SELECTIVE INDEPENDENCE

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Abstract. We show that consistently $\mathfrak{i} < \mathfrak{a}_T$, where $\mathfrak{i}$ is the minimal cardinality of a maximal independent family and $\mathfrak{a}_T$ is the minimal cardinality of a maximal family of pairwise almost disjoint subtrees of $2^{<\omega}$.

Moreover, we introduce the notion of selective independence and show that a variant of Sacks forcing, which preceded and is closely related to Miller's rational perfect set forcing, preserves selective independent families, P-points and Ramsey ultrafilters. In particular, we establish the consistency of $\mathfrak{u} = \mathfrak{i} < \mathfrak{a}_T$. Additionally, we show that consistently $\mathfrak{i} < \mathfrak{u} = \mathfrak{a}_T$.

1. Introduction

In this paper, we focus on the following infinitary combinatorial families of reals: independent families, almost disjoint families, and ultrafilter bases. A family $\mathcal{A}$ of $[\omega]^\omega$ is said to be independent if for every two disjoint finite non-empty subfamilies $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ the set $\bigcap \mathcal{B} \setminus \bigcup \mathcal{C}$ is infinite. It is maximal independent if it is in addition maximal under inclusion. The minimal cardinality of a maximal independent family is denoted $\mathfrak{i}$. The almost disjointness number, denoted $\mathfrak{a}$, is defined as the minimal cardinality of a maximal (under inclusion) infinite family of pairwise disjoint infinite subsets of ω. The consistency of $\mathfrak{a} < \mathfrak{i}$ is well-known, as it holds in the Cohen model. However, the consistency of $\mathfrak{i} < \mathfrak{a}$ is a long-standing open problem. Since $\mathfrak{d} \leq \mathfrak{i}$ (see [7]), a model of $\mathfrak{i} < \mathfrak{a}$ is necessarily a model of $\mathfrak{d} < \mathfrak{a}$. Note that, in all known models of $\mathfrak{d} < \mathfrak{a}$ (using ultrapowers, or Shelah's template construction, [14]) the value of $\mathfrak{i}$ coincides with the value of $\mathfrak{a}$. For more recent studies on the set of possible cardinalities of maximal independent families, see [4, 5].

Building upon earlier work [2], we define a class of independent families (originally appearing in [12]), to which we refer as selective independent families (see Definition 22). Selective independent families exist under CH (shown originally in [12] and later [2]). We show that selective independent families are preserved under the countable support iteration of a modification of Sacks forcing, to which we refer as partition forcing (see Definition 7 and Theorem 23) and so in particular, we show that the existence of selective independent families is consistent with the negation of CH. Moreover, the use of partition forcing allows us to control the value of $\mathfrak{a}_T$, a close

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relative of the almost disjointness number, defined as the minimal cardinality of a maximal family of pairwise disjoint subtrees of $2^{<\omega}$. This leads to our main result:

Theorem. Assume CH. It is relatively consistent that $\mathfrak{i} < \mathfrak{a}_T$.

Since $\mathfrak{d} \leq \mathfrak{i}$, in the above model $\mathfrak{d} = \omega_1 < \mathfrak{a}_T = \omega_2$, result which was first obtained by O. Spinas in [16]. We show that partition forcing preserves P-points, which leads to the following:

Theorem. Assume CH. It is relatively consistent that

$$\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} < \mathfrak{a}_T.$$

The preservation properties of selective independent families, allow to simultaneously increase $\mathfrak{u}$ and $\mathfrak{a}_T$, while preserving a small witness to $\mathfrak{i}$ (see Theorem 26) and so, additionally we establish the consistency of $\mathfrak{i} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2$.

Good surveys on the cardinal invariant $\mathfrak{a}_T$ can be found in [16, 11]. Based on an earlier result by A. Miller [9], O. Spinas [16] shows that $\mathfrak{a}_T$ is the minimal cardinality of a maximal family of pairwise disjoint closed subsets of an uncountable Polish space, while L. Newelski [11] and O. Spinas [16] established $\mathfrak{d} \leq \mathfrak{a}_T$. The consistency of $\mathrm{cof}(\mathcal{N}) < \mathfrak{a}_T$ was obtained in [16] via an iteration of partition forcings. The notion of a partition forcing was introduced by A. Miller in [9] and is pointed out as the origin of Miller's well known rational perfect set forcing [10] (see the historical comments on Miller forcing in Helbeisen's [7]). Recently, the consistency of $\mathfrak{a} < \mathfrak{a}_T$ was established by O. Guzmán, M. Hrušák and O. Téllez [6]. J. Stern [15] and independently K. Kunen (see [16]) established the consistency of $\mathfrak{a}_T < \mathrm{cof}(\mathcal{M})$, by showing that the inequality holds in the random real model. Hence, by the results of our paper and the fact that $\mathfrak{a}_T = \omega_1 < \mathfrak{c} = \mathfrak{i} = \mathfrak{u} = \mathfrak{r}$ holds in the random real model, the invariant $\mathfrak{a}_T$ is independent from $\mathfrak{i}$, $\mathfrak{u}$ and $\mathfrak{r}$. Furthermore, $\mathfrak{a}_T$ is independent from $\mathrm{cof}(\mathcal{M})$, $\mathrm{cof}(\mathcal{N})$, $\mathrm{non}(\mathcal{M})$, and $\mathrm{cov}(\mathcal{N})$.

The conditions of a partition forcing are special perfect trees and so the question about the value of $\mathfrak{a}_T$ in Sacks model becomes of interest. L. Newelski [11] shows that $\mathfrak{a}_T = \omega_1$ in the Sacks model[1] and so the partition forcing appears optimal in raising the value of $\mathfrak{a}_T$. Note that the **ZFC** relation of $\mathfrak{a}_T$ with $\mathfrak{a}$, or $\mathrm{non}(\mathcal{N})$, is unclear. That is, neither $\mathfrak{a} \leq \mathfrak{a}_T$ nor $\mathrm{non}(\mathcal{N}) \leq \mathfrak{a}_T$ is known (see [6]).

In our study of the partition forcing, we introduce the notion of a $\mathcal{C}$-branching trees (see Definition 8), which itself underlines the notion of a fusion with witnesses (see Definition 10 and Lemma 11) and plays an important role in establishing the preservation properties of selective independent families. Moreover, our notion of a fusion with witnesses generalizes earlier fusion arguments appearing in [6, 16, 11].

