universität wien

1. BACHELORARBEIT

Titel der Bachelorarbeit

Martin's Axiom, Measure and Category

Author

Alexander Boris Wendlinger

angestrebter akademischer Grad

Bachelor of Science (BSc.)

Wien, im Juni 2019

Studienkennzahl lt. Studienblatt:	A 01632256
Studienrichtung lt. Studienblatt:	Mathematik
Betreut von:	Vera Fischer, Privatdoz., PhD.

Abriss

Diese Arbeit behandelt Martins Axiom und einige klassische Ergebnisse in kombinatorscher Mengenlehre, deren Existenz Cohens fundamentaler Entdeckung der Erzwingung (1963) und der intensiven Forschung von Martin, Solovay und anderen geschuldet ist. Weiters wird Einsicht in resultierende Implikationen in Topologie, Maßtheore und Kategorie gegeben, und schlussendlich noch die Beziehung zu Suslin's Hypothese erläutert.

Abstract

This thesis introduces Martin's Axiom and some classical results in combinatorial set theory, which developed as a result of Cohen's groundbreaking invention of forcing in 1963 and the following efforts of Martin, Solovay and others. Further insights into topological, measure-theoretical, and categorical implications are given and we close with Martin's axiom's relation to Suslin's hypothesis.

Introduction

In the last century the development of mathematical logic has made rapid advancements and created a more solid foundation for mathematical theory as a whole. The first introduction of the Axiom of choice (Zermelo in 1904), as a tool to well-order any set, garnered criticism. Today we are still careful with the usage of (AC) and often explicitly state whenever it is used, yet it has become essential to many branches and important theorems of mathematics. Similarly the axioms of Zermelo-Fraenkel and choice (ZFC), while having been criticized, both for being too strong and too weak, have become the pillar for a unified set theory. Interestingly there are statements, quite naive ones like the continuum's hypothesis (CH) at that, whose truth or untruth is simply undetermined in ZFC. Therefore both $\mathrm{ZFC} + (\mathrm{CH})$ and $\mathrm{ZFC} + \neg(\mathrm{CH})$ remain consistent, as long as ZFC was consistent to begin with. Of course, a conclusion from Gödel's famous second incompleteness theorem is that consistency of ZFC could only be proven within ZFC if it were inconsistent to begin with. From this the dilemma of modern set theory is quite clear, while nonetheless opening opportunities to study different, (relatively) consistent assumptions and their implications. More specifically, this paper will introduce Martin's Axiom, a weaker form of (CH) if you will, and some conclusions we can come to in the theory of $\mathrm{ZFC} + (\mathrm{MA}) + \neg(\mathrm{CH})$. This includes regularity of 2^ω, and assertions about the smallest nonzero-measure set and the smallest non-meager set, as well as statements concerning cardinals which suffice MA, like Solovay's lemma. Besides these theorems, another important conclusion is drawn in topology with the product of (ccc)-topological-spaces being (ccc). Finally a brief introduction of Suslin's hypothesis is given and how it relates to MA, specifically $\mathrm{MA}(\omega_1) \to \mathrm{SH}$.

Contents

1 Basics and background material

The nowadays common axiomatic set theory of Zermelo-Fraenkel, was first developed by Zermelo in 1907. In a paper published in 1908 Zermelo listed seven basic axioms, much like the ones below. This list was later found to be incomplete or at least in some cases inadequate to describe sets in a desired way. Without the Axiom of Foundation nothing forbids a cycle of sets containing one another. There also needs to be a formal and precise way to create sets containing certain elements, for that Fraenkel proposed to use formulas from first order logic and the comprehension scheme was put forth. This was also necessary to avoid cases like Russel's paradox. Therefore these Axioms were added and became the cornerstones of a widely accepted axiomatic set theory.[1]

Definition 1.1 (ZFC)**.** The axioms themselves will play a subsidiary role and are only states for completeness, since ZFC will be referenced multiple times. For simplicity we understand the free variables in axioms to be universally quantified. Thus the axioms of ZFC are as follows:

(i). **Extensionality**

$$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$$

(ii). **Foundation**

$$\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \nexists z(z \in x \wedge z \in y))$$

(iii). **Comprehension Scheme** For each $\mathcal{L}_\in$ formula ϕ where y is not free

$$\exists y \forall x(x \in y \leftrightarrow x \in v \wedge \phi(x))$$

(iv). **Pairing**
Write $z = \{x, y\}$ for $x \in z \wedge y \in z \wedge \forall u(u \in z \rightarrow (u = x \vee u = y)$. Then the axiom states:

$$\forall x \forall y \exists z z = \{x, y\}$$

(v). **Union**
Write $z = \bigcup x$ for $y \in z \leftrightarrow \exists w(w \in x \wedge y \in w)$. Then the axiom states:

$$\forall x \exists z\ z = \bigcup x$$

[1] see [6].

(vi). **Replacement Scheme**

$$\forall x \in a \exists! y \phi(x,y) \to \exists b \forall x \in a \leftrightarrow \exists y \in b\, \phi(x,y)$$

(vii). **Existence of the empty set**

$$\exists \emptyset \forall x \neg (x \in \emptyset)$$

(viii). **Power Set**

$$\exists y \forall z (z \subseteq x \to z \in y)$$

(ix). **Choice**

For simplicity we state it as the well-ordering theorem. R refers not to a set but to a binary relation.

$$\forall x \exists R (R \text{ well-orders } x)$$

(x). **Infinity**

$S(y) = y \cup \{y\}$, then the axiom states:

$$\exists x (\emptyset \in x \wedge \forall y \in x (S(y) \in x))$$

Remark. (a) In the above set of axioms, the existence of the empty set is redundant, as long as we formulate the axiom of infinity a little differently. By the axiom of infinity there exists some (infinite) set x and thus by the comprehension scheme $\text{Aus}_{\neg(x=x)}$ there exists $\{x \in a : x \neq x\} = \emptyset$. However (A7) is usually still named for completeness purposes, as one could also contemplate ZFC without infinity.

(b) The axiom of foundation (A2) ensures that there are no loops of sets containing one another.

(c) ZFC=ZF+AC+∞, ZF=ZFC\AC, $\text{ZFC}^{-\infty} = \text{ZFC} \setminus \text{(A7)}$

1.1 Ordinals

This next section gives a brief recollection about ordinals. For more details, see [3] which we will mainly follow.

Definition 1.2 (Transitive set)**.** A set x is called transitive iff every element of x is also a subset.

$$\iff \forall y \in x\, (y \subseteq x)$$

Remark. Of course, by the axiom of foundation, for a transitive set every element must be a proper subset.

Definition 1.3 (Well order)**.** A total order $\leq$ on a set x is called a well order iff every nonempty subset has a $\leq$-least element.

$$\iff \forall_{y \subseteq x}(y \neq \emptyset \to \exists_{z \in y} \forall_{z' \in y} z \leq z')$$

Definition 1.4 (Ordinal)**.** A set α is called an ordinal iff α is transitive and well-ordered by $\in$.

Definition 1.5 (Von Neumann-numbers)**.** For a set n we define $n+1 \coloneqq n \cup \{n\}$. A set x is called inducitve iff $\emptyset \in x$ and $y \in x \to y+1 \in x$.

Now von Neumann-numbers are defined as follows: $0 = \emptyset$, $1 = 0+1 = \emptyset \cup \{\emptyset\}$,..., $n+1 = n \cup \{n\}$ and so essentially we identify every natural number with a set. We can also define addition, multiplication and exponentiation on the von Neumann-numbers in the natural way, and ultimately show that it suffices Peano's axioms.

Remark. (a) Infinity can be rephrased as $\exists x(x \text{ is inductive})$.

(b) We introduce these numbers in order to be able to speak of the set of natural numbers, as well as to emphasize the duality of $n \in \mathbb{N}$ referring to both a number and a set, something which is left to context for the following chapters.

Definition 1.6. $\omega = \bigcap\{n : n \text{ is inductive}\}$

Remark. (a) Existence for ω follows the axiom of infinity and (Aus_ϕ) for the appropriate $\phi = (\forall_x(x \text{ inductive} \to y \in x)$.

(b) ω is essentially just $\mathbb{N}$ the set of all von Neumann-numbers. $\omega = \{n : n \in \mathbb{N}\}$.

(c) ω is inductive and ω is an ordinal:

Proof. Let $n \in \omega$, let x be any inductive set, then $n \in x$ and therefore $n+1 \in x$, x was arbitrary and so $x+1 \in \omega$. From (b) it is clear that it is an ordinal. □

(d) We need not stop at ω and can consider $\omega + 1$ and so on $\omega + \omega$, $\omega \cdot \omega$, ω^n, ω^ω. (With the replacement scheme and the assertion that every set has an ordinal isomorphic, we can rest assured of well-definiteness).

(e) The 'set' of all ordinals would be an ordinal, this contradicts foundation and there is no such set. We would need a more general theory of classes and speak of the class of ordinals ON.

Theorem 1.1. *The class ON is well ordered by* $\in$.

Notation: We also say $a < b$ for ordinals $a \in b$.

Definition 1.7 (Successor and limit ordinals)**.** An ordinal β is called

(i). a successor ordinal iff $\beta = \alpha + 1$ for some ordinal α.

(ii). a limit ordinal iff $\beta \neq \emptyset$ and beta is no successor ordinal.

Remark. This gives a partition on $ON \setminus \{0\}$

Definition 1.8. We call two sets A, B with their respective orderings R, R' isomorphic iff $(A, R) \cong (B, R')$, meaning there exists some bijection f that is compatible with the ordering: $aRb \iff f(a)R'f(b)$.

Theorem 1.2. *For every set A well ordered by some relation R, there exits some unique ordinal* α *such that* $(A, R) \cong (\alpha, \in)$*.*

Definition 1.9 (Type of a set)**.** For some set A which is well ordered by the relation R, type(A) is the unique ordinal a such that $(A, R) \cong (a, \in)$.

With this we can define multiplication and addition of ordinals in a very natural way.

Definition 1.10. $\alpha \cdot \beta = \text{type}(\alpha \times \beta)$ and $\alpha + \beta = \text{type}(\{0\} \times \alpha \cup \{1\} \times \beta)$

Now with much more interesting chapters to come, a more detailed motivation would go beyond the scope of this bachelor-thesis and we will close with the following: We can use both transfinite induction as well as transfinite recursion on well ordered sets. This justifies the powerful method of taking κ-many steps for any cardinal κ, breaking a lot of ground and opening new possibilities for proofs that were previously limited to induction on ω.

1.2 Cardinals

We compare sizes of sets A, B by finding injective ($A \preccurlyeq B$) or surjective ($A \succcurlyeq B$) functions from A to B. Sets have equal 'size' (later cardinality) iff there exists a bijection between them and we write $|A| = |B|$. We say $A \prec B$ iff $A \preccurlyeq B$ and $B \not\preccurlyeq A$.

This subsection has the purpose of recapitulating main aspects concerning cardinals, again an excellent reference is [3].

