The hereditarily ordinal definable sets in inner models with finitely many Woodin cardinals

Sandra Uhlenbrock

August 14th, 2017

Joint work with Grigor Sargsyan

Logic Colloquium 2017, Special session on set theory
The hereditarily ordinal definable sets in inner models with finitely many Woodin and strong cardinals

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Some like it HOD

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Drawing by Martin Zeman.
HOD is the class of all hereditarily ordinal definable sets,

\[ \text{HOD} = \{ x \mid \text{TC}(\{x\}) \subset \text{OD} \} \].
Motivation

Definition

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- Aim: Understand \( \text{HOD}^M \) for various canonical inner models \( M \) like \( L(\mathbb{R}) \), \( L[x] \) or \( M_n(x) \) (assuming determinacy).
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- Aim: Understand $\text{HOD}^M$ for various canonical inner models $M$ like $L(\mathbb{R})$, $L[x]$ or $M_n(x)$ (assuming determinacy).
- Test question: Is $\text{HOD}^M$ a model of GCH?
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- **Aim:** Understand \( \text{HOD}^M \) for various canonical inner models \( M \) like \( L(\mathbb{R}), L[x] \) or \( M_n(x) \) (assuming determinacy).
- **Test question:** Is \( \text{HOD}^M \) a model of GCH?
- **Method:** Show that \( \text{HOD}^M \) is a fine structural model.
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Definition

HOD is the class of all hereditarily ordinal definable sets,

\[ \text{HOD} = \{ x \mid \text{TC}(\{x\}) \subset \text{OD} \}. \]

- Aim: Understand HOD\(^M\) for various canonical inner models \(M\) like \(L(\mathbb{R})\), \(L[x]\) or \(M_n(x)\) (assuming determinacy).
- Test question: Is HOD\(^M\) a model of GCH?
- Method: Show that HOD\(^M\) is a fine structural model.
- This would imply that we have GCH, ♦, □, ... in HOD\(^M\).
Motivation

Why are these models $\text{HOD}^M$ interesting?

Under determinacy hypotheses, these models can contain large cardinals, e.g. Woodin cardinals.

Theorem (Woodin) Assume $\Delta^1_2$-determinacy. Then for a Turing cone of reals $x$, 

$$\text{HOD}^L[x] \models \omega^L[x]^2$$

is a Woodin cardinal.

Moreover Woodin showed analogous results for $\text{HOD}^M(x, g)$ and $\text{HOD}^L(R)$. 

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\[
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Moreover Woodin showed analogous results for \( \text{HOD}^{M_n}(x) \) and \( \text{HOD}^L(\mathbb{R}) \).
What is known about $\text{HOD}^{L(\mathbb{R})}$ under $\text{AD}^{L(\mathbb{R})}$
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**Theorem (Becker, 1980)**

$\text{HOD}^{L(\mathbb{R})} \models \text{GCH}_\alpha \text{ for all } \alpha < \omega^V_1$. 
What is known about $\text{HOD}^{L(\mathbb{R})}$ under $\text{AD}^{L(\mathbb{R})}$

Theorem (Steel, Woodin, 1993)

$\text{HOD}^{L(\mathbb{R})} \cap \mathbb{R} = \mathcal{M}_\omega \cap \mathbb{R}$. 

$\mathcal{M}_\omega$ $\mathcal{HOD}^{L[\mathbb{R}]}$
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**Theorem (Steel, Woodin, 1993)**

\[
\text{HOD}^{L(\mathbb{R})} \cap \mathcal{P}(\omega_1^V) = N \cap \mathcal{P}(\omega_1^V),
\]

where $N$ is the $\omega_1^V$-th iterate of $M_\omega$ by its least measure.
What is known about $\text{HOD}^{L(\mathbb{R})}$ under $\text{AD}^{L(\mathbb{R})}$

**Theorem (Steel, 1995)**

Let $M_\infty$ be a direct limit of iterates of $M_\omega$, then

$$\text{HOD}^{L(\mathbb{R})} \cap V(\delta_1^2)^{L(\mathbb{R})} = M_\infty \cap V(\delta_1^2)^{L(\mathbb{R})},$$

where $(\delta_1^2)^{L(\mathbb{R})} = \sup\{\alpha \mid \exists f (f : \mathbb{R} \to \alpha \text{ and } f \text{ is surjective and } \Delta_1^{L(\mathbb{R})})\}$. 

\[ \begin{array}{cccc}
\kappa & M_\omega & N & M_\infty & \text{HOD}^{L(\mathbb{R})} \\
\delta_1^2 & \mathbb{R} & \mathcal{P}(\omega_1^Y) & \mathbb{R}
\end{array} \]
Theorem (Woodin, \(\approx 1996\))

\[
\text{HOD}^{L(\mathbb{R})} = L[M_\infty, \Lambda],
\]

where \(\Lambda\) is a partial iteration strategy for \(M_\infty\).
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**Question**

Assume $\Delta^1_2$-determinacy. Do we have

$$\text{HOD}^{L[x]} \models \text{GCH}$$

for a Turing cone of reals $x$?
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**Question**
Assume $\Delta^1_2$-determinacy. Do we have

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What we can do is (under the right determinacy assumption) analyze $\text{HOD}^L[x][G]$ for a Turing cone of reals $x$, where

- $G$ is $\text{Col}(\omega, < \kappa_x)$-generic over $L[x]$, and
- $\kappa_x = \text{least inaccessible cardinal in } L[x]$. 
HOD$^{L[x,G]}$ as a core model

For every real $x$ let $\kappa_x$ denote the least inaccessible cardinal in $L[x]$.