Outline of the paper: In section 2 we study dense maximality for independent families and provide three equivalent charactrisations of such densely maximal independent families. In section 3 we give further analysis on the partition forcing, establish the notion of a fusion with witnesses, and apply it to give an alternative proof of Spinas' theorem on the ${}^{\omega}\omega$-boundedness of the partition forcing. In Section 4 we establish preservation properties of the density ideal

[1]It is the consequence (C7) of Covering Property Axiom, see [1].

associated to an independent family with respect to the partition forcing and its iterations. In Section 5, we introduce the notion of selective independence, which is captured by dense maximality and properties of the density filter. In Section 6 we establish our main results, by showing that the partition forcing and countable support iterations of partition forcing preserve selective independent families, P-points and Ramsey ultrafilters. We conclude the paper with stating two remaining open problems.

2. Dense Maximality

For more on the notions in the following definition, we refer the reader to [2].

Definition 1. Let $\mathcal{A}$ be an infinite independent family. Then:

(1) The density ideal of $\mathcal{A}$, denoted $\mathrm{id}(\mathcal{A})$ consists of all $X \subseteq \omega$ with the property that $\forall h \in \mathrm{FF}(\mathcal{A})$ there is $h' \in \mathrm{FF}(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \cap X = \emptyset$.

(2) The density filter of $\mathcal{A}$, denoted $\mathrm{fil}(\mathcal{A})$, is the dual filter of $\mathrm{id}(\mathcal{A})$. Thus $Y \in \mathrm{fil}(\mathcal{A})$ if and only if $\forall h \in \mathrm{FF}(\mathcal{A}) \exists h' \in \mathrm{FF}(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \subseteq Y$.

Lemma 2. Let $\mathcal{A}$ be an infinite independent family. The following are equivalent:

(a) For all $X \in \mathcal{P}(\omega)\backslash\mathcal{A}$ and all $h \in \mathrm{FF}(\mathcal{A})$ there is an extension h' of h such that $\mathcal{A}^{h'} \cap X$ or $\mathcal{A}^{h'}\backslash X$ is finite (and so empty).

(b) For all $h \in \mathrm{FF}(\mathcal{A})$ and all $X \subseteq \mathcal{A}^h$ either $\mathcal{A}^h\backslash X \in \mathrm{id}(\mathcal{A})$, or there is $h' \in \mathrm{FF}(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash X$.

(c) For each $X \in \mathcal{P}(\omega)\backslash \mathrm{fil}(\mathcal{A})$ there is $h \in \mathrm{FF}(\mathcal{A})$ such that $X \subseteq \omega\backslash\mathcal{A}^h$.

Proof. The equivalence of (a) and (b) can be found in [3, Lemma 31]. The equivalence of (b) and (c) implicitly appears in [2, Theorem 29], as well as [12]. In interest of completeness of the presentation, we give a detailed proof.

((a) $\Rightarrow$ (b)) Fix $h \in \mathrm{FF}(\mathcal{A})$, $X \subseteq \mathcal{A}^h$ and suppose $\mathcal{A}^h\backslash X \notin \mathrm{id}(\mathcal{A})$. Thus, in particular $\mathcal{A}^h\backslash X \notin \mathrm{id}(X)$. Thus, there is $h' \in \mathrm{FF}(\mathcal{A})$ such that for all $h'' \supseteq h'$ the set $\mathcal{A}^{h''} \cap (\mathcal{A}^h\backslash X) \neq \emptyset$.

If $h \perp h'$, then $\mathcal{A}^{h'} \cap (\mathcal{A}^h\backslash X) = \emptyset$, which is a contradiction.

Therefore, $h' \not\perp h$. Wlg $h' \supseteq h$ and so for all $h'' \supseteq h'$, $\mathcal{A}^{h''}\backslash X \neq \emptyset$. Apply the fact that (a) holds. Then, there is $h'' \supseteq h'$ such that $\mathcal{A}^{h''} \cap X = \emptyset$ or $\mathcal{A}^{h''}\backslash X = \emptyset$. However, the latter can not happen by the choice of h' and so there is $h'' \supseteq h'$ such that $\mathcal{A}^{h''} \cap X = \emptyset$. Therefore $\mathcal{A}^{h''} \subseteq \mathcal{A}^{h'}\backslash X \subseteq \mathcal{A}^h\backslash X$, which establishes (b).

((b) $\Rightarrow$ (a)) Suppose $\mathcal{A}$ satisfies (b). Let $X \in [\omega]^\omega\backslash\mathcal{A}$, $h \in \mathrm{FF}(\mathcal{A})$. We have to show that there is $h' \supseteq h$ such that either $\mathcal{A}^{h''} \cap X$, or $\mathcal{A}^{h'}\backslash X$ is empty. Consider $Y = X \cap \mathcal{A}^h$.

If $\mathcal{A}^h\backslash Y \in \mathrm{id}(\mathcal{A})$, then since $Y \subseteq X$, $\omega\backslash X \subseteq \omega\backslash Y$ and so $\mathcal{A}^h\backslash X \subseteq \mathcal{A}^h\backslash Y$. Therefore, in this case $\mathcal{A}^h\backslash X \in \mathrm{id}(\mathcal{A})$. But then there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap (\mathcal{A}^h\backslash X) = \mathcal{A}^{h'}\backslash X = \emptyset$.

Otherwise, there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash Y$ and so $\mathcal{A}^{h'} \cap Y = \emptyset$. However, $\mathcal{A}^{h'} \cap Y = \mathcal{A}^{h'} \cap (X \cap \mathcal{A}^h) = \mathcal{A}^{h'} \cap X$, i.e. $\mathcal{A}^{h'} \cap X = \emptyset$. Therefore (a) holds.

((b) $\Rightarrow$ (c)) : Consider any $Z \notin \mathrm{fil}(\mathcal{A})$. Then $\omega\backslash Z \notin \mathrm{id}(\mathcal{A})$ and so there is $h \in \mathrm{FF}(\mathcal{A})$ such that for all $h' \supseteq h$

$$|\mathcal{A}^{h'} \cap (\omega\backslash Z)| = |\mathcal{A}^{h'}\backslash Z| = \omega.$$

Consider the set $Y = \mathcal{A}^h\backslash Z$. Thus $Y \subseteq \mathcal{A}^h$. By (b) either $\mathcal{A}^h\backslash Y \in \mathrm{id}(\mathcal{A})$, or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash Y$. In the former case $\exists h' \supseteq h$ such that $\mathcal{A}^{h'} \cap (\mathcal{A}^h\backslash Y) = \mathcal{A}^{h'}\backslash Y = \emptyset$. However $\mathcal{A}^{h'}\backslash Y = \mathcal{A}^{h'}\backslash(\mathcal{A}^h \cap \omega\backslash Z) = \mathcal{A}^{h'} \cap [\omega\backslash\mathcal{A}^h \cup Z] = \mathcal{A}^{h'} \cap Z$. Thus there is h' such that $Z \subseteq \omega\backslash\mathcal{A}^{h'}$. In the latter case, there is $h' \supseteq h$ such that

$$\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash Y = \mathcal{A}^h \cap [\omega\backslash\mathcal{A}^h \cup Z] = \mathcal{A}^h \cap Z.$$

But then $\mathcal{A}^{h'}\backslash Z = \emptyset$, which is a contradiction to the choice of h.