Theorem 1.3. $A \prec \mathcal{P}(A)$

Definition 1.11. We call a set A countable iff $A \prec \omega$. We call it infinite iff $\omega \prec A$. A is countably infinite iff it is countable and infinte. A is uncountable infinite or just uncountable iff A is not countable.

Definition 1.12 (Von Neumann-cardinals)**.** Ordinals α where $\forall\beta < \alpha\ (\beta \prec \alpha)$ are called (von Neumann) cardinals.

Example. ω is a cardinal. $\omega+1$ is not a cardinal. In this sense every finite ordinal 1,2,3,4,..,n are cardinals. However beyond ω cardinals and ordinals somewhat diverge, as then a successor ordinal is no successor cardinal (or even just a cardinal).

Definition 1.13. Let A be a well order-able set. $|A|$ is the least ordinal α such that $A \preccurlyeq \alpha$ and $\alpha \preccurlyeq A$.

Remark. In ZF, the axiom of choice is equivalent to the assertion that every set has a well order. Whenever $|A|$ exists it is a cardinal.

Theorem 1.4. *For every cardinal α there is a cardinal β such that $\alpha \prec \beta$.*

Definition 1.14 (Successor cardinal)**.**

(i) For a cardinal κ we write κ^+ for the cardinality of the least ordinal o such that $\kappa \prec o$. That is $\kappa^+ := |\inf\{o \in \mathrm{ON} : \kappa < |o|\}|$.

(ii) We write ω_1 for the smallest uncountable ordinal and $\aleph_1$ for the smallest uncountable cardinal. Similarly $\aleph_0 = |\omega|$ for the smallest countable, infinite cardinal.

(iii) Let o be some ordinal.
$\aleph_{o+1} = \min\limits_{\aleph_o < \kappa} \kappa$.
$\aleph_\lambda = \sup\limits_{o<\lambda} \aleph_o$ where λ is a limit ordinal.

Definition 1.15. Let κ be some limit ordinal.

$$\mathrm{cf}(\kappa) := \min\{\mathrm{type}(X) : X \subseteq \kappa, \sup X = \kappa\}$$

We say $\mathrm{cf}(\kappa)$ is the cofinality of κ.

(i) κ is regular iff $\mathrm{cf}(\kappa) = \kappa$

(ii) κ is singular iff $\mathrm{cf}(\kappa) < \kappa$

The continuum hypothesis (CH) is the assertion that the set of all real numbers has minimal possible cardinality for any uncountable set. We call the cardinality of $\mathbb{R}$ the size of the continuum, $\mathfrak{c} := |\mathbb{R}| = 2^{\aleph_0}$. $\aleph_1$ is the smallest uncountable cardinal. Now (CH) states $\mathfrak{c} = \aleph_1$. Cantor speculated that (CH) might be true and spent many years trying to prove it. The continuum hypothesis even became the very first item on David Hilbert's list of twenty three problems, which enumerated important, and at the time unresolved, questions concerning many fields of mathematics. More than a century later many of these problems could at least be partially answered, and some like the Riemann hypothesis still remain unsolved. Today (CH) is listed as solved, but it was only decades after Cantor's death in 1918, that it was proven to be impossible prove or disprove (CH) within standard ZFC set-theory. In fact it was Kurt Gödel who showed $\neg$(CH) to be impossible to prove in 1940. Finally in 1963 Paul Cohen developed the methods of forcing and showed the same to be the case for (CH), rendering all endeavours to try and prove it futile. This outlines how far we can go in ZFC and provides a complete answer to the original problem, albeit not in an expected fashion.[2]

Now with consistency of ZFC $+ \neg$(CH) we may introduce weaker axioms than (CH), like Martin's axiom (MA) and look at their results and implications.

2 Martins Axiom

After Cohen developed the method of forcing in 1963, with which he also showed the consistency of ZFC$+\neg$(CH), people began to extend his methods considerably. Martin, Solovay and Tennenbaum, amongst others proved many statements to be independent of ZFC $+ \neg$(CH). Martin managed to distill the essence of some of their results into one axiom, today called Martin's Axiom. Once we are able to formulate MA we will see that (CH) $\rightarrow$ (MA). In particular this also implies that (MA), much like (CH), cannot be disproven in ZFC showing its relative consistency. Consistency of ZFC+(MA)+$\neg$(CH) will not be proven, we will however show relative consistency of (MA) with $\mathfrak{c}$ being arbitrarily large.[3]

To formulate Martin's Axiom we need some prerequisites to specify exactly what we are talking about. Throughout the entire thesis we will conform to the more modern constructions in [3], which is also a great reference for more details and entails most of what will be discussed.

2.1 Prerequisites

Definition 2.1. A binary relation $\leq$ on some set X is a preorder, iff it is transitive and reflexive.

$$x \leq y \wedge y \leq z \rightarrow x \leq z \text{ (transitivity)}$$
$$x \leq x \text{ (reflexivity)}$$

Remark. Note that two elements of P must not be comparable, and the preorder differs from the partial order in that $y \leq x \wedge x \leq y \not\rightarrow x = y$. However many preorders tend to be a partial orders anyways, especially in this thesis were many

[2] see [1].

[3] see pages 171 and 172 of [3]

orders are induced by subset-orderings and antisymmetry ($y \leq x \wedge x \leq y \to x = y$) follows from antisymmetry of $\subseteq$.

Definition 2.2. A triple $(\mathbb{X}, \leq, \mathbb{1})$ is called a forcing poset, iff $\leq$ is (at least) a preorder on $\mathbb{X}$ and $\mathbb{1} \in \mathbb{X}$ is a largest element ($\forall x \in \mathbb{X}\, x \leq \mathbb{1}$). Whenever the ordering and the largest element is clear from context, we refer to the forcing poset $\mathbb{X}$ by abuse of notation. Let $\mathbb{X}$ be a forcing poset:

(i) Elements of $\mathbb{X}$ are called forcing conditions. $p \leq q$ is read "p extends q".

(ii) $p, q \in \mathbb{X}$ are incompatible ($p \perp q$) iff $\nexists r \in \mathbb{X}(r \leq p \wedge r \leq q)$. Otherwise they are called compatible ($p \not\perp q$)

(iii) A subset $A \subseteq \mathbb{X}$ whose elements are pairwise incompatible is called an antichain.

(iv) Iff every antichain in $\mathbb{X}$ is countable, $\mathbb{X}$ is said to have the countable chain condition (ccc).

Definition 2.3. Let $(\mathbb{X}, \leq, \mathbb{1})$ be a forcing poset. A filter on $\mathbb{X}$ is a set $F \subseteq \mathbb{X}$ such that

- $\mathbb{1} \in F$
- $\forall p, q \in F\, \exists r \in F(r \leq p \wedge r \leq q)$
- $\forall p, q \in \mathbb{X}(q \leq p \wedge q \in F \to p \in F)$

Definition 2.4. Let $(\mathbb{X}, \leq, \mathbb{1})$ be a forcing poset.
$D \subseteq \mathbb{X}$ is dense iff $\forall p \in \mathbb{X}\, \exists q \in D\, q \leq p$.

2.2 Delta-System

Definition 2.5. (Delta-system) A family $\mathcal{A}$ of sets forms a delta-system (Δ-system or in some literature also called sunflower-system) iff the intersection of any two sets is constant:

$$\exists k\, \forall a, b \in \mathcal{A}\, (a \neq b \to a \cap b = k)$$

Then k is called the root or kernel of the delta system.

The name of the delta-system supposedly comes from the fact that one could visualize the system as a river delta, where all branches originate from the same source, before splitting into a multitude of streams. Similarly all petals of the sunflower are arranged around and share the same disk. We are mainly interested in delta-systems for the following lemma and its application in the proves of some main theorems concerning this thesis. As a matter of fact there are however many statements about the existence of delta-systems for sets of certain cardinals as well as unproven conjectures.

Lemma 2.1 (Delta-system lemma)**.** *For an uncountable, regular cardinal κ and a family $\mathcal{A}$ of finite sets such that $|\mathcal{A}| = \kappa$ there exists a subset $\mathcal{B} \in [A]^\kappa$ such that $\mathcal{B}$ forms a delta-system.*

Proof. For $n \in \omega$ consider $A_n := \{A \in \mathcal{A} : |A| = n\}$. By this construction $\mathcal{A} = \dot{\bigcup}_{n\in\omega} A_n$. Since κ is regular and by assumption $|A| = \kappa$ there exists some $n \in \omega$ with $|A_n| = \kappa$.

Claim: For every $n \in \omega$ and every family of κ sets of size n there exists a κ-sized delta-system.

Proof by induction over $n \in \omega$. If $n = 1$ we see that A_1 has cardinality κ and since it contains only singular sets $\forall x, y \in A_1\ (x \neq y \to x \cap y = \emptyset)$. Therefore it forms a delta-system with empty kernel. For an element p in a set $A' \in \mathcal{A}$ define $D_p := \{A \in A_n : p \in A\}$. Assume $n > 1$.
$n - 1 \to n$:
Case 1: $\exists p\ (|D_p| = \kappa)$. Now consider a new family $\mathcal{C} := \{A \setminus \{p\} : A \in D_p\}$. Because $n > 1$ we find that $|\mathcal{C}| = \kappa$. Therefore $\mathcal{C}$ forms a family of κ sets with size $n - 1$. By the induction hypothesis $\exists \mathcal{C}' \in [\mathcal{C}]^\kappa$ ($\mathcal{C}'$ is a delta-system). Let K be the kernel of $\mathcal{C}'$, Then $\{A \cup \{p\} : A \in \mathcal{C}'\}$ is a delta system of κ-many sets of size n, with kernel $K \cup \{p\}$.
Case 2: $\forall p\ (|D_p| < \kappa)$. We make the following observation:
For some arbitrary set $S \in A_n$ we find that $\bigcup_{p\in S} D_p$ has cardinality strictly less than κ as long as $|S| < \kappa$. This furthermore implies $A_n \setminus \bigcup_{p\in S} D_p \neq \emptyset$.

With this knowledge fix some $X_0 \in A_n$. Naturally $|X_0| < \kappa$ and we can find $X_1 \in A_n$ such that $X_0 \cap X_1 = \emptyset$. Assume that we have defined $\{X_\beta\}_{\beta<\alpha}$ in a similar manner such that $\forall \beta < \alpha\ (X_\beta \cap (\bigcup_{\gamma<\beta} X_\gamma) = \emptyset)$. By regularity of κ we again see that $|\bigcup_{\beta<\alpha} X_\beta| < \kappa$ and according to our prior observation we can again find $X_\alpha \in A_n$ such that $X_\alpha \cap \bigcup_{\beta<\alpha} X_\beta = \emptyset$. After κ steps of finding such X_ξ, we obtain the delta system $\{X_\xi\}_{\xi\in\kappa}$ with cardinality κ and empty kernel. □

Therefore we can find a κ-sized delta system for $|A_n| = \kappa$ which is also a delta-system for $A \supseteq A_n$. □

2.3 Martin's Axiom

Definition 2.6 (Martins Axiom)**.** (i) $MA_{\mathbb{X}}(\kappa)$ states that for every family $\mathcal{D}$ of dense subsets $D \subseteq \mathbb{X}$ and $|\mathcal{D}| \leq \kappa$, there exists a filter $F \subseteq \mathbb{X}$ such that

$\forall D \in \mathcal{D}\,(F \cap D \neq \emptyset)$.

