DISCUSSION ON RAMSEY FILTERS

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Abstract. Clarifying characterizations of Ramsey filters.

Definitions in [1]:

Definition 1. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$.

(1) $\mathcal{F}$ is centered if for every $\mathcal{H} \in [\mathcal{F}]^{<\omega}$ the intersection $\bigcap \mathcal{H} \in \mathcal{F}$.
(2) $\mathcal{F}$ is said to be a P-set if for every countable subfamily $\mathcal{H} \subseteq \mathcal{F}$ there is $A \in \mathcal{F}$ such that $A \subseteq^* H$ for every $H \in \mathcal{H}$.
(3) $\mathcal{F}$ is a Q-set if for every bounded partition $\mathcal{E}$ of ω there is $X \in \mathcal{F}$ such that $|X \cap E| \leq 1$ for every $E \in \mathcal{E}$. We say that X is a semi-selector for $\mathcal{E}$.

Definition 2. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is said to be Ramsey if $\mathcal{F}$ is a centered family containing the co-finite sets which is both, a P-set and a Q-set.

We prove in [1]:

Lemma 3. Let $\mathcal{F}$ be a filter. The following are equivalent:

(a) $\mathcal{F}$ is a Q-filter.
(b) For any increasing function $f \in {}^{\omega}\omega$ there is $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ such that $f(k(n)) < k(n+1)$.

Proof. ((a) $\Rightarrow$ (b)) Inductively, choose a sequence $\{n(l)\}_{l\in\omega}$ such that $n(0) = 0$ and

$$n(l+1) = \min\{n \colon n_l < n \text{ and } \forall m \leq n_l (f(m) \leq n)\}.$$

We consider the partition $\mathcal{E}_0 = \{[n_{3l}, n_{3l+3})\}_{l\in\omega}$. There is $C_1 \in \mathcal{F}$ such that C_1 is a semi-selector for $\mathcal{E}_0$. Now, define an equivalence relation $\mathcal{E}_1$ on C_1 as follows:

$$m \sim_{\mathcal{E}_1} k \text{ iff } m = k \vee m < k \leq f(m) \vee k < m \leq f(k).$$

Each $\mathcal{E}_1$ equivalence relation has at most two members. Indeed, if there were three numbers $m_1 < m_2 < m_3$ in one equivalence class of $\mathcal{E}_1$ then $m_1 < m_2 < m_3 \leq f(m_1)$. There are $l_1 < l_2 < l_3$ such that $m_i \in [n_{3l_i}, n_{3l_i+3})$. Then $m_1 < n_{3l_2} \leq m_2 < n_{3l_3} \leq m_3 \leq f(m_1)$. However, on the other hand by the definition of sequence $\{n(l)\}_{l\in\omega}$ we have $f(m_1) \leq n_{3l_2+1} < n_{3l_3}$, a contradiction.

Extend $\mathcal{E}_1$ to an equivalence relation $\mathcal{E}_2$ on ω by defining

$$m \sim_{\mathcal{E}_2} k \text{ iff } m = k \vee m \sim_{\mathcal{E}_1} k.$$

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There is C_2 in $\mathcal{F}$ such that C_2 is a semi-selector for $\mathcal{E}_2$. Without loss of generality $C_2 \subseteq C_1$ and $0 \in C_2$. Let $\{k(n)\}_{n\in\omega}$ enumerate in increasing order C_2. Thus for all n, n' we have that $k(n) \not\sim_{\mathcal{E}_2} k(n')$. Thus, if $n < n'$ then $k(n') \not\leq f(k(n))$ and so for all $n \in \omega$, $f(k(n)) < k(n+1)$.

((b) $\Rightarrow$ (a)) Let $\mathcal{E}$ be a bounded partition of ω. We set

$$f(n) = \max \bigcup \{E \in \mathcal{E} \colon (\exists i \leq n)\ i \in E\}.$$

There is $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ such that $f(k(n)) < k(n+1)$ for each $n \in \omega$. The set $\{k(n) \colon n \in \omega\}$ is a semi-selector for $\mathcal{E}$. Indeed, $k(n) \leq f(k(n)) < k(n+1)$ and therefore $k(n+1)$ is from different set of partition $\mathcal{E}$ than all $k(i)$ for $i \leq n$. □

The equivalent formulations of Ramseyness:

Lemma 4. Let $\mathcal{F}$ be a filter. The following are equivalent:

(a) $\mathcal{F}$ is a Ramsey filter.

(b) For any sequence $\{\mathcal{G}_i\}_{i\in\omega}$ of finite subsets of $\mathcal{F}$ there is $a \in \mathcal{F}$ such that

$$a(n+1) \in \bigcap \mathcal{G}_{a(n)}.$$

(c) For any increasing sequence $\{\mathcal{G}_i\}_{i\in\omega}$ of finite subsets of $\mathcal{F}$ there is $a \in \mathcal{F}$ such that

$$a(n+1) \in \bigcap \mathcal{G}_{a(n)}.$$

(d) For any sequence $\{F_i\}_{i\in\omega}$ in $\mathcal{F}$ there is $a \in \mathcal{F}$ such that $a(n+1) \in F_{a(n)}$.

Proof. ((a) $\Rightarrow$ (b)) $\mathcal{F}$ is a P-set and therefore there is $C_0 \in \mathcal{G}$ such that $C_0 \subseteq^* G$ for each $G \in \bigcup\{\mathcal{G}_n \colon n \in \omega\}$. Thus, for some function $f \in {}^\omega\omega$

$$(\forall n \in \omega)\ C_0 \setminus f(n) \subseteq \bigcap \mathcal{G}_n.$$

Let us take $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ from Lemma 3 such that $\{k(n+1) \colon n \in \omega\} \subseteq C_0$. Hence, we have $k(n+1) \in C_0 \setminus f(k(n))$, and so

$$k(n+1) \in \bigcap \mathcal{G}_{k(n)}.$$

((b) $\Rightarrow$ (c)) Special case.

((c) $\Rightarrow$ (d)) It is enough to take $\mathcal{G}_i = \{F_0, F_1, \ldots, F_i\}$.

((d) $\Rightarrow$ (a)) First we shall show that $\mathcal{F}$ is a P-set. Let $\{G_i\}_{i\in\omega}$ be a sequence in $\mathcal{F}$. We set $F_i = G_0 \cap G_1 \cap \cdots \cap G_i$, and we take $a \in \mathcal{F}$ such that $a(n+1) \in F_{a(n)}$. The set a is a pseudointersection of $\{G_i\}_{i\in\omega}$. Indeed, if G_j is such that $j \leq a(n)$ then $\{a(k) \colon k \geq n+1\} \subseteq G_j$.

We shall show that $\mathcal{F}$ is a Q-set using Lemma 3. Indeed, let the function $f \in {}^\omega\omega$ be increasing. We consider sets $F_i = (f(i), +\infty)$, and we take $a \in \mathcal{F}$ such that $a(n+1) \in F_{a(n)}$. Hence, $a(n+1) > f(a(n))$. □

References

[1] V. Fischer, J.Šupina, *Selective independence*, preprint.

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