HOD in $M_n(x, g)$

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work in progress with Grigor Sargsyan

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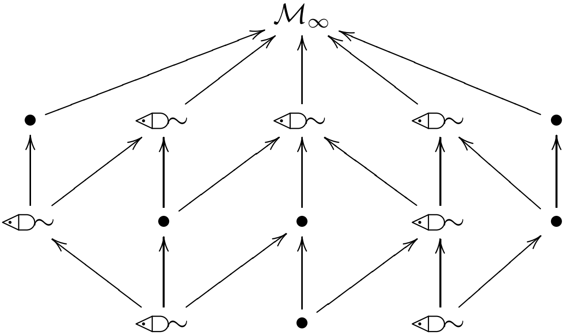

Some like it HOD

UC Irvine, July 18 – 29, 2016

Drawing by Martin Zeman.

Motivation

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- Test question: Is HOD^M a model of GCH?
- Goal: Show that HOD^M is a core model (i.e. a fine structural model).
- This would imply that we have $\mathrm{GCH}, \diamondsuit, \square, \ldots$ in HOD^M.

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- (Steel, 1995)

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_{(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}} = M_\infty \cap V_{(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})}},$$

  where M_∞ is a direct limit of iterates of M_ω, and $(\boldsymbol{\delta}_1^2)^{L(\mathbb{R})} = \sup\{\alpha \mid \exists f(f : \mathbb{R} \to \alpha \text{ and } f \text{ is surjective and } \boldsymbol{\Delta}_1^{L(\mathbb{R})})\}$.

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- (Woodin, ≈1996)

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Question

Assume Δ^1_2-determinacy. Do we have

$$\mathrm{HOD}^{L[x]} \models \mathrm{GCH}$$

for a Turing cone of reals x?

What we can do is (under the right determinacy assumption) analyze $\mathrm{HOD}^{L[x][G]}$ for a Turing cone of reals x, where

- G is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $L[x]$, and
- $\kappa_x =$ least inaccessible cardinal in $L[x]$.

$\mathrm{HOD}^{L[x,G]}$ as a core model

For every real x let κ_x denote the least inaccessible cardinal in $L[x]$.

Theorem (Woodin, 90's)

Assume Δ^1_2-determinacy. For a Turing cone of x,

$$\mathrm{HOD}^{L[x,G]} = L[M_\infty, \Lambda],$$

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- g is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $M_n(x)$, and
- $\kappa_x < \delta_0^{M_n(x)}$ is an inaccessible strong cutpoint cardinal of $M_n(x)$ such that κ_x is a limit of strong cutpoint cardinals in $M_n(x)$.

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- Sargsyan: $\delta^{M_\infty} = (\kappa_x^+)^{M_n(x)}$.
- By definability of the internal direct limit system we have that

$$M_\infty \subseteq \mathrm{HOD}^{M_n(x)[g]}.$$

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Let κ_∞ be the least inaccessible cardinal of M_∞ strictly above δ_∞.

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- In fact, $\pi_\infty \restriction \alpha \in M_\infty$ for all $\alpha < \delta$.

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Using this we can show:

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Moreover we are optimistic to show:

Lemma

For some $M_n(x)[g]$-definable set $A \subseteq \omega_2^{M_n(x)[g]}$ we have that

$$\mathrm{HOD}^{M_n(x)[g]} = M_n(A).$$

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This should then give that

$$\mathrm{HOD}^{M_n(x)[g]} = M_n(M_\infty, \Lambda),$$

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Question

Is $\mathrm{HOD}^{M_n(x)}$ (without the generic g) a core model?

It is not even known if $\mathrm{HOD}^{L[x]}$ and $\mathrm{HOD}^{M_n(x)}$ are models of GCH.

Thank you for your attention!