(ii) $MA(\kappa)$ states that $MA_{\mathbb{X}}(\kappa)$ is true for all countable chain condition posets $\mathbb{X}$.

(iii) MA states that $\forall \kappa < \mathfrak{c}\; MA(\kappa)$.

Lemma 2.2. *If $\lambda \leq \kappa$ then $MA_{\mathbb{X}}(\kappa) \to MA_{\mathbb{X}}(\lambda)$ and $MA(\kappa) \to MA(\lambda)$.*

Proof. By definition we can find generic filters (filters that have nonempty intersection with dense sets) for all $|\mathcal{D}| \leq \kappa$ and so in particular for all $|\mathcal{D}| \leq \lambda$. □

Lemma 2.3 (Generic Filter Existence Lemma)**.** *For a countable family $D = \{D_n : n \in \omega,\ D_n \subseteq \mathbb{X}$ dense$\}$ of dense subsets of a forcing poset $\mathbb{X}$, there exists a filter $F \subseteq \mathbb{X}$ such that $F \cap D_n \neq \emptyset\ \forall n \in \omega$*

Proof. Since $\mathbb{X}$ is a forcing poset it is nonempty. Choose for example $\mathbb{1} \in \mathbb{X}$ now because D_0 is dense there $\exists d_0 \in D_0(d_0 \leq \mathbb{1})$. For D_{n+1} choose $d_{n+1} \in D_{n+1}$ such that $d_{n+1} \leq d_n$. This is always possible since $d_n \in \mathbb{X}$ and D_n are dense for all n.

Claim: $F = \{x \in \mathbb{X} : \exists n(d_n \leq x)\}$ is a filter and $F \cap D_n \neq \emptyset$.
Because $d_n \in F \cap D_n$ the intersections are never empty. To show that F is a filter we check all requirements.

- $\mathbb{1} \in F$
- Let $p, q \in F$ we must find some $r \in \mathbb{X}$ such that $r \leq p \wedge r \leq q$. Find therefore $n, m \in \omega$ such that $d_n \leq p$ and $d_m \leq q$. W.l.o.g $n \in m$ and by recursively applying the transitive property of our preorder we get $d_m \leq q \wedge d_m \leq d_n \leq p$.
- For $p \in F$ and $q \in \mathbb{X}(p \leq q)$ there $\exists d_n \leq p \leq q \to d_n \leq q$ and by definition of our filter $q \in F$.

□

Remark. Note that the only requirement, for our family of dense subsets, is to be countable. Therefore $MA_{\mathbb{X}}(\aleph_0)$.

In the following we will show $\neg MA(\mathfrak{c})$ where $\mathfrak{c}$ is the cardinality of the continuum. To show this it is sufficient to find some countable chain condition poset $\mathbb{X}$ and dense sets that do not suffice $MA(\mathfrak{c})$, but we will in fact proof a more powerful statement.

Definition 2.7. For some sets A, B, denote $\mathrm{Fn}(A, B)$ as the set of all finite partial functions from A to B. For the following consider $\mathrm{Fn}(A, B)$ together with $\supseteq$ as $\leq$ and $\mathbb{1} = \emptyset$ as a forcing poset.

It follows that any two such finite functions p, q are compatible iff they agree on $\mathrm{dom}(p) \cup \mathrm{dom}(q)$.

Lemma 2.4. *$Fn(A, B)$ has the (ccc) $\iff$ $A = \emptyset$ or B is countable*

Proof. Consider $A = \emptyset$ or $B = \emptyset$. It follows that $\mathrm{Fn}(A, B) = \emptyset$ and therefore has (ccc). Now let $A, B \neq \emptyset$
$\rightarrow$: Let B be uncountable, we will construct an uncountable antichain. A is nonempty therefore fix some $a \in A$. The set of singleton functions $\{\{(a, b) : b \in B)\}\} \subseteq \mathrm{Fn}(A, B)$ forms an uncountable antichain.
$\leftarrow$: Let B be countable. Take some subset $P' \subseteq \mathrm{Fn}(A, B)$ with $|P'| = \omega_1$. (If there is no such subset, then of course every antichain is countable). We write $\mathrm{Dom}(P') = \{\mathrm{dom}(p) : p \in P'\} \in [A]^{<\omega}$. By lemma 2.1 there exists a delta-system $P \subseteq \mathrm{Dom}(P')$ such that $|P| = \omega_1$. Denote the kernel as k and since $p \cap q = k$ for $p \neq q \in P$ the kernel is finite. Therefore $|B^k| = n$ for some $n \in \omega$ and there exist finite functions $p, q \in P'$ $(p \neq q)$ that agree on the kernel and $\mathrm{dom}(p) \cap \mathrm{dom}(q) = k$. Then in particular $p \cup q$ is a finite function and, in particular, a common extension for p, q. Thus $p \not\perp q$ and P' cannot be an antichain. $\square$

Proposition 2.5. *Let $|A| \geq \omega$ and $B \neq \emptyset$, then for arbitrary $a \in A$, $b \in B$, the following sets are dense in $Fn(A, B)$.*
(i) $D_a = \{d \in Fn(A, B) : a \in dom(d)\}$
(ii) $R_b = \{r \in Fn(A, B) : b \in im(d)\}$
If in addition we have $|B| \geq 2$ then the set $F_h = \{f \in Fn(A, B) : f \nsubseteq h\}$ is also dense $(\forall h \in B^A)$.

Proof. Let $x \in Fn(A, B)$.
If $a \in \mathrm{dom}(x)$ we get $x \in D_a$. If $a \notin \mathrm{dom}(x)$ the we can choose some arbitrary $\hat{b} \in B$ such that $x \subseteq x \cup \{(a, \hat{b})\}$ and $x \cup \{(a, \hat{b})\} \in D_a$ which proves (i).

Again we only need consider $b \notin \mathrm{im}(x)$, then by assumption $|A| \geq \omega$ and $|\mathrm{dom}(x)| < \omega$. It follows that $A \setminus \mathrm{dom}(x) \neq \emptyset$. Therefore $\forall \hat{a} \in A \setminus \mathrm{dom}(x) : x \subseteq x \cup \{(\hat{a}, b)\}$ and $x \cup \{(\hat{a}, b)\} \in R_b$ which proves (ii).

For the last set we similarly see that if $x \nsubseteq h$ then by definition $x \in F$. If $x \subseteq h$ like before choose $\hat{a} \in A \setminus \mathrm{dom}(x)$ and $\hat{b} \in B \setminus h(\hat{a})$. Then $x \subseteq x \cup \{(\hat{a}, \hat{b})\} \in F_h$ $\square$

Lemma 2.6. *$MA(\kappa) \rightarrow \kappa < \mathfrak{c}$.*

Proof. Let $2 \leq |B| \leq \omega \leq |A|$

Claim: If $G \subseteq \mathrm{Fn}(A, B)$ is a filter, then $f_G = \bigcup G$ is a (not necessarily finite) function with $\mathrm{dom}(f) \subseteq A$ and $\mathrm{im}(f) \subseteq B$

Proof of claim: If f_G were not a function, there would have to be some point at which f_G is not the graph of a function, more explicitly some $(a, b), (a, b') \in f_G$ such that $b \neq b'$. $\rightarrow \exists g, g' \in G[(a, b) \in g \wedge (a, b') \in g']$
$\rightarrow g \perp g'$ which is a contradiction to G being a filter. $\lightning$ $\square$

Having established that f_G is in fact a function, we want to make it have certain properties. This can be accomplished by letting our Filter G hit some dense sets. For this assume $MA(\kappa)$ for some $\kappa \geq \mathfrak{c}$.
Take $D = \{D_a : a \in A\} \cup \{E_h : h \in B^A\}$, $|D| = \mathfrak{c}$ and is a family of dense sets in $\mathrm{Fn}(A, B)$. By $MA(\kappa)$ we find a filter G such that $\forall a \in A(G \cap D_a \neq \emptyset)$ and $\forall h \in B^A(G \cap E_h \neq \emptyset)$. Since $G \cap D_a \neq \emptyset$ for arbitrary $a \in A$ it follows that $\mathrm{dom}(f_G) = A$. It furthermore follows that $f_G \in B^A$. But because $G \cap E_h \neq \emptyset$ it follows that $f \neq h$. A contradiction, since $h \in B^A$ was arbitrary. $\lightning$ $\square$

Example. The countable chain condition in the formulation of Martin's Axiom cannot be dropped. As an example let $|A| \leq \omega < \omega_1 \leq |B|$, then by lemma 2.4 $\mathrm{Fn}(A, B)$ is not (ccc). If we had a filter F such that $F \cap D_a \neq \emptyset$ and $F \cap R_b \neq \emptyset$ for all $a \in A, b \in B$, then f_G would be a surjective correspondence $A \longrightarrow B$. $\lightning$

2.4 Solovay's lemma and almost disjointness

Definition 2.8. For some $A \subseteq \mathcal{P}(A)$ we define the almost disjoint sets partial order on $\mathbb{X}_A = [\omega]^{<\omega} \times [A]^{<\omega} = \{\langle u, C\rangle : u \subseteq \omega, C \subseteq A, |C| + |u| < \omega\}$ as follows:

$$\langle u', C'\rangle \leq \langle u, C\rangle \iff u \subseteq u', C \subseteq C', \forall c \in C(c \cap u' \subseteq u)$$

Remark. (a) The almost disjoint sets partial order is in fact a partial order.

(b) With this partial order and the largest element $\mathbb{1} = \langle \emptyset, \emptyset\rangle$, we can speak of $\mathbb{X}_A$ as a forcing poset.

Proof of partial order. Reflexivity is clear by the fact that $x \cap s \subseteq s$. To show transitivity assume $\langle u, C\rangle \leq \langle u', C'\rangle$ and $\langle u', C'\rangle \leq \langle u'', C''\rangle$. Then $u \supseteq u' \supseteq u''$, $C \supseteq C' \supseteq C''$ and $\forall c \in C''$ $(c \cap u \subseteq c \cap u' \subseteq u'')$. Antisymmetry of $\leq$ follows by antisymmetry of inclusion $\subseteq$, or rather reverse inclusion $\supseteq$. $\square$

Lemma 2.7. *Let $\langle u, C\rangle, \langle u', C'\rangle \in \mathbb{X}_A$. Then these conditions are compatible $\langle u, C\rangle \not\perp \langle u', C'\rangle$ iff $[\forall c \in C(c \cap u' \subseteq u)$ and $\forall c' \in C'(c' \cap u \subseteq u')]$.*

Proof. $\leftarrow$: Under the above assumptions, a natural extension of both conditions is $\langle u \cup u', C \cup C' \rangle$.
$\rightarrow$: follows directly from the definition. For a common extension $\langle \hat{u}, \hat{C} \rangle$ satisfies $\forall c \in C(\hat{u} \cap c \subseteq u)$ and $\forall c' \in C'(\hat{u} \cap c' \subseteq u')$. Now consider w.l.o.g. that we can find $c \in C$ such that $u' \cap c \nsubseteq u$, this implies $\exists c \in C\,[(c \cap u) \cup (c \cap u') \subseteq c \cap \hat{u} \nsubseteq u]$. ↯ $\square$

Definition 2.9. For some filter $F \subseteq \mathbb{X}_A$ let $d_F := \{u \in w : \exists C(\langle u, C \rangle \in F)\}$.

