HOD in $M_n(x, g)$

Sandra Uhlenbrock

January 25th-30th, 2017

work in progress with Grigor Sargsyan

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Some like it HOD

UC Irvine, July 18 – 29, 2016

Drawing by Martin Zeman.
Want to understand $\text{HOD}^M$ for various inner models $M$ like $L(\mathbb{R})$, $L[x]$ or $M_n(x)$ (assuming determinacy).
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Test question: Is $\text{HOD}^M$ a model of GCH?
Motivation

- Want to understand $\text{HOD}^M$ for various inner models $M$ like $L(\mathbb{R})$, $L[x]$ or $M_n(x)$ (assuming determinacy).
- Test question: Is $\text{HOD}^M$ a model of GCH?
- Goal: Show that $\text{HOD}^M$ is a core model (i.e. a fine structural model).
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- Test question: Is $\text{HOD}^M$ a model of GCH?
- Goal: Show that $\text{HOD}^M$ is a core model (i.e. a fine structural model).
- This would imply that we have GCH, ♦, □, ... in $\text{HOD}^M$. 
What is known about $\text{HOD}^L(\mathbb{R})$

Assume $\text{AD}^L(\mathbb{R})$.
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- (Becker, 1980) $\text{HOD}^{L(\mathbb{R})} \models \text{GCH}_\alpha$ for all $\alpha < \omega_1^V$. 

(Steel, Woodin, 1993) $\text{HOD}^{L(\mathbb{R})} \cap \mathbb{R} = M_\omega \cap \mathbb{R}$.

(Steel, Woodin, 1993) $\text{HOD}^{L(\mathbb{R})} \cap P(\omega_{V_1}) = N \cap P(\omega_{V_1})$, where $N$ is the $\omega_{V_1}$-th iterate of $M_\omega$ by its least measure.

(Steel, 1995) $\text{HOD}^{L(\mathbb{R})} \cap V(\delta_{21}) = M_\infty \cap V(\delta_{21})$, where $M_\infty$ is a direct limit of iterates of $M_\omega$, and $(\delta_{21})_{L(\mathbb{R})} = \sup \{ \alpha | \exists f (f: \mathbb{R} \to \alpha \text{ and } f \text{ is surjective and } \Delta_{L(\mathbb{R})}^1) \}$.

(Woodin, ≈ 1996) $\text{HOD}^{L(\mathbb{R})} = L[M_\infty, \Lambda]$, where $\Lambda$ is a partial iteration strategy for $M_\infty$. 

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where \( N \) is the \( \omega_1^V \)-th iterate of \( M_\omega \) by it’s least measure.

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\text{HOD}^{L(\mathbb{R})} \cap V(\delta_1^2)^{L(\mathbb{R})} = M_\infty \cap V(\delta_1^2)^{L(\mathbb{R})},
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where \( M_\infty \) is a direct limit of iterates of \( M_\omega \), and

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(\delta_1^2)^{L(\mathbb{R})} = \sup\{ \alpha \mid \exists f (f : \mathbb{R} \to \alpha \text{ and } f \text{ is surjective and } \Delta_1^{L(\mathbb{R})}) \}.
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What is known about $\text{HOD}^{L[x]}$

... very little.

Question

Assume $\Delta^1_2$-determinacy. Do we have $\text{HOD}^{L[x]} \models \text{GCH}$ for a Turing cone of reals $x$?

What we can do is (under the right determinacy assumption) analyze $\text{HOD}^{L[x]}[G]$ for a Turing cone of reals $x$, where $G$ is $\text{Col}(\omega, <\kappa_x)$-generic over $L[x]$, and $\kappa_x$ is the least inaccessible cardinal in $L[x]$. 

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**Question**

Assume $\Delta^1_2$-determinacy. Do we have

$$\text{HOD}^{L[x]} \models \text{GCH}$$

for a Turing cone of reals $x$?

What we can do is (under the right determinacy assumption) analyze $\text{HOD}^{L[x][G]}$ for a Turing cone of reals $x$, where

- $G$ is $\text{Col}(\omega, <\kappa_x)$-generic over $L[x]$, and
- $\kappa_x = \text{least inaccessible cardinal in } L[x]$. 
For every real $x$ let $\kappa_x$ denote the least inaccessible cardinal in $L[x]$.

**Theorem (Woodin, 90's)**

Assume $\Delta^1_2$-determinacy. For a Turing cone of $x$,

$$\text{HOD}^{L[x,G]} = L[M_\infty, \Lambda],$$

where $G$ is $\text{Col}(\omega, \langle \kappa_x \rangle)$-generic over $L[x]$, $M_\infty$ is a direct limit of mice, and $\Lambda$ is a partial iteration strategy for $M_\infty$. 
Assume $\Pi_1$-determinacy.

Goal: Generalize this analysis to $\text{HOD}_{M_n(x, g)}$ for a Turing cone of reals $x$, where $M_n(x)$ denotes the least proper class iterable premouse with $n$ Woodin cardinals, $g$ is $\text{Col}(\omega, <\kappa_x)$-generic over $M_n(x)$, and $\kappa_x < \delta_{M_n(x)}$ is an inaccessible strong cutpoint cardinal of $M_n(x)$ such that $\kappa_x$ is a limit of strong cutpoint cardinals in $M_n(x)$. 
Assume $\Pi^1_{n+2}$-determinacy.

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Assume $\Pi_{n+2}^1$-determinacy.

