal{N}$ is $(n+1)$-small,
(4) for every cutpoint $\gamma < \delta$ of $\mathcal{N}$, γ is not Woodin in $Lp^n(\mathcal{N}|\gamma)$,
(5) $\mathcal{N}|(\delta^{+(i+1)})^{\mathcal{N}} = Lp^n(\mathcal{N}|(\delta^{+i})^{\mathcal{N}})$ for all $i < \omega$, and
(6) for all $\eta < \delta$, $\mathcal{N} \vDash$ "$\mathcal{N}|\delta$ is (ω, η, η)-iterable".

[3] We say a transitive set a is *self-wellordered* iff a is wellordered in $L_\omega[a]$.

If $\mathcal{N}$ is an n-suitable premouse we denote the ordinal δ from Definition 3.2 by $\delta^{\mathcal{N}}$. Moreover, we write $\hat{\mathcal{N}} = M_n(\mathcal{N}|\delta^{\mathcal{N}})$ for any n-suitable premouse $\mathcal{N}$. Then $\mathcal{N} = \hat{\mathcal{N}}|((\delta^{\mathcal{N}})^{+\omega})^{\hat{\mathcal{N}}}$ for every n-suitable premouse $\mathcal{N}$ by well-known properties of the lower part model Lp^n. We now give some definitions indicating how n-suitable premice can be iterated.

Definition 3.3. *Let $\mathcal{N}$ be an arbitrary premouse and let $\mathcal{T}$ be an iteration tree on $\mathcal{N}$ of limit length.*

*(1) We say a premouse $\mathcal{Q} = \mathcal{Q}(\mathcal{T})$ is a $\mathcal{Q}$-*structure for $\mathcal{T}$ *iff $\mathcal{M}(\mathcal{T}) \trianglelefteq \mathcal{Q}$, $\mathcal{Q}$ is sound, $\delta(\mathcal{T})$ is a cutpoint of $\mathcal{Q}$, $\mathcal{Q}$ is $(\omega, \omega_1, \omega_1)$-iterable above $\delta(\mathcal{T})$, and if $\mathcal{Q} \neq \mathcal{M}(\mathcal{T})$*

$$\mathcal{Q} \models \text{“}\delta(\mathcal{T}) \text{ is a Woodin cardinal”},$$

and

(i) over $\mathcal{Q}$ there exists an $r\Sigma_n$-definable set $A \subset \delta(\mathcal{T})$ such that there is no $\kappa < \delta(\mathcal{T})$ such that κ is strong up to $\delta(\mathcal{T})$ with respect to A as being witnessed by extenders on the sequence of $\mathcal{Q}$ for some $n < \omega$, or

(ii) $\rho_n(\mathcal{Q}) < \delta(\mathcal{T})$ for some $n < \omega$.

*(2) Let b be a cofinal well-founded branch through $\mathcal{T}$. Then we say a premouse $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$ is a $\mathcal{Q}$-*structure for b in $\mathcal{T}$ *iff $\mathcal{Q}$ is sound and $\mathcal{Q} = \mathcal{M}_b^{\mathcal{T}}|\gamma$, where $\gamma \leq \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord}$ is the least ordinal such that either*

$$\gamma < \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord} \text{ and } \mathcal{M}_b^{\mathcal{T}}|(\gamma+1) \models \text{“}\delta(\mathcal{T}) \text{ is not Woodin”},$$

or

$$\gamma = \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord} \text{ and } \rho_n(\mathcal{M}_b^{\mathcal{T}}) < \delta(\mathcal{T})$$

for some $n < \omega$ or over $\mathcal{M}_b^{\mathcal{T}}$ there exists an $r\Sigma_n$-definable set $A \subset \delta(\mathcal{T})$ such that there is no $\kappa < \delta(\mathcal{T})$ such that κ is strong up to $\delta(\mathcal{T})$ with respect to A as being witnessed by extenders on the sequence of $\mathcal{M}_b^{\mathcal{T}}$ for some $n < \omega$.

If no such ordinal $\gamma \leq \mathcal{M}_b^{\mathcal{T}} \cap \mathrm{Ord}$ exists, we let $\mathcal{Q}(b, \mathcal{T})$ be undefined.

Remark. If it exists, $M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$ is n-suitable, where δ_0 is the least Woodin cardinal in M_{n+1}. We denote this premouse by M_{n+1}^- and write $\Sigma_{M_{n+1}^-}$ for its iteration strategy induced by the canonical $\mathcal{Q}$-structure guided iteration strategy $\Sigma_{M_{n+1}}$ for M_{n+1} for countable stacks of normal trees without drops on the main branches.

Our goal is to approximate the iteration strategy $\Sigma_{M_{n+1}^-}$ inside $\mathrm{HOD}^{M_n(x)[g]}$. Analogous to [SchlTr, Definition 5.32] we define the following requirement, which will be used in Definition 3.6 to make the proof of Lemmas 3.8 and 3.9 work.

Definition 3.4. *Let $\mathcal{N}$ be an n-suitable premouse and let $\mathcal{T}$ be a normal iteration tree on $\mathcal{N}$ of length $< \omega_1^V$. Then we say that $\mathcal{T}$ is* suitability strict *iff for all $\alpha < \mathrm{lh}(\mathcal{T})$,*

(*i*) *if* $[0, \alpha]_T$ *does not drop then* $\mathcal{M}_\alpha^\mathcal{T}$ *is* n*-suitable, and*
(*ii*) *if* $[0, \alpha]_T$ *drops then no* $\mathcal{R} \trianglelefteq \mathcal{M}_\alpha^\mathcal{T}$ *is* n*-suitable.*

Definition 3.5. *Let* $\mathcal{N}$ *be an* n*-suitable premouse and let* $\mathcal{T}$ *be a normal iteration tree on* $\mathcal{N}$ *of length* $< \omega_1^V$.

(1) $\mathcal{T}$ *is* correctly guided *iff for every limit ordinal* $\lambda < \mathrm{lh}(\mathcal{T})$*, if* b *is the branch choosen for* $\mathcal{T} \upharpoonright \lambda$ *in* $\mathcal{T}$*, then* $\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda)$ *exists and* $\mathcal{Q}(b, \mathcal{T} \upharpoonright \lambda) \trianglelefteq M_n(\mathcal{M}(\mathcal{T} \upharpoonright \lambda))$.
(2) $\mathcal{T}$ *is* short *iff* $\mathcal{T}$ *is correctly guided and in case* $\mathcal{T}$ *has limit length* $\mathcal{Q}(\mathcal{T})$ *exists and* $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_n(\mathcal{M}(\mathcal{T}))$.
(3) $\mathcal{T}$ *is* maximal *iff* $\mathcal{T}$ *is correctly guided and not short.*

Definition 3.6. *Let* $\mathcal{N}$ *be an* n*-suitable premouse. We say* $\mathcal{N}$ *is* short tree iterable *iff whenever* $\mathcal{T}$ *is a short tree on* $\mathcal{N}$,

(*i*) $\mathcal{T}$ *is suitability strict,*
(*ii*) *if* $\mathcal{T}$ *has a last model, then every putative*[4] *iteration tree* $\mathcal{U}$ *extending* $\mathcal{T}$ *such that* $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T}) + 1$ *has a well-founded last model, and*
(*iii*) *if* $\mathcal{T}$ *has limit length, then there exists a cofinal well-founded branch* b *through* $\mathcal{T}$ *such that* $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(\mathcal{T})$.

This can be generalized to stacks of correctly guided normal trees.

Definition 3.7. *Let* $\mathcal{N}$ *be an* n*-suitable premouse and* $m < \omega$*. Then we say* $(\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ *is a* correctly guided finite stack *on* $\mathcal{N}$ *iff*

(*i*) $\mathcal{N}_0 = \mathcal{N}$,
(*ii*) $\mathcal{N}_i$ *is* n*-suitable and* $\mathcal{T}_i$ *is a correctly guided normal iteration tree on* $\mathcal{N}_i$ *which acts below* $\delta^{\mathcal{N}_i}$ *for all* $i \leq m$,
(*iii*) *for every* $i < m$ *either* $\mathcal{T}_i$ *has a last model which is equal to* $\mathcal{N}_{i+1}$ *and the iteration embedding* $i^{\mathcal{T}_i} : \mathcal{N}_i \rightarrow \mathcal{N}_{i+1}$ *exists or* $\mathcal{T}_i$ *is maximal and* $\mathcal{N}_{i+1} = M_n(\mathcal{M}(\mathcal{T}_i)) | (\delta(\mathcal{T}_i)^{+\omega})^{M_n(\mathcal{M}(\mathcal{T}_i))}$.

Moreover, we say that $\mathcal{M}$ *is the last model of* $(\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ *iff either*

(*i*) $\mathcal{T}_m$ *has a last model which is equal to* $\mathcal{M}$ *and the iteration embedding* $i^{\mathcal{T}_m} : \mathcal{N}_m \rightarrow \mathcal{M}$ *exists,*
(*ii*) $\mathcal{T}_m$ *is of limit length and short and there is a non-dropping cofinal well-founded branch* b *through* $\mathcal{T}_m$ *such that* $\mathcal{Q}(b, \mathcal{T})$ *exists,* $\mathcal{T}_m{}^\frown b$ *is correctly guided, and* $\mathcal{M} = \mathcal{M}_b^\mathcal{T}$*, or*
(*iii*) $\mathcal{T}_m$ *is maximal and* $\mathcal{M} = M_n(\mathcal{M}(\mathcal{T}_m)) | (\delta(\mathcal{T}_m)^{+\omega})^{M_n(\mathcal{M}(\mathcal{T}_m))}$.

Finally, we say that $\mathcal{M}$ *is a* correct iterate *of* $\mathcal{N}$ *iff there is a correctly guided finite stack on* $\mathcal{N}$ *with last model* $\mathcal{M}$*. In case there is a correctly guided finite stack on* $\mathcal{N}$ *with last model* $\mathcal{M}$ *of length* 1*, i.e., such that* $m = 0$*, we say that* $\mathcal{M}$ *is a* pseudo-normal iterate *(or just* pseudo-iterate*) of* $\mathcal{N}$.

[4] An iteration tree $\mathcal{U}$ is a *putative iteration tree* if $\mathcal{U}$ satisfies all properties of an iteration tree, but in case $\mathcal{U}$ has a last model we allow this last model to be ill-founded.

Analogous to Theorem 3.14 in [StW16] we also have a version of the comparison lemma for short tree iterable premice and pseudo-normal iterates.

Lemma 3.8 (Pseudo-comparison lemma)**.** *Let $\mathcal{N}$ and $\mathcal{M}$ be n-suitable premice which are short tree iterable. Then there is a common pseudo-normal iterate $\mathcal{R} \in M_n(y)$ such that $\delta^{\mathcal{R}} \leq \omega_1^{M_n(y)}$, where y is a real coding $\mathcal{N}$ and $\mathcal{M}$.*

The proof of Lemma 3.8 is similar to the proof of Theorem 3.14 in [StW16], so we omit it. Similarly, we have an analogue to the pseudo-genericity iteration (see Theorem 3.16 in [StW16]).

Lemma 3.9 (Pseudo-genericity iterations)**.** *Let $\mathcal{N}$ be an n-suitable premouse which is short tree iterable and let z be a real. Then there is a pseudo-normal iterate $\mathcal{R}$ of $\mathcal{N}$ in $M_n(y,z)$ such that z is $\mathbb{B}^{\mathcal{R}}$-generic over $\mathcal{R}$ and $\delta^{\mathcal{R}} \leq \omega_1^{M_n(y,z)}$, where y is a real coding $\mathcal{N}$ and $\mathbb{B}^{\mathcal{R}}$ denotes Woodin's extender algebra inside $\mathcal{R}$.*

For the definition of the direct limit system converging to HOD we need the notion of s-iterability. To define this, we first introduce some notation. For an n-suitable premouse $\mathcal{N}$, a finite sequence of ordinals s, and some $k < \omega$ let

$$T_{s,k}^{\mathcal{N}} = \{(t, \ulcorner\phi\urcorner) \in [((\delta^{\mathcal{N}})^{+k})^{\mathcal{N}}]^{<\omega} \times \omega \mid \phi \text{ is a } \Sigma_1\text{-formula and } M_n(\mathcal{N}|\delta^{\mathcal{N}}) \vDash \phi[t,s]\},$$

where $\ulcorner\phi\urcorner$ denotes the Gödel number of ϕ. Let $Hull_1^{\mathcal{N}}$ denote an uncollapsed Σ_1 hull in $\mathcal{N}$. Then we let

$$\gamma_s^{\mathcal{N}} = \sup(Hull_1^{\mathcal{N}}(\{T_{s,k}^{\mathcal{N}} \mid k < \omega\}) \cap \delta^{\mathcal{N}})$$

and

$$H_s^{\mathcal{N}} = Hull_1^{\mathcal{N}}(\gamma_s^{\mathcal{N}} \cup \{T_{s,k}^{\mathcal{N}} \mid k < \omega\}).$$

Then $\gamma_s^{\mathcal{N}} = H_s^{\mathcal{N}} \cap \delta^{\mathcal{N}}$. For $s_m = (u_1, \dots, u_m)$ the sequence of the first m uniform indiscernibles, we write $\gamma_m^{\mathcal{N}} = \gamma_{s_m}^{\mathcal{N}}$ and $H_m^{\mathcal{N}} = H_{s_m}^{\mathcal{N}}$. Then we have that $\sup_{m\in\omega} \gamma_m^{\mathcal{N}} = \delta^{\mathcal{N}}$ (see Lemma 5.3 in [Sa13]).