Lemma 2.8. *For a filter $F \subseteq \mathbb{X}_A$ let $\langle u, C \rangle \in F$. Then $d_F \cap c \subseteq u\ (\forall c \in C)$.*

Proof. It is sufficient to show that $(d_F \setminus u) \cap c = \emptyset$ for any and all $c \in C$. For this let $c \in C$ and $v \in d_F \setminus u$ (if there is no such v then there is nothing left to show). By definition there $\exists \langle u', C' \rangle \in F\ (v \in u')$ and since F is a filter we can find some element $\langle \hat{u}, \hat{C} \rangle \in F$ which extends both conditions $\langle \hat{u}, \hat{C} \rangle \leq \langle u, C \rangle, \langle u', C' \rangle$. Therefore $v \in \hat{u} \supseteq u'$ and $v \in \hat{u} \setminus u$. Furthermore by the condition for extension $\hat{u} \cap c \subseteq u$ and similarly $(\hat{u} \setminus u) \cap c = \emptyset$, which implies $v \notin c$. Since $v \in d_F \setminus u$ was arbitrary it follows that $(d_F \setminus u) \cap c = \emptyset$. Since $c \in C$ was arbitrary it follows that $\forall c \in C\ (d_F \cap c \subseteq u)$. $\square$

Corollary 2.9. *For $x \in A$ the set $D_x := \{\langle u, C \rangle \in \mathbb{X}_A : x \in C\}$ is dense in $\mathbb{X}_A$. Whenever $F \cap D_x \neq \emptyset$ for a Filter F, then $|d_F \cap x| < \omega$.*

Proof. To show that D_x is dense, let $p \in \mathbb{X}_A$ arbitrary. $p = \langle u, C \rangle$ if $x \in C$ then $p \in D_x$. If $x \notin C$ then $\langle u, C \cup \{x\} \rangle \leq \langle u, C \rangle$ and $\langle u, C \cup \{x\} \rangle \in D_x$.
The second part of the corollary directly follows from the previous lemma in that if $F \cap D_x \neq \emptyset$ there exists some $\langle u, C \rangle \in F$ such that $x \in C$ and therefore $|d_F \cap x| \leq |u| < \omega$. $\square$

To use our construction of the forcing poset $\mathbb{X}_A$ meaningfully with Martin's axiom we need it to suffice the countable chain condition. It is not by chance that $\mathbb{X}_A$ was constructed in a way such that it is a (ccc)-poset but to show this we will need a new class of posets called σ-centered.

Definition 2.10. Let $\mathbb{X}$ be a forcing poset. Iff $\forall n \in \omega(\exists \mathbb{X}_n \subseteq \mathbb{X})$ such that
(i) $\mathbb{X} = \dot{\bigcup}_{n \in \omega} \mathbb{X}_n$
(ii) $\forall p, q \in \mathbb{X}_n \exists r \in \mathbb{X}_n (r \leq p, q)$
then we call $\mathbb{X}$ σ-centered. If the forcing poset just satisfies (ii) we only call it centered.

Proposition 2.10. *For $A \subseteq P(\omega)$ the forcing poset $\mathbb{X}_A$ is (ccc) and any σ-centered forcing poset $\mathbb{X}$ is (ccc).*

Proof. We first prove that any σ-centered forcing poset $\mathbb{X}$ has the countable chain condition. For this assume our poset $X = \dot{\bigcup}_{n\in\omega}\mathbb{X}_n$ and $\mathbb{X}_n$ centered. Since every antichain in $\mathbb{X}_n$ has to be singular, it is sufficient to show that any uncountable set $A \subseteq \mathbb{X}$ shares at least two elements with some $\mathbb{X}_n$.
Let $A \subseteq \mathbb{X}$ such that $|A| \geq \omega_1$. If $|A \cap \mathbb{X}_n| \leq 1$ $(\forall n \in \omega)$ then $|A| = |\mathbb{X} \cap A| = |\dot{\bigcup}_{n\in\omega}\mathbb{X}_n \cap A| \leq \omega$. This is a contradiction $\lightning$ and therefore any set A which has uncountable cardinality cannot be an antichain.

Now it is sufficient to show that $\mathbb{X}_A$ is in fact σ-centered. For $n \in [\omega]^{<\omega}$ we define $\hat{\mathbb{X}}_n \coloneqq \{\langle u, C\rangle \in \mathbb{X}_A : u = n\}$.

Claim: $\hat{\mathbb{X}}_n$ is centered.

Proof of claim. Let $\langle u, C\rangle$, $\langle u', C'\rangle \in \hat{\mathbb{X}}_n$. Since $u = n = u'$ $\langle n, C \cup C'\rangle$ is a common extension. □

We know $|[\omega]^{<\omega}| = \omega$ and there is a bijection $f : \omega \longrightarrow [\omega]^{<\omega}$. Then $\mathbb{X}_A = \dot{\bigcup}_{n\in w}\hat{\mathbb{X}}_{f(n)}$. (Or more simply put every subset of ω is also an element and the assertion follows directly). □

Lemma 2.11 (Solovay's lemma)**.** *Assume that $MA(\kappa)$ holds where κ is some infinite cardinal.*
Let $\mathcal{A}, \mathcal{B} \subset P(\omega)$ such that $|\mathcal{A}|, |\mathcal{B}| \leq \kappa$. If $\forall b \in \mathcal{B}\ \forall C \in [\mathcal{A}]^{<\omega}$ $(|b \setminus \bigcup C| = \omega)$, then there exists some $d \in [\omega]^\omega$ such that $\forall a \in \mathcal{A}(|a \cap d| < \omega)$ and $\forall b \in \mathcal{B}(|b \cap d| = \omega)$.

Proof. Let $b \in \mathcal{B}$, $n \in \omega$, we define $E_n^b \coloneqq \{\langle u, C\rangle \in \mathbb{X}_A : u \cap b \nsubseteq n\}$.

Claim 1: E_n^b is dense.

Proof. For n, b fix let $\langle u, C\rangle \in \mathbb{X}_A$ arbitrary. By assumption $|b \setminus \bigcup C| = \omega$. This implies the existence of a sufficiently large element $m \in b \setminus \bigcup C : n < m$. Since this is the case we see that $\langle u \cup \{m\}, C\rangle \in E_n^b$ as well as $\langle u \cup \{m\}, C\rangle \leq \langle u, C\rangle$ which shows density. □

As we saw in corollary 2.9 $D = \{D_a\}_{a\in A}$ is family of dense sets. Now we consider the superset $D \cup \{E_n^b\}_{b\in B, n\in w}$ which forms a family of dense sets with cardinality $|D \cup \{E_n^b\}_{b\in B,n\in w}| \leq |\omega||\kappa| \leq |\kappa|$. By assumption $MA(\kappa)$ holds and there exists a filter $F \subseteq \mathbb{X}_A$ such that the intersections with every dense member is nonempty.

Claim 2: $d = d_F$ satisfies $\forall a \in \mathcal{A}$ $(|a \cap d| < \omega)$ and $\forall b \in \mathcal{B}$ $(|b \cap d| = \omega)$.

Proof. Let $a \in A$, since $F \cap D_x \neq \emptyset$ by the second part of corollary 2.9 we see that $|d_F \cap a| < \omega$. Let $b \in B$, $\forall n \in \omega\ (F \cap E^b_n \neq \emptyset)$ therefore $\forall n \in \omega\ \exists \langle u, C\rangle \in F\ \exists m\ (n < m$ and $m \in u \cap b)$. It follows that $m \in d_F \cap b$ and $\omega \geq |b| \geq |b \cap d_F| \geq \omega$. □

□

Definition 2.11 (Almost disjointness)**.** For an infinite cardinal κ we define κ-almost disjointness (κ-a.d.).

(i) Elements in $x, y \in [\kappa]^\kappa = \{k \subseteq \kappa : |k| = \kappa\}$ are called κ-a.d (or whenever the cardinal is clear from context only a.d.) iff $|x \cap y| < \kappa$.

(ii) A family $\mathcal{A} \subseteq [\kappa]^\kappa$ of such subsets of κ is called almost disjoint iff $\forall x, y \in \mathcal{A}\ (x \neq y \rightarrow x, y$ are a.d.)

(iii) $\mathcal{A}$ is called κ-maximal almost disjoint iff $\mathcal{A}$ is almost disjoint and maximal under inclusion: $\nexists B \subseteq [\kappa]^\kappa (A \subsetneq B)$

For any arbitrary κ there must always be a maximal κ-almost disjoint family in ZFC, since Zorn's lemma naturally applies here.

Corollary 2.12. *Assume MA(κ) where $\omega \leq \kappa < 2^\omega$. Then an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ of cardinality κ is not maximal.*

Proof. We wish to apply Solovay's lemma to $\mathcal{A}$ and $\mathcal{B} = \{\omega\}$. To do so we first show the following claim:

Claim: $\forall C \in [\mathcal{A}]^{<\omega}\ (|\omega \setminus \bigcup C| = \omega)$

Proof of claim. Suppose our claim does not hold. Then we can find some $C \in [\mathcal{A}]^{<\omega}$ such that $|\omega \setminus \bigcup C| < \omega$. Observe that $\omega = \big(\bigcup C\big) \dot\cup \big(\omega \setminus \bigcup C\big)$ where $\omega \setminus \bigcup C$ is finite per assumption. For $A \in \mathcal{A} \setminus C$ it is clear that $A \subseteq \big(\bigcup C\big) \dot\cup \big(\omega \setminus \bigcup C\big)$. Since the second part is finite and $|A| = \omega$ we must have infinite intersection $|\bigcup C \cap A| = \omega$. Furthermore C is a finite family and so there exits some $C_0 \in C$ such that $(|C_0 \cap A| = \omega)$. However $C_0 \neq A \in \mathcal{A}$ contradicting that $\mathcal{A}$ is almost disjoint. □

Since the condition for Solovay's lemma is satisfied, we can find some $d \in [\omega]^\omega$ where $|d| = \omega$ and $\forall x \in \mathcal{A}\ (|x \cap d| < \omega)$. Therefore $\mathcal{A} \cup \{d\}$ is a.d. and $\mathcal{A}$ is not maximal. □

Theorem 2.13. *Let κ be an infinte regular cardinal and let $\mathcal{A} \subseteq [\kappa]^\kappa$ be a.d. such that $|\mathcal{A}| = \kappa$, then $\mathcal{A}$ is not maximal almost disjoint.*

Proof. Because our a.d.-family has cardinality κ we shall rewrite it as $\mathcal{A} = \bigcup_{\xi\in\kappa} A_\xi = \{A_\xi : \xi < \kappa\}$ and for every $\xi < \kappa$ define $B_\xi = A_\xi \setminus \bigcup_{\eta<\xi} A_\eta = A_\xi \setminus \bigcup_{\eta<\xi} (A_\xi \cap A_\eta)$. Per definition we have $|A_\xi| = \kappa$ and $\forall\eta \neq \xi(|A_\eta \cap A_\xi| < \kappa)$. But κ is regular and therefore $B_\xi \neq \emptyset$ and for each $\xi \in \kappa$ we can pick some $b_\xi \in B_\xi$, with this we define $B = \{b_\xi : \xi < \kappa\}$.