((c) $\Rightarrow$ (b)) Take any $X, h \in \mathrm{FF}(\mathcal{A})$ such that $X \subseteq \mathcal{A}^h$. If $X \in \mathrm{fil}(\mathcal{A})$ then $X \cup \omega\backslash\mathcal{A}^h \in \mathrm{fil}(\mathcal{A})$ and so

$$\omega\backslash(X \cup \omega\backslash\mathcal{A}^h) = \mathcal{A}^h\backslash X \in \mathrm{id}(\mathcal{A}).$$

If $X \notin \mathrm{fil}(\mathcal{A})$ then by (c) there is $h \in \mathrm{FF}(\mathcal{A})$ such that $X \subseteq \omega\backslash\mathcal{A}^h$. Take any $h' \supseteq h$. Then $\mathcal{A}^{h'} \subseteq \mathcal{A}^h$ and so $\omega\backslash\mathcal{A}^h \subseteq \omega\backslash\mathcal{A}^{h'}$, which implies that $X \subseteq \omega\backslash\mathcal{A}^{h'}$. Therefore $\mathcal{A}^{h'} \cap X = \emptyset$ and so $\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash X$. $\square$

The following crucial notion for indestructible maximal independent families was introduced in [2].

Definition 3. An independent family $\mathcal{A}$ is said to be densely maximal if any of the above three equivalent characterisations holds.

Remark 4. We shall use the notation $\langle\mathcal{G}\rangle_{\mathrm{up}} = \{X \in \mathcal{P}(\omega) \colon \exists G \in \mathcal{G}(G \subseteq X)\}$ and $\langle\mathcal{G}\rangle_{\mathrm{dn}} = \{X \in \mathcal{P}(\omega) \colon \exists G \in \mathcal{G}(X \subseteq G)\}$.

Corollary 5. A family $\mathcal{A}$ is densely maximal if and only if

$$\mathcal{P}(\omega) = \mathrm{fil}(\mathcal{A}) \cup \langle\{\omega\backslash\mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}}.$$

Lemma 6. Let $\mathcal{A}$ be an infinite independent family. Then

$$\mathrm{id}(\mathcal{A}) = \bigcup\{\mathrm{id}(\mathcal{B}) : \mathcal{B} \in [\mathcal{A}]^{\leq\omega}\}.$$

Proof. Let $\mathcal{B} \in [\mathcal{A}]^{\leq\omega}$ and let $X \in \mathrm{id}(\mathcal{B})$. Take any $h \in \mathrm{FF}(\mathcal{A})$ and let $h^* = h \restriction \mathcal{B}$. Then $h^* \in \mathrm{FF}(\mathcal{B})$ and so there is $h^{**} \in \mathrm{FF}(\mathcal{B})$ such that $h^{**} \supseteq h^*$ and $X \cap \mathcal{B}^{h^{**}} = \emptyset$. Let $h' = h \restriction (\mathcal{A}\backslash\mathcal{B}) \cup h^{**}$. Then $h' \supseteq h$ and $\mathcal{A}^{h'} \cap X \subseteq \mathcal{B}^{h^{**}} \cap X = \emptyset$. Thus $X \in \mathrm{id}(\mathcal{A})$.

Suppose $X \in \mathrm{id}(\mathcal{A})$. Take any $\mathcal{B} \in [\mathcal{A}]^{\leq\omega}$. Inductively, construct an increasing chain $\{\mathcal{B}_n\}_{n\in\omega}$ of countable independent families, contained in $\mathcal{A}$ such that

$$\forall h \in \mathrm{FF}(\mathcal{B}_n)\exists h' \in \mathrm{FF}(\mathcal{B}_{n+1})$$

such that $\mathcal{B}_{n+1}^{h'}(= \mathcal{A}^{h'}) \cap X = \emptyset$. Let $\mathcal{B} = \bigcup_{n\in\omega}\mathcal{B}_n$. Then $\mathcal{B} \in [\mathcal{A}]^{\leq\omega}$ and $X \in \mathrm{id}(\mathcal{B})$. $\square$

3. Partition Forcing

Let us recall that a set A which is contained in $[p]$ for some perfect subtree p of $2^{<\omega}$ is nowhere dense in $[p]$ if for every $s \in p$ there is $t \in p$ such that $s \subseteq t$ and $\{f \in [p] \colon t \subseteq f\} \cap A = \emptyset$.

Definition 7 (Partition forcing)**.** Let $\mathcal{C} = \{C_\alpha\}_{\alpha\in\omega_1}$ be an uncountable partition of 2^ω into closed sets and let $\mathbb{Q}(\mathcal{C})$ be the set of perfect trees $p \subseteq 2^{<\omega}$ such that each C_α is nowhere dense in $[p]$. The order of $\mathbb{Q}(\mathcal{C})$ is inclusion.

A. Miller, [9], showed that the poset $\mathbb{Q}(\mathcal{C})$ has the Laver property, while O. Spinas [16] established the ${}^\omega\omega$-bounding property of $\mathbb{Q}(\mathcal{C})$. Thus, $\mathbb{Q}(\mathcal{C})$ has the Sacks property. Of particular importance for our construction is the existence of fusion sequences in $\mathbb{Q}(\mathcal{C})$. We begin with auxiliary notions.

Definition 8. Let $\mathcal{C} = \{C_\alpha\}_{\alpha\in\omega_1}$ be an uncountable partition of 2^ω into closed sets.

(1) We say that $x, y \in {}^\omega 2$ are $\mathcal{C}$-different if x, y belong to different elements of $\mathcal{C}$.
(2) A tree $p \subseteq 2^{<\omega}$ is said to be $\mathcal{C}$-branching if for any $s \in p$ there are $\mathcal{C}$-different branches in $[p]$ extending s.

