UE Grundzüge der mathematischen Logik (SS 2017)
Ausgewählte Lösungen - Übungsblatt 10

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8. Juni 2017

Aufgabe 1. Sei $\mathcal{L}$ eine beliebige Sprache und sei T eine widerspruchsfreie Theorie. Zeigen Sie, dass T eine vollständige Erweiterung hat.

Proof. Let φ be an $\mathcal{L}$-sentence. In class we proved that $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ cannot both be simultaneously inconsistent.

Assume $\mathcal{L}$ is an enumerable language. Consider, for each $n \in \mathbb{N}$, the set W_n of finite words of length n using the alphabet of $\mathcal{L}$ and the symbols in $\{\neg, \wedge, \doteq, \exists, (,), v_0, v_1, v_2, \dots\}$. Using **Übungsblatt 8 Aufgabe 8 (2)** we can prove that W_n is enumerable, and using **Übungsblatt 8 Aufgabe 8 (1)** we can prove that $\bigcup\{W_n : n \in \mathbb{N}\}$ is enumerable. Then any $\mathcal{L}$-sentence φ is an element of $\bigcup\{W_n : n \in \mathbb{N}\}$, and therefore the set of $\mathcal{L}$-sentences is enumerable. We can now construct a complete theory T^* as done in class, by attaching to T either φ_i or $\neg\varphi_i$ from our listing of $\mathcal{L}$-sentences $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$.

Now assume $\mathcal{L}$ is not enumerable. Let A be a set and let $<$ be an ordering on A (an anti-reflexive and transitive relation). A set $C \subseteq A$ is called a *chain* if and only if $<\restriction C$ is a linear ordering on C (an anti-reflexive and transitive relation with trichotomy law). Recall the statement of

> **Zorn's lemma.** Let A be a nonempty set and let $<$ be an ordering on A, such that for any chain $C \subseteq A$ there exists an $x \in A$ such that there is no $y \in C$ with $x < y$. Then there exists an $x_{max} \in A$ such that no $y \in A$ satisfies $x_{max} < y$. In words, if every chain in A has an upper bound in A, then A contains at least one maximal element.

We will prove in class that Zorn's lemma is equivalent to (AC). Consider the set S of all $\mathcal{L}$-sentences. Then the set of all consistent extensions of T, call it A, satisfies $A \subseteq \mathcal{P}(S)$. The set A has a natural ordering given by $\subsetneq$. For $C \subseteq A$ a chain, define $T_C := \bigcup C$. Then for any $y \in C$ it is true that $y \subseteq T_C$, i.e., T_C is an upper bound for C, provided that $T_C \in A$. Indeed, suppose that T_C is inconsistent, then we can find $\psi_1, \dots, \psi_n \in T_C$ such that $\vdash_{\mathcal{L}} \neg(\psi_1 \wedge \dots \wedge \psi_n)$. But then, since we have finitely many sentences $\psi_1, \dots, \psi_n$ and C is a chain, we could find $y \in C$ such that $\psi_1, \dots, \psi_n \in y$, so $y \in A$ would be inconsistent, and that contradicts the definition of A. Using Zorn's lemma, there exists $T^* \in A$ maximal. That is, we found an extension T^* of T which is consistent and such that any other consistent extension of T is a subset of T^*. To show that T^* is complete, it remains to argue why T^* contains any sentence or its negation. Indeed, for any $\mathcal{L}$-sentence φ, if $T^* \cup \{\varphi\}$ is consistent, then $\varphi \in T^*$ by the maximality of T^*, and if $T^* \cup \{\varphi\}$ is inconsistent, then we proved in class that $T^* \cup \{\neg\varphi\}$ is consistent, and we get $\neg\varphi \in T^*$ again by the maximality of T^*. □

Aufgabe 2. Zeigen Sie mit Hilfe des Auswahlsaxioms: Eine lineare Ordnung $<$ auf einer Menge M ist genau dann eine Wohlordnung, wenn es keine Folge $\langle x_n : n \in \mathbb{N}\rangle$ von Elementen von M mit $x_{n+1} < x_n$ für alle $n \in \mathbb{N}$ gibt.

Proof. Let $<$ be a linear ordering (anti-reflexive, transitive relation with trichotomy law) on a set M. Assume that M is not well-ordered by $<$. By definition, this means that there exists a nonempty set $A \subseteq M$ with no $<$-minimal element, which implies that, for each $a \in A$, the set $B_a = \{x \in A \mid x < a\} \subseteq A$ is nonempty. Applying the axiom of choice to the nonempty set of nonempty sets $C = \{B_a \subseteq A \mid a \in A\}$ we can find a choice function $f : C \to \bigcup C$ such that, for each $a \in A$, $f(B_a) \in B_a$ and therefore $f(B_a) < a$. Now we can inductively define a sequence $\langle x_n : n \in \mathbb{N}\rangle$ of elements of A by setting $x_0 = a_0$ and $x_{n+1} = f(B_{x_n})$, where $a_0 \in A$ is some arbitrary element of A. By definition of f we have that $x_{n+1} < x_n$. This proves the direction of the statement which we did not prove in class. □

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Now assume $\mathcal{L}$ is not enumerable. Recall the statement of the

> **Hausdorff maximal principle.** Let F be a set and let $\mathcal{F}$ be a collection of subsets of F with the property that for any chain $\overline{\mathcal{F}} \subseteq \mathcal{F}$ (i.e. $X \subseteq Y$ or $Y \subseteq X$ for any $X, Y \in \overline{\mathcal{F}}$) it is true that $\bigcup \overline{\mathcal{F}} = \{a \in F \,|\, a \in X \text{ for some } X \in \overline{\mathcal{F}}\}$ is an element of $\mathcal{F}$. Then there exists an $x_{max} \in \mathcal{F}$, such that no proper superset of x_{max} is also an element of $\mathcal{F}$.