Definition 3.10. *Let $\mathcal{N}$ be an n-suitable premouse and s a finite sequence of ordinals. Then $\mathcal{N}$ is* s-iterable *iff every correct iterate of $\mathcal{N}$ is short tree iterable and for every correctly guided finite stack $(\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ on $\mathcal{N}$ with last model $\mathcal{M}$ there is a sequence of non-dropping branches $(b_i \mid i \leq m)$ and a sequence of embeddings $(\pi_i \mid i \leq m)$ such that*

- *(i) if $\mathcal{T}_i$ has successor length $\alpha + 1$, then $b_i = [0, \alpha]_{T_i}$ and $\pi_i = i_{0,\alpha}^{\mathcal{T}_i}$ is the corresponding iteration embedding for $i \leq m$,*
- *(ii) if $\mathcal{T}_m$ is short, then b_m is the unique cofinal well-founded branch through $\mathcal{T}_m$ such that $\mathcal{Q}(b_m, \mathcal{T}_m)$ exists and $\mathcal{T}_m{}^\frown b_m$ is correctly guided and $\pi_m = i_{b_m}^{\mathcal{T}_m}$ is the corresponding iteration embedding,*

(iii) *if $\mathcal{T}_i$ is maximal, then b_i is a cofinal well-founded branch through $\mathcal{T}_i$ such that $\mathcal{M}_{b_i}^{\mathcal{T}_i} = \mathcal{N}_{i+1}$ if $i < m$ or $\mathcal{M}_{b_i}^{\mathcal{T}_i} = \mathcal{M}$ if $i = m$, and $\pi_i = i_{b_i}^{\mathcal{T}_i}$ is the corresponding iteration embedding for $i \leq m$, and*

(iv) *if we let $\pi = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_0$ then for every $k < \omega$,*
$$\pi(T_{s,k}^{\mathcal{N}}) = T_{s,k}^{\mathcal{M}}.$$

In this case we say that the sequence $\vec{b} = (b_i \mid i \leq m)$ witnesses s-iterability *for $\vec{\mathcal{T}} = (\mathcal{T}_i, \mathcal{N}_i \mid i \leq m)$ or that $\vec{b}$ is an s-*iterability branch *for $\vec{\mathcal{T}}$ and we write $\pi_{\vec{\mathcal{T}},\vec{b}} = \pi$.*

Now for every two s-iterability branches for $\vec{\mathcal{T}}$ on $\mathcal{N}$ their corresponding iteration embeddings agree on $H_s^{\mathcal{N}}$.

Lemma 3.11 (Uniqueness of s-iterability embeddings, Lemma 5.5 in [Sa13])**.** *Let $\mathcal{N}$ be an n-suitable premouse, s a finite sequence of ordinals, and $\vec{\mathcal{T}}$ a correctly guided finite stack on $\mathcal{N}$. Moreover let $\vec{b}$ and $\vec{c}$ be s-iterability branches for $\vec{\mathcal{T}}$. Then*
$$\pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_s^{\mathcal{N}} = \pi_{\vec{\mathcal{T}},\vec{c}} \upharpoonright H_s^{\mathcal{N}}.$$

The uniqueness of s-iterability embeddings yields that for every n-suitable, s-iterable $\mathcal{N}$, every correctly guided finite stack $\vec{\mathcal{T}}$ on $\mathcal{N}$ and every s-iterability branch $\vec{b}$ for $\vec{\mathcal{T}}$, the embedding $\pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_s^{\mathcal{N}}$ is independent of the choice of $\vec{b}$, but it might still depend on $\vec{\mathcal{T}}$. This motivates the following definition.

Definition 3.12. *Let $\mathcal{N}$ be an n-suitable premouse and s a finite sequence of ordinals. Then $\mathcal{N}$ is* strongly s-iterable *iff for every correct iterate $\mathcal{R}$ of $\mathcal{N}$, $\mathcal{R}$ is s-iterable and for every two correctly guided finite stacks $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ on $\mathcal{R}$ with common last model $\mathcal{M}$ and s-iterability witnesses $\vec{b}$ and $\vec{c}$ for $\vec{\mathcal{T}}$ and $\vec{\mathcal{U}}$ respectively, we have that*
$$\pi_{\vec{\mathcal{T}},\vec{b}} \upharpoonright H_s^{\mathcal{R}} = \pi_{\vec{\mathcal{U}},\vec{c}} \upharpoonright H_s^{\mathcal{R}}.$$

A so-called *bad sequence argument* shows the following lemma, which yields the existence of strongly s-iterable premice.

Lemma 3.13 (Lemma 5.9 in [Sa13])**.** *For every finite sequence of ordinals s and any short tree iterable n-suitable premouse $\mathcal{N}$ there is a pseudo-normal iterate $\mathcal{M}$ of $\mathcal{N}$ such that $\mathcal{M}$ is strongly s-iterable.*

If $\mathcal{N}$ is strongly s-iterable and $\vec{\mathcal{T}}$ is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$, let $\pi_{\mathcal{N},\mathcal{M},s} : H_s^{\mathcal{N}} \rightarrow H_s^{\mathcal{M}}$ denote the embedding given by any s-iterability branch $\vec{b}$ for $\vec{\mathcal{T}}$. As $\mathcal{N}$ is strongly s-iterable, the embedding $\pi_{\mathcal{N},\mathcal{M},s}$ does not depend on the choice of $\vec{\mathcal{T}}$ and $\vec{b}$.

Recall that we write $M_{n+1}^- = M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$, where δ_0 is the least Woodin cardinal in M_{n+1}, and $\Sigma_{M_{n+1}^-}$ for the canonical iteration strategy for M_{n+1}^-

induced by $\Sigma_{M_{n+1}}$. Moreover, recall that for $m < \omega$, we write s_m for the set of the first m uniform indiscernibles. Then M_{n+1}^- is n-suitable and strongly s_m-iterable for every m. Moreover, if $\vec{\mathcal{T}}$ is a correctly guided finite stack on M_{n+1}^- with last model $\mathcal{M}$, then $\pi_{M_{n+1}^-,\mathcal{M},s_m}$ agrees with the iteration embedding according to $\Sigma_{M_{n+1}^-}$ on $H_{s_m}^{M_{n+1}^-}$. The first direct limit system we define will consist of iterates of M_{n+1}^-.

Definition 3.14. *Let*

$$\tilde{\mathcal{F}}^+ = \{\mathcal{N} \mid \mathcal{N} \textit{ is an iterate of } M_{n+1}^- \textit{ via } \Sigma_{M_{n+1}^-} \textit{ by a finite stack of trees}\}$$

and for $\mathcal{N}, \mathcal{M} \in \tilde{\mathcal{F}}^+$ let $\mathcal{N} \leq^+ \mathcal{M}$ iff $\mathcal{M}$ is an iterate of $\mathcal{N}$ via the tail strategy $\Sigma_{\mathcal{N}}$ as witnessed by some finite stack of iteration trees. Then we let $\tilde{\mathcal{M}}_\infty^+$ be the direct limit of $(\tilde{\mathcal{F}}^+, \leq^+)$ under the iteration maps.

Remark. The prewellordering $\leq^+$ on $\tilde{\mathcal{F}}^+$ is directed and the direct limit $\tilde{\mathcal{M}}_\infty^+$ is well-founded as the limit system $(\tilde{\mathcal{F}}^+, \leq^+)$ only consists of iterates of M_{n+1}^- via the canonical iteration strategy $\Sigma_{M_{n+1}^-}$.

Since $\tilde{\mathcal{F}}^+$ is not definable enough for our purposes, we now introduce another direct limit system which has the same direct limit $\tilde{\mathcal{M}}_\infty^+$.

Definition 3.15. *Let*

$$\tilde{\mathcal{I}} = \{(\mathcal{N}, s) \mid \mathcal{N} \textit{ is } n\textit{-suitable}, s \in [\mathrm{Ord}]^{<\omega}, \textit{ and } \mathcal{N} \textit{ is strongly } s\textit{-iterable}\}$$

and

$$\tilde{\mathcal{F}} = \{H_s^{\mathcal{N}} \mid (\mathcal{N}, s) \in \tilde{\mathcal{I}}\}.$$

For $(\mathcal{N}, s), (\mathcal{M}, t) \in \tilde{\mathcal{I}}$ we let $(\mathcal{N}, s) \leq_{\tilde{\mathcal{I}}} (\mathcal{M}, t)$ iff there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ and $s \subseteq t$. In this case we let $\pi_{(\mathcal{N},s),(\mathcal{M},t)} : H_s^{\mathcal{N}} \to H_t^{\mathcal{M}}$ denote the canonical corresponding embedding.

Remark. The prewellordering $\leq_{\tilde{\mathcal{I}}}$ on $\tilde{\mathcal{I}}$ is directed: Let $(\mathcal{N}, s), (\mathcal{M}, t) \in \tilde{\mathcal{I}}$. By Lemma 3.13 there exists an n-suitable premouse $\mathcal{R}$ which is strongly $(s \cup t)$-iterable. Let $\mathcal{S}$ be the result of simultaneously comparing $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{R}$ in the sense of Lemma 3.8. Then $(\mathcal{S}, s \cup t) \in \tilde{\mathcal{I}}$, $(\mathcal{N}, s) \leq_{\tilde{\mathcal{I}}} (\mathcal{S}, s \cup t)$, and $(\mathcal{M}, t) \leq_{\tilde{\mathcal{I}}} (\mathcal{S}, s \cup t)$, as desired.

Definition 3.16. *Let $\tilde{\mathcal{M}}_\infty$ be the direct limit of $(\tilde{\mathcal{F}}, \leq_{\tilde{\mathcal{I}}})$ under the embeddings $\pi_{(\mathcal{N},s),(\mathcal{M},t)}$. For $(\mathcal{N}, s) \in \tilde{\mathcal{I}}$ let $\pi_{(\mathcal{N},s),\infty} : H_s^{\mathcal{N}} \to \tilde{\mathcal{M}}_\infty$ denote the corresponding direct limit embedding.*

The fact that $\tilde{\mathcal{M}}_\infty$ is well-founded follows from the next lemma.

Lemma 3.17 (Lemma 5.10 in [Sa13])**.** $\tilde{\mathcal{M}}_\infty = \tilde{\mathcal{M}}_\infty^+$.

The second direct limit system. To obtain HOD of some inner model from the direct limit, we in particular need to show that the direct limit is in fact contained in HOD of that inner model. In our setting we therefore need to internalize the direct limit system into the inner model $M_n(x)[g]$ fixed above. We first aim to define a direct limit system similar to $(\tilde{\mathcal{F}}, \leq_{\tilde{\mathcal{I}}})$ in $M_n(x)$ analogous to [Sa13]. In a second step, we then modify the system to obtain direct limit systems with the same direct limit which are definable in $M_n(x)[g]$.

The notion of n-suitability from Definition 3.2 is already internal to $M_n(x)$ and $M_n(x)[g]$, i.e., if $\mathcal{N} \in M_n(x)|\kappa$ then $\mathcal{N}$ is n-suitable in V iff $\mathcal{N}$ is n-suitable in $M_n(x)$ by the following lemma.

Lemma 3.18. *Let δ_0 denote the least Woodin cardinal in $M_n(x)$.*

(1) For all $y \in V_{\delta_0}^{M_n(x)[g]}$, $Lp^n(y) \in \mathrm{HOD}_y^{M_n(x)[g]}$.
(2) $V_{\delta_0}^{M_n(x)}$ and $V_{\delta_0}^{M_n(x)[g]}$ are closed under the operation $y \mapsto Lp^n(y)$.

Proof. Let $y \in V_{\delta_0}^{M_n(x)[g]}$ be arbitrary. The model $M_n(x)[g]$ can be organized as a $V_\kappa^{M_n(x)[g]}$-premouse and as such it inherits the iterability from $M_n(x)$ and is in fact equal to $M_n(V_\kappa^{M_n(x)[g]})$. Consider $L[E](y)^{M_n(x)[g]}$, the result of a fully backgrounded extender construction above y using extenders from the sequence of $M_n(x)[g]$ organized as a $V_\kappa^{M_n(x)[g]}$-premouse, and compare it with $Lp^n(y)$. First, we argue that $Lp^n(y)$ does not move. If it would move, the $Lp^n(y)$-side of the coiteration would have to drop because $Lp^n(y)$ does not have any total extenders. Moreover, it would have to iterate to a proper class model which is equal to an iterate of $L[E](y)^{M_n(x)[g]}$. As $L[E](y)^{M_n(x)[g]}$ has n Woodin cardinals, this would imply that $Lp^n(y)$ has a level which is not n-small, contradicting the definition of $Lp^n(y)$.