Note that, a $\mathcal{C}$-branching tree is perfect. We will use the following notation: whenever $\mathcal{C}$ as above is given, for each $x \in 2^\omega$ we denote by α_x the unique ordinal such that $x \in C_{\alpha_x}$.

Lemma 9. Let $p \subseteq 2^{<\omega}$ be a tree. The following are equivalent:

(a) $p \in \mathbb{Q}(\mathcal{C})$.
(b) p is $\mathcal{C}$-branching.
(c) p is perfect and $[p]$ contains a countable dense subset with $\mathcal{C}$-different branches.

Proof. ((a) $\Rightarrow$ (c)) Let $p \in \mathbb{Q}(\mathcal{C})$. p is a perfect tree by the definition. Thus arrange split(p) and assign by induction, to each splitting node s, a real x from $[p]$ extending s which was either already considered or belongs to different set from $\mathcal{C}$ than all previously selected reals. This is possible since any $s \in$ split(p) may be extended to $t \in$ split(p) with $[p(t)]$ being disjoint with finitely many sets from $\mathcal{C}$ containing all previously selected reals. The set of all assigned branches is the required dense set.

((c) $\Rightarrow$ (b)) Trivial.

((b) $\Rightarrow$ (a)) Let $\beta < \omega_1$ and $s \in p$. There are $x, y \in [p]$ such that $s \subseteq x, y$ and $\alpha_x \neq \alpha_y$. We take $z \in \{x, y\}$ such that $\alpha_z \neq \beta$. Since $z \in [p] \setminus C_\beta$ and C_β is closed, there is $s \subseteq t \subseteq z$ such that $[p_t] \cap C_\beta = \emptyset$. □

The particular enumeration constructed in Lemma 9 will be applied several times. Therefore we state explicitely that we may assume the dense set in Lemma 9 is enumerated as $\{x_t \colon t \in p\}$ such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

Definition 10. [Fusion sequence with witnesses]

(1) Let p be a condition in $\mathbb{Q}(\mathcal{C})$. We say that a set $X \subseteq {}^\omega 2$ is a p-witness for the n-th level if $X \subseteq [p]$, for each $s \in \text{split}_n(p)$ there is $x \in X$ extending s, and X has $\mathcal{C}$-different elements.

Note that if X is a p-witness for the $(n+1)$-st level then each node from n-th splitting level of p is contained in $\mathcal{C}$-different branches.

(2) Let $(p,X),(q,Y)$ be couples with p,q being conditions in $\mathbb{Q}(\mathcal{C})$, and sets X,Y being p-witness for the $(n+1)$-st level, q-witness for the n-th level, respectively. Then
$$(p,X)\leq^n (q,Y) \text{ if and only if } p\leq q \text{ and } X\supseteq Y.$$
Note that if $(p,X)\leq^n (q,Y)$ then $\mathrm{split}_{<n}(p)=\mathrm{split}_{<n}(q)$.

(3) A sequence $\{(p_n,X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses if $(p_{n+1},X_{n+1})\leq^n (p_n,X_n)$ for each n.

Lemma 11. If a sequence $\{(p_n,X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses then the fusion $\bigcap\{p_n : n\in\omega\}$ is a condition in $\mathbb{Q}(\mathcal{C})$.

Proof. We denote $p=\bigcap\{p_n\colon n\in\omega\}$, $X=\bigcup\{X_n : n\in\omega\}$, and we assume that we have $s\in p$. We take $n\in\omega$ and $t\in\mathrm{split}_n(p)$ such that t extends s. Since $\mathrm{split}_n(p)=\mathrm{split}_n(p_{n+1})$, the set X_{n+1} contains $\mathcal{C}$-different branches extending t. Hence, X is dense in $[p]$. One can easily see that X is contained in $[p]$. Finally, by Lemma 9 we conclude that $p\in\mathbb{Q}(\mathcal{C})$. □

A. Miller [9] and O. Spinas [16] applied separate fusion arguments in their proofs, while A. Miller [9] introduced the notion of a fusion even formally. The partial order $\mathbb{Q}(\mathcal{C})$ was recently used in [6], where the notion of a nice sequence was isolated from O. Spinas's fusion arguments. Our definition of fusion sequence covers both approaches. The sequence $\{X_n\}_{n\in\omega}$ in our definition may be obtained as sets of leftmost branches in Miller's fusion argument, and as certain terms of nice sequence in Spinas's approach. In fact, nice sequence may be obtained reenumerating our dense set $\{x_t\colon t\in p\}$ in Lemma 9.

In addition to fusion sequences, we shall use two basic schemas to amalgamate conditions. Let us have a condition $p\in\mathbb{Q}(\mathcal{C})$, and for each $s\in\mathrm{split}_n(p)$, $i\in\{0,1\}$, a condition $q(s,i)$ extending $p(s^\frown i)$. Using Lemma 9, one can easily see that the tree
$$q=\bigcup\{q(s,i)\colon s\in\mathrm{split}_n(p), i\in\{0,1\}\}$$
is a condition in $\mathbb{Q}(\mathcal{C})$ as well. In the second amalgamation technique, we are given a decreasing sequence $\{q_i\}_{i\in\omega}$ of extensions of p with strictly increasing stems $s_n=\mathrm{stem}\, q_n$. We set $x=\bigcup_{i\in\omega}s_i$ and take $q=\bigcup_{i\in\omega}q_i(s_i^\frown\langle 1-x(|s_i|)\rangle)$. Again, using Lemma 9, one can easily see that q is a condition in $\mathbb{Q}(\mathcal{C})$.

The proof of the fact that $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding is underlying many of the fusion arguments to follow. For convenience of the reader, we repeat it here. We need two auxiliary assertions.

Lemma 12. Let $\dot f$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$ and let h be a function in ${}^\omega\omega\cap V$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$: There is a real $x\in[q]$ and a sequence $\{f_s\}_{s\in x\restriction\mathrm{split}(q)}$ of functions in ${}^{<\omega}\omega$ such that for any $s=x\restriction\mathrm{split}_n(q)$ we have $q(s)\Vdash\dot f\restriction h(n)=f_s$.

