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Theorem: DLO is not κ-categorical for any uncountable cardinal κ.

Proof: Consider $(\mathbb{Q}, <_{\mathbb{Q}})$ with the usual order $<_{\mathbb{Q}}$ on $\mathbb{Q}$.

Claim: Let $(L, <_L)$ be any linear order. Then $<_{lex}$ on $(L,<_L)\times(\mathbb{Q},<_{\mathbb{Q}})$ is dense, where $<_{lex}$ is defined by $(a,b) <_{lex} (a',b')$ iff

$$a <_L a' \text{ or}$$
$$(a=a' \text{ and } b <_{\mathbb{Q}} b').$$

Easy to see: $<_{lex}$ is a linear order again.

Proof: Let $(a,b),(a',b') \in L\times\mathbb{Q}$ with $(a,b) <_{lex} (a',b')$.

If $a <_L a'$, pick some $q \in \mathbb{Q}$ with $b < q$.

Then $(a,b) <_{lex} (a,q) <_{lex} (a',b')$.

If $a=a'$ and $b <_{\mathbb{Q}} b'$, choose $q \in \mathbb{Q}$ with $b <_{\mathbb{Q}} q \leq_{\mathbb{Q}} b'$. Then $(a,b) <_{lex} (a,q) <_{lex} (\overset{a}{\underset{\|}{a'}},b')$ $\square$

A similar argument shows that $<_{lex}$ has no endpoints as $<_{\mathbb{Q}}$ has no endpoints.

To prove the theorem let κ be the set of all ordinals $< \kappa$ with the usual linear order $<$ (in fact well-order). Let $<^*$ be the reverse order on κ, i.e. $\alpha <^* \beta$ iff $\alpha > \beta$.

Now let $<_{lex}$ be the order on $(\kappa, <)\times(\mathbb{Q},<_{\mathbb{Q}})$ and $<^*_{lex}$ the order on $(\kappa, <^*)\times(\mathbb{Q},<_{\mathbb{Q}})$.

Then $\mathfrak{A}=(\kappa\times\mathbb{Q}, <_{lex})$ and $\mathfrak{B}=(\kappa\times\mathbb{Q}, <^*_{lex})$ are both models of DLO of card. κ.

Want to show that they are not isomorphic:

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$\mathfrak{A}$ contains a strictly increasing sequence of length κ, e.g. $((\alpha,0) \mid \alpha<\kappa)$.

if $\mathfrak{A} \cong \mathfrak{B}$, then there is such a sequence in $(\kappa\times\mathbb{Q}, <^*_{lex})$, call it $((\alpha_\beta, q_\beta) \mid \beta<\kappa)$.

By the def'n of $<^*_{lex}$ this means that $(\alpha_\beta \mid \beta<\kappa)$ is not increasing in the usual order $<$ on κ.

Since $<$ on κ is a well-order, it cannot contain an infinite decreasing sequence. That means there is some $\beta_0<\kappa$ s.t. $\alpha_{\beta_0} = \alpha_\gamma$ for all γ s.t. $\beta_0 \le \gamma < \kappa$. By the def'n of $<^*_{lex}$, this implies that the sequence $(q_\gamma \mid \beta_0 \le \gamma < \kappa)$ is strictly decreasing and uncountable. But $\mathbb{Q}$ is countable. $\lightning$

$\square$