The hereditarily ordinal definable sets in inner models with finitely many Woodin cardinals

Sandra Uhlenbrock

August 14th, 2017

Joint work with Grigor Sargsyan

Logic Colloquium 2017, Special session on set theory

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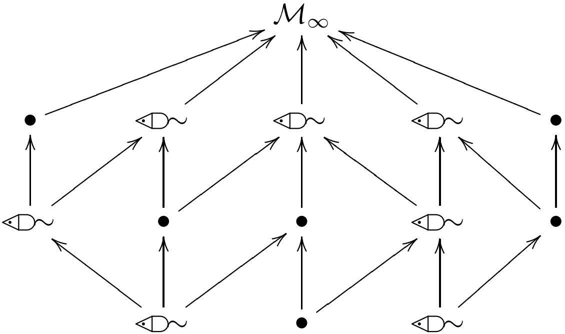

Some like it HOD

UC Irvine, July 18 – 29, 2016

Drawing by Martin Zeman.

Motivation

Definition

HOD is the class of all hereditarily ordinal definable sets,

$$\mathrm{HOD} = \{x \mid \mathrm{TC}(\{x\}) \subset \mathrm{OD}\}.$$

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- Test question: Is HOD^M a model of GCH?
- Method: Show that HOD^M is a fine structural model.
- This would imply that we have $\mathrm{GCH}, \diamondsuit, \square, \dots$ in HOD^M.

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Under determinacy hypotheses, these models can contain large cardinals, e.g. Woodin cardinals.

Theorem (Woodin)

Assume Δ^1_2-determinacy. Then for a Turing cone of reals x,

$$\mathrm{HOD}^{L[x]} \models \omega_2^{L[x]} \text{ is a Woodin cardinal.}$$

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Moreover Woodin showed analogous results for $\mathrm{HOD}^{M_n(x)}$ and $\mathrm{HOD}^{L(\mathbb{R})}$.

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Theorem (Becker, 1980)

$\mathrm{HOD}^{L(\mathbb{R})} \vDash \mathrm{GCH}_\alpha$ *for all* $\alpha < \omega_1^V$.

$\mathrm{HOD}^{L[\mathbb{R}]}$

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Theorem (Steel, Woodin, 1993)

$\mathrm{HOD}^{L(\mathbb{R})} \cap \mathbb{R} = M_\omega \cap \mathbb{R}$.

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Theorem (Steel, Woodin, 1993)

$$\mathrm{HOD}^{L(\mathbb{R})} \cap \mathcal{P}(\omega_1^V) = N \cap \mathcal{P}(\omega_1^V),$$

where N is the ω_1^V-th iterate of M_ω by its least measure.

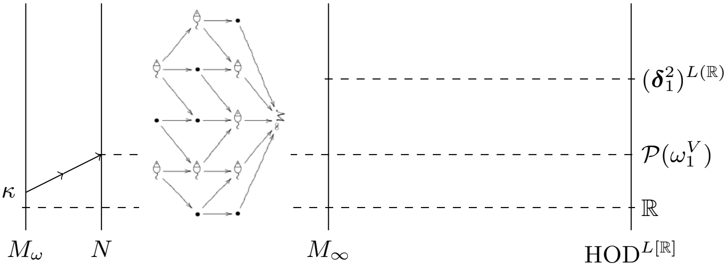
What is known about $\mathrm{HOD}^{L(\mathbb{R})}$ under $\mathrm{AD}^{L(\mathbb{R})}$

Theorem (Steel, 1995)

Let M_∞ be a direct limit of iterates of M_ω, then

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_{(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}} = M_\infty \cap V_{(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}},$$