Therefore, $Lp^n(y) \lhd \mathcal{R}$ for some iterate $\mathcal{R}$ of $L[E](y)^{M_n(x)[g]}$. The iteration from $L[E](y)^{M_n(x)[g]}$ to $\mathcal{R}$ resulting from the comparison process can be defined over $L[E](y)^{M_n(x)[g]}$ from the extender sequence of $L[E](y)^{M_n(x)[g]}$ and a finite sequence of ordinals as it cannot leave any total measures behind and thus can only use measures of order 0. The extender sequence of the $V_\kappa^{M_n(x)[g]}$-premouse $M_n(x)[g]$ is in $\mathrm{HOD}^{M_n(x)[g]}_{V_\kappa^{M_n(x)[g]}} = \mathrm{HOD}^{M_n(x)[g]}$. Therefore $Lp^n(y) \in \mathrm{HOD}^{M_n(x)[g]}$ by the definability of the $L[E]$-construction.

For (2), the closure of $V_{\delta_0}^{M_n(x)[g]}$ follows immediately from (1). For $V_{\delta_0}^{M_n(x)}$ notice that for $y \in V_{\delta_0}^{M_n(x)}$, $Lp^n(y) \in \mathrm{HOD}_y^{M_n(x)[g]} \subseteq \mathrm{HOD}_y^{M_n(x)}$ by homogeneity of the forcing. □

By *stacking* the Lp^n-operation, this lemma in fact shows that for all $y \in V_{\delta_0}^{M_n(x)}$, $M_n(y)|\kappa_0 \in \mathrm{HOD}_y^{M_n(x)[g]}$, where κ_0 denotes the least measurable cardinal in $M_n(y)$.

The definitions of short tree, maximal tree, and correctly guided finite stack we gave above are internal to $M_n(x)$ and $M_n(x)[g]$ as well, as they can be defined only using the Lp^n-operation. The only notion we have to take care of is s-iterability since it is not even clear how the sets $T^{\mathcal{N}}_{s,k}$ can be identified inside $M_n(x)$. This obstacle is solved by shrinking the direct limit system $(\tilde{\mathcal{F}}, \leq_{\tilde{\mathcal{I}}})$ to a dense subset as follows.

Definition 3.19. *Let*

$$\begin{aligned}\mathcal{G} = \{&\mathcal{N} \in M_n(x)|\kappa \mid \mathcal{N} \textit{ is } n\textit{-suitable and } M_n(x) \vDash \textit{“for some cardinal}\\ &\textit{cutpoint } \eta, \delta^{\mathcal{N}} = \eta^+, \mathcal{N}|\delta^{\mathcal{N}} \in \mathcal{P}(M_n(x)|\eta^+) \cap M_n(x)|(\eta^+ + \omega),\\ &\textit{and } M_n(x)|\eta \textit{ is generic over } \mathcal{N} \textit{ for the } \delta^{\mathcal{N}}\textit{-generator version of}\\ &\quad \textit{the extender algebra at } \delta^{\mathcal{N}}\textit{”}\}.\end{aligned}$$

See for example Section 4.1 in [Fa] for an introduction to the δ-generator version of the extender algebra at some Woodin cardinal δ. The following lemma shows how we can use the fact that $\mathcal{N} \in \mathcal{G}$ to detect $M_n(\mathcal{N}|\delta^{\mathcal{N}})$ inside $M_n(x)$. For some premouse $\mathcal{R} \in \mathcal{G}$ we denote the last model of a $\mathcal{P}$-construction above $\mathcal{R}|\delta^{\mathcal{R}}$ performed inside $M_n(x)$ as introduced in [SchSt09] (see also Proposition 2.3 and Definition 2.4 in [Sa13]) by $\mathcal{P}^{M_n(x)}(\mathcal{R}|\delta^{\mathcal{R}})$.

Lemma 3.20 (Lemma 5.11 in [Sa13])**.** *Let $\mathcal{N} \in M_n(x)|\kappa$ be an n-suitable premouse such that for some cardinal cutpoint $\eta < \delta^{\mathcal{N}}$ of $M_n(x)$, we have that $\mathcal{N}|\delta^{\mathcal{N}} \in \mathcal{P}(M_n(x)|\eta^+) \cap M_n(x)|(\eta^+ + \omega)$ and $M_n(x)|\eta$ is generic over $\mathcal{N}$ for the $\delta^{\mathcal{N}}$-generator version of the extender algebra at $\delta^{\mathcal{N}}$. Then $\mathcal{N} \in \mathcal{G}$ and*

$$\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}}) = M_n(\mathcal{N}|\delta^{\mathcal{N}}).$$

In particular, $M_n(\mathcal{N}|\delta^{\mathcal{N}})[M_n(x)|\eta] = M_n(x)$.

Using pseudo-genericity iterations (see Lemma 3.9) we can obtain the following corollary.

Corollary 3.21. *Let $\mathcal{N}$ be a short tree iterable n-suitable premouse such that $\mathcal{N} \in M_n(x)|\kappa$. Then there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ such that $\mathcal{M} \in \mathcal{G}$ and $\mathcal{P}^{M_n(x)}(\mathcal{M}|\delta^{\mathcal{M}}) = M_n(\mathcal{M}|\delta^{\mathcal{M}})$.*

Now the following definition of s-iterability agrees with the previous one given in Definition 3.10 for n-suitable premice in $\mathcal{G}$.

Definition 3.22. *For $\mathcal{N} \in \mathcal{G}$, $s \in [\mathrm{Ord}]^{<\omega}$, and $k < \omega$ let*

$$\begin{aligned}T^{\mathcal{N},*}_{s,k} = \{(t, \ulcorner\phi\urcorner) \in [((\delta^{\mathcal{N}})^{+k})^{\mathcal{N}}]^{<\omega} \times \omega \mid \phi \textit{ is a } \Sigma_1\textit{-formula and}\\ \mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}}) \vDash \phi[t, s]\}.\end{aligned}$$

Then we say for $\mathcal{N} \in \mathcal{G}$ and $s \in [\mathrm{Ord}]^{<\omega}$ that $M_n(x) \vDash$ “$\mathcal{N}$ is s-iterable below κ” iff for every $\mathrm{Col}(\omega, {<}\kappa)$-generic G over $M_n(x)$ and every correctly guided finite stack $\vec{\mathcal{T}} = (\mathcal{T}_i, \mathcal{N}_i \mid i \leq m) \in \mathrm{HC}^{M_n(x)[G]}$ on $\mathcal{N}$ with last model

$\mathcal{M} \in \mathcal{G}$, there is a sequence of branches $\vec{b} = (b_i \mid i \leq m) \in M_n(x)[G]$ and a sequence of embeddings $(\pi_i \mid i \leq m)$ satisfying $(i) - (iii)$ in Definition 3.10 such that if we let $\pi_{\vec{\mathcal{T}},\vec{b}} = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_0$, then for every $k < \omega$,

$$\pi_{\vec{\mathcal{T}},\vec{b}}(T_{s,k}^{\mathcal{N},*}) = T_{s,k}^{\mathcal{M},*}.$$

In addition, we define $M_n(x) \vDash$ "$\mathcal{N}$ is strongly s-iterable below κ" analogous to Definition 3.12 for all $\mathrm{Col}(\omega, {<}\kappa)$-generic G and stacks $\vec{\mathcal{T}}, \vec{\mathcal{U}} \in M_n(x)[G]$. For $\mathcal{N} \in \mathcal{G}$, $s \in [\mathrm{Ord}]^{<\omega}$, and $k < \omega$, we have $T_{s,k}^{\mathcal{N},*} = T_{s,k}^{\mathcal{N}}$, so we will omit the $*$ for $\mathcal{N} \in \mathcal{G}$. Using this, $\gamma_s^{\mathcal{N}}$ and $H_s^{\mathcal{N}}$ are defined as before. Then we can define the internal direct limit system as follows.

Definition 3.23. *Let*

$$\mathcal{I} = \{(\mathcal{N}, s) \mid \mathcal{N} \in \mathcal{G}, s \in [\mathrm{Ord}]^{<\omega}, \text{ and } M_n(x) \vDash \text{"}\mathcal{N} \text{ is strongly } s\text{-iterable below } \kappa\text{"}\}$$

and

$$\mathcal{F} = \{H_s^{\mathcal{N}} \mid (\mathcal{N}, s) \in \mathcal{I}\}.$$

Moreover, for $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}$ we let $(\mathcal{N}, s) \leq (\mathcal{M}, t)$ iff there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ and $s \subseteq t$. In this case we let as before $\pi_{(\mathcal{N},s),(\mathcal{M},t)} : H_s^{\mathcal{N}} \to H_t^{\mathcal{M}}$ denote the canonical corresponding embedding.

For clarity, we sometimes write $\leq_{\mathcal{I}}$ for $\leq$. Similar as before we have that for every $\mathcal{N} \in \mathcal{G}$ and $s \in [\mathrm{Ord}]^{<\omega}$ there is a normal correct iterate $\mathcal{M}$ of $\mathcal{N}$ such that $(\mathcal{M}, s) \in \mathcal{I}$. Using the fact that κ is inaccessible and a limit of cutpoints in $M_n(x)$ we can obtain the following lemma.

Lemma 3.24 (Lemma 5.14 in [Sa13])**.** *$\leq$ is directed.*

Therefore we can again define the direct limit.

Definition 3.25. *Let $\mathcal{M}_\infty$ be the direct limit of $(\mathcal{F}, \leq)$ under the embeddings $\pi_{(\mathcal{N},s),(\mathcal{M},t)}$. Moreover, let $\delta_\infty = \delta^{\mathcal{M}_\infty}$ be the Woodin cardinal in $\mathcal{M}_\infty$ and $\pi_{(\mathcal{N},s),\infty} : H_s^{\mathcal{N}} \to \mathcal{M}_\infty$ be the direct limit embedding for all $(\mathcal{N}, s) \in \mathcal{I}$.*

An argument similar to the one for Lemma 3.17 shows that this direct limit is well-founded as well. As we will use ideas from this proof in the next section, we will give some details here. We again first define another direct limit system which consists of iterates of M_{n+1}^- and then show that its direct limit $\mathcal{M}_\infty^+$ is equal to $\mathcal{M}_\infty$.

Definition 3.26. *Let*

$$\mathcal{F}^+ = \{\mathcal{Q} \in \mathcal{G} \mid \mathcal{Q} \text{ is the last model of a correctly guided finite stack on } M_{n+1}^- \text{ via } \Sigma_{M_{n+1}^-}\}.$$

Moreover, let $\mathcal{P} \leq^+ \mathcal{Q}$ for $\mathcal{P}, \mathcal{Q} \in \mathcal{F}^+$ iff there is a correctly guided finite stack on $\mathcal{P}$ according to the tail strategy $\Sigma_{\mathcal{P}}$ with last model $\mathcal{Q}$. In this case we let $i_{\mathcal{P},\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ denote the corresponding iteration embedding.

Then $\leq^+$ on $\mathcal{F}^+$ is directed, so we can define the direct limit.

Definition 3.27. *Let $\mathcal{M}_\infty^+$ be the direct limit of $(\mathcal{F}^+, \leq^+)$ under the embeddings $i_{\mathcal{P},\mathcal{Q}}$. Moreover, let $i_{\mathcal{Q},\infty} : \mathcal{Q} \to \mathcal{M}_\infty^+$ denote the direct limit embedding for all $\mathcal{Q} \in \mathcal{F}^+$.*

Then it is easy to see that $\mathcal{M}_\infty^+$ is well-founded as $\mathcal{F}^+$ only consists of iterates of M_{n+1}^- according to the canonical iteration strategy $\Sigma_{M_{n+1}^-}$.

Lemma 3.28 (Lemma 5.15 in [Sa13])**.** *$\mathcal{M}_\infty^+ = \mathcal{M}_\infty$ and hence $\mathcal{M}_\infty$ is well-founded.*

Proof. We construct a sequence $(\mathcal{Q}_i \mid i < \omega)$ of iterates of M_{n+1}^- such that $\mathcal{Q}_i \in \mathcal{F}^+$ for every $i < \omega$ and $(\mathcal{Q}_i \mid i < \omega)$ is cofinal in $\mathcal{G}$, i.e., for every $\mathcal{N} \in \mathcal{G}$ there is an $i < \omega$ such that $\mathcal{Q}_i$ is the last model of a correctly guided finite stack on $\mathcal{N}$.