Claim: B is a.d. from every element of A.

Proof of claim. By our construction of B, we see that for every $\xi < \kappa$ $|B \cap A_\xi| \leq \xi$. Therefore B is almost disjoint from every element of $\mathcal{A}$. □

Now we find that $\mathcal{A}$ cannot be maximal since $\mathcal{A} \cup B$ is a strict a.d. superset of $\mathcal{A}$. □

In the following theorem we will see that in some cases we can in particular find almost disjoint families of a certain size. This will become relevant in the proof of a main theorem near the end of this chapter.

Theorem 2.14. *For an infinite cardinal κ such that $2^{<\kappa} = \kappa$, there exists an a.d. family $\mathcal{A} \subseteq [\kappa]^\kappa$ with cardinality $|\mathcal{A}| = 2^\kappa$.*

Proof. For $x \subseteq \kappa$ we define $A_x \coloneqq \{x \cap \alpha : \alpha \in \kappa\}$ and $\mathcal{A} \coloneqq \{A_x : x \in [\kappa]^\kappa\}$.

Claim: $\mathcal{A}$ is κ-almost disjoint.

Proof of claim. Let $x, y \subseteq \kappa$ be distinct $x \neq y$. Then w.l.o.g. $x \setminus y \neq \emptyset$ and for some fix $\xi \in x \setminus y$ we find that $\xi \in x \cap \eta$ $(\forall\eta \in \kappa \setminus (\xi + 1))$. Similarly for any $\eta' \in \kappa$ we find that $\xi \notin y \cap \eta'$ since $\xi \notin y$. It follows that $|A_x \cap A_y| \leq |\{x \cap \alpha : \alpha \leq \xi\}| \leq \xi < \kappa$. □

$|\mathcal{A}| = |P(\kappa) \setminus [2]^{<\kappa}|$ and $2^{<\kappa} = \kappa < 2^\kappa = |P(\kappa)|$. So $\mathcal{A}$ is of cardinality 2^κ. To find a subset of $P(\kappa)$ with such cardinality, we use a bijection. Define $I \coloneqq \{x \subseteq \kappa : \bigcup x < \kappa\}$. $|I| = 2^{<\kappa} = \kappa$. Therefore there exists some bijection $f : I \longrightarrow \kappa$. For $x \in [\kappa]^\kappa$ we define $A'_x \coloneqq \{f(x \cap \xi) : \alpha < \kappa\}$. With this we find an induced bijection $F : \mathcal{A} \longrightarrow \mathcal{A}'$ where $F(A_x) = A'_x$. And so $\mathcal{A}' \coloneqq \{A'_x : x \in [\kappa]^\kappa\} \subseteq P(\kappa)$ satisfies the theorem. □

Now we can proceed to a major theorem concerning Martin's Axiom.

Theorem 2.15. *Assume MA(κ) where $\omega \leq \kappa < 2^\omega$. Then $2^\kappa = 2^\omega$.*

Proof. Firstly since $\omega \leq \kappa$ we find that by monotonicity of exponentiation $2^\omega \leq 2^\kappa$. Therefore it is sufficient to find a surjective function $f : 2^\omega \longrightarrow 2^\kappa$.

Now by theorem 2.14 we can always find an almost disjoint family of cardinality 2^ω. By taking subsets (AC and existence of well order) we can find an

a.d. family $\mathcal{A}$ with cardinality κ ($\forall \kappa < 2^\omega$). Define $h : \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\mathcal{A})$ where $h(n) := \{A \in \mathcal{A} : |A \cap n| < \omega\}$. We show that h is onto:
For $\mathcal{B} \subsetneq \mathcal{A}$, we apply solovay's lemma 2.11 to $\mathcal{B}, \mathcal{A} \setminus \mathcal{B}$. Since we have almost disjointness $\forall A \in \mathcal{A} \setminus \mathcal{B}\ \forall C \in [\mathcal{B}]^{<\omega}$ $(|A \setminus \bigcup C| = \omega)$. Therefore we can find $d \in [\omega]^\omega \subseteq \mathcal{P}(\omega)$ such that $\forall B \in \mathcal{B}\ |B \cap d| < \omega$ and $\forall A \in \mathcal{A} \setminus \mathcal{B}\ |A \cap d| = \omega$. It follows that $h(d) = B$.

Now to find some d such that $h(d) = \mathcal{A}$. Since $\mathcal{A}$ has size κ it is not maximal. This means we can find a.d. $\hat{\mathcal{A}} \supsetneq \mathcal{A}$ and for some $\hat{A} \in \hat{\mathcal{A}} \setminus \mathcal{A}$ we find $\forall A \in \mathcal{A}\ (|\hat{A} \cap A| < \omega)$ because of almost disjointness. It follows that $h(\hat{A}) = \mathcal{A}$. Therefore h is onto. We know that there exist bijections $g : 2^\omega \longrightarrow \mathcal{P}(\omega)$, $g' : \mathcal{P}(\mathcal{A}) \longrightarrow 2^\kappa$. Then the function $f = g' \circ h \circ g$ is onto as well, implying $2^\kappa = |\mathcal{P}(\omega)| \geq |\mathcal{P}(\mathcal{A})| = 2^\kappa$ $\square$

For the following excursion into König's theorem we reference [2] for more details.

Lemma 2.16. *For a family A of indices let $\{\kappa_a\}_{a\in A}$ and $\{\lambda_a\}_{a\in A}$ be two sets of cardinals $\kappa_a < \lambda_a$. Then*

$$\sum_{a\in A} \kappa_a < \prod_{a\in A} \lambda_a$$

Proof. Let $\prod_{a\in A} \lambda_a = \left|\prod_{a\in A} I_a\right|$ for pairwise disjoint sets of cardinality $|I_a| = \lambda_a$. W.lo.g. we find $J_a \subsetneq I_a$ such that $|J_a| = \kappa_a$ and $\sum_{a\in A} \kappa_a = \left|\dot{\bigcup}_{a\in A} J_a\right|$. For $j \in \dot{\bigcup}_{a\in A} J_a$ there exists some unique $b \in A$ such that $j \in J_b \subset I_b$. We further define elements $x_a \in I_a \setminus J_a$. For $j \in J_b$ define $f_j : A \longrightarrow \bigcup_{a\in A} I_a$ as follows:

$$f_j(a) = \begin{cases} j & a = b \\ x_a & a \neq b \end{cases}$$

This induces the injection $F : \dot{\bigcup}_{a\in A} J_a \longrightarrow \prod_{a\in A} I_a$ were $F(j) = f_j$ (suppose $f_j = f_{j'}$ then we find some $b \in A$ such that $j = f_j(b) = f_{j'}(b) = j'$). It follows that $\sum_{a\in A} \kappa_a \leq \prod_{a\in A} \lambda_a$. Now suppose we had any arbitrary $G : \dot{\bigcup}_{a\in A} J_a \longrightarrow \prod_{a\in A} I_a$. Then for every $a \in A$ choose $y_a \in I_a \setminus \{G[j](a) : j \in J_a\}$ (this is possible since $J_a \subsetneq I_a$). We observe that $g : A \longrightarrow \bigcup_{a\in A} I_a$ where $g(a) = y_a$ is not met by G, and even the image of g is not once met by any $G[j]$, but $g \in \prod_{a\in A} I_a$. Thus G cannot be surjective, and since G was arbitrary there is no bijection between $\dot{\bigcup}_{a\in A} J_a$ and $\prod_{a\in A} I_a$. The strict inequality follows. $\square$

Remark. The above prove of lemma 2.16 uses (AC) implicitly. Furthermore if we have some index set A and the family $\lambda = \{\lambda_a\}_{a\in A}$, of nonempty sets $\lambda_a \neq \emptyset$, then the lemma states that $0 = \sum_{a\in A} \emptyset < \prod_{a\in A} \lambda_a$. This means that $\prod_{a\in A} \lambda_a$ is not empty and there is a choice function, which is just another formulation for (AC).

Corollary 2.17. *Let $\kappa > \aleph_0$. Then $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.*

Proof. By definition of $\mathrm{cf}(\kappa)$ we find some familiy $\lambda = \{\lambda_i\}_{i\in\mathrm{cf}(\kappa)}$ such that $\lambda_i < \kappa$ and $\bigcup \lambda = \kappa$. Thus $\kappa = \sum_{i\in\mathrm{cf}(\kappa)} \lambda_i < \prod_{i\in\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)}$. □

Theorem 2.18 (König's theorem)**.** *Let κ, λ be such that $\kappa \geq 2$ and $\lambda \geq \omega$. Then* $\mathrm{cf}(\kappa^\lambda) > \lambda$.

Proof by contradiction. Let $\tau = \kappa^\lambda$ and suppose $\lambda \geq \mathrm{cf}(\tau)$. Then by the above corollary 2.17 we find that $\tau < \tau^{\mathrm{cf}(\tau)} \leq \tau^\lambda = \kappa^{\lambda\lambda} = \kappa^\lambda$.

□

Corollary 2.19. *$MA \to 2^\omega$ is regular.*

Proof. Let $\omega \leq \lambda < 2^\omega$. By theorem 2.15 $2^\lambda = 2^\omega$. This implies $\lambda < \mathrm{cf}(2^\lambda) = \mathrm{cf}(\omega)$. Since λ was arbitrary $2^\omega \leq \mathrm{cf}(2^\omega)$ and by definition $\mathrm{cf}(2^\omega) \leq 2^\omega$. □

3 Application in Measure Theory

It is already known that presuppositions can yield very interesting and sometimes unwanted results. A good example of this would be non-Lebesgue measurable sets and their existence in ZFC. Another closely related example is the Banach-Tarski paradox, showing that a sphere $S \subset \mathbb{R}^n$ can be dissected into a finite number of pieces, which can then be rearranged (translated and rotated) into two identical disjoint copies of the original sphere. Just five pieces are said to be sufficient and the paradox holds for dimensions $n > 2$. [4]

Both constructions are non-constructive and require (AC). More specifically the existence of non-Lebesgue measurable sets is a direct consequence of (AC) and cannot be proven without it.[5]

It follows in particular that the mathematical concept of volume is unlike an intuitive physical one (under the assumptions of ZFC). Similarly it is interesting to see a few more consequences of (MA).