**Goal:** Generalize this analysis to $\text{HOD}^{M_n(x)[g]}$ for a Turing cone of reals $x$, where

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- $g$ is $\text{Col}(\omega, < \kappa_x)$-generic over $M_n(x)$, and
- $\kappa_x < \delta_0^{M_n(x)}$ is an inaccessible strong cutpoint cardinal of $M_n(x)$ such that $\kappa_x$ is a limit of strong cutpoint cardinals in $M_n(x)$. 

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Let $x$ be a real such that $M_{n+1}^\# \in M_n(x)$. 
The idea of the proof (very sketchy!)

Let $x$ be a real such that $M_{n+1}^\# \in M_n(x)$.

- Define a direct limit system of iterates of $M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$ which have a Woodin cardinal that is countable in $M_n(x)[g]$ together with iteration embeddings, call the direct limit $M^+_\infty$.

$M^\infty_\infty$ is well-founded as $M_{n+1}$ is sufficiently iterable.

Define an internal direct limit system of suitable strongly $s$-iterable premice in $M_n(x)[g]$ and call its direct limit $M^\infty_\infty$.

Sargsyan: $M^\infty_\infty = M^\infty_\infty$, so in particular $M^\infty_\infty$ is well-founded.

Sargsyan: $\delta M^\infty_\infty = (\kappa + x)_{M_n(x)}$.

By definability of the internal direct limit system we have that $M^\infty_\infty \subseteq HOD_{M_n(x)[g]}$. 

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Let $x$ be a real such that $M^\#_{n+1} \in M_n(x)$.

- Define a direct limit system of iterates of $M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$ which have a Woodin cardinal that is countable in $M_n(x)[g]$ together with iteration embeddings, call the direct limit $M^+_{\infty}$.

- $M^+_{\infty}$ is well-founded as $M_{n+1}$ is sufficiently iterable.
Let $x$ be a real such that $M_{n+1}^# \in M_n(x)$.

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- $M_+^\infty$ is well-founded as $M_{n+1}$ is sufficiently iterable.

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- $M_\infty^+$ is well-founded as $M_{n+1}$ is sufficiently iterable.
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Let \( x \) be a real such that \( M_{n+1}^\# \in M_n(x) \).

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- \( M^+_\infty \) is well-founded as \( M_{n+1} \) is sufficiently iterable.
- Define an internal direct limit system of suitable strongly \( s \)-iterable premice in \( M_n(x)[g] \) and call its direct limit \( M_\infty \).
- Sargsyan: \( M_\infty = M^+_\infty \), so in particular \( M_\infty \) is well-founded.
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- By definability of the internal direct limit system we have that

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M_\infty \subseteq \text{HOD}^{M_n(x)[g]}.
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The idea of the proof (very sketchy!)

Let $\kappa_\infty$ be the least inaccessible cardinal of $M_\infty$ strictly above $\delta_\infty$.  
- $M_\infty[H]$ for a $\text{Col}(\omega, <\kappa_\infty)$-generic $H$ is the derived model of $M_\infty$. 

Lemma (Derived model resemblance, Woodin) 
The derived model $M_\infty[H]$ (adding the theory of $M_n$ on top) is elementary equivalent to $M_n(x)[g]$. 

Therefore $M_\infty[H]$ has its own version of the direct limit system, call the direct limit model $M^*_\infty = \left(M_\infty\right)_{M_\infty[H]}$. 

$M_\infty$ shows up in this direct limit system, let $\pi_\infty: M_\infty \to M^*_\infty$ be the corresponding map. 

In fact, $\pi_\infty|_\alpha \in M_\infty$ for all $\alpha < \delta_\infty$. 

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- Use the derived model as a surrogate for $M_n(x)[g]$ to compute $\text{HOD}^{M_n(x)[g]}$.

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- $M_\infty$ shows up in this direct limit system, let $\pi_\infty : M_\infty \to M^*_\infty$ be the corresponding map.

- In fact, $\pi_\infty \upharpoonright \alpha \in M_\infty$ for all $\alpha < \delta$. 

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Using this we can show:

**Theorem**

\[ \text{HOD}^{M_n(x,g)} \cap V_{\delta_{\infty}} = M_{\infty} \cap V_{\delta_{\infty}}. \]
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\text{HOD}^{M_n(x)[g]} \cap V_{\delta_\infty} = M_\infty \cap V_{\delta_\infty}.
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Moreover we are optimistic to show:

**Lemma**

For some \( M_n(x)[g] \)-definable set \( A \subseteq \omega_2^{M_n(x)[g]} \) we have that

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\text{HOD}^{M_n(x)[g]} = M_n(A).
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\[ \text{HOD}^{M_n(x)[g]} = M_n(A). \]

This should then give that

\[ \text{HOD}^{M_n(x)[g]} = M_n(M_{\infty}, \Lambda), \]

where \( \Lambda \) is a partial iteration strategy for \( M_{\infty} \).
Open questions

Question

Is $\text{HOD}^{L[x]}$ (without the generic $G$) a core model?
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Proposition (Schlutzenberg, 2016)
Given sufficient large cardinals, there is a cone of reals $x$ such that if $\mathcal{F}$ is a natural candidate for a limit system to analyze HOD$^L[x]$, then $\mathcal{F}$ is not closed under pseudo-comparison of pairs.
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Proposition (Schlutzenberg, 2016)

Given sufficient large cardinals, there is a cone of reals \( x \) such that if \( \mathcal{F} \) is a natural candidate for a limit system to analyze \( \text{HOD}^{L[x]} \), then \( \mathcal{F} \) is not closed under pseudo-comparison of pairs.

Question

Is \( \text{HOD}^{M_n(x)} \) (without the generic \( g \)) a core model?

It is not even known if \( \text{HOD}^{L[x]} \) and \( \text{HOD}^{M_n(x)} \) are models of GCH.
Thank you for your attention!