Now we can start defining the ultraproduct of structures

Let $\mathcal{L}$ be a language and $(\mathfrak{A}_i)_{i\in I}$ $\mathcal{L}$-structures, $I \neq \emptyset$.

Moreover let $\mathcal{U}$ be an ultrafilter on I.

Then $\prod_{i\in I} A_i = \{ f : I \to \bigcup_{i\in I} A_i \mid f \text{ is a function s.t. } f(i) \in A_i \text{ for all } i \in I \}$

Next we define an equivalence relation $\sim_{\mathcal{U}}$ on the set $\prod_{i\in I} A_i$:

reflexive, symmetric & transitive

For $f, g \in \prod_{i\in I} A_i$ let $f \sim_{\mathcal{U}} g$ iff

$$\{ i \in I \mid f(i) = g(i) \} \in \mathcal{U}$$

"f & g agree on a large set"

Exercise: Check that $\sim_{\mathcal{U}}$ is indeed an equivalence relation on $\prod_{i\in I} A_i$.

Write $[f]_{\sim\mathcal{U}} = \{ g \in \prod_{i\in I} A_i \mid f \sim_{\mathcal{U}} g \}$ for the equivalence class of f modulo $\sim_{\mathcal{U}}$.

Moreover write $\prod_{i\in I} A_i / \mathcal{U}$ for the set of all eq. classes $[f]_{\sim\mathcal{U}}$.

We define the ultraproduct $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i / \mathcal{U}$ of the structures $(\mathfrak{A}_i)_{i\in I}$ as follows:

- the universe A of $\mathfrak{A}$ is $\prod_{i\in I} A_i / \mathcal{U}$, i.e. it consists of equivalence classes $[f]_{\sim\mathcal{U}}$.
- for every constant symbol $c \in \mathcal{L}$ let
  $c^{\mathfrak{A}} = [f_c]_{\sim\mathcal{U}}$, where $f_c : I \to \bigcup_{i\in I} A_i$ is the function with $f_c(i) = c^{\mathfrak{A}_i}$ for every $i \in I$.
- for every n-ary function symbol $F \in \mathcal{L}$ let
  $F^{\mathfrak{A}}([f_1]_{\sim\mathcal{U}}, \dots, [f_n]_{\sim\mathcal{U}}) = [g]_{\sim\mathcal{U}}$,
  where $[f_1]_{\sim\mathcal{U}}, \dots, [f_n]_{\sim\mathcal{U}} \in \prod_{i\in I} A_i/\mathcal{U}$ and $g : I \to \bigcup_{i\in I} A_i$ is defined by $g(i) = F^{\mathfrak{A}_i}(\underbrace{f_1(i)}_{\in A_i}, \dots, \underbrace{f_n(i)}_{\in A_i}) \in A_i$.

Note: This definition of $F^{\mathfrak{A}}$ does not depend on the choice of the representatives $f_1, \dots, f_n$:

Let $f_1', \dots, f_n' \in \prod_{i \in I} A_i$ with $[f_k]_{\sim_\mathcal{U}} = [f_k']_{\sim_\mathcal{U}}$ for $1 \le k \le n$,

i.e. $\{ i \in I \mid f_k(i) = f_k'(i) \} \in \mathcal{U}$ for all $1 \le k \le n$.

~~As~~ Let $g' : I \to \bigcup_{i \in I} A_i$ be defined by

$$g'(i) = F^{\mathfrak{A}_i}(f_1'(i), \dots, f_n'(i)).$$

Then
$$\underbrace{\bigcap_{1 \le k \le n} \{ i \in I \mid f_k(i) = f_k'(i) \}}_{\substack{\text{i.e. } \{ i \in I \mid f_k(i) = f_k'(i) \text{ f.a. } 1 \le k \le n \} \\ \in \\ \mathcal{U}}} \subseteq \underbrace{\{ i \in I \mid g(i) = g'(i) \}}_{\substack{\in \\ \mathcal{U}}}$$

Hence $g \sim_\mathcal{U} g'$.

