Our next goal is the compactness theorem, one of the fundamental results in first-order logic.

<u>Def</u>: A (possibly infinite) theory T is called <u>finitely satisfiable</u> iff every finite subset of T is consistent.

<u>Theorem 3</u> (Compactness Theorem): A theory T is consistent iff it is finitely satisfiable.

Idea of the proof: We have models for every finite subset of T. Want to "combine" them into a single model which is a model of T.

We need the concepts of filters and ultrafilters.

<u>Def</u>: Let I be a nonempty set. Then $\mathcal{F} \subseteq \mathcal{P}(I)$ (i.e. $\mathcal{F}$ consists of subsets of I) is called a <u>filter</u> on I iff

(1) $\emptyset \notin \mathcal{F}$, $I \in \mathcal{F}$ ("$\mathcal{F}$ is nontrivial")

(2) If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$ ("$\mathcal{F}$ is closed under supersets")

(3) If $X_1, X_2 \in \mathcal{F}$, then $X_1 \cap X_2 \in \mathcal{F}$ ("$\mathcal{F}$ is closed under intersections")

We can think of elements in a filter as "large" sets.

<u>Examples</u>:
- $\mathcal{F} = \{I\}$ is always a filter on I
- $\mathcal{F} = \{\{a,b\}, \{a\}\}$ is a filter on $I = \{a,b\}$
- Let $A \subseteq I$, $A \neq \emptyset$. Then $\mathcal{F} = \{X : A \subseteq X\}$ is a filter on I, called <u>principal filter</u>.
- $\mathcal{F} = \{X \subseteq \mathbb{N} \mid \mathbb{N} \setminus X \text{ is finite}\}$ is a filter on $\mathbb{N}$, called the <u>Fréchet filter</u>.

Def: Let $\mathcal{F} \subseteq \mathcal{P}(I)$ be a filter on I.

(1) $\mathcal{F}$ is called maximal if there is no filter $\mathcal{F}'$ on I which properly extends $\mathcal{F}$. (We say $\mathcal{F}'$ extends $\mathcal{F}$ iff $\mathcal{F} \subseteq \mathcal{F}'$.)

(2) $\mathcal{F}$ is called an ultrafilter if for all $X \subseteq I$, $X \in \mathcal{F}$ or $I \setminus X \in \mathcal{F}$.

Lemma 4: Let $\mathcal{F} \subseteq \mathcal{P}(I)$ be a filter. Then $\mathcal{F}$ is maximal iff $\mathcal{F}$ is an ultrafilter.

proof: Exercise.

Proposition 5: Let $\mathcal{F} \subseteq \mathcal{P}(I)$ be a filter on I. Then there ex. an ultrafilter $\hat{\mathcal{F}} \subseteq \mathcal{P}(I)$ on I such that $\mathcal{F} \subseteq \hat{\mathcal{F}}$.

For the proof we need the Axiom of Choice or equivalently Zorn's Lemma. We will use it in the following form:

Hausdorff's Maximality Principle (HMP):
Let A be a set and $\mathcal{A}$ a set of subsets of A (i.e. $\mathcal{A} \subseteq \mathcal{P}(A)$).
(*) Assume that for all $\mathcal{T} \subseteq \mathcal{A}$ such that for all $X, Y \in \mathcal{T}$, $X \subseteq Y$ or $Y \subseteq X$ (i.e. for all $\subseteq$-chains $\mathcal{T}$), we have
$\bigcup \mathcal{T} = \{a \in A \mid a \in X \text{ for some } X \in \mathcal{T}\} \in \mathcal{A}$.
Then there ex. an $X_{max} \in \mathcal{A}$ such that no proper superset of X_{max} is an element of $\mathcal{A}$.

proof of Prop. 5: We use (HMP) to show that every filter $\mathcal{F}$ on I can be extended to a maximal filter (thus ultrafilter) $\hat{\mathcal{F}}$ on I.

Let $\mathcal{A} = \{\mathcal{F}' \mid \mathcal{F}' \text{ is a filter on } I \text{ with } \mathcal{F} \subseteq \mathcal{F}'\}$.
We check that the condition (*) in (HMP) is true for this $\mathcal{A}$. So let $\mathcal{T}$ be a $\subseteq$-chain in $\mathcal{A}$, we have to show that $\bigcup \mathcal{T}$ is a filter on I extending $\mathcal{F}$.

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First of all notice that $\mathcal{F} \subseteq \bigcup \mathcal{A}$ as $\mathcal{F} \subseteq \mathcal{F}'$ for every $\mathcal{F}' \in \mathcal{A}$. We now prove that $\bigcup \mathcal{A}$ is a filter:

(1) $\emptyset \notin \bigcup \mathcal{A}$ b/c otherwise $\emptyset \in \mathcal{F}'$ for some $\mathcal{F}' \in \mathcal{A}$, contradicting that $\mathcal{F}'$ is a filter.

& $I \in \bigcup \mathcal{A}$ b/c $I \in \mathcal{F}'$ for some (in fact any) $\mathcal{F}' \in \mathcal{A}$ since $\mathcal{F}'$ is a filter on I

(2) Closure under supersets:
Let $X \in \bigcup \mathcal{A}$, i.e. $X \in \mathcal{F}'$ for some $\mathcal{F}' \in \mathcal{A}$.
Let $Y \subseteq I$ with $X \subseteq Y$. Since $\mathcal{F}'$ is a filter it follows that $Y \in \mathcal{F}'$. Hence $Y \in \bigcup \mathcal{A}$, as desired.

(3) Closure under intersections:
Let $X_1, X_2 \in \bigcup \mathcal{A}$, i.e. there are $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}$ such that $X_1 \in \mathcal{F}_1$ and $X_2 \in \mathcal{F}_2$.
As $\mathcal{A}$ is a chain it follows that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ or $\mathcal{F}_2 \subseteq \mathcal{F}_1$.
Assume without loss of generality that $\mathcal{F}_1 \subseteq \mathcal{F}_2$.
Then $X_1, X_2 \in \mathcal{F}_2$ and since $\mathcal{F}_2$ is a filter, $X_1 \cap X_2 \in \mathcal{F}_2$. Therefore $X_1 \cap X_2 \in \bigcup \mathcal{A}$.

We have shown that $\bigcup \mathcal{A} \in \mathcal{A}$.
Therefore (HMP) implies that there is an $\mathcal{F}_{max} \in \mathcal{A}$ such that no proper superset of $\mathcal{F}_{max}$ is an element of $\mathcal{A}$.
This $\mathcal{F}_{max}$ is the maximal filter extending $\mathcal{F}$ we were looking for.

$\square$