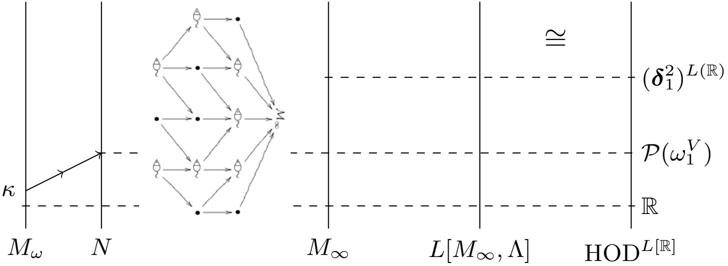
where $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})} = \sup\{\alpha \mid \exists f (f : \mathbb{R} \to \alpha$ and f is surjective and $\boldsymbol{\Delta}^{L(\mathbb{R})}_1)\}$.



Theorem (Woodin, $\approx$1996)

$$\mathrm{HOD}^{L(\mathbb{R})} = L[M_\infty, \Lambda],$$

where Λ is a partial iteration strategy for M_∞.



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Question

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What we can do is (under the right determinacy assumption) analyze $\mathrm{HOD}^{L[x][G]}$ for a Turing cone of reals x, where

- G is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $L[x]$, and
- $\kappa_x =$ least inaccessible cardinal in $L[x]$.

$\mathrm{HOD}^{L[x,G]}$ as a core model

For every real x let κ_x denote the least inaccessible cardinal in $L[x]$.

Theorem (Woodin, 90's)

Assume Δ^1_2-determinacy. For a Turing cone of x,

$$\mathrm{HOD}^{L[x,G]} = L[M_\infty, \Lambda],$$

where G is $\mathrm{Col}(\omega, {<}\kappa_x)$-generic over $L[x]$, M_∞ is a direct limit of mice, and Λ is a partial iteration strategy for M_∞.

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Theorem (Sargsyan, U.)

Assume $\mathbf{\Pi}^1_{n+2}$-determinacy. Then for a Turing cone of reals x,

$$\mathrm{HOD}^{M_n(x,g)} = M_n(M_\infty, \Lambda),$$

where M_∞ is a direct limit of iterates of an initial segment of M_{n+1} and Λ is a partial iteration strategy for M_∞.

The idea of the proof (very sketchy!)

Let x be a real such that $M_{n+1}^{\#} \in M_n(x)$.

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- Define a direct limit system of iterates of $M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$ which have a Woodin cardinal that is countable in $M_n(x,g)$ together with iteration embeddings, call the direct limit M_∞.

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- This will yield $\mathrm{HOD}^{M_n(x,g)} \subseteq M_n(M_\infty, \Lambda)$.

HOD in other canonical inner models

We can generalize this to other canonical minimal inner models with some Woodin and/or strong cardinals, e.g. $M_{\omega+17}$ ($\omega + 17$ Woodin cardinals) or M_{ws} (a strong cardinal above a Woodin cardinal).

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Theorem (Sargsyan, U.)

Let $M(x)$ be the minimal proper class iterable x-premouse with a fixed number of Woodin and strong cardinals (in a fixed order). Assume enough determinacy. Then for a Turing cone of reals x,

$$\mathrm{HOD}^{M(x,g)} = M(M_\infty, \Lambda),$$

where g is generic over $M(x)$ for the Levy collapse of the bottom inaccessible to ω, M_∞ is a direct limit of premice and Λ is a partial iteration strategy for M_∞.

Open questions

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Proposition (Schlutzenberg, 2016)

Given sufficient large cardinals, there is a cone of reals x such that if $\mathcal{F}$ is a natural candidate for a limit system to analyze $\mathrm{HOD}^{L[x]}$, then $\mathcal{F}$ is not closed under pseudo-comparison of pairs.

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Question

Is $\mathrm{HOD}^{M_n(x)}$ (without the generic g) a fine structural model?

It is not even known if $\mathrm{HOD}^{L[x]}$ and $\mathrm{HOD}^{M_n(x)}$ are models of GCH.

Thank you for your attention!

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Remark: Uhlenbrock $\rightarrow$ Müller