Definition 3.1. Let $\mathcal{N}$ be the set of all Lebesgue null sets. We write μ for the Lebesgue measure.

$$\mathrm{add}(\mathcal{N}) \coloneqq \min\{|E| : E \subseteq \mathcal{N} \wedge \bigcup E \notin \mathcal{N}\})$$

[4] see §2.2 of [5] *(An epsilon of room, Vol. 1.)*

[5] Solovay wrote an article concerning this topic, see [4]

Definition 3.2. Let $\epsilon \in \mathbb{R}^+$. We define the poset $\mathbb{X}_\epsilon$ as follows:

$$(\mathbb{X}_\epsilon, \leq) := (\{U \subseteq \mathbb{R} : U \text{ open, } \mu(U) < \epsilon\}, \supseteq)$$

Lemma 3.1. *Let $\epsilon > 0$. For $p, q \in \mathbb{X}_\epsilon$ and a Filter $F \subseteq \mathbb{X}_\epsilon$ the following hold.*

(i) $p \not\perp q \iff \mu(p \cup q) < \epsilon$.

(ii) $\mu(\bigcup F) \leq \epsilon$

Proof. (i): $\leftarrow$: Assume $\mu(p \cup q) < \epsilon$. Then $p \cup q \in \mathbb{X}_\epsilon$ is a common extension. $\rightarrow$: Suppose $\mu(p \cup q) \geq \epsilon$. Let $p, q \geq s \in \mathbb{X}_\epsilon$ then $p \cup q \subseteq s$ and it follows by nature of the measure $\epsilon \leq \mu(p \cup q) \leq \mu(s)$. Contradiction $\lightning$.

(ii): We first show that for finitely many elements of the filter, their union is also in the filter. For this, it is sufficient to show that the union of any two elements is still the filter. Let $p, q \in F$, then there exists some $s \leq p, q$. Observe that $p \cup q \subseteq s$ implying $s \leq p \cup q$. But a filter is upward-closed (closed with respect to weaker conditions) and so $p \cup q \in F$. It follows that for $P \in [F]^{<\omega}$, the countable union $\bigcup P$ is an element of F and $\mu(\bigcup P) < \epsilon$. Now if we had $|F| = \omega$ we would be finished, but our filter could easily be uncountable.
Let $\mathcal{B}$ denote the countable base of our metric topology on $\mathbb{R}$ consisting of open intervals of rational endpoints (implying rational length). For $x \in \bigcup F$ we can find a set $p \in F$ such that $x \in p$. By the property of a base we can also find some $B \in \mathcal{B}$ with $x \in B \subseteq p$ implying both $\mu(B) \leq \mu(U) < \epsilon$ and very importantly $B \in F$ (F is upward-closed). $\mathcal{B}$ is countable and therefore we find the countable set $\{p_i\}_{i\in\omega} = F \cap \mathcal{B}$. Again for $x \in \bigcup F$ we can find some $i \in \omega$ such that $x \in p_i$ and therefore $\bigcup F \subseteq \bigcup_{i\in\omega} p_i$. In order to stay precise let us define $\{\hat{p}_i\}_{i\in\omega}$ where $\hat{p}_i := \bigcup\limits_{j=1}^{i} p_j$. Since $\forall n \in \omega\ \mu(\bigcup_{i=1}^{n} \hat{p}_i) < \epsilon$ and $\{\hat{p}_i\}_{i\in\omega}$ is an ascending sequence of sets we see that $\mu(\bigcup\limits_{i\in\omega} p_i) = \lim\limits_{n\to\infty} \mu(\bigcup\limits_{i\in n} \hat{p}_i) \leq \epsilon$. $\square$

Lemma 3.2. *Let $\epsilon \in \mathbb{R}^+$. Then $\mathbb{X}_\epsilon$ has the countable chain condition.*

Proof by contradiction. Suppose $\mathbb{X}_\epsilon$ is not (ccc) and let $\{p_\alpha\}_{\alpha\in\omega_1}$ be an uncountable antichain.

Claim: $\exists n \in \mathbb{N} \left(\frac{1}{n} < \epsilon \text{ and } A_n = \{\alpha \in \omega_1 : \mu(p_\alpha) \leq \epsilon - \frac{3}{n}\} \text{ is uncountable}\right)$.

Proof of claim. Suppose not. Thus $\forall_{n\in\mathbb{N}} A_n$ is countable. But $\omega_1 = \lim\limits_{n\to\infty} A_n = \bigcup\limits_{n\in\mathbb{N}} A_n$. Contradiction $\lightning$ since countable unions of countable sets are countable. $\square$

For some n such that the above claim holds denote $\delta = \frac{1}{n}$. For every $\alpha \in A = A_n$ choose, in accordance to Littlewood's first principle, $B_\alpha \in B =$

$\{\bigcup \mathcal{B}' : \mathcal{B}' \in [\mathcal{B}]^{<\omega}\}$ such that $\mu(p_\alpha \triangle B_\alpha) \leq \delta$. Now for $\alpha \neq \beta \in A$ we get $\mu(p_\alpha \cup p_\beta) \geq \epsilon$ because $p_\alpha \perp p_\beta$. Furthermore $\mu(p_\alpha \cap p_\beta) \leq \mu(p_\alpha) \leq \epsilon - 3\delta$ and $p_\alpha \cup p_\beta = (p_\alpha \triangle p_\beta) \dot\cup (p_\alpha \cap p_\beta)$. This means $\mu(p_\alpha \triangle p_\beta) \geq 3\delta$. However since the symmetric set difference $\triangle$ suffices a kind of triangle inequality we get $\mu(p_\alpha \triangle p_\beta) \leq \mu(p_\alpha \triangle B_\alpha) + \mu(B_\alpha \triangle B_\beta) + \mu(p_\beta \triangle B_\beta)$. Observe that $\delta \leq \mu(B_\alpha \triangle B_\beta)$ and in particular $B_\alpha \neq B_\beta$. Ultimately we see that $\{B_\alpha\}_{\alpha \in A}$ is an uncountable subset of B. But B itself is countable. This is a contradiction. $\lightning$ $\square$

The premise of this thesis was to find the smallest set of nonzero measure and we will be able to give a partial answer shortly. With the inclusion $\subseteq$ being our relation it is, of course, useless to speak of 'the' smallest set of nonzero measure. This is because we can always remove a countable number of elements with (AC) and still have a non-null set. However it is possible to give a meaningful answer as to which is the minimum number of Lebesgue-null sets such that their union has measure greater zero. More formally we want to determine $\text{add}(\mathcal{N}) := \min\{|E| : E \subseteq \mathcal{N} \wedge \bigcup E \notin \mathcal{N}\})$. We know that $\bigcup_{x \in \mathbb{R}} x = \mathbb{R}$ is not null but any countable union of null sets is null. Therefore $\aleph_0 < \text{add}(\mathcal{N}) \leq 2^{\aleph_0}$. With our culmination of knowledge so far we prove a main theorem of this paper.

Theorem 3.3. *$MA(\kappa) \rightarrow add(\mathcal{N}) > \kappa$*

Proof. Fix null sets $\{N_\alpha\}_{\alpha \in \kappa} \subseteq \mathcal{N}$ and $\epsilon > 0$. For $\alpha \in \kappa$ define $D_\alpha := \{p \in \mathbb{X}_\epsilon : N_\alpha \subseteq p\} \neq \emptyset$. We show that D_α is dense in $\mathbb{X}_\epsilon$ $(\forall \alpha \in \kappa)$. Let $q \in \mathbb{X}_\epsilon$ then $\mu(q) = \epsilon_q < \epsilon$ for some ϵ_q. Since N_α is a null set we can find $s \in \mathbb{X}_\epsilon$ such that $N_\alpha \subseteq s$ and $\mu(s) < \epsilon - \epsilon_q$. By taking $p := q \cup s$ we see $\mu(p) = \mu(q \cup s) \leq \mu(q) + \mu(s) < \epsilon_q + \epsilon - \epsilon_q = \epsilon$. Since q, s are open their union is also open and so in particular $p \in D_\alpha$, $p \leq q$. But q was arbitrary so D_α is dense.
Now since X_ϵ is (ccc) we apply $MA(\kappa)$ and find a filter $F \subseteq \mathbb{X}_\epsilon$ such that $F \cap D_\alpha \neq \emptyset$ $(\forall \alpha \in \kappa)$. This means for every $\alpha \in \kappa$, $N_\alpha \subseteq \bigcup F$ implying $\bigcup\limits_{\alpha \in \kappa} N_\alpha \subseteq \bigcup F$. We remember that by lemma 3.1 $\mu(\bigcup F) \leq \epsilon$. Since $\epsilon > 0$ was arbitrary we find $\mu(\bigcup\limits_{\alpha \in \kappa} N_\alpha) = 0$. $\square$

4 Application in Topology

Definition 4.1. Let (X, T) be a topological space.

(i) We say $A \subset X$ is nowhere dense iff $(\overline{A})^{\circ} = \emptyset$.

(ii) Reversely we call $D \subseteq X$ dense iff $\overline{D} = X$. Equivalently, by definition of closure, a set D is dense iff for all nonempty open sets $U \subseteq X$, $U \cap D \neq \emptyset$.

(iii) We say $B \subset X$ is meagre iff $\exists A_i$ nowhere dense ($i \in \omega$) such that $B \subseteq \bigcup_{i\in\omega} A_i$.

Remark. (a) If A is nowhere dense then $\overline{A}$ is closed and nowhere dense.

(b) By the Baire category theorem a meager subset of the metric space $\mathbb{R}$ has empty interior.

4.1 Categorical analogy

Definition 4.2. We write $\mathcal{M}$ for the family of all meager subsets of the metric space $\mathbb{R}$.

$$add(\mathcal{M}) \coloneqq \min\{|M| : M \subseteq \mathcal{M}, \bigcup M \notin \mathcal{M}\}$$

Remark. Note that $\mathcal{M}$ is a σ-ideal, much like $\mathcal{N}$ in the previous chapter (closed under countable union of elements and $\subseteq$, and contains $\emptyset$).

Assuming $MA(\kappa)$ we want to show that for every family $M = \{M_\alpha\}_{\alpha<\kappa}$ of meager subsets $M_\alpha \subset \mathbb{R}$, the union $\bigcup M = \bigcup\limits_{\alpha<\kappa} M_\alpha$ is meager as well. This means we have to find a countable family of nowhere dense sets $H = \{H_n\}_{n\in\omega}$ such that $\bigcup M \subseteq \bigcup H$ or equivalently $\mathbb{R} \setminus \bigcup H \subseteq \mathbb{R} \setminus \bigcup M$ which is also equivalent to $\bigcap\limits_{n\in\omega} \mathbb{R}\setminus H_n \subseteq \bigcap\limits_{\alpha<\kappa} \mathbb{R}\setminus M_\alpha$. We remember that the closure of nowhere dense sets is nowhere dense, and so w.l.o.g. H_n are closed nowhere dense sets. Therefore it is enough to show that for every family of κ-many dense open sets $U_\alpha = (\mathbb{R} \setminus M_\alpha)^{\mathrm{o}}$, we are able to find countably many dense open sets $V_n = \mathbb{R} \setminus H_n$ such that $\bigcap\limits_{n\in\omega} V_n \subseteq \bigcap\limits_{\alpha<\kappa} U_\alpha$.