In V, fix some sequence $(\xi_i \mid i < \omega)$ of ordinals cofinal in κ. We define $(\mathcal{Q}_i \mid i < \omega)$ together with a strictly increasing sequence $(\eta_i \mid i < \omega)$ of cardinal cutpoints of $M_n(x)|\kappa$ by induction on $i < \omega$. So let $\mathcal{Q}_0 = M_{n+1}^-$ and let $\eta_0 < \kappa$ be a cardinal cutpoint of $M_n(x)$. Moreover assume that we already constructed $(\mathcal{Q}_i \mid i \leq j)$ and $(\eta_i \mid i \leq j)$ with the above mentioned properties such that in addition $(\mathcal{Q}_i \mid i \leq j) \in M_n(x)|\eta_j$. Let $\mathcal{Q}_{j+1}^*$ be the result of simultaneously pseudo-comparing (in the sense of Lemma 3.8) all n-suitable premice $\mathcal{M}$ such that $\mathcal{M} \in \mathcal{G} \cap M_n(x)|\eta_j$. Then in particular $\mathcal{Q}_{j+1}^*$ is a normal iterate of $\mathcal{Q}_j$ according to the canonical tail iteration strategy $\Sigma_{\mathcal{Q}_j}$, but $\mathcal{Q}_{j+1}^*$ might not be in $\mathcal{G}$. Let ν be a cardinal cutpoint of $M_n(x)$ such that $\eta_j < \nu < \kappa$ and $\mathcal{Q}_{j+1}^* \in M_n(x)|\nu$. Note that such a ν exists as κ is inaccessible and a limit of cardinal cutpoints in $M_n(x)$. Let $\mathcal{Q}_{j+1}$ be the normal iterate of $\mathcal{Q}_{j+1}^*$ according to the canonical tail strategy $\Sigma_{\mathcal{Q}_{j+1}^*}$ of $\Sigma_{\mathcal{Q}_j}$ obtained by Woodin's genericity iteration such that $M_n(x)|\nu$ is generic over $\mathcal{Q}_{j+1}$ for the $\delta^{\mathcal{Q}_{j+1}}$-generator version of the extender algebra (see for example Section 4.1 in [Fa]). Then $\mathcal{Q}_{j+1} \in \mathcal{G}$ is as desired. Finally choose $\eta_{j+1} < \kappa$ such that $\eta_{j+1} > \max(\eta_j, \xi_j)$, η_{j+1} is a cardinal cutpoint in $M_n(x)$ and $(\mathcal{Q}_i \mid i \leq j+1) \in M_n(x)|\eta_{j+1}$.

Now we define an embedding $\sigma : \mathcal{M}_\infty \to \mathcal{M}_\infty^+$ as follows. Let $x \in \mathcal{M}_\infty$. Since $(\mathcal{Q}_i \mid i < \omega)$ is cofinal in $\mathcal{G}$, there are $i, m < \omega$ such that $(\mathcal{Q}_i, s_m) \in \mathcal{I}$ and $x = \pi_{(\mathcal{Q}_i, s_m),\infty}(\bar{x})$ for some $\bar{x} \in H_{s_m}^{\mathcal{Q}_i} \subseteq \mathcal{Q}_i$. Then we let $\sigma(x) = i_{\mathcal{Q}_i,\infty}(\bar{x})$. It follows as in the proof of Lemma 5.10 in [Sa13] that the definition of σ does not depend on the choice of $i, m < \omega$ and in fact $\sigma = \mathrm{id}$. $\square$

Moreover, it is possible to compute δ_∞.

Lemma 3.29 (Lemma 5.16 in [Sa13])**.** $\delta_\infty = (\kappa^+)^{M_n(x)}$.

Direct limit systems in $\mathrm{HOD}^{M_n(x)[g]}$. Finally, we will argue that $\mathcal{M}_\infty \in \mathrm{HOD}^{M_n(x)[g]}$ by first defining direct limit systems in various premice $M(y)$ satisfying certain properties definable in $M_n(x)[g]$ and then showing that the direct limits $\mathcal{M}_\infty^{M(y)}$ are equal to $\mathcal{M}_\infty$. A similar approach but in a completely different setting can be found in [SaSch18].

In what follows, we will let $(K(z))^N$ denote the core model constructed above a real z inside some n-small model N with n Woodin cardinals in the sense of [Sch06], i.e., the core model $K(z)$ is constructed between consecutive Woodin cardinals. Lemma 1.1 in [Sch06] (due to John Steel) implies that $(K(x))^{M_n(x)} = M_n(x)$. We will use this fact and consider more arbitrary premice with this property in what follows. We state the following definitions in V, but we will later apply them inside $M_n(x)[g]$.

Definition 3.30. *Let $y \in {}^\omega\omega \cap M_n(x)[g]$. Then we say y is* pre-dlm-suitable *iff there is a proper class y-premouse $M(y)$ satisfying the following properties.*

(i) *$M(y)$ is n-small and has n Woodin cardinals,*
(ii) *the least inaccessible cardinal in $M(y)$ is κ,*
(iii) *$M(y) = (K(y))^{M(y)}$, and*
(iv) *there is a $\mathrm{Col}(\omega, {<}\kappa)$-generic h over $M(y)$ such that*

$$M(y)[h] = M_n(x)[g].$$

We also call such a y-premouse $M(y)$ pre-dlm-suitable *and say that $M(y)$* witnesses that y is pre-dlm-suitable*.*

Using this, we can define a version of the direct limit system $\mathcal{F}$ inside arbitrary pre-dlm-suitable y-premice $M(y)$.

Definition 3.31. *Let $y \in {}^\omega\omega$ be pre-dlm-suitable as witnessed by $M(y)$. Then we let*

$$\begin{aligned}\mathcal{G}^{M(y)} = \{\mathcal{N} \in M(y)|\kappa \mid \mathcal{N} &\text{ is } n\text{-suitable and } M(y) \vDash \text{“for some cardinal}\\ &\text{cutpoint } \eta, \delta^{\mathcal{N}} = \eta^+, \mathcal{N}|\delta^{\mathcal{N}} \in \mathcal{P}(M(y)|\eta^+) \cap M(y)|(\eta^+ + \omega),\\ &\text{and } M(y)|\eta \text{ is generic over } \mathcal{N} \text{ for the } \delta^{\mathcal{N}}\text{-generator version of}\\ &\quad \text{the extender algebra at } \delta^{\mathcal{N}}\text{”}\}.\end{aligned}$$

Analogous as before, we can now define when for an n-suitable premouse $\mathcal{N}$, $M(y) \vDash$ “$\mathcal{N}$ is strongly s-iterable below κ” by referring to $\mathcal{P}^{M(y)}(\mathcal{N}|\delta^{\mathcal{N}})$ in the definition of $(T_{s,k}^{\mathcal{N},*})^{M(y)}$. Let $\gamma_s^{\mathcal{N},M(y)}$ and $H_s^{\mathcal{N},M(y)}$ be defined analogous to $\gamma_s^{\mathcal{N}}$ and $H_s^{\mathcal{N}}$ inside $M(y)$ using $(T_{s,k}^{\mathcal{N},*})^{M(y)}$. For $M(y) = M_n(x)$ and $\mathcal{N} \in \mathcal{G}$ this agrees with our previous definition of strong s-iterability.

Definition 3.32. *Let $y \in {}^\omega\omega$ be pre-dlm-suitable as witnessed by $M(y)$. Then we let*

$$\mathcal{I}^{M(y)} = \{(\mathcal{N}, s) \mid \mathcal{N} \in \mathcal{G}^{M(y)}, s \in [\mathrm{Ord}]^{<\omega}, \text{ and } M(y) \models \text{“}\mathcal{N} \text{ is strongly } s\text{-iterable below } \kappa\text{”}\}$$

and

$$\mathcal{F}^{M(y)} = \{H_s^{\mathcal{N},M(y)} \mid (\mathcal{N}, s) \in \mathcal{I}^{M(y)}\}.$$

Moreover, for $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}^{M(y)}$ we let $(\mathcal{N}, s) \leq_{\mathcal{I}^{M(y)}} (\mathcal{M}, t)$ iff there is a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M}$ and $s \subseteq t$. In this case we let $\pi^{M(y)}_{(\mathcal{N},s),(\mathcal{M},t)} : H_s^{\mathcal{N},M(y)} \to H_t^{\mathcal{M},M(y)}$ denote the canonical corresponding embedding. Finally, let $\mathcal{M}_\infty^{M(y)}$ denote the direct limit of $(\mathcal{F}^{M(y)}, \leq_{\mathcal{I}^{M(y)}})$ under these embeddings.

We will now strengthen this and define when a real $y \in {}^\omega\omega$ (or a y-premouse $M(y)$) is dlm-suitable.

Definition 3.33. *Let $y \in {}^\omega\omega \cap M_n(x)[g]$ be pre-dlm-suitable as witnessed by some y-premouse $M(y)$. We say that y is* dlm-suitable *(witnessed by $M(y)$) iff*

- *(i) for every $s \in [\mathrm{Ord}]^{<\omega}$ there is a premouse $\mathcal{N}$ such that $(\mathcal{N}, s) \in \mathcal{I}^{M(y)}$, and*
- *(ii) for every $\mathcal{N} \in \mathcal{G}^{M(y)}$,*

$$\mathcal{P}^{M(y)}(\mathcal{N}|\delta^{\mathcal{N}}) = K^{M_n(x)[g]}(\mathcal{N}|\delta^{\mathcal{N}}).$$

Lemma 3.34. *$M_n(x)$ witnesses that x is dlm-suitable.*

Proof. The fact that $M_n(x)$ satisfies (i) follows from Lemma 3.13 and Corollary 3.21, so we only have to show (ii). Let $\mathcal{N} \in \mathcal{G}$. Then $\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}}) = M_n(\mathcal{N}|\delta^{\mathcal{N}})$ by Lemma 3.20. Moreover, there is some G generic over the result of the $\mathcal{P}$-construction $\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}})$ for the $\delta^{\mathcal{N}}$-generator version of the extender algebra at $\delta^{\mathcal{N}}$ with $\mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}})[G] = M_n(x)$. That means

$$M_n(\mathcal{N}|\delta^{\mathcal{N}})[G] = M_n(x).$$

Now,

$$\begin{aligned} K^{M_n(x)[g]}(\mathcal{N}|\delta^{\mathcal{N}}) &= K^{M_n(\mathcal{N}|\delta^{\mathcal{N}})[G][g]}(\mathcal{N}|\delta^{\mathcal{N}}) = K^{M_n(\mathcal{N}|\delta^{\mathcal{N}})}(\mathcal{N}|\delta^{\mathcal{N}}) \\ &= M_n(\mathcal{N}|\delta^{\mathcal{N}}) = \mathcal{P}^{M_n(x)}(\mathcal{N}|\delta^{\mathcal{N}}), \end{aligned}$$

by generic absoluteness of the core model and Lemma 1.1 in [Sch06] (due to Steel). □

Condition (ii) in Definition 3.33 will ensure that for any dlm-suitable y-premouse $M(y)$ and $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I} \cap \mathcal{I}^{M(y)}$ with $(\mathcal{N}, s) \leq_{\mathcal{I}} (\mathcal{M}, t)$ and $(\mathcal{N}, s) \leq_{\mathcal{I}^{M(y)}} (\mathcal{M}, t)$, the induced embeddings $\pi_{(\mathcal{N},s),(\mathcal{M},t)}$ and $\pi^{M(y)}_{(\mathcal{N},s),(\mathcal{M},t)}$ agree. Hence we can show in the following lemma that the direct limit

$\mathcal{M}_\infty^{M(y)}$ defined inside some dlm-suitable $M(y)$ will in fact be the same as the direct limit $\mathcal{M}_\infty$ defined inside $M_n(x)$.

Lemma 3.35. *Let $y \in {}^\omega\omega$ be dlm-suitable as witnessed by $M(y)$. Then $\mathcal{F}$ and $\mathcal{F}^{M(y)}$ have cofinally many points in common and $\mathcal{M}_\infty = \mathcal{M}_\infty^{M(y)}$.*

Proof. Let h be $\mathrm{Col}(\omega, {<}\kappa)$-generic over $M(y)$ such that $M(y)[h] = M_n(x)[g]$. Let $(\mathcal{N}, s) \in \mathcal{I}$ and $(\mathcal{N}', s') \in \mathcal{I}^{M(y)}$. We aim to show that there is some $(\mathcal{M}, t) \in \mathcal{I} \cap \mathcal{I}^{M(y)}$ such that $(\mathcal{N}, s) \leq_{\mathcal{I}} (\mathcal{M}, t)$ and $(\mathcal{N}', s') \leq_{\mathcal{I}^{M(y)}} (\mathcal{M}, t)$. As condition (ii) in Definition 3.33 yields that the embeddings associated to $\mathcal{F}$ and $\mathcal{F}^{M(y)}$ agree, this suffices to show that $\mathcal{M}_\infty = \mathcal{M}_\infty^{M(y)}$.

Let $t = s \cup s'$. By assumption, there is a t-iterable premouse $\mathcal{R}$ in $M_n(x)$ and a t-iterable premouse $\mathcal{R}'$ in $M(y)$. Therefore we can assume that $\mathcal{N}$ and $\mathcal{N}'$ are both t-iterable in $M_n(x)$ and $M(y)$ respectively as we can replace them by the result of their coiteration with $\mathcal{R}$ and $\mathcal{R}'$ respectively.

By the choice of $M(y)$ and generic absoluteness of the core model we have

$$\begin{aligned} M(y) &= (K(y))^{M(y)} = (K(y))^{M(y)[h]} \\ &= (K(y))^{M_n(x)[g]} = (K(y))^{M_n(x)[g\restriction\xi]}, \end{aligned} \tag{1}$$

where $\xi < \kappa$ is such that $y \in M_n(x)[g \restriction \xi]$. Analogously, using Lemma 1.1 in [Sch06] due to Steel and generic absoluteness of the core model again,

$$\begin{aligned} M_n(x) &= (K(x))^{M_n(x)} = (K(x))^{M_n(x)[g]} \\ &= (K(x))^{M(y)[h]} = (K(x))^{M(y)[h\restriction\xi']}, \end{aligned} \tag{2}$$

where $\xi' < \kappa$ is such that $x \in M(y)[h \restriction \xi']$. Now we can obtain the following claim.