**Theorem (Woodin, 90’s)**

Assume $\Delta^1_2$-determinacy. For a Turing cone of $x$,

$$\text{HOD}^{L[x,G]} = L[M_\infty, \Lambda],$$

where $G$ is $\text{Col}(\omega, <\kappa_x)$-generic over $L[x]$, $M_\infty$ is a direct limit of mice, and $\Lambda$ is a partial iteration strategy for $M_\infty$. 
Goal: Generalize this analysis to $\text{HOD}^{M_n(x, g)}$ for a Turing cone of reals $x$. 

Theorem (Sargsyan, U.)

Assume $\Pi_1^{n+2}$-determinacy. Then for a Turing cone of reals $x$,

$$\text{HOD}^{M_n(x, g)} = M_n(M_\infty, \Lambda),$$

where $M_\infty$ is a direct limit of iterates of an initial segment of $M_n^{+1}$ and $\Lambda$ is a partial iteration strategy for $M_\infty$. 

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HOD in $M_n(x, g)$

**Goal:** Generalize this analysis to $\text{HOD}^{M_n(x,g)}$ for a Turing cone of reals $x$, where

- $M_n(x)$ denotes the least proper class iterable $x$-premouse with $n$ Woodin cardinals,
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**Theorem (Sargsyan, U.)**

Assume $\Pi^1_{n+2}$-determinacy. Then for a Turing cone of reals $x$,

$$\text{HOD}^{M_n(x,g)} = M_n(M_\infty, \Lambda),$$

where $M_\infty$ is a direct limit of iterates of an initial segment of $M_{n+1}$ and $\Lambda$ is a partial iteration strategy for $M_\infty$. 
Let $x$ be a real such that $M_{n+1}^\# \in M_n(x)$. 
The idea of the proof (very sketchy!)

Let \( x \) be a real such that \( M_{n+1}^\# \in M_n(x) \).
- Define a direct limit system of iterates of \( M_{n+1}|(\delta_0^+\omega)^{M_{n+1}} \) which have a Woodin cardinal that is countable in \( M_n(x, g) \) together with iteration embeddings, call the direct limit \( M_\infty \).

\[ \text{HOD}_{M_n(x, g)} \cap V_{\delta_\infty} = M_\infty \cap V_{\delta_\infty} \]

For some \( M_n(x, g) \)-definable set \( A \subseteq \omega \), we have that \( \text{HOD}_{M_n(x, g)} = M_n(A) \).

This will yield \( \text{HOD}_{M_n(x, g)} \subseteq M_n(M_\infty, \Lambda) \).
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- We can do this such that
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  M_\infty \subseteq \text{HOD}^{M_n(x, g)}.
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- Using the direct limit system we can show
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  \text{HOD}^{M_n(x, g)} \cap V_{\delta_\infty} = M_\infty \cap V_{\delta_\infty}.
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- We can do this such that $M_{\infty} \subseteq \text{HOD}^{M_n(x, g)}$.

- Using the direct limit system we can show
  $$\text{HOD}^{M_n(x, g)} \cap V_{\delta_{\infty}} = M_{\infty} \cap V_{\delta_{\infty}}.$$  

- For some $M_n(x, g)$-definable set $A \subseteq \omega_2^{M_n(x, g)}$ we have that
  $$\text{HOD}^{M_n(x, g)} = M_n(A).$$

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HOD in $M_n(x, g)$  
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The idea of the proof (very sketchy!)

Let $x$ be a real such that $M_{n+1}^\# \in M_n(x)$.

- Define a direct limit system of iterates of $M_{n+1}|(\delta_0^{+\omega})^{M_{n+1}}$ which have a Woodin cardinal that is countable in $M_n(x, g)$ together with iteration embeddings, call the direct limit $M_\infty$.

- We can do this such that $M_\infty \subseteq HOD^{M_n(x, g)}$.

- Using the direct limit system we can show

  $$HOD^{M_n(x, g)} \cap V_{\delta_\infty} = M_\infty \cap V_{\delta_\infty}.$$ 

- For some $M_n(x, g)$-definable set $A \subseteq \omega_2^{M_n(x, g)}$ we have that

  $$HOD^{M_n(x, g)} = M_n(A).$$

- This will yield $HOD^{M_n(x, g)} \subseteq M_n(M_\infty, \Lambda)$. 

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We can generalize this to other canonical minimal inner models with some Woodin and/or strong cardinals, e.g. $M_{\omega+17}$ ($\omega + 17$ Woodin cardinals) or $M_{ws}$ (a strong cardinal above a Woodin cardinal).
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**Theorem (Sargsyan, U.)**

Let $M(x)$ be the minimal proper class iterable $x$-premouse with a fixed number of Woodin and strong cardinals (in a fixed order). Assume enough determinacy. Then for a Turing cone of reals $x$,

$$\text{HOD}^{M(x,g)} = M(M_\infty, \Lambda),$$

where $g$ is generic over $M(x)$ for the Levy collapse of the bottom inaccessible to $\omega$, $M_\infty$ is a direct limit of premice and $\Lambda$ is a partial iteration strategy for $M_\infty$. 
Question

Is $\text{HOD}^{L[x]}$ (without the generic $G$) a fine structural model?
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Proposition (Schlutzenberg, 2016)

Given sufficient large cardinals, there is a cone of reals $x$ such that if $\mathcal{F}$ is a natural candidate for a limit system to analyze $\text{HOD}^{L[x]}$, then $\mathcal{F}$ is not closed under pseudo-comparison of pairs.
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Question

Is $\text{HOD}^{M_n(x)}$ (without the generic $g$) a fine structural model?

It is not even known if $\text{HOD}^{L[x]}$ and $\text{HOD}^{M_n(x)}$ are models of GCH.
Thank you for your attention!
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Remark: Uhlenbrock → Müller