Theorem 4.1. *$MA(\kappa) \to add(\mathcal{M}) > \kappa$*

Proof. Let $\{U_\alpha\}_{\alpha<\kappa}$ be a family of dense open subset of $\mathbb{R}$ with the metric topology. We enumerate the countable base $B = \{B_i\}_{i\in\omega} \cup \emptyset = \{(p, q) : p, q \in \mathbb{Q} \wedge p < q\} \cup \emptyset$ containing all nonempty open intervals with rational start- and endpoints (we emphasize $B_i \neq \emptyset$). Now for every $\alpha < \kappa$ and $j \in \omega$ we define

(1) $a_\alpha = \{i \in \omega : B_i \nsubseteq U_\alpha\}$ and $\mathcal{A} = \{a_\alpha\}_{\alpha<\kappa}$.

(2) $c_j = \{i \in \omega : B_i \subseteq B_j\}$ and $\mathcal{B} = \{c_j\}_{j\in\omega}$.

We wish to apply Solovay's lemma. For $c_j \in \mathcal{B}$ and $C \in [\mathcal{A}]^{<\omega}$ we need to show $|c_j \setminus \bigcup C| = \omega$. Suppose $C = \{a_\alpha : \alpha \in \hat{C}\}$ for some $\hat{C} \in [\kappa]^{<\omega}$. Then $c_j \setminus \bigcup\limits_{\alpha\in\hat{C}} a_\alpha = \{i \in \omega : B_i \subseteq B_j \wedge B_i \subseteq \bigcap\limits_{\alpha\in\hat{C}} U_\alpha\} = \{i \in \omega : B_i \subseteq \big(B_j \cap \bigcap\limits_{\alpha\in\hat{C}} U_\alpha\big)\}$. We find that $B_j = (p_j, q_j)$ and $\bigcap\limits_{\alpha\in\hat{C}} U_\alpha$ is open and dense and thus, by definition of a base, $\emptyset \neq B_k \subseteq \big(B_j \cap \bigcap\limits_{\alpha\in\hat{C}} U_\alpha\big)$ for some $k \in \omega$. But B_k contains infinitely many rational intervals meaning infinitely many B_i. It follows that $|c_j \setminus \bigcup C| = \omega$. The condition for Solovay's lemma is satisfied, and by application of the latter we find some $d \in [\omega]^\omega$ such that $\forall_{\alpha<\kappa}(|d \cap a_\alpha| < \omega)$ and $\forall_{j\in\omega}(|c_j \cap d = \omega|)$. With this d we define $V_n = \bigcup\{B_i : i \in d \wedge n < i\}$.

Claim: $\forall_{n\in\omega} V_n$ is dense and open.

Proof of claim. Let $n \in \omega$ fix. V_n is open as the union of open sets. For density it is sufficient to show $\forall_{i\in\omega}(V_n \cap B_i \neq \emptyset)$. For $j \in \omega$ it follows by construction that $|d \cap c_j| = \omega$ and therefore we can find $i \in \omega$ such that $n < i$ and $i \in d \cap c_j$. This implies $B_i \subseteq B_j$ and $B_i \subseteq V_n$. Ultimately $\emptyset \neq B_i \subseteq B_j \cap V_n$. Since $j \in \omega$ was arbitrary the assertion follows. □

Now for $\alpha < \kappa$ fix, $|d \cap a_\alpha| < \omega$. This means we can find some $n \in \omega$ such that $d \cap a_\alpha \subseteq \omega$. Then for $i \in d$ with $i > n$, $B_i \subseteq U_\alpha$. This is equivalent to $V_n \subseteq U_\alpha$. Since $\alpha < \kappa$ was arbitrary $\bigcap\limits_{n\in\omega} V_n \subseteq \bigcap\limits_{\alpha<\kappa} U_\alpha$ and the assertion follows because V_n are dense. □

4.2 Product of (ccc)-spaces

Definition 4.3. A topological space (X, T) has the countable chain condition iff every family of pairwise disjoint, (nontrivial) open subsets of X is countable.

$$X \text{ (ccc)} \iff \forall A \subseteq T\big((x, y \in A \to x, y \neq \emptyset \wedge x \cap y = \emptyset) \to |A| \leq \omega\big)$$

We call a family A of pairwise disjoint, nontrivial open subsets of X an antichain.

A somewhat uninteresting example of a space without (ccc) is $\mathbb{R}$ with the discrete topology. Every $x \in \mathbb{R}$ is open and $|\mathbb{R}| \geq \omega_1$.

However the metric space $\mathbb{R}$ is (ccc) since the base $\mathcal{B}$ is countable: Suppose there was an uncountable $\{A_i\}_{i\in\omega_1}$ family of nonempty open sets A_i. Then by the definition of the base we can find an uncountable set $\{B_i\}_{i\in\omega_1}$ of nonempty pairwise disjoint elements of $B \in \mathcal{B}$ and $B_i \subseteq A_i$. Then $\omega = |B| \geq |\{B_i\}_{i\in\omega_1}| = \omega_1$ which is a contradiction. We can summarize this as X (AA2) $\to X$ (ccc). In fact an even stronger implication holds, as the following lemma shows.

Lemma 4.2. *If the topological space X is separable, then X is (ccc).*

Proof by contradiction. Suppose X is separable and not (ccc). Let $\{A_i\}_{i\in\omega_1}$ be an uncountable family of pairwise disjoint, nonempty, open sets. Since X is separable we find a countable dense set D. Because of density we can find $x_i \in A_i \cap D \neq \emptyset$, for each $i \in \omega_1$. Since A_i are disjoint this produces an uncountable family $\{a_i\}_{i\in\omega_1}$. Contradiction to $|D| = \omega$. $\lightning$ □

Theorem 4.3. *Let I be an index set and $\{X_i\}_{i\in I}$ a set of topological spaces. If $\forall \tau \in [I]^{<\omega}$ $\prod_{i\in\tau} X_i$ is (ccc), then $\prod_{i\in I} X_i$ is (ccc).*

Proof by contradiction. Suppose there exists an uncountable antichain $\{U_\alpha\}_{\alpha\in\omega_1}$ of sets in $\prod_{i\in I} X_i$. W.l.o.g. $U_\alpha = \big(\prod_{s\in k_\alpha} U_\alpha^s\big) \times \big(\prod_{\alpha\in I\setminus k_\alpha} X_\alpha\big)$ for some $k_\alpha \in [I]^{<\omega}$ and basic open subsets $U_\alpha^s \subseteq X_\alpha$ for $s \in k_\alpha$. Take the set of all such k_α.
Case 1: $\{k_\alpha\}_{\alpha\in\omega_1}$ is countable. Then we can find some uncountable $A \subseteq \omega_1$ with $\forall\alpha \in A\, (k_\alpha = k)$ for some $k \in [I]^{<\omega}$. This will be made use of later.
Case 2: $\{k_\alpha\}_{\alpha\in\omega_1}$ is uncountable. Then the conditions of lemma 2.1 are met and we can find $A \subseteq \omega_1$ such that $\{k_\alpha\}_{\alpha\in A}$ forms a delta system with kernel $k \in [I]^{<\omega}$.
Now we proceed similarly for both cases. Let $\alpha, \beta \in A$. Because $U_\alpha \cap U_\beta = \emptyset$ by the property of the product topology there exists at least one $s \in k$ with $U_\alpha^s \cap U_\beta^s = \emptyset$. In particular $\{\prod_{s\in k} U_\alpha^s\}_{\alpha\in\omega_1}$ is an uncountable family of pairwise disjoint, nonempty, open subsets of $\prod_{\alpha\in k} X_\alpha$. This is a contradiction since $\prod_{\alpha\in k} X_\alpha$ is (ccc) per assumption. □

Remark. If the topological product of any two topological spaces X, Y were to be (ccc), then the above lemma would imply that any product of (ccc) spaces were (ccc) as well. While this might be very desirable we have yet to show such a property.

Definition 4.4. We say some family $\{U_\alpha\}_{\alpha\in A}$ has the finite intersection property iff $\forall B \in [A]^{<\omega}\, \big(\bigcap B \neq \emptyset\big)$.

Lemma 4.4. *Assume $MA(\omega_1)$ holds. Let the topological space X be (ccc). If $\{U_\alpha\}_{\alpha\in\omega_1}$ is a family of nonempty, open subsets $U_\alpha \subseteq X$, then $\exists A \in [\omega_1]^{\omega_1}$ such that $\forall B \in [A]^{<\omega}\, \big(\bigcap_{\alpha\in B} U_\alpha \neq \emptyset\big)$.*

Proof. For $\alpha \in \omega_1$ we define $V_\alpha \coloneqq \bigcup_{\gamma>\alpha} U_\gamma$.

Claim 1: $\exists\alpha \in \omega_1 \forall\beta > \alpha\, (\overline{V}_\alpha = \overline{V}_\beta)$.

Proof of claim 1. Suppose by contradiction that the claim is false. This means that for every $\alpha \in \omega_1$ we can find some $\beta_\alpha > \alpha$ such that $\overline{V}_{\beta_\alpha+1} \subsetneq \overline{V}_{\beta_\alpha})$ (since $V_\beta \subseteq V_\alpha$ for $\beta > \alpha$) and $\beta_\alpha \neq \beta_{\alpha+1}$. This leaves us with the uncountable set $\{\beta_\alpha\}_{\alpha\in\omega_1}$. Fix some β_α and let $x \in \overline{V}_{\beta_\alpha} \setminus \overline{V}_{\beta_\alpha+1} = \overline{V}_{\beta_\alpha} \cap (X \setminus \overline{V}_{\beta_\alpha+1})$. We observe that $X \setminus \overline{V}_{\beta_\alpha+1}$ is open and $x \in \overline{V}_{\beta_\alpha} \iff \forall U \subseteq X \text{open}, x \in U\, (U \cap V_{\beta_\alpha} \neq \emptyset)$. This implies $\forall U \subseteq X\text{open}, x \in U\, \big(\emptyset \neq (U \cap X \setminus \overline{V}_{\beta_\alpha+1}) \cap \overline{V}_{\beta_\alpha} \subseteq V_{\beta_\alpha} \setminus \overline{V}_{\beta_\alpha+1}\big)$ since the finite intersection of open sets is open. Now because any union of open sets is open, for any β_α the resulting set $W_\alpha = V_{\beta_\alpha} \setminus \overline{V}_{\beta_\alpha+1}$ is open and nonempty. In particular for $\alpha_1 \neq \alpha_2$ w.l.o.g. $\alpha_1+1 \leq \alpha_2$ and thus $W_{\alpha_1} \cap W_{\alpha_2} = \emptyset$. Therefore $\{W_\alpha\}_{\alpha\in\omega_1}$ is an uncountable family of pairwise disjoint, nonempty open subsets of X. This is a contradiction since X is (ccc). $\lightning$ $\square$

Now let $\alpha \in \omega_1$ such that $\forall\beta > \alpha\, (\overline{V}_\alpha = \overline{V}_\beta)$. We define $\mathbb{P} = \{U \subseteq X : \emptyset \neq U \text{ open}, U \subseteq V_\alpha\}$ as a poset with the inclusion $\subseteq$ as our relation. Then elements of $\mathbb{P}$ are incompatible iff they are disjoint. It follows that $\mathbb{P}$ is (ccc) in the poset sense, because X is (ccc) in the topological sense.