Proof. Let $p\in\mathbb{Q}(\mathcal{C})$. One can construct a decreasing sequence $\{q_i\}_{i\in\omega}$ of extensions of p with strictly increasing stems such that $q_n\Vdash\dot f\restriction h(n)=f_n$ for some $f_n\in{}^{h(n)}\omega$. We denote $s_n=$

stem q_n and we set $x = \bigcup_{i\in\omega} s_i$. Finally, we take the amalgamation $q = \bigcup_{i\in\omega} q_i(s_i^\frown\langle 1 - x(|s_i|)\rangle)$. $\square$

Lemma 13. Let $\dot f$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$: For all $m \in \omega$, for all $t \in \mathrm{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot f \restriction (m+1) = \check f_t.$$

Proof. Let $p \in \mathbb{Q}(\mathcal{C})$. We build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \le q$ such that its fusion q has the required property. Let the condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_0)}$ be obtained from Lemma 12 for p and $h(n) = n+1$. We set $X_0 = \{x\}$.

Let $0 \le n < \omega$. Suppose we have defined $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$. Let $s \in \mathrm{split}_n(q_n)$. Take the unique branch $x \in X_n$ extending s, node $r = x \restriction \mathrm{split}_{n+1}(q_n)$, and number $i = x(|s|)$ in $\{0,1\}$. We set $q(s,i) = q_n(r)$. Let $t \supseteq s^\frown\langle 1-i\rangle$ be such that $[q_n(t)] \cap C_{\alpha_x} = \emptyset$ for all already considered branches x (i.e., all branches in X_n and those assigned to previous nodes in some order of $\mathrm{split}_n(q_n)$). Use Lemma 12 for $q_n(t)$ and $h(j) = n+j+2$ to obtain $q(s, 1-i) \le q_n(t)$, branch x and sequence $\{f_s\}_{s\in x\restriction \mathrm{split}(q_n)}$.

Finally, let X_{n+1} be the set of all considered branches in this step, and

$$q_{n+1} = \bigcup\{q(s,i)\colon s \in \mathrm{split}_n(q_n), i \in \{0,1\}\}.$$

One can verify that the sequence $\{(q_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses. $\square$

Lemma 14 (O. Spinas [16])**.** The poset $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding.

Proof. Let $\dot f$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$ and let $p \in \mathbb{Q}(\mathcal{C})$. We will show that there is $q \le p$ and $g \in V \cap {}^\omega\omega$ such that $q \Vdash \dot f \le^* \check g$.

By Lemma 13 we can assume that for all $m \in \omega$, for all $t \in \mathrm{split}_m(p)$ there is $f_t \in {}^{m+1}\omega$ such that $p(t) \Vdash \dot f \restriction (m+1) = \check f_t$. Define $g \in {}^\omega\omega$ as follows:

$$g(n) = \max\{f_s(n) + 1\colon s \in \mathrm{split}_n(q)\}.$$

Then $q \Vdash \forall n(\dot f(n) < g(n))$. $\square$

4. Preservation Theorems

Lemma 15. Assume CH. Let $\mathcal{A}$ be an independent family, and let H be a $\mathbb{Q}(\mathcal{C})$-generic filter. Then $\mathrm{id}(\mathcal{A})^{V[H]}$ is generated by $\mathrm{id}(\mathcal{A}) \cap V$, where V denotes the ground model. With other words for each $X \in \mathrm{id}(\mathcal{A}) \cap V[H]$ there is $Y \in \mathrm{id}(\mathcal{A}) \cap V$ such that $X \subseteq Y$.

Proof. Fix $p \in \mathbb{Q}(\mathcal{C})$ such that $p \Vdash \dot X \in \mathrm{id}(\mathcal{A})$. By Lemma 6 we can find $q \le p$ and a name $\dot{\mathcal{B}}$ for a countable subset of $\mathcal{A}$ such that $q \Vdash \dot X \in \mathrm{id}(\mathcal{B})$. Identifying $\mathrm{FF}(\dot B)$ with $2^{<\omega}$ in $V^{\mathbb{Q}(\mathcal{C})}$, we get a $\mathbb{Q}(\mathcal{C})$-name for a dense subset of $2^{<\omega}$ defined by

$$q \Vdash h \in \dot D \text{ iff } \dot B^h \cap \dot X = \emptyset.$$

Claim. Let $\{v_i\}_{i\in\omega} \subseteq {}^{<\omega}2$. The set of all conditions r satisfying the following property is dense below q: There is $x \in [r]$ and $\{u_s\}_{s\in x\restriction \mathrm{split}(r)}$ such that for $s = x \restriction \mathrm{split}_n(r)$ we have $u_s \supseteq v_n$ and $r(s) \Vdash \check{u}_s \in \dot{D}$.

Proof. Let $q' \leq q$. Since $q \Vdash$ "$\dot{D}$ is dense open", one can construct a decreasing sequence $\{q_i\}_{i\in\omega}$ of subconditions of q' with strictly increasing stems such that $u_n \supseteq v_n$ and $q_n \Vdash \check{u}_n \in \dot{D}$ for some $u_n \in {}^{<\omega}2$. We denote $s_n = \operatorname{stem} q_n$ and we set $x = \bigcup_{i\in\omega} s_i$. Finally, we take $r = \bigcup_{i\in\omega} q_i(s_i^\frown\langle 1 - x(|s_i|)\rangle)$. $\square$

Claim. The set Δ of $r \in \mathbb{Q}(\mathcal{C})$ such that $r \Vdash \check{K} \subseteq \dot{D}$ for some dense $K \subseteq 2^{<\omega}$ is dense below q.

Proof. Let $\langle w_n \colon n \in \omega\rangle$ enumerate $2^{<\omega}$. We build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$. Let q_0, x and $\{u_s\}_{s\in x\restriction \mathrm{split}(q_0)}$ be obtained from the claim for r and $\{w_i\}_{i\in\omega}$. We set $X_0 = \{x\}$.

Let $0 \leq n < \omega$. Suppose we have defined $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$. Let $s \in \mathrm{split}_n(q_n)$. Take the unique branch $x \in X_n$ extending s and denote the set of all already considered branches as X_s (i.e., all branches in X_n and those assigned to previous nodes in some order of $\mathrm{split}_n(q_n)$). We denote $r = x \restriction \mathrm{split}_{n+1}(q_n)$, $i = x(|s|) \in \{0, 1\}$, and we set $q(s, i) = q_n(r)$. Let $t \supseteq s^\frown\langle 1 - i\rangle$ be such that $[q_n(t)] \cap C_{\alpha_x} = \emptyset$ for all already considered branches $x \in X_s$. Use previous claim for $q_n(t)$ and $\{v_j\}_{j\in\omega}$ such that v_j is defined as the maximum among $u_{x\restriction \mathrm{split}_{n+j+1}(q_n)}$ for all already considered branches $x \in X_s$. We obtain condition $q(s, 1 - i) \leq q_n(t)$, branch x and sequence $\{u_s\}_{s\in x\restriction \mathrm{split}(q(s,1-i))}$.