Claim 1. *$M(y)$ and $M_n(x)$ have cofinally many common cardinal cutpoints below κ.*

Proof. As $M(y) = K(y)^{M_n(x)[g\restriction\xi]}$ is an inner model of $M_n(x)[g \restriction \xi]$, every cardinal above ξ in $M_n(x)$ is a cardinal in $M(y)$. Now let $\eta > \xi, \xi'$ be a cutpoint of $M_n(x)$ which is large enough such that in $M(y)$ there is some cutpoint between ξ' and η. Suppose η is not a cutpoint of $M(y)$, say there is an extender E overlapping η in $M(y)$. Since there is some cutpoint in $M(y)$ between ξ' and η, it follows that $\mathrm{crit}(E) > \xi'$. Then, as $M_n(x) = K(x)^{M(y)[h\restriction\xi']}$, by maximality of the core model there is also an extender on the $M_n(x)$-sequence overlapping η, contradicting the assumption that η is a cutpoint of $M_n(x)$. □

Moreover, Equations (1) and (2) yield

$$M(y) \subseteq M_n(x)[g \restriction \xi] \subseteq M(y)[h \restriction \zeta],$$

where $\xi' < \zeta < \kappa$ is such that $g \restriction \xi \in M(y)[h \restriction \zeta]$. By the intermediate model theorem (see for example Lemma 15.43 in [Je03]) this implies that

$M_n(x)[g \upharpoonright \xi]$ is a generic extension of $M(y)$ for a forcing of size less than κ.[5] Since $M_n(x)[g \upharpoonright \xi]$ is a generic extension of $M_n(x)$ for a forcing of size less than κ as well, this implies by Theorem 1.3 in [Us17] that there is some common inner model $W \subseteq M_n(x) \cap M(y)$ such that $M_n(x)[g \upharpoonright \xi]$ is a generic extension of W for a forcing of size less than κ.

As every generic extension via a forcing of size less than κ can be absorbed by the collapse of some ordinal $\beta < \kappa$, this yields that we can fix some ordinal $\beta < \kappa$ and some $\mathrm{Col}(\omega, \beta)$-generic $b \in M_n(x)[g]$ over W such that $x, y, \mathcal{N}, \mathcal{N}' \in W[b]$. Then $M_n(x)$ and $M(y)$ exist in $W[b]$ as definable subclasses because

$$(K(x))^{W[b]} = (K(x))^{M_n(x)[g \upharpoonright \xi]} = (K(x))^{M_n(x)} = M_n(x)$$

and similarly

$$(K(y))^{W[b]} = (K(y))^{M_n(x)[g \upharpoonright \xi]} = (K(y))^{M_n(x)[g]} = (K(y))^{M(y)[h]} = M(y)$$

by generic absoluteness of the core model again. Let $\dot{x}, \dot{y}, \dot{\mathcal{N}}$ and $\dot{\mathcal{N}}'$ be $\mathrm{Col}(\omega, \beta)$-names for $x, y, \mathcal{N}$ and $\mathcal{N}'$ in W. Moreover, let $p \in \mathrm{Col}(\omega, \beta)$ force all properties we need about $\dot{x}, \dot{y}, \dot{\mathcal{N}}$ and $\dot{\mathcal{N}}'$. For $q \leq_{\mathrm{Col}(\omega,\beta)} p$ let b_q be the $\mathrm{Col}(\omega, \beta)$-generic filter over W such that $\bigcup b_q$ agrees with q on $\mathrm{dom}(q)$ and with $\bigcup b$ everywhere else.

Now we construct $(\mathcal{M}, t) \in \mathcal{I} \cap \mathcal{I}^{M(y)}$. Let $\eta < \kappa$ be a cardinal cutpoint of both $M(y)$ and $M_n(x)$ such that $\xi, \xi' < \eta$, which exists by Claim 1. Then in fact $(\eta^+)^{M_n(x)} = (\eta^+)^{M(y)}$ as by Equations (1) and (2) at the beginning of the proof

$$(\eta^+)^{M(y)} \leq (\eta^+)^{M_n(x)[g \upharpoonright \xi]} = (\eta^+)^{M_n(x)} \leq (\eta^+)^{M(y)[h \upharpoonright \xi']} = (\eta^+)^{M(y)}.$$

By the same argument, $(\eta^+)^{K(\dot{x}^{b_q})} = (\eta^+)^{K(\dot{y}^{b_q})}$ for all $q \leq_{\mathrm{Col}(\omega,\beta)} p$.

Work in $W[b]$. Using Lemmas 3.8 and 3.9, we obtain an inner model $\mathcal{M}$ by pseudo-comparing all $(\dot{\mathcal{N}})^{b_q}$ and $(\dot{\mathcal{N}}')^{b_q}$ for $q \leq_{\mathrm{Col}(\omega,\beta)} p$ and simultaneously pseudo-genericity iterating such that $K(\dot{x}^{b_q})|\eta$ and $K(\dot{y}^{b_q})|\eta$ are generic over $\mathcal{M}$ and $\delta^{\mathcal{M}} = (\eta^+)^{K(\dot{x}^{b_q})} = (\eta^+)^{K(\dot{y}^{b_q})}$. Since $\mathcal{M}$ is definable in $W[b]$ from $\{b_q \mid q \leq_{\mathrm{Col}(\omega,\beta)} p\}$ and parameters from W, we have that in fact $\mathcal{M} \in W \subseteq M_n(x) \cap M(y)$, as $\mathcal{M}$ does not depend on the choice of the generic b. Moreover, $\mathcal{M}$ is a correct iterate of $\mathcal{N}$ in $M_n(x)$ and a correct iterate of $\mathcal{N}'$ in $M(y)$.

As argued above, we can assume that $\mathcal{N}$ and $\mathcal{N}'$ are t-iterable in $M_n(x)$ and $M(y)$ respectively for $t = s \cup s'$. Therefore $\mathcal{M}$ is t-iterable in both, $M_n(x)$ and $M(y)$. Hence, $(\mathcal{M}, t) \in \mathcal{I} \cap \mathcal{I}^{M(y)}$, $(\mathcal{N}, s) \leq_{\mathcal{I}} (\mathcal{M}, t)$, and $(\mathcal{N}', s') \leq_{\mathcal{I}^{M(y)}} (\mathcal{M}, t)$, as desired. $\square$

This yields that $\mathcal{M}_\infty \in \mathrm{HOD}^{M_n(x)[g]}$.

[5] I.e. $M(y)$ is a ground of $M_n(x)[g \upharpoonright \xi]$. See for example [FHR15] or [Us17] for an introduction to the theory of grounds.

4. HOD below δ_∞

In this section we will show that $\mathrm{HOD}^{M_n(x)[g]}$ and $\mathcal{M}_\infty$ agree up to δ_∞ by generalizing the arguments in Section 3.4 in [StW16]. We show a version of Woodin's derived model resemblance for our setting. For this, we do not need to talk about generic extensions such as $M(y)[h]$ and work with $M(y)$ directly instead. Choose for any ordinal α an arbitrary $(\mathcal{N}, s) \in \mathcal{I}$ such that $\alpha \in s$ and let $\alpha^* = \pi_{(\mathcal{N},s),\infty}(\alpha)$. Note that the value of α^* does not depend on the choice of $(\mathcal{N}, s)$. We also let $t^* = \{\alpha^* \mid \alpha \in t\}$ for $t \in [\mathrm{Ord}]^{<\omega}$.

Lemma 4.1. *Let $\mathcal{N}$ be an n-suitable premouse and $s \in [\mathrm{Ord}]^{<\omega}$ such that $(\mathcal{N}, s) \in \mathcal{I}$. Let $\bar\xi < \gamma_s^{\mathcal{N}}$, $\xi = \pi_{(\mathcal{N},s),\infty}(\bar\xi)$ and $t \in [\mathrm{Ord}]^{<\omega}$. Moreover, let $\varphi(v_0, v_1)$ be a formula in the language of premice, i.e., we allow the extender sequence as a predicate. Then the following are equivalent.*

(*a*) *$M_n(\mathcal{M}_\infty|\delta_\infty) \vDash \varphi(\xi, t^*)$,*
(*b*) *in $M_n(x)[g]$, there is some dlm-suitable $y \in {}^\omega\omega$ witnessed by $M(y)$ with $(\mathcal{N}, s) \in \mathcal{I}^{M(y)}$ and a correctly guided finite stack on $\mathcal{N}$ with last model $\mathcal{M} \in M(y)$ such that whenever $\mathcal{R} \in \mathcal{G}^{M(y)}$ is the last model of a correctly guided finite stack on $\mathcal{M}$, then $\mathcal{P}^{M(y)}(\mathcal{R}|\delta^{\mathcal{R}}) \vDash \varphi(\pi^{M(y)}_{(\mathcal{N},s),(\mathcal{R},s)}(\bar\xi), t)$.*

Proof. To prove that (*a*) implies (*b*) we assume toward a contradiction that (*b*) is false. So in $M_n(x)[g]$ for all dlm-suitable $y \in {}^\omega\omega$ and $M(y)$ witnessing this with $(\mathcal{N}, s) \in \mathcal{I}^{M(y)}$ and all correctly guided finite stacks on $\mathcal{N}$ with last model $\mathcal{M} \in M(y)$, there is a correctly guided finite stack on $\mathcal{M}$ with last model $\mathcal{R} \in \mathcal{G}^{M(y)}$ such that $\mathcal{P}^{M(y)}(\mathcal{R}|\delta^{\mathcal{R}}) \vDash \neg\varphi(\pi^{M(y)}_{(\mathcal{N},s),(\mathcal{R},s)}(\bar\xi), t)$.

We can assume without loss of generality that $\mathcal{N} \in M_n(x)$ is the last model of a correctly guided finite stack on M^-_{n+1} via the canonical iteration strategy $\Sigma_{M^-_{n+1}}$ and strongly s-iterable below κ with respect to branches choosen by $\Sigma_{M^-_{n+1}}$. Moreover, we can assume that $max(s)$ is a uniform indiscernible. If this is not already the case, we replace $\mathcal{N}$ by a pseudo-iterate of the result of the pseudo-comparison of $\mathcal{N}$ with M^-_{n+1} using Lemma 3.8 and Corollary 3.21.

Claim 1. *There are n-suitable premice $\mathcal{N}_k \in \mathcal{F}^+$ for $k < \omega$ which are cofinal in $\mathcal{F}^+$ such that $\mathcal{N}_0 = \mathcal{N}$ and for all $k < \omega$,*

$$M_n(\mathcal{N}_k|\delta^{\mathcal{N}_k}) \vDash \neg\varphi(\bar\xi_k, t),$$

where $\bar\xi_k = i_{\mathcal{N}_0,\mathcal{N}_k}(\bar\xi)$ is the image of $\bar\xi$ under the iteration map induced by $\Sigma_{M^-_{n+1}}$.

Proof. Let $(\mathcal{Q}_i \mid i < \omega)$ be an enumeration of $\mathcal{F}^+$ and $\mathcal{N}_0 = \mathcal{N}$. Then we construct $\mathcal{N}_{k+1}$ inductively. So assume that we already constructed $\mathcal{N}_k$ and pseudo-coiterate $\mathcal{N}_k$ with $\mathcal{Q}_k$ to some model $\mathcal{N}^*_k$ (see Lemma 3.8). By assumption (*b*) is false, so let $\mathcal{R}$ be a counterexample witnessing this for $\mathcal{N}^*_k$ and the dlm-suitable premouse $M_n(x)$. That means $\mathcal{R} \in \mathcal{G}$ is the last model of a correctly guided finite stack on $\mathcal{N}_k^*$ such that $M_n(\mathcal{R}|\delta^{\mathcal{R}}) = \mathcal{P}^{M_n(x)}(\mathcal{R}|\delta^{\mathcal{R}}) \models \neg\varphi(i_{\mathcal{N},\mathcal{R}}(\bar{\xi}), t)$ as $i_{\mathcal{N},\mathcal{R}} \upharpoonright H_s^{\mathcal{N}} = \pi_{(\mathcal{N},s),(\mathcal{R},s)}$. But $\mathcal{R} \in \mathcal{F}^+$ since $\mathcal{R} \in \mathcal{G}$ and it is a correct iterate of $\mathcal{Q}_k$. Thus we can let $\mathcal{N}_{k+1} = \mathcal{R}$. □

Since $(\mathcal{N}_k \mid k < \omega)$ is cofinal in $\mathcal{F}^+$, it follows that the direct limit of $(\mathcal{N}_k, i_{\mathcal{N}_k,\mathcal{N}_l} \mid k < l < \omega)$ is equal to $\mathcal{M}_\infty^+$. Let $\hat{\mathcal{N}_k} = M_n(\mathcal{N}_k|\delta^{\mathcal{N}_k})$ and let $\hat{i}_{\hat{\mathcal{N}}_k,\infty} : \hat{\mathcal{N}_k} \to M_n(\mathcal{M}_\infty^+|\delta^{\mathcal{M}_\infty^+}) = \hat{\mathcal{M}}_\infty^+$ be the corresponding extension of the direct limit map $i_{\mathcal{N}_k,\infty}$. Then we have for all sufficiently large k that

$$M_n(\mathcal{M}_\infty^+|\delta^{\mathcal{M}_\infty^+}) \models \neg\varphi(i_{\mathcal{N}_k,\infty}(\bar{\xi}_k), \hat{i}_{\hat{\mathcal{N}}_k,\infty}[t]).$$

Since we assumed that $\mathcal{N}$ is strongly s-iterable below κ with respect to branches choosen by $\Sigma_{M_{n+1}^-}$ and $\bar{\xi} < \gamma_s^{\mathcal{N}}$, it follows that $i_{\mathcal{N}_k,\infty}(\bar{\xi}_k) = i_{\mathcal{N},\infty}(\bar{\xi}) = \pi_{(\mathcal{N},s),\infty}(\bar{\xi}) = \xi$ as $\bar{\xi}_k = i_{\mathcal{N},\mathcal{N}_k}(\bar{\xi})$.