Claim 2: For $\omega_1 > \beta > \alpha$ let $D_\beta \coloneqq \{p \in \mathbb{P} : \exists\gamma > \beta\, (p \subseteq U_\gamma)\}$. Then $\{D_\beta\}_{\alpha<\beta<\omega_1}$ is a family of dense subsets (in the poset sense) with at most ω_1 many elements.

Proof of claim 2. We only need to show density of D_β for $\alpha < \beta < \omega$ in the poset sense. For this choose some arbitrary $p \in \mathbb{P}$. Per definition $p \neq \emptyset$ and we can fix $x \in p$. Now $p \subseteq \overline{V}_\alpha = \overline{V}_\beta$ and so $x \in \overline{V}_\beta$ which again means $\forall U \subseteq X \text{open}, x \in U\, (U \cap V_\beta \neq \emptyset)$. p is exactly such an open set and therefore $\emptyset \neq p \cap V_\beta = p \cap \big(\bigcup_{\gamma>\beta} U_\gamma\big)$. Thus $\exists\gamma > \beta\, (p \cap U_\gamma \neq \emptyset)$. We see that $p \cap U_\gamma \in D_\beta$ and $p \cap U_\gamma \subseteq p$. But $p \in \mathbb{P}$ was arbitrary and therefore density follows. $\square$

Now by $MA(\omega_1)$ we can find a filter $F \subseteq \mathbb{P}$ such that $\forall\beta \in \omega_1, (\alpha < \beta)\, (F \cap D_\beta \neq \emptyset)$. We write $A \coloneqq \{\gamma \in \omega_1 : \exists p \in F\, (p \subseteq U_\gamma)\}$. F has nonempty intersection with all D_β which means $\forall\beta \in \omega_1, (\alpha < \beta)\, \exists p \in F\, (p \subseteq U_\gamma)$ and therefore implies $|A| = \omega_1$ The last, relatively forward claim completes the proof.

Claim 3: $\{U_\xi\}_{\xi\in A}$ has the finite intersection property.

Proof of claim 3. For $n \in [A]^{<\omega}$ let $\{U_i\}_{i\in n} \subseteq \{U_\xi\}_{\xi\in A}$. For each U_i we can find some $\hat{p}_i \in F$ such that $\hat{p}_i \subseteq U_i$. We arrange all $\hat{p}_i$ in some arbitrary manner and rename them $p_1, ..., p_n$. Since F is a filter we can find common extensions $q_{n-1} \subseteq p_1, p_2$; $q_{n-2} \subseteq q_{n-1}, p_3$;...;$q_1 \subseteq q_2, p_n$. Thus q_1 extends all p_i and $\emptyset \neq q_1 \subseteq p_1 \cap ... \cap_n \subseteq \bigcap_{i\in n} U_i$. Since $n \in [A]^{<\omega}$ was arbitrary $\{U_\xi\}_{\xi\in A}$ has the finite intersection property. $\square$

In summary $A = \{\gamma \in \omega_1 : \exists p \in F\, (p \subseteq U_\gamma)\}$ suffices the assertion of the lemma. $\square$

Lemma 4.5. *Assume $MA(\omega_1)$ holds true. Then any product of (ccc)-spaces is also (ccc).*

Proof. As stated in an earlier remark, it is sufficient to show that the product $X \times Y$ is (ccc), for any topological spaces X, Y with (ccc). Now suppose by contradiction that there were X, Y (ccc) such that $X \times Y$ is not (ccc). By

assumption there exists some uncountable antichain $\{\widehat{W}_\alpha\}_{\alpha\in\omega_1}$. Since all $\widehat{W}_\alpha$ are nonempty and open, we can find $U_\alpha \subseteq X$ open, $V_\alpha \subseteq Y$ open and subsets $W_\alpha = U_\alpha \times V_\alpha \subseteq \widehat{W}_\alpha$. Having established this we can consider the family $\{U_\alpha\}_{\alpha\in\omega_1}$ of nonempty, open subsets of X. By lemma 4.4 there exists some $A \in [\omega_1]^{\omega_1}$ such that $\{U_\alpha\}_{\alpha\in A}$ has the finite intersection property. In particular for $\alpha \neq \beta \in A$ the intersection $U_\alpha \cap U_\beta \neq \emptyset$. But $W_\alpha \cap W_\beta$ are disjoint and so $V_\alpha \cap V_\beta = \emptyset$. Since all V_α are nonempty this further means $V_\alpha \neq V_\beta$. Therefore $\{V_\alpha\}_{\alpha\in A}$ is an antichain in Y. This is a contradiction since Y is (ccc). $\lightning$
We summarize that any countable product of (ccc)-spaces is (ccc) and therefore any product of (ccc)-spaces must be (ccc), as by theorem 4.3. $\square$

5 Suslin's Hypothesis

Definition 5.1 (Linearly ordered topological space)**.** A partial order $\leq$ which satisfies comparability ($a \leq b \vee b \leq a$) is called a total order. For some totally ordered set $(X, \leq)$ we define the order topology as the topology uniquely determined by the base $\mathcal{B} = \{(x, y) \subseteq X : x, y \in X\} \cup \{(-\infty, y), (x, \infty) : x, y \in X\}$ containing intervals $(x, y) \coloneqq \{z \in X : z \neq x, y \text{ and } x \leq z \leq y\}$. A linearly ordered topological space is a totally ordered set $(X, <)$ with the order topology.

Definition 5.2 (Suslin's Hypothesis)**.** A Suslin line is a linearly ordered topological space that is (ccc) and not separable. Suslin's Hypothesis (SH) is the statement that there are no Suslin lines.

Remark. A Suslin line is uncountable. Any linearly ordered topological space $(X, <)$ which satisfies $(a), (b), (c)$ is isomorphic to $(\mathbb{R}, <)$. Historically the Suslin hypothesis developed from Suslin's question if (c) could be replaced by (d), while it is clear that $(c) \to (d)$.

(a) X has no first or least element

(b) X is connected in the order topology

(c) X is separable in the order topology

(d) X is (ccc) in the order topology

Lemma 5.1. *If X is a Suslin line, then $X^2 = X \times X$ is not (ccc).*

Proof. For elements $a, b \in X$ we denote $(a, b) \coloneqq \{x \in X : a < x < b\}$.

Claim 1: For $\alpha < \omega_1$ it is always possible to find $a_\alpha, b_\alpha, c_\alpha \in X$ such that:

(i) $a_\alpha < b_\alpha < c_\alpha$

(ii) $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$

(iii) $(a_\alpha, b_\alpha) \cap \{b_\xi : \xi < \alpha\} = \emptyset$

Proof of claim 1. Since X is not separable we can find a_0, b_0, c_0 that suffice the above conditions. Now suppose that for all $\xi < \alpha$ we have found according elements a_ξ, b_ξ, c_ξ. The set S of all isolated points in X has to be countable. Otherwise all elements of S are pairwise incompatible and so S is an uncountable antichain, a contradiction since X is (ccc). Therefore $S \cup \{b_\xi : \xi < \alpha\}$ is countable and cannot be dense. This means $X_\alpha = X \setminus \overline{S \cup \{b_\xi : \xi < \alpha\}} \neq \emptyset$ and in particular X_α is a nontrivial open set for which we can find a nontrivial open subset $(a_\alpha, c_\alpha) \subseteq X_\alpha$ in our base. Suppose $(a_\alpha, c_\alpha) = \{p_i\}_{i \leq n}$ for $n \in \mathbb{N}$. If $n = 1$ then (a_α, c_α) is a neighbourhood containing only p_1, otherwise $\{(a_\alpha, p_2)\}$ is a neighbourhood containing only p_1. In any case it follows that p_1 is an isolated point, a contradiction. Therefore (a_α, c_α) is infinite, and we can find $b_\alpha \in (a_\alpha, c_\alpha)$ such that $\emptyset \neq (a_\alpha, b_\alpha), (b_\alpha, c_\alpha)$. $a_\alpha, b_\alpha, c_\alpha$ suffice the conditions (i),(ii) and (iii). □

Now for every $\alpha < \omega_1$, we write $U_\alpha := (a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha)$.

Claim 2: $U = \{U_\alpha\}_{\alpha < \omega_1} \subseteq X \times X$ is an antichain.

Proof of claim 2. Let $\alpha, \xi \in \omega_1$, w.l.o.g $\xi < \alpha$. By the construction in claim 1, $b_\xi \notin (a_\alpha, c_\alpha)$. This leaves us with two cases, either $b_\xi \leq a_\alpha$ and $(a_\xi, b_\xi) \cap (a_\alpha, b_\alpha) = \emptyset$, or $c_\alpha \leq b_\xi$ and $(b_\xi, c_\xi) \cap (b_\alpha, c_\alpha) = \emptyset$. It follows that in both cases $U_\alpha \cap U_\xi = \emptyset$. But any element in U is nonempty and open and so U is an antichain. □

Since $U \subseteq X \times X$ forms an uncountable antichain, X^2 cannot be (ccc). □

Corollary 5.2. $MA(\omega_1) \to SH$

Proof. Under $MA(\omega_1)$ the product of (ccc)-spaces is always (ccc) itself. Suppose there existed some Suslin line X. Then X is (ccc) but $X \times X$ is not (ccc) which is a contradiction. □

References

[1] *Continuum hypothesis*. Apr. 2019. URL: `https://en.wikipedia.org/wiki/Continuum_hypothesis`.

[2] *König's theorem (set theory)*. 2019. URL: `https://en.wikipedia.org/wiki/K%C3%B6nig's_theorem_(set_theory)`.

[3] Kenneth Kunen. *Set theory*. Vol. 34. College Publications, 2013.

[4] Robert M. Solovay. "A Model of Set-Theory in Which Every Set of Reals is Lebesgue Measurable". In: *The Annals of Mathematics* 92.1 (1970).

[5] Terence Tao. *An epsilon of room*. Vol. 1. American Mathematical Society, 2010.

[6] *Zermelo–Fraenkel set theory*. 2019. URL: `https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory`.