Finally, let X_{n+1} be the set of all considered branches in this step, and

$$q_{n+1} = \bigcup\{q(s, i) \colon s \in \mathrm{split}_n(q_n), i \in \{0, 1\}\}.$$

One can see that the sequence $\{(q_n, X_n)\}_{n\in\omega}$ is a fusion sequence, so the fusion q' is a condition. The set K is defined as the set of all $u_n = \bigcup\{u_s \colon s \in \mathrm{split}_n(q')\}$. One can check that $u_n \supseteq w_n$ and $q' \Vdash \check{u}_n \in \dot{D}$ for all n. $\square$

Then for some dense $K \subseteq 2^{<\omega}$ we have $V[H] \vDash K \subseteq \dot{D}[H]$. Take $Y = \bigcap_{t\in K}(\omega\setminus\mathcal{A}^t)$. Then $Y \in \mathrm{id}(\mathcal{A}) \cap V$ and $V[H] \vDash \dot{X}[H] \subseteq Y$ as desired. $\square$

Lemma 16. Assume CH. Let $\mathcal{A}$ be an independent family and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ be a countable support iteration such that for each α there is a partition $\mathcal{C}$ of 2^ω such that

$$V^{\mathbb{P}_\alpha} \vDash \mathbb{Q}_\alpha = \mathbb{Q}(\mathcal{C}).$$

Then for every $\alpha \leq \omega_2$ the ideal $\mathrm{id}(\mathcal{A})^{V^{\mathbb{P}}_\alpha}$ is generated by $\mathrm{id}(\mathcal{A}) \cap V$. That is,

$$V^{\mathbb{P}_\alpha} \vDash (\forall X \in \mathrm{id}(\mathcal{A}) \exists Y \in \mathrm{id}(\mathcal{A}) \cap V \text{ such that } X \subseteq Y).$$

Proof. The proof is a straightforward corollary to the above Lemma and [13, Ch. VI, Theorem 0.A.3]. $\square$

5. Selective independence

Definition 17. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$.

(1) $\mathcal{F}$ is centered if for every $\mathcal{H} \in [\mathcal{F}]^{<\omega}$ the intersection $\bigcap \mathcal{H} \in \mathcal{F}$.
(2) $\mathcal{F}$ is said to be a P-set if for every countable subfamily $\mathcal{H} \subseteq \mathcal{F}$ there is $A \in \mathcal{F}$ such that $A \subseteq^* H$ for every $H \in \mathcal{H}$.
(3) $\mathcal{F}$ is a Q-set if for every bounded partition $\mathcal{E}$ of ω there is $X \in \mathcal{F}$ such that $|X \cap E| \leq 1$ for every $E \in \mathcal{E}$. We say that X is a semi-selector for $\mathcal{E}$.

Definition 18. A family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is said to be Ramsey if $\mathcal{F}$ is a centered family containing the co-finite sets which is both, a P-set and a Q-set.

Theorem 19. [12, Lemma 3.2] (CH) *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ be a countable support iteration of proper ${}^\omega\omega$-bounding posets. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a Ramsey set and let $\mathcal{H} \subseteq \mathcal{P}(\omega) \backslash \langle \mathcal{F} \rangle_{up}$. Suppose for each $\alpha < \delta$, $V^{\mathbb{P}_\alpha} \models \mathcal{P}(\omega) = \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn}$. Then, the same property holds at δ, i.e.*

$$V^{\mathbb{P}_\delta} \models \mathcal{P}(\omega) = \langle \mathcal{F} \rangle_{up} \cup \langle \mathcal{H} \rangle_{dn}.$$

We use a combinatorial characterization of Q-filters, a similar one to a characterization of happy families, see Proposition 0.7 by A. Mathias [8] or Proposition 11.6 in [7].

Lemma 20. Let $\mathcal{F}$ be a filter. The following are equivalent:

(a) $\mathcal{F}$ is a Q-filter.
(b) For any increasing function $f \in {}^\omega\omega$ there is $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ such that $f(k(n)) < k(n+1)$.

Proof. ((a) $\Rightarrow$ (b)) Inductively, choose a sequence $\{n(l)\}_{l \in \omega}$ such that $n(0) = 0$ and

$$n(l+1) = \min\{n \colon n_l < n \text{ and } \forall m \leq n_l (f(m) \leq n)\}.$$

We consider the partition $\mathcal{E}_0 = \{[n_{3l}, n_{3l+3})\}_{l \in \omega}$. There is $C_1 \in \mathcal{F}$ such that C_1 is a semi-selector for $\mathcal{E}_0$. Now, define an equivalence relation $\mathcal{E}_1$ on C_1 as follows:

$$m \sim_{\mathcal{E}_1} k \text{ iff } m = k \vee m < k \leq f(m) \vee k < m \leq f(k).$$

Each $\mathcal{E}_1$ equivalence relation has at most two members. Indeed, if there were three numbers $m_1 < m_2 < m_3$ in one equivalence class of $\mathcal{E}_1$ then $m_1 < m_2 < m_3 \leq f(m_1)$. There are $l_1 < l_2 < l_3$ such that $m_i \in [n_{3l_i}, n_{3l_i+3})$. Then $m_1 < n_{3l_2} \leq m_2 < n_{3l_3} \leq m_3 \leq f(m_1)$. However, on the other hand by the definition of sequence $\{n(l)\}_{l \in \omega}$ we have $f(m_1) \leq n_{3l_2+1} < n_{3l_3}$, a contradiction.

Extend $\mathcal{E}_1$ to an equivalence relation $\mathcal{E}_2$ on ω by defining

$$m \sim_{\mathcal{E}_2} k \text{ iff } m = k \vee m \sim_{\mathcal{E}_1} k.$$

There is C_2 in $\mathcal{F}$ such that C_2 is a semi-selector for $\mathcal{E}_2$. Without loss of generality $C_2 \subseteq C_1$ and $0 \in C_2$. Let $\{k(n)\}_{n \in \omega}$ enumerate in increasing order C_2. Thus for all n, n' we have that $k(n) \not\sim_{\mathcal{E}_2} k(n')$. Thus, if $n < n'$ then $k(n') \not\leq f(k(n))$ and so for all $n \in \omega$, $f(k(n)) < k(n+1)$.