Let $k < \omega$ be large enough such that $(\mathcal{N}_k, s \cup t) \in \mathcal{I}$ and $\hat{i}_{\hat{\mathcal{N}_l},\hat{\mathcal{N}}_{l+1}}(s) = s$ for all $l \geq k$. Such a k exists by a so-called bad sequence argument similar to the one in the proof of Lemma 5.8 in [Sa13]. For $l \geq k$ and $s^- = s \setminus \{max(s)\}$, let

$$\gamma_s^{\hat{\mathcal{N}}_l} = \sup(Hull_1^{\hat{\mathcal{N}}_l|max(s)}(s^-) \cap \delta^{\mathcal{N}_l}),$$

and

$$H_s^{\hat{\mathcal{N}}_l} = Hull_1^{\hat{\mathcal{N}}_l|max(s)}(\gamma_s^{\hat{\mathcal{N}}_l} \cup s^-).$$

Now we let for $l < j$, $\hat{\pi}_{(\mathcal{N}_l,s),(\mathcal{N}_j,t)} : H_s^{\hat{\mathcal{N}}_l} \to H_t^{\hat{\mathcal{N}}_j}$ denote the canonical corresponding embedding extending $\pi_{(\mathcal{N}_l,s^-),(\mathcal{N}_j,t^-)}$ given by the iteration embedding via a tail of the iteration strategy $\Sigma_{M_{n+1}}$. Let $\hat{\mathcal{M}}_\infty$ be the direct limit under these embeddings and let $\hat{\pi}_{(\mathcal{N}_l,s),\infty} : H_s^{\hat{\mathcal{N}}_l} \to \hat{\mathcal{M}}_\infty$ for $l \geq k$ denote the direct limit embedding.

Now consider the map

$$\hat{\sigma} : \hat{\mathcal{M}}_\infty \to \hat{\mathcal{M}}_\infty^+ = M_n(\mathcal{M}_\infty^+|\delta^{\mathcal{M}_\infty^+})$$

which is the canonical extension of the map $\sigma : \mathcal{M}_\infty \to \mathcal{M}_\infty^+$ defined in the proof of Lemma 3.28, i.e., for $x \in \hat{\mathcal{M}}_\infty$, say $x = \hat{\pi}_{(\mathcal{N}_l,s),\infty}(\bar{x})$ for some $\bar{x} \in H_s^{\hat{\mathcal{N}}_l}$ and $k \leq l < \omega$, let $\hat{\sigma}(x) = \hat{i}_{\hat{\mathcal{N}}_l,\infty}(\bar{x})$. Then it follows as in the proof of Lemma 5.10 in [Sa13] that $\hat{\sigma} = \mathrm{id}$ and $\hat{\mathcal{M}}_\infty = M_n(\mathcal{M}_\infty|\delta_\infty)$. Moreover, we have that $\hat{\sigma}[t^*] = \hat{\sigma}(\hat{\pi}_{(\mathcal{N}_k,s\cup t),\infty}[t]) = \hat{i}_{\hat{\mathcal{N}}_k,\infty}[t]$. Therefore pulling back under $\hat{\sigma}$ yields that

$$M_n(\mathcal{M}_\infty|\delta_\infty) \models \neg\varphi(\xi, t^*).$$

This is the desired contradiction to (a).

To show that (b) implies (a) we now assume that (b) is true. Let $M(y)$ be the dlm-suitable premouse with $(\mathcal{N}, s) \in \mathcal{I}^{M(y)}$ given by (b). As before we can assume without loss of generality that $\mathcal{N}$ is the last model of a correctly guided finite stack on M_{n+1}^- via the canonical iteration strategy $\Sigma_{M_{n+1}^-}$, that $\mathcal{N}$ is strongly s-iterable below κ with respect to branches choosen by $\Sigma_{M_{n+1}^-}$, that $max(s)$ is a uniform indiscernible, and that $\mathcal{N} \in \mathcal{G} \cap \mathcal{G}^{M(y)}$ using Lemma 3.35.

Claim 2. *There are n-suitable premice $\mathcal{N}_k \in \mathcal{F}^+$ for $k < \omega$ which are cofinal in $\mathcal{F}^+$ such that $\mathcal{N}_0 = \mathcal{N}$ and for all $k < \omega$,*

$$M_n(\mathcal{N}_k|\delta^{\mathcal{N}_k}) \vDash \varphi(\bar{\xi}_k, t),$$

where $\bar{\xi}_k = i_{\mathcal{N}_0,\mathcal{N}_k}(\bar{\xi})$ is the image of $\bar{\xi}$ under the iteration map induced by $\Sigma_{M^-_{n+1}}$.

Proof. By the proof of Lemma 3.35, we can pick a sequence $(\mathcal{Q}_i \mid i < \omega)$ of premice cofinal in $\mathcal{F}^+$ such that $\mathcal{Q}_i \in \mathcal{F}^{M(y)}$ for all $i < \omega$. Let $\mathcal{N}_0 = \mathcal{N}$ and construct $\mathcal{N}_{k+1} \in M(y)$ inductively. Assume that we already constructed $\mathcal{N}_k$ and let $\mathcal{M} \in M(y)$ be the last model of a correctly guided finite stack on $\mathcal{N}$ witnessing that (b) is true. Simultaneously pseudo-coiterate $\mathcal{M}$ with $\mathcal{N}_k$ and $\mathcal{Q}_k$ to some premouse $\mathcal{N}^*_k$. Using genericity iterations and Lemma 3.35, there is a pseudo-iterate $\mathcal{R}$ of $\mathcal{N}^*_k$ such that $\mathcal{R} \in \mathcal{G} \cap \mathcal{G}^{M(y)}$ (see also Corollary 3.21). In particular, we have by dlm-suitability of $M(y)$ that

$$\begin{aligned}\mathcal{P}^{M(y)}(\mathcal{R}|\delta^{\mathcal{R}}) &= K^{M_n(x)[g]}(\mathcal{R}|\delta^{\mathcal{R}}) = K^{M_n(\mathcal{R}|\delta^{\mathcal{R}})[G][g]}(\mathcal{R}|\delta^{\mathcal{R}}) \\ &= K^{M_n(\mathcal{R}|\delta^{\mathcal{R}})}(\mathcal{R}|\delta^{\mathcal{R}}) = M_n(\mathcal{R}|\delta^{\mathcal{R}})\end{aligned}$$

for some G generic over $M_n(\mathcal{R}|\delta^{\mathcal{R}})$ for the extender algebra and therefore $M_n(\mathcal{R}|\delta^{\mathcal{R}}) \vDash \varphi(i_{\mathcal{N},\mathcal{R}}(\bar{\xi}), t)$ using (b) as $i_{\mathcal{N},\mathcal{R}}(\bar{\xi}) = \pi_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\xi}) = \pi^{M(y)}_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\xi})$. Moreover, $\mathcal{R}$ is the last model of a correctly guided finite stack on $\mathcal{Q}_k$ and thus $\mathcal{R} \in \mathcal{F}^+$, so we can let $\mathcal{N}_{k+1} = \mathcal{R}$. □

As before we can use this claim to obtain that

$$M_n(\mathcal{M}_\infty|\delta_\infty) \vDash \varphi(\xi, t^*),$$

which proves (a). □

Let κ_∞ be the least inaccessible cardinal above δ_∞ in $\hat{\mathcal{M}}_\infty = M_n(\mathcal{M}_\infty|\delta_\infty)$ and fix some H which is $\mathrm{Col}(\omega, {<}\kappa_\infty)$-generic over $\hat{\mathcal{M}}_\infty$. Then Lemma 4.1 implies for example that $\hat{\mathcal{M}}_\infty[H]$ and $M_n(x)[g]$ are elementary equivalent (for formulae in the language of set theory) as for $\mathcal{R}$ as in the statement of Lemma 4.1, there is some $\mathrm{Col}(\omega, {<}\kappa^{\mathcal{R}})$-generic G, where $\kappa^{\mathcal{R}}$ is the least inaccessible cardinal above $\delta^{\mathcal{R}}$ in $\mathcal{P}^{M(y)}(\mathcal{R}|\delta^{\mathcal{R}})$, such that $\mathcal{P}^{M(y)}(\mathcal{R}|\delta^{\mathcal{R}})[G] = M_n(x)[g]$.

We defined a direct limit system $\mathcal{F}^{M(y)}$ for all dlm-suitable $M(y)$ in $M_n(x)[g]$. Therefore, there is a direct limit system $\mathcal{F}^{*,M(y)}$ with the same properties for each dlm-suitable $M(y)$ in $\hat{\mathcal{M}}_\infty[H]$ (adapting the definition of dlm-suitable to $\hat{\mathcal{M}}_\infty[H]$). It is easy to see that Lemma 4.1 implies that $\mathcal{M}_\infty$ is strongly

s^*-iterable in all dlm-suitable $M(y)$ in $\hat{\mathcal{M}}_\infty[H]$ for all $s \in [\mathrm{Ord}]^{<\omega}$. So we can consider its direct limit embedding

$$\pi_\infty^{M(y)} = \bigcup\{(\pi^{M(y)}_{(\mathcal{M}_\infty, s^*),\infty})^{\mathcal{F}^*, M(y)} \mid s \in [\mathrm{Ord}]^{<\omega}\}$$

in the system $\mathcal{F}^{*,M(y)}$. But in fact, $\pi_\infty^{M(y)} = \pi_\infty^{M(z)}$ for two dlm-suitable models $M(y)$ and $M(z)$, so we call this unique embedding π_∞.

Lemma 4.2. *For all $\eta < \delta_\infty$ we have that $\pi_\infty(\eta) = \eta^*$.*

Proof. This is again a consequence of Lemma 4.1. Consider the dlm-suitable premouse $M_n(x)$. Let $\eta = \pi_{(\mathcal{N},s),\infty}(\bar{\eta})$ for some $(\mathcal{N}, s) \in \mathcal{I}$ and $\bar{\eta} < \gamma_s^{\mathcal{N}}$ and consider the formula

$$\varphi(v_0, v_1, v_2, v_3) = \text{“}1 \Vdash_{\mathrm{Col}(\omega, <\kappa^*)} \text{for all dlm-suitable } y \text{ with } v_0 \in \mathcal{G}^{M(y)},$$
$$\text{we have } (v_0, v_1) \in \mathcal{I}^{M(y)}, v_2 < \gamma_{v_1}^{v_0, M(y)}, \text{ and } \pi^{M(y)}_{(v_0,v_1),\infty}(v_2) = v_3\text{”},$$

where κ^* refers to the least inaccessible cardinal above the least Woodin cardinal of the current model. Recall that for any dlm-suitable y and z witnessed by $M(y)$ and $M(z)$, for any $(\mathcal{N}, s), (\mathcal{M}, t) \in \mathcal{I}^{M(y)} \cap \mathcal{I}^{M(z)}$ with $(\mathcal{N}, s) \leq_{\mathcal{I}^{M(y)}} (\mathcal{M}, t)$ and $(\mathcal{N}, s) \leq_{\mathcal{I}^{M(z)}} (\mathcal{M}, t)$ the induced embeddings $\pi^{M(y)}_{(\mathcal{N},s),(\mathcal{M},t)}$ and $\pi^{M(z)}_{(\mathcal{N},s),(\mathcal{M},t)}$ agree. Hence, in $M_n(x)[g]$, we have for every $\mathcal{R} \in \mathcal{G}$ which is the last model of a correctly guided finite stack on $\mathcal{N}$ that for all dlm-suitable y such that $\mathcal{R} \in \mathcal{G}^{M(y)}$, in fact $(\mathcal{R}, s) \in \mathcal{I}^{M(y)}$, $\pi_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\eta}) < \gamma_s^{\mathcal{R},M(y)}$, and $\pi^{M(y)}_{(\mathcal{R},s),\infty}(\pi_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\eta})) = \eta$. Therefore, $\mathcal{P}^{M(y)}(\mathcal{R}|\delta^{\mathcal{R}}) \vDash \varphi(\mathcal{R}, s, \pi_{(\mathcal{N},s),(\mathcal{R},s)}(\bar{\eta}), \eta)$. So Lemma 4.1 yields that $\hat{\mathcal{M}}_\infty \vDash \varphi(\mathcal{M}_\infty, s^*, \eta, \eta^*)$. So in $\hat{\mathcal{M}}_\infty[H]$, for all dlm-suitable y, we have $(\pi^{M(y)}_{(\mathcal{M}_\infty,s^*),\infty})^{\mathcal{F}^*,M(y)}(\eta) = \eta^*$, as desired. □

Theorem 4.3. $V_{\delta_\infty}^{\mathrm{HOD}^{M_n(x)[g]}} = V_{\delta_\infty}^{\mathcal{M}_\infty}$.

