LECTURE INTRODUCTION TO MATHEMATICAL LOGIC

WINTERSEMESTER 2017

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Diese Vorlesung ist eine Einführung in verschiedene Teilgebiete der mathematischen Logik und deren Zusammenhänge. Wir beginnen mit einer Wiederholung von Formeln und Strukturen und führen im Anschluss Ultrapotenzkonstruktionen von Strukturen ein. Dies führt uns zu einer genaueren Untersuchung des Gebietes der Modelltheorie. In diesem Rahmen betrachten wir insbesondere Typen und Eigenschaften von abzählbaren Modellen. Zudem werden wir Spiele betrachen, welche in der Modelltheorie benutzt werden und auch mit der Mengenlehre verwandt sind. Danach werden wir uns der Berechenbarkeitstheorie zuwenden und grundlegende Begriffe diskutieren, welche uns ermöglichen werden den ersten Gödelschen Unvollständigkeitssatz in allgemeiner Form zu beweisen. Abschließend werden wir den zweiten Gödelschen Unvollständigkeitssatz diskutieren.

Diese Vorlesung benötigt keine Vorkenntnisse. Dennoch können Kenntnisse wie sie zum Beispiel im Rahmen der Vorlesung "Grundzüge der mathematischen Logik" (siehe `http://boolesrings.org/sandrauhlenbrock/teaching/grundzuege-2017/` oder auch `http://www.logic.univie.ac.at/~muellem3/teaching.html` für die vorherigen Semester) erworben werden nützlich sein.

This lecture will be an introduction to different areas of mathematical logic and their connections. We start with a review of formulas and structures and introduce the method of ultrapower construction. This will lead us to a closer look into the area of model theory which we will augment with the study of types and the structure of countable models. Moreover we will study classical games which are used in model theory and are related to set theory. Afterwards we will introduce some basic concepts from the area of computability theory which will enable us to prove Gödel's first incompleteness theorem in full generality. Finally this lecture will be completed with a discussion of Gödel's second incompleteness theorem.

This lecture will be self-contained. Nevertheless some familiarity with the contents of the lecture "Grundzüge der mathematischen Logik" (e.g. see `http://boolesrings.org/sandrauhlenbrock/teaching/grundzuege-2017/` or `http://www.logic.univie.ac.at/~muellem3/teaching.html` for the previous semesters) might be helpful.

Date: January 13, 2018.

Time and place: Wednesdays 4:30pm - 6pm and Fridays 3pm - 4:30pm in the KGRC lecture room.

The following is a brief summary of the content of every lecture. This is just an overview and there is no guarentee for correctness or completeness.

Lecture 1 (Wed Oct 4th) Overview, languages, structures, substructures, embeddings, isomorphisms (1.1 to 1.1.7 in [TZ12])

Lecture 2 (Fri Oct 6th) Terms, assignments, formulas (1.1.9 to 1.2.8 in [TZ12])

Lecture 3 (Wed Oct 11th) Satisfaction relation, free variables, substitution lemma, negation normal form, universal and existential formulas (1.2.9 to 1.2.15 in [TZ12])

Lecture 4 (Fri Oct 13th) Universal and existential formulas (1.2.16 and 1.2.17 in [TZ12]), theories, elementary equivalence (Section 1.3 in [TZ12])

Lecture 5 (Wed Oct 18th) Motivation (see Example 3.1.8 in [GJ98]), elementary embeddings and elementary substructures, Tarski's Test (Tarski-Vaught criterion) (Section 2.1 in [TZ12] up to 2.1.2)

Lecture 6 (Fri Oct 20th) Tarski's Chain Lemma (2.1.4 in [TZ12]), statement of the compactness theorem, filter, maximal filter, ultrafilter, examples, Hausdorff's Maximality Principle (HMP) (Section 3.2 in [GJ98] up to 3.2.5, see also handwritten notes on my webpage)

Lecture 7 (Wed Oct 25th) Every filter can be extended to an ultrafilter (handwritten notes on my webpage), definition of the ultraproduct, Łoś's theorem, compactness theorem as a corollary (3.2.9 to 3.2.12 in [GJ98], see also handwritten notes on my webpage)

Lecture 8 (Fri Oct 27th) Compactness theorem as a corollary (3.2.12 in [GJ98], see also handwritten notes on my webpage), Theorem of Löwenheim-Skolem (2.1.3 and 2.3.1 in [TZ12]), Vaught's Test, κ-categorical theories (Section 2.3 in [TZ12])

Allerheiligen (Wed Nov 1st)

Lecture 9 (Fri Nov 3rd) DLO is $\aleph_0$-categorical (Section 2.3 in [TZ12]), partial isomorphisms, back-and-forth sets (Section 5.4 in [Vä11]), idea of the Ehrenfeucht-Fraïssé game (Section 5.5 in [Vä11])