((b) $\Rightarrow$ (a)) Let $\mathcal{E}$ be a bounded partition of ω. We set

$$f(n) = \max \bigcup \{E \in \mathcal{E} \colon (\exists i \leq n)\ i \in E\}.$$

There is $\{k(n)\colon n\in\omega\}\in\mathcal{F}$ such that $f(k(n))<k(n+1)$ for each $n\in\omega$. The set $\{k(n)\colon n\in\omega\}$ is a semi-selector for $\mathcal{E}$. Indeed, $k(n)\le f(k(n))<k(n+1)$ and therefore $k(n+1)$ is from different set of partition $\mathcal{E}$ than all $k(i)$ for $i\le n$. $\square$

Lemma 21. An ${}^{\omega}\omega$-bounding forcing notion preserves Q-filters.

Proof. If $\mathbb{P}$ is an ${}^{\omega}\omega$-bounding forcing notion, $\mathcal{F}$ a Q-filter in V, then we use part (2) of Lemma 20 for $f\in V\cap{}^{\omega}\omega$ dominating function $g\in V^{\mathbb{P}}\cap{}^{\omega}\omega$. $\square$

Definition 22. An independent family $\mathcal{A}$ is said to be selective if it is densely maximal and $\mathrm{fil}(\mathcal{A})$ is Ramsey.

Selective independent families exist under CH (see [12, 2]).

6. Indestructibility

Theorem 23. (CH) *Let $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\beta\colon\alpha\le\omega_2,\beta<\omega_2\rangle$ be a countable support iteration such that for each α, $\mathbb{Q}_\alpha=\mathbb{Q}(\mathcal{C}_\alpha)$ for some partition $\mathcal{C}_\alpha$ of 2^ω. If $\mathcal{A}$ is a selective independent family then ($\mathcal{A}$ is a selective independent family)$^{V^{\mathbb{P}_{\omega_2}}}$.*

Proof. We begin with a proof that $\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}$ has a property similar to being a happy family by A. Mathias [8], see [7] as well. Note that A. Mathias [8, Proposition 0.10] has shown that an ultrafilter $\mathcal{G}$ is Ramsey if and only if $\mathcal{G}$ is happy (see Proposition 11.7 in [7] as well).

Claim 1. In $V^{\mathbb{P}_\alpha}$, let $\{\mathcal{G}_n\}_{n\in\omega}$ be a sequence of finite subsets of $\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}$. There is $\{k(n)\colon n\in\omega\}\in\mathrm{fil}(\mathcal{A})\cap V$ such that

$$k(n+1)\in\bigcap\mathcal{G}_{k(n)}.$$

Proof. $\mathcal{G}$ is a p-set and therefore there is $C_0\in\mathcal{G}$ such that $C_0\subseteq^* G$ for each $G\in\bigcup\{\mathcal{G}_n\colon n\in\omega\}$. Thus, for some function $f\in{}^{\omega}\omega$

$$(\forall n\in\omega)\ C_0\setminus f(n)\subseteq\bigcap\mathcal{G}_n.$$

Since $\mathbb{P}_\alpha$ is ${}^{\omega}\omega$-bounding, without loss of generality $f\in V\cap{}^{\omega}\omega$, f is strictly increasing, and $n+2<f(n)$. Let us take $\{k(n)\colon n\in\omega\}\in\mathrm{fil}(\mathcal{A})\cap V$ from Lemma 20 such that $C=\{k(n+1)\colon n\in\omega\}\subseteq C_0$. Hence, we have $k(n+1)\in C_0\setminus f(k(n))$, and so

$$k(n+1)\in\bigcap\mathcal{G}_{k(n)}.$$

$\square$

We will prove by induction on $\alpha\le\omega_2$ that the family $\mathcal{A}$ remains densely maximal in $V^{\mathbb{P}_\alpha}$. Suppose first that α is a limit. Note that for each $\beta\le\alpha$

$$(\mathrm{fil}(\mathcal{A}))^{V^{\mathbb{P}_\beta}}=\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}.$$

Now, suppose for each $\beta<\alpha$

$$V^{\mathbb{P}_\beta}\models\mathcal{A}\text{ is densely maximal.}$$

That is $V^{\mathbb{P}_\beta} \vDash \operatorname{fil}(\mathcal{A}) \cup \langle\{\omega\backslash\mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}} = \mathcal{P}(\omega)$, i.e.

$$V^{\mathbb{P}_\beta} \vDash \langle\operatorname{fil}(\mathcal{A}) \cap V\rangle_{\mathrm{up}} \cup \langle\{\omega\backslash\mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}} = \mathcal{P}(\omega).$$

However, by Shelah's preservation theorem

$$V^{\mathbb{P}_\alpha} \vDash \langle\operatorname{fil}(\mathcal{A}) \cap V\rangle_{\mathrm{up}} \cup \langle\{\omega\backslash\mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}} = \mathcal{P}(\omega).$$

Thus $\mathcal{A}$ remains densely maximal in $V^{\mathbb{P}_\alpha}$.

Suppose $V^{\mathbb{P}_\alpha} \vDash \mathcal{A}$ is densely maximal. We will show that

$$V^{\mathbb{P}_{\alpha+1}} \vDash \mathcal{A} \text{ is densely maximal.}$$

In $V^{\mathbb{P}_{\alpha+1}}$, take any $Y \in \mathcal{P}(\omega)\backslash\langle\operatorname{fil}(\mathcal{A}) \cap V\rangle_{\mathrm{up}}$. Suppose $Y \notin \langle\{\omega\backslash\mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}}$. Thus, for all $h \in \mathrm{FF}(\mathcal{A})$, $Y \not\subseteq \omega\backslash\mathcal{A}^h$ and so for all $h \in \mathrm{FF}(\mathcal{A})$, $|Y \cap \mathcal{A}^h| = \omega$. Therefore in $V^{\mathbb{P}_\alpha}$ we can fix $p \in \mathbb{Q}_\alpha$ and a $\mathbb{Q}_\alpha$-name $\dot{Y}$ for Y such that for all $h \in \mathrm{FF}(\mathcal{A})$,

$$p \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty.$$

By Lemma 13 we can assume that for all $m \in \omega$, for all $t \in \operatorname{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that

$$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t.$$

Now, in $V^{\mathbb{P}_\alpha}$ for each $t \in p$, let

$$Y_t = \{m \in \omega : p(t) \nVdash \check{m} \notin \dot{Y}\}.$$

Claim 2. (i) $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$.
(ii) If $s \subseteq t$ then $Y_t \subseteq Y_s$.
(iii) $Y_t \in \operatorname{fil}(\mathcal{A}) \cap V^{\mathbb{P}_\alpha}$.
(iv) If $m \in Y_s$ for $s \in \operatorname{split}_n(p)$, and $n < m$ then there is $t \in \operatorname{split}_m(p)$ extending s such that $p(t) \Vdash \check{m} \in \dot{Y}$.