Proof. By the internal definition of $\mathcal{M}_\infty$ from Lemma 3.35 we have that $V_{\delta_\infty}^{\mathcal{M}_\infty} \subseteq V_{\delta_\infty}^{\mathrm{HOD}^{M_n(x)[g]}}$. For the other inclusion we first show the following claim.

Claim 1. $\pi_\infty \restriction \alpha \in \hat{\mathcal{M}}_\infty$ *for all $\alpha < \delta_\infty$.*

Proof. As $\alpha < \delta_\infty$, there exists an $s \in [\mathrm{Ord}]^{<\omega}$ such that for all dlm-suitable $M(y)$ in $\hat{\mathcal{M}}_\infty[H]$, $\alpha < \gamma_{s^*}^{\mathcal{M}_\infty, M(y)}$. For any such s and $M(y)$ we have by definition that $\pi_\infty \restriction \alpha = (\pi^{M(y)}_{(\mathcal{M}_\infty,s^*),\infty})^{\mathcal{F}^*,M(y)} \restriction \alpha$. Therefore $\pi_\infty \restriction \alpha \in \mathrm{HOD}_{\mathcal{M}_\infty}^{\hat{\mathcal{M}}_\infty[H]}$ and thus $\pi_\infty \restriction \alpha \in \hat{\mathcal{M}}_\infty$ by homogeneity of the forcing $\mathbb{P} = \mathrm{Col}(\omega, <\kappa_\infty)$. □

Now let $A \in V_{\delta_\infty}^{\mathrm{HOD}^{M_n(x)[g]}}$ be arbitrary. Let $\alpha < \delta_\infty$ be such that $A \subset \alpha$ is defined over $M_n(x)[g]$ by a formula φ with ordinal parameters from $t \in [\mathrm{Ord}]^{<\omega}$ and let $\beta < \alpha$ be arbitrary. That means $\beta \in A$ iff $M_n(x)[g] \vDash$

$\varphi(\beta, t)$. Lemma 4.1 yields that this is the case iff $\hat{\mathcal{M}}_\infty[H] \models \varphi(\beta^*, t^*)$. Since $\beta < \alpha < \delta_\infty$, we have that $\beta^* = \pi_\infty(\beta)$ by Lemma 4.2. Moreover, we have by Claim 1 that $\pi_\infty \restriction \alpha \in \hat{\mathcal{M}}_\infty$. Therefore, it follows by homogeneity of the forcing $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa_\infty)$ that $A \in \hat{\mathcal{M}}_\infty$ since t^* is a fixed parameter in $\hat{\mathcal{M}}_\infty$. Thus $A \in V_{\delta_\infty}^{\mathcal{M}_\infty}$, as desired. $\square$

5. The full HOD in $M_n(x)[g]$

To compute the full model $\mathrm{HOD}^{M_n(x)[g]}$, i.e., prove Theorem 1.1, we first show the following lemma.

Lemma 5.1. *$\mathrm{HOD}^{M_n(x)[g]} = M_n(A)$ for some set $A \subseteq \omega_2^{M_n(x)[g]}$ with $A \in \mathrm{HOD}^{M_n(x)[g]}$.*

Proof. Let $\mathbb{V}$ denote the Vopěnka algebra in $M_n(x)[g]$ for making a real generic over $\mathrm{HOD}^{M_n(x)[g]}$. By Vopěnka's theorem (see for example Theorem 15.46 in [Je03] or Theorem 9.0.1 in [La17]) there is a $\mathbb{V}$-generic G_x over $\mathrm{HOD}^{M_n(x)[g]}$ such that $x \in \mathrm{HOD}^{M_n(x)[g]}[G_x]$ and in fact $\mathrm{HOD}^{M_n(x)[g]}[G_x] = \mathrm{HOD}_x^{M_n(x)[g]}$.

Claim 1. *There is some $\tilde{\mathbb{V}} \in \mathrm{HOD}^{M_n(x)[g]}$ which is isomorphic to $\mathbb{V}$ and a subset of $\omega_2^{M_n(x)[g]}$.*

Proof. Work in $M_n(x)[g]$. Each real, i.e., element of $\mathcal{P}(\omega)$, can be coded by a countable ordinal and each set of reals can be coded by an ordinal $< \omega_2$. Forcing with the Vopěnka algebra $\mathbb{V}$ is ω_2-c.c. in $M_n(x)[g]$ as otherwise there would be an ω_2 sequence of pairwise distinct non-empty sets of reals, contradicting CH. The Vopěnka algebra is in $\mathrm{HOD}^{M_n(x)[g]}$ and when considering $\mathrm{HOD}^{M_n(x)[g]}[G_x]$ cardinals $\geq (\kappa^+)^{M_n(x)}$ are preserved. Since $(\kappa^{+\omega})^{M_n(x)}$ is below the least measurable cardinal of $M_n(x)$, $M_n(x)|(\kappa^{+\omega})^{M_n(x)}$ can be written as the Lp^n-stack of height $(\kappa^{+\omega})^{M_n(x)}$ above x and is therefore by the argument in Lemma 3.18 an element of $\mathrm{HOD}_x^{M_n(x)[g]} = \mathrm{HOD}^{M_n(x)[g]}[G_x]$. But the Vopěnka algebra is a subset of some ordinal $\alpha < (\kappa^{++})^{M_n(x)} = (\kappa^{++})^{\mathrm{HOD}^{M_n(x)[g]}[G_x]} = (\kappa^{++})^{\mathrm{HOD}^{M_n(x)[g]}}$, so there is some $\tilde{\mathbb{V}} \in \mathrm{HOD}^{M_n(x)[g]}$ which is isomorphic to $\mathbb{V}$ and a subset of $(\kappa^+)^{M_n(x)}$. $\square$

For the rest of this proof we write $\mathbb{V}$ for the $\tilde{\mathbb{V}}$ from the previous claim and show that $M_n(\mathbb{V}) = \mathrm{HOD}^{M_n(x)[g]}$.

Claim 2. *G_x is $\mathbb{V}$-generic over $M_n(\mathbb{V})$.*

Proof. The dense sets in question are elements of $\mathcal{P}(\mathbb{V})^{M_n(\mathbb{V})}$ and hence elements of $Lp^n(\mathbb{V}) = \bigcup\{N \mid N$ is a countable $\mathbb{V}$-premouse with $\rho_\omega(N) = \mathbb{V}$ which is n-small, sound, and $(\omega, \omega_1, \omega_1)$-iterable$\}$. As $\mathbb{V} \in \mathrm{HOD}^{M_n(x)[g]}$, Lemma 3.18 yields that $Lp^n(\mathbb{V}) \in \mathrm{HOD}^{M_n(x)[g]}$, which implies the claim. $\square$

Claim 3. *$M_n(x)$ and $M_n(\mathbb{V})[G_x]$ have the same least measurable cardinal κ_0. Moreover, $V_{\kappa_0}^{M_n(x)} = V_{\kappa_0}^{M_n(\mathbb{V})[G_x]}$.*

Proof. Write $\kappa_0^{M_n(x)}$ and $\kappa_0^{M_n(\mathbb{V})}$ for the least measurable cardinal of $M_n(x)$ and $M_n(\mathbb{V})$ respectively. $M_n(\mathbb{V})$ and $M_n(\mathbb{V})[G_x]$ have the same least measurable cardinal. The proof of Lemma 3.18 shows that $Lp^n(z) \in M_n(\mathbb{V})[G_x]$ for any $z \in V_{\delta_0}^{M_n(\mathbb{V})[G_x]}$, where δ_0 denotes the least Woodin cardinal in $M_n(\mathbb{V})[G_x]$. As $M_n(x)|\kappa_0^{M_n(x)}$ is equal to the Lp^n-stack of height $\kappa_0^{M_n(x)}$ over x, it follows that $M_n(x)|\kappa_0^{M_n(x)} \subseteq M_n(\mathbb{V})[G_x]$. Analogously, $M_n(\mathbb{V})|\kappa_0^{M_n(\mathbb{V})} \subseteq M_n(x)$ and in fact $M_n(\mathbb{V})[G_x]|\kappa_0^{M_n(\mathbb{V})} \subseteq M_n(x)$.
Suppose $\kappa_0^{M_n(\mathbb{V})} < \kappa_0^{M_n(x)}$ and let $\mathcal{U}$ be the measure on $\kappa_0^{M_n(\mathbb{V})}$ in $M_n(\mathbb{V})$ (which we identify with the lift of this measure to $M_n(\mathbb{V})[G_x]$). In particular, $\mathcal{U}$ measures all subsets of $\kappa_0^{M_n(\mathbb{V})}$ in $M_n(x)$ as well and $\kappa_0^{M_n(\mathbb{V})}$ is a cardinal in $M_n(x)$ because otherwise the function witnessing this would be an element of $M_n(x)|\kappa_0^{M_n(x)}$ and hence in $M_n(\mathbb{V})[G_x]$. So

$$V_{\kappa_0^{M_n(\mathbb{V})}}^{M_n(\mathbb{V})[G_x]} = V_{\kappa_0^{M_n(\mathbb{V})}}^{M_n(x)}.$$

Now $M_n(x)|(\kappa_0^{M_n(\mathbb{V})})^+ = Lp^n(M_n(x)|\kappa_0^{M_n(\mathbb{V})})$ and hence it contains a mouse with the measure $\mathcal{U}$ on $\kappa_0^{M_n(\mathbb{V})}$. This contradicts the fact that $\kappa_0^{M_n(x)}$ is the least measurable cardinal in $M_n(x)$. The argument in the case $\kappa_0^{M_n(\mathbb{V})} > \kappa_0^{M_n(x)}$ is analogous. Therefore, $M_n(x)$ and $M_n(\mathbb{V})[G_x]$ have the same least measurable cardinal κ_0 and the argument above shows $V_{\kappa_0}^{M_n(x)} = V_{\kappa_0}^{M_n(\mathbb{V})[G_x]}$. $\square$

For what follows, it suffices to work with the least inaccessible $\lambda < \kappa_0$ of $M_n(x)$ which is above κ, so we restrict ourselves to this situation.

Claim 4. $V_\lambda^{M_n(\mathbb{V})} \subseteq \mathrm{HOD}^{M_n(x)[g]}$.

Proof. This follows from the proof of Lemma 3.18 as $M_n(\mathbb{V})|\lambda$ can be obtained as the Lp^n-stack of height λ over $\mathbb{V}$ and $\mathbb{V} \in \mathrm{HOD}^{M_n(x)[g]}$. $\square$

Now we can show that the lemma holds below λ.

Claim 5. $V_\lambda^{M_n(\mathbb{V})} = V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}$.

Proof. We first show that $V_\lambda^{M_n(\mathbb{V})}[G_x] = V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}[G_x]$. The inclusion $\subseteq$ follows from Claim 4. For the other inclusion we have that

$$\mathrm{HOD}^{M_n(x)[g]}[G_x] = \mathrm{HOD}_x^{M_n(x)[g]} \subseteq \mathrm{HOD}_x^{M_n(x)} \subseteq M_n(x),$$

using the homogeneity and ordinal definability of the forcing $\mathrm{Col}(\omega, {<}\kappa)$. Therefore by Claim 3

$$V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}[G_x] \subseteq V_\lambda^{M_n(x)} = V_\lambda^{M_n(\mathbb{V})}[G_x].$$

Finally, we argue that the equality $V_\lambda^{M_n(\mathbb{V})}[G_x] = V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}[G_x]$ also holds true without adding the generic G_x. As by Claim 4 we have $V_\lambda^{M_n(\mathbb{V})} \subseteq V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}$, we are again left with proving the other inclusion. Let $\mathbb{P} = \mathbb{V} \times \mathrm{Col}(\omega, {<}\kappa)$. Then (G_x, g) is $\mathbb{P}$-generic over both $V_\lambda^{M_n(\mathbb{V})}$ and $V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}$, and $V_\lambda^{M_n(\mathbb{V})}[G_x, g] = V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}[G_x, g]$. Let $a \in V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}$ be a set of ordinals. Then there is a $\mathbb{P}$-name $\sigma \in V_\lambda^{M_n(\mathbb{V})}$ such that $\sigma_{(G_x,g)} = a$. This is forced over $V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}}$, i.e., there is a $p \in \mathbb{P}$ such that $V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}} \vDash$ "$p \Vdash \sigma = \check{a}$". Thus $V_\lambda^{M_n(\mathbb{V})}$ can compute the elements of a using the forcing relation for $\mathbb{P}$ below p. Hence $a \in V_\lambda^{M_n(\mathbb{V})}$, as desired. $\square$

Now we are able to extend Claim 3 to the full models.

Claim 6. $M_n(\mathbb{V})[G_x] = M_n(x)$.

Proof. Consider $M_n(x)[g]$ as a $V_\lambda^{M_n(x)[g]}$-premouse and note that it equals $M_n(V_\lambda^{M_n(x)[g]})$. We use $\mathcal{P}^{M_n(x)[g]}(M_n(\mathbb{V})|\lambda)$ to denote the result of a $\mathcal{P}$-construction in the sense of [SchSt09] above $M_n(\mathbb{V})|\lambda$ inside the $V_\lambda^{M_n(x)[g]}$-premouse $M_n(x)[g]$. By Claim 3, $V_\lambda^{M_n(\mathbb{V})}[G_x] = V_\lambda^{M_n(x)}$, so $V_\lambda^{M_n(\mathbb{V})}[G_x][g] = V_\lambda^{M_n(x)[g]}$ and this $\mathcal{P}$-construction is well-defined. Moreover, the following argument shows that the construction never projects across λ.