Let F be the set of all $\mathcal{L}$-sentences. Then the set of all consistent extensions of T, call it $\mathcal{F}$, satisfies $\mathcal{F} \subseteq \mathcal{P}(F)$. For $\overline{\mathcal{F}} \subseteq \mathcal{F}$ a chain, we want to show that $\bigcup \overline{\mathcal{F}} \in \mathcal{F}$, that is, we want to show that the theory $\bigcup \overline{\mathcal{F}}$ is consistent (the fact that it contains T is clear). Indeed, suppose that $\bigcup \overline{\mathcal{F}}$ is inconsistent, then we can find $\psi_1, \dots, \psi_n \in \bigcup \overline{\mathcal{F}}$ such that $\vdash_{\mathcal{L}} \neg(\psi_1 \wedge \dots \wedge \psi_n)$. But then, since we have finitely many sentences $\psi_1, \dots, \psi_n$ and $\overline{\mathcal{F}}$ is a chain, we could find $y \in \overline{\mathcal{F}}$ such that $\psi_1, \dots, \psi_n \in y$, so $y \in \mathcal{F}$ would be inconsistent, and that contradicts the definition of $\mathcal{F}$. By the Hausdorff maximal principle, there exists $T^* \in \mathcal{F}$ maximal. That is, we found an extension T^* of T which is consistent and such that any other consistent extension of T is a subset of T^*. To show that T^* is complete, it remains to argue why T^* contains any sentence or its negation. Indeed, for any $\mathcal{L}$-sentence φ, if $T^* \cup \{\varphi\}$ is consistent, then $\varphi \in T^*$ by the maximality of T^*, and if $T^* \cup \{\varphi\}$ is inconsistent, then we proved in class that $T^* \cup \{\neg\varphi\}$ is consistent, and we get $\neg\varphi \in T^*$ again by the maximality of T^*. $\square$

Michael König

Bonus Blatt:

z.z. eine unendliche Karte $\langle C, N\rangle$ ist vier färbbar gdw. jede endl. Teilkarte vierfärbbar ist.

Hinweis: $\langle C, N\rangle$, C ... Menge der Länder
Nachbarschafts-beziehung N ... Menge der Paare $\{c, d\}$ ≙ angrenzende Länder.

vierfärbbar wenn: $\exists F: C \to \{1,2,3,4\}$, s.d. $F(c) \neq F(d)$ wenn $\{c,d\} \in N$

Betrachte: unendl. Menge $\{A_{i,j}\}_{i,j \in \mathbb{N}}$ mit c_i ist Land, j ist Farbe.

Menge Φ aller Aussagenlogische Formeln:

1) $A_{i,j} \to \neg A_{i',j}$ und $A_{i',j} \to \neg A_{i,j}$ wobei $j \in \{1,\dots,4\}$ und $\{c_j, c_{j'}\} \in N$

2) $A_{i,1} \vee A_{i,2} \vee A_{i,3} \vee A_{i,4}$ für $i \in \mathbb{N}$

3) $A_{i,j} \to \neg A_{i,j'}$ für $j' \in \{1,\dots,4\} \setminus \{j\}$, $i \in \mathbb{N}$

In der Graphentheorie bedeuten die Aussagen 1-3 folgendes:
1) ... Nachbarn haben unterschiedliche Farbe
2) ... jedes Land muss gefärbt werden mit einer der 4-Farben
3) ... kein Land kann 2 Farben bekommen.

Bew: „⇒" trivial wenn unendliche Karte viergefärbt ist, dann auch jede endl. Teilkarte
„⇐" Ang. jede endliche Teilkarte von $\langle C, N\rangle$ ist vierfärbbar.

⇒ •) jede endliche TM von Φ ist erfüllbar, da Φ lediglich die Aussagen aus Graphentheorie in Logik übersetzt. (Siehe 1-3)

•) Betrachte eine Belegung $\mathcal{B}$ ~~...~~, mit ~~...~~. $\mathcal{B} \models \Phi$.
Dies entspricht ~~ist~~ einer zulässige Färbung von $\langle C, N\rangle$, da (1, 2, 3) korrekt ist.

⇒ Φ ist erfüllbar gdw. es existiert eine Vierfärbung von $\langle C, N\rangle$
dies gilt laut Annahme für jede endl. Teilkarte also auch für jede endl. TM Φ.

•) Kompaktheitssatz: Wenn jede endliche TM von Φ erfüllbar ist, dann auch Φ

⇒ die Behauptung, da Φ widerum in die Graphentheorie übersetzt werden kann und somit die unendliche Karte vierfärbbar ist.