- for every n-ary relation symbol $R \in \mathcal{L}$ let

$R^{\mathfrak{A}}([f_1]_{\sim_\mathcal{U}}, \dots, [f_n]_{\sim_\mathcal{U}})$ iff

$$\{ i \in I \mid R^{\mathfrak{A}_i}(f_1(i), \dots, f_n(i)) \} \in \mathcal{U}$$

for $[f_1]_{\sim_\mathcal{U}}, \dots, [f_n]_{\sim_\mathcal{U}} \in \prod_{i \in I} A_i / \mathcal{U}$.

Similar as for function symbols it follows that the definition of $R^{\mathfrak{A}}$ does not depend on the choice of representatives for $[f_1]_{\sim_\mathcal{U}}, \dots, [f_n]_{\sim_\mathcal{U}}$.

The main reason for considering ultraproducts is given by the following theorem.

<u>Theorem</u> (<u>Łoś's theorem</u>):

Let $I \neq \emptyset$ be some set and $(\mathfrak{A}_i)_{i\in I}$ a family of $\mathcal{L}$-structures for some language $\mathcal{L}$. Let $\mathcal{U}$ be an ultrafilter on I.

Then for any $\mathcal{L}$-formula $\varphi(x_1,\dots,x_n)$:

$$\prod_{i\in I} \mathfrak{A}_i/\mathcal{U} \models \varphi([f_1]_{\sim\mathcal{U}},\dots,[f_n]_{\sim\mathcal{U}}) \quad \text{iff}$$

$$\{i\in I \mid \mathfrak{A}_i \models \varphi(f_1(i),\dots,f_n(i))\} \in \mathcal{U}.$$

(That means, the theory of an ultraproduct can be computed from the theory of the individual models $\mathfrak{A}_i$.)

<u>proof</u>: Exercise, induction on φ using the properties of the ultrafilter $\mathcal{U}$. □

Now we can prove the compactness theorem.

Recall:

<u>Thm</u>: Let Σ be a set of sentences such that every finite subset of Σ has a model.

Then Σ has a model.

<u>proof</u>: Let I be the set of all finite subsets of Σ, i.e. every $i\in I$ is a finite set of sentences.

For each $i\in I$ let $\mathfrak{A}_i$ be a model such that $\mathfrak{A}_i \models \varphi$ for every $\varphi\in i$ (we write $\mathfrak{A}_i \models i$).

For $i \in I$, let $X_i = \{j \in I \mid i \subseteq j\}$ be the set of all supersets j of i which are in I.

Let $\mathcal{F} = \{X \mid \exists i \in I \;\; X_i \subseteq X\}$.

Then $\mathcal{F}$ is a filter on I as $X_i \cap X_j = X_{i \cup j}$ and

sets in I which are supersets of both i and j

$i \cup j \in I$ for $i, j \in I$.

(i, j are finite subsets of Σ', so $i \cup j$ is a finite subset of Σ' as well.)

Let $\mathcal{U} \supseteq \mathcal{F}$ be an ultrafilter.

Then in particular $X_i \in \mathcal{U}$ for all $i \in I$ (as $X_i \in \mathcal{F}$).

Now we claim that $\mathfrak{A} := \prod_{i \in I} \mathfrak{A}_i / \mathcal{U}$ is a model of Σ'.

Let $\varphi \in \Sigma'$. As $\mathfrak{A}_i \models i$ (by choice of $\mathfrak{A}_i$) for all $i \in I$, we have $\mathfrak{A}_i \models \varphi$ if $\varphi \in i$.

Hence $\{i \in I \mid \mathfrak{A}_i \models \varphi\} \supseteq \{i \in I \mid \varphi \in i\} = \{i \in I \mid \{\varphi\} \subseteq i\} = X_{\{\varphi\}} \in \mathcal{U}$.

Therefore $\{i \in I \mid \mathfrak{A}_i \models \varphi\} \in \mathcal{U}$,

so by Łoś's theorem $\mathfrak{A} \models \varphi$.

$\square$