Assume toward a contradiction that there is a level $\mathcal{P}$ of the $\mathcal{P}$-construction above $M_n(\mathbb{V})|\lambda$ inside $M_n(x)[g]$ such that $\rho_\omega(\mathcal{P}) = \rho < \lambda$. That means there is an $r\Sigma_{k+1}(\mathcal{P})$-definable set $a \subseteq \rho$ for some $k < \omega$ such that $a \notin \mathcal{P}$. As by the proof of Claim 4, $M_n(\mathbb{V})|\lambda \in \mathrm{HOD}^{M_n(x)[g]}$ it follows by definability of the $\mathcal{P}$-construction and of the extender sequence of $M_n(V_\lambda^{M_n(x)[g]})$ (see Lemma 1.1 in [Sch06] due to J. Steel) that $\mathcal{P} \in \mathrm{HOD}^{M_n(x)[g]}$. This means that in particular $a \in \mathrm{HOD}^{M_n(x)[g]}$. But $a \subseteq \rho < \lambda$ and by Claim 5, $V_\lambda^{\mathrm{HOD}^{M_n(x)[g]}} = V_\lambda^{M_n(\mathbb{V})} = V_\lambda^{\mathcal{P}}$, so $a \in \mathcal{P}$. Contradiction.

Now it follows by construction (see [SchSt09]) that

$$\mathcal{P}^{M_n(x)[g]}(M_n(\mathbb{V})|\lambda)[G_x][g] = M_n(x)[g].$$

But this yields that $\mathcal{P}^{M_n(x)[g]}(M_n(\mathbb{V})|\lambda)[G_x] = M_n(x)$, without adding the generic g, by an argument as the one at the end of the previous claim. Moreover, $\mathcal{P}^{M_n(x)[g]}(M_n(\mathbb{V})|\lambda) = M_n(\mathbb{V})$ and thus $M_n(\mathbb{V})[G_x] = M_n(x)$, as desired. $\square$

This argument also shows the following claim.

Claim 7. $M_n(\mathbb{V}) \subseteq \mathrm{HOD}^{M_n(x)[g]}$.

Now, the next claim follows from the first half of the proof of Claim 5.

Claim 8. $M_n(\mathbb{V})[G_x] = \mathrm{HOD}^{M_n(x)[g]}[G_x]$.

Finally, the statement of Claim 8 also holds true without adding the generic G_x by the argument at the end of the proof of Claim 5. Hence $M_n(\mathbb{V}) = \mathrm{HOD}^{M_n(x)[g]}$, as desired. $\square$

Corollary 5.2. *Let $F(s) = s^*$ for $s \in [\mathrm{Ord}]^{<\omega}$. Then*

$$\mathrm{HOD}^{M_n(x)[g]} = M_n(\mathcal{M}_\infty|\delta_\infty, F \restriction \delta_\infty).$$

Proof. Note that $\mathcal{M}_\infty|\delta_\infty$ and $F \restriction \delta_\infty$ are elements of $\mathrm{HOD}^{M_n(x)[g]}$ by construction. Let $\eta = \sup F``\delta_\infty$ and let γ be the least inaccessible cardinal of $M_n(\mathcal{M}_\infty|\delta_\infty, F \restriction \delta_\infty)$ above η. Let $A \subseteq \omega_2^{M_n(x)[g]}$ be as in the statement of Lemma 5.1, i.e., such that $\mathrm{HOD}^{M_n(x)[g]} = M_n(A)$. Moreover, let φ be a formula defining A, i.e., $\xi \in A$ iff $M_n(x)[g] \vDash \varphi(\xi)$. Then, as $F(\xi) = \pi_\infty(\xi)$ for $\xi < \delta_\infty$ by Lemma 4.2,

$$\xi \in A \text{ iff } M_n(\mathcal{M}_\infty|\delta_\infty) \vDash \text{“}1 \Vdash^{\mathbb{P}} \varphi(\pi_\infty(\xi)), \text{where } \pi_\infty \text{ is the direct limit embedding from the systems on } \mathcal{M}_\infty\text{”}$$

for $\mathbb{P} = \mathrm{Col}(\omega, {<}\kappa_\infty)$. Consider $L[E](\mathcal{M}_\infty|\delta_\infty)^{M_n(\mathcal{M}_\infty|\delta_\infty, F\restriction\delta_\infty)}$, the result of a fully backgrounded extender construction in the sense of [MS94] inside $M_n(\mathcal{M}_\infty|\delta_\infty, F \restriction \delta_\infty)$ above $\mathcal{M}_\infty|\delta_\infty$. The premice $M_n(\mathcal{M}_\infty|\delta_\infty)$ and $L[E](\mathcal{M}_\infty|\delta_\infty)^{M_n(\mathcal{M}_\infty|\delta_\infty, F\restriction\delta_\infty)}$ successfully compare to a common proper class premouse without drops on the main branches. Since the iterations take place above δ_∞, $\xi < \delta_\infty$ is not moved and we have by elementarity

$$\xi \in A \text{ iff } L[E](\mathcal{M}_\infty|\delta_\infty)^{M_n(\mathcal{M}_\infty|\delta_\infty, F\restriction\delta_\infty)} \vDash \text{“}1 \Vdash^{\mathbb{P}} \varphi(\pi_\infty(\xi)), \text{where } \pi_\infty \text{ is the direct limit embedding from the systems on } \mathcal{M}_\infty\text{”}.$$

Therefore it follows that $A \in M_n(\mathcal{M}_\infty|\delta_\infty, F \restriction \delta_\infty)$.

By the same argument as in the proof of Claim 3 in the proof of Lemma 5.1 we now obtain that the universes of $M_n(\mathcal{M}_\infty|\delta_\infty, F \restriction \delta_\infty)$ and $M_n(A)$ agree up to their common least measurable cardinal. In particular, γ is also the least inaccessible cardinal above η of $M_n(A)$ and we can rearrange $M_n(\mathcal{M}_\infty|\delta_\infty, F \restriction \delta_\infty)$ and $M_n(A)$ as $V_\gamma^{M_n(\mathcal{M}_\infty|\delta_\infty, F\restriction\delta_\infty)}$-premice. As such it follows that the following equalities for classes (not structures) hold:

$$\begin{aligned} M_n(\mathcal{M}_\infty|\delta_\infty, F \restriction \delta_\infty) &= M_n(V_\gamma^{M_n(\mathcal{M}_\infty|\delta_\infty, F\restriction\delta)}) \\ &= M_n(A) = \mathrm{HOD}^{M_n(x)[g]}. \end{aligned}$$

$\square$

The following corollary follows immediately from Lemma 4.2 and Corollary 5.2.

Corollary 5.3. $\mathrm{HOD}^{M_n(x)[g]} = M_n(\mathcal{M}_\infty|\delta_\infty, \pi_\infty \restriction \delta_\infty)$.

We now consider the iteration strategy for $\mathcal{M}_\infty$. Let Λ be the restriction of $\Sigma_{M_{n+1}^-}$ to correctly guided finite stacks $\vec{\mathcal{T}}$ on $\mathcal{M}_\infty|\delta_\infty$ such that $\vec{\mathcal{T}} \in \hat{\mathcal{M}}_\infty|\kappa_\infty$, where κ_∞ is the least inaccessible cardinal in $\hat{\mathcal{M}}_\infty$ above δ_∞.

Lemma 5.4. $\Lambda \in \mathrm{HOD}^{M_n(x)[g]}$.

Proof. Let $\mathcal{T}$ be a maximal tree on $\mathcal{M}_\infty|\delta_\infty$ with $\mathcal{T} \in \hat{\mathcal{M}}_\infty|\kappa_\infty$. Moreover, let $b = \Lambda(\mathcal{T})$. Let $\mathcal{R} = \mathcal{M}_b^\mathcal{T}$ be the last model of $\mathcal{T}^\frown b$. Then $\mathcal{R} \in \mathrm{HOD}^{M_n(x)[g]}$. Moreover, let $\delta_\infty^{\mathcal{F}^*}$ be the least Woodin cardinal in $\mathcal{M}_\infty^{\mathcal{F}^*}$, the direct limit of the system $\mathcal{F}^{*,M(y)}$ for some/all dlm-suitable $M(y)$ in $\hat{\mathcal{M}}_\infty[H]$. Then $\mathcal{M}_\infty^{\mathcal{F}^*}|\delta_\infty^{\mathcal{F}^*}$ is an iterate of $\mathcal{R}$. As $\pi_\infty \restriction \delta_\infty \in \mathrm{HOD}^{M_n(x)[g]}$, we can identify b inside $M_n(x)[g]$ as the unique branch through $\mathcal{T}$ which is $(\pi_\infty \restriction \delta_\infty)$-realizable, i.e., such that there is an elementary embedding $\sigma : \mathcal{M}_b^\mathcal{T} \rightarrow \mathcal{M}_\infty^{\mathcal{F}^*}|\delta_\infty^{\mathcal{F}^*}$ with $\pi_\infty \restriction \delta_\infty = \sigma \circ i_b^\mathcal{T}$.
The same argument applies to pseudo-normal iterates $\mathcal{N}$ of $\mathcal{M}_\infty$ with $\mathcal{N}|\delta^\mathcal{N} \in \hat{\mathcal{M}}_\infty|\kappa_\infty$ and maximal iteration trees $\mathcal{T}$ on $\mathcal{N}|\delta^\mathcal{N}$ such that $\mathcal{T} \in \hat{\mathcal{M}}_\infty|\kappa_\infty$, hence $\Lambda \in \mathrm{HOD}^{M_n(x)[g]}$. □

Similarly to Lemma 3.47 in [StW16] we finally need a method of Boolean-valued comparison. As the proof is analogous we omit it.

Lemma 5.5. *Let H be $\mathrm{Col}(\omega, {<}\kappa_\infty)$-generic over $\hat{\mathcal{M}}_\infty$, and let $\mathcal{Q}$ be such that $\hat{\mathcal{M}}_\infty[H] \vDash$ "$\mathcal{Q}$ is countable and n-suitable". Then there is an $\mathcal{R}$ such that*

(1) $\mathcal{R}$ is a pseudo-normal iterate of $\mathcal{Q}$,
(2) $\mathcal{R}$ is a $\Sigma_{M_{n+1}^-}$-iterate of $\mathcal{M}_\infty$, and
(3) $\mathcal{R} \in \hat{\mathcal{M}}_\infty$.

Finally, we can finish the proof of Theorem 1.1.

Theorem 5.6.
$$\mathrm{HOD}^{M_n(x)[g]} = M_n(\mathcal{M}_\infty|\delta_\infty, \pi_\infty \restriction \delta_\infty) = M_n(\hat{\mathcal{M}}_\infty|\kappa_\infty, \Lambda).$$

Proof. $\mathrm{HOD}^{M_n(x)[g]} = M_n(\mathcal{M}_\infty|\delta_\infty, \pi_\infty \restriction \delta_\infty)$ is Corollary 5.3. Moreover, $M_n(\mathcal{M}_\infty|\delta_\infty, \pi_\infty \restriction \delta_\infty) = M_n(\hat{\mathcal{M}}_\infty|\kappa_\infty, \Lambda)$ follows from Lemma 5.5 as follows. First, $\Lambda \in M_n(\mathcal{M}_\infty|\delta_\infty, \pi_\infty \restriction \delta_\infty) = \mathrm{HOD}^{M_n(x)[g]}$ and $\hat{\mathcal{M}}_\infty|\kappa_\infty \in M_n(\mathcal{M}_\infty|\delta_\infty, \pi_\infty \restriction \delta_\infty)$ by considering the Lp^n-stack on $\mathcal{M}_\infty|\delta_\infty$. The direct limit of $\mathcal{F}^{*,M(y)}$ for some $M(y)$ is the same as the direct limit of all Λ-iterates of $\mathcal{M}_\infty$ which are an element of $\hat{\mathcal{M}}_\infty|\kappa_\infty$ via the comparison maps. Moreover, we have that π_∞ is the canonical direct limit map of this system and therefore definable from $\hat{\mathcal{M}}_\infty|\kappa_\infty$ and Λ. So $\pi_\infty \restriction \delta_\infty \in M_n(\hat{\mathcal{M}}_\infty|\kappa_\infty, \Lambda)$. Now, $M_n(\mathcal{M}_\infty|\delta_\infty, \pi_\infty \restriction \delta_\infty) = M_n(\hat{\mathcal{M}}_\infty|\kappa_\infty, \Lambda)$ follows analogous to the proof of Corollary 5.2. □

Note that Theorem 4.3 and Lemma 5.1 together imply Corollary 1.2, i.e., that the GCH holds in $\mathrm{HOD}^{M_n(x)[g]}$. Finally, most of the arguments we gave in this and the previous sections generalize with only small changes to more arbitrary canonical self-iterable inner models, e.g. M_ω, $M_{\omega+42}$. We leave the details to the reader.

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