Lecture 10 (Wed Nov 8th) The Ehrenfeucht-Fraïssé game (Section 5.5 in [Vä11]), Separation lemma (Lemma 3.1.1 in [TZ12]), statement and easy implication of Lemma 3.1.2 in [TZ12]

Lecture 11 (Fri Nov 10th) DLO is not κ-categorical for any $\kappa > \aleph_0$ (see handwritten notes on my webpage), proof of Lemma 3.1.2 in [TZ12], Theorem 3.1.3 to Definition 3.1.4, statement of Corollary 3.1.5 (1) in [TZ12]

Lecture 12 (Wed Nov 15th) Corollary 3.1.5 in [TZ12], Quantifier elimination (3.2.1, 3.2.3 and 3.2.4 in [TZ12])

Lecture 13 (Fri Nov 17th) Quantifier elimination (3.2.5), prime structures (3.2.2), model completeness (3.2.6) in [TZ12]

Lecture 14 (Wed Nov 22nd) Robinson's test and model companions (3.2.7 to 3.2.9 in [TZ12]), types (2.2.8 and 2.2.9 in [TZ12])

Lecture 15 (Fri Nov 24th) Omitting types theorem (Section 4.1 in [TZ12]), every two countable, ω-saturated models which are elementarily equivalent are isomorphic (see Lemma in Section 4.3 in [TZ12])

Lecture 16 (Wed Nov 29th) Theorem 4.3.1 (Ryll-Nardzewski), Remark 4.3.4, Corollary 4.3.7, Example DLO is $\aleph_0$-categorical as a consequence of 4.3.1 + QE, Definition 4.3.8 (small theories), Lemma 4.3.9, Theorem 4.3.10 (Vaught's never two theorem, beginning of the proof) in [TZ12]

Lecture 17 (Fri Dec 1st) 4.3.10 (Vaught's never two theorem, end of the proof), skeletons, $\mathcal{K}$-saturation (4.4.1), any two countable $\mathcal{K}$-saturated structures are isomorphic (4.4.2), ω-homogeneity (4.3.6), Remark 4.3.5, ultrahomogeneity (4.4.3), the amalgamation method (4.4.4, beginning of the proof) in [TZ12]

Lecture 18 (Wed Dec 6th) The amalgamation method, Fraïssé limits (end of proof of 4.4.4 in [TZ12])

Maria Empfängnis (Fri Dec 8th)

Lecture 19 (Wed Dec 13th) Finite relational languages (4.4.5 to 4.4.7 in [TZ12]), Example: The Fraïssé limit of the class of finite linear orders is a model of DLO

Lecture 20 (Fri Dec 15th) Peano Arithmetic PA, discussion of the incompleteness theorem and plan of proof, coding finite sequences of numbers (4.0, 4.1, 4.2, 4.3 and 4.4 in [GJ98])

Lecture 21 (Wed Jan 10th) A formal notion of proof, derivation and theorem (1.4 in [GJ98])

Lecture 22 (Fri Jan 12th) The completeness theorem (2.2 in [GJ98])

Lecture 23 (Wed Jan 17th) Gödel numbers (4.5 in [GJ98])

Lecture 24 (Fri Jan 19th) Proof of Gödel's incompleteness theorem for PA (4.6 and 4.7 in [GJ98])

Lecture 25 (Wed Jan 24th) Gödel's incompleteness theorem for other axiom systems, Gödel's self-referential lemma, the undefinability of truth (4.8 in [GJ98]), Gödel's second incompleteness theorem

Lecture 26 (Fri Jan 26th) Time for summary, questions, etc...

Lecture 27 (Wed Jan 31th) Oral Exams. Please write an e-mail with your full name and Matrikelnummer to make an appointment.

References

[BBJ07] Boolos, G.S. ; Burgess, J.P. ; Jeffrey, R.C.: *Computability and Logic*. Cambridge University Press, 2007

[Git] Gitman, Victoria: Course notes: Logic I. `http://boolesrings.org/victoriagitman/files/2013/05/logicnotespartial.pdf`

[GJ98] Goldstern, M. ; Judah, H.: *The Incompleteness Phenomenon*. Taylor & Francis, 1998 (Ak Peters Series)

[Mü] Müller, Moritz: Skript zur Unvollständigkeit. `http://www.logic.univie.ac.at/~muellem3/arithmetik.pdf`

[TZ12] Tent, Katrin ; Ziegler, Martin: *A Course in Model Theory*. Cambridge University Press, 2012 (Lecture Notes in Logic)

[Vä11] Väänänen, Jouko: *Models and Games*. Cambridge University Press, 2011 (Cambridge Studies in Advanced Mathematics)