Proof. (i) Let $m \in \dot{Y}[G]$ for a generic G containing $p(t)$. If $p(t) \Vdash \check{m} \notin \dot{Y}$ then $m \notin \dot{Y}[G]$, a contradiction.

(ii) Since $p(t) \subseteq p(s)$, from $p(t) \nVdash \check{m} \notin \dot{Y}$ we obtain $p(s) \nVdash \check{m} \notin \dot{Y}$.

(iii) If $Y_t \notin \operatorname{fil}(\mathcal{A}) \cap V^{\mathbb{P}_\alpha}$ then there is $h \in \mathrm{FF}(\mathcal{A})$ such that $Y_t \subseteq \omega\backslash\mathcal{A}^h$, i.e. $Y_t \cap \mathcal{A}^h = \emptyset$. Since $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$, then $p(t) \Vdash \mathcal{A}^h \cap \dot{Y} = \emptyset$. However, $p(t) \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty$, which is a contradiction.

(iv) Since $p(s) \nVdash \check{m} \notin \dot{Y}$ there is a condition $q \leq p(s)$ such that $q \Vdash \check{m} \in \dot{Y}$. However, by our assumption on p due to Lemma 13, for any $t \in \operatorname{split}_m(p)$ we have either $p(t) \Vdash \check{m} \in \dot{Y}$ or $p(t) \Vdash \check{m} \notin \dot{Y}$. Since $\{p(t) : t \in \operatorname{split}_m(p), t \supseteq s\}$ is pre-dense in $p(s)$, there is $t \in \operatorname{split}_m(p)$ extending s such that $p(t) \Vdash \check{m} \in \dot{Y}$. □

Claim 3. We can assume that a dense set $X \subseteq [p]$ with $\mathcal{C}$-different elements has the associated family $\{y_x : x \in X\}$ of sets in $\operatorname{fil}(\mathcal{A})$ such that if $t = x \restriction \operatorname{split}_n(p)$ then

$$p(t) \Vdash y_x(n) \in \dot{Y}.$$

Proof. Hence, $Y_t \in \langle \mathrm{fil}(\mathcal{A}) \cap V\rangle_{\mathrm{up}}$ for each $t \in \mathrm{split}(p)$. By Claim 1 for $\mathcal{G}_n$ being the family of all Y_t's with $t \in \mathrm{split}_{\leq n+2}(p)$, we obtain $\{k(n) \colon n \in \omega\} \in \mathrm{fil}(\mathcal{A})$ such that

$$k(n+1) \in \bigcap\{Y_t \colon t \in \mathrm{split}_{\leq k(n)+2}(p)\}.$$

Moreover, by part (4) of Claim 2, for any $s \in \mathrm{split}_{k(n)+1}(p)$ there is $t \in \mathrm{split}_{k(n+1)}(p)$ extending s such that $p(t) \Vdash \check{k}(n+1) \in \dot{Y}$. For each branch $x \in [p]$ we consider set

$$i(x) = \{i \colon p(t) \Vdash \check{k}(i+1) \in \dot{Y} \text{ for } t = x \restriction \mathrm{split}_{k(i+1)}(p)\}.$$

We say that $x \in [p]$ is acceptable branch if $i(x)$ is cofinite. The smallest n with $i(x) \supseteq [n, \infty)$ is called a degree of acceptability of x. Due to part (4) of Claim 2 there are acceptable branches extending each $s \in p$. Note that for each acceptable branch x, $y_x = \{k(i+1) \colon i \in i(x)\} \in \mathrm{fil}(\mathcal{A})$. We continue using a fusion argument. We build a fusion sequence $\{(p_n, X_n)\}_{n\in\omega}$.

To define p_0, take some acceptable branch x extending some node in $\mathrm{split}_{k(0)+1}(p)$ with degree of acceptability at most 1, and a node $s = x \restriction \mathrm{split}_{k(1)}(p)$. We set $p_0 = p(s)$ and $X_0 = \{x\}$.

Let us assume that p_n and X_n are defined, and consider $s \in \mathrm{split}_{k(n)}(p) \cap \mathrm{split}(p_n)$. Take the unique acceptable branch $x \in X_n$ extending s. We set $q(s,i) = q_n(r)$. Define $i = x(|s|) \in \{0,1\}$ and $s_i = x \restriction \mathrm{split}_{k(n+1)}(p)$. Then we set s_{1-i} to be an extension of $s^\frown\langle 1-i\rangle$ such that:

- (i) $[p(s_{1-i})] \cap C_{\alpha_x} = \emptyset$ for all already considered acceptable branches x (i.e., all branches in X_n and those assigned to previous nodes in some order of $\mathrm{split}_{k(n)}(p) \cap \mathrm{split}(p_n)$).
- (ii) $s_{1-i} \in \mathrm{split}_{k(m+1)}(p)$ with $m \in i(x)$ for some acceptable branch $x \in [p]$ with degree of acceptability at most m.

Finally, let X_{n+1} be the set of all considered acceptable branches in this step, and

$$p_{n+1} = \bigcup\{p(s_i) \colon s \in \mathrm{split}_{k(n)}(p) \cap \mathrm{split}(p_n), i \in \{0,1\}\}.$$

One can see that the sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses. Moreover, the fusion $q = \bigcap\{p_n \colon n \in \omega\}$ satisfies the requirements. We set $X = \bigcup\{X_n \colon n \in \omega\}$.

We shall show that the family $\{y_x \colon x \in X\}$ possesses the desired properties. Indeed, let $x \in X$. For each $n \in \omega$ we have $y_x(n) = k(i(x)(n)+1)$. Due to construction of q we have $x \restriction \mathrm{split}_n(q) = x \restriction \mathrm{split}_{k(j_n+1)}(p)$ for some increasing sequence $\{j_i\}_{i\in\omega}$, and if $t = x \restriction \mathrm{split}_n(q)$ then $p(t) \Vdash \check{k}(j_n+1) \in \dot{Y}$. Thus $k(j_n+1) \in y_x$ and consequently $j_n \geq i(x)(n)$ for each n. Let us now fix n and consider $t = x \restriction \mathrm{split}_n(q)$. The definition of y_x guaranties that $p(s) \Vdash \check{k}(i(x)(n)+1) \in \dot{Y}$ for $s = x \restriction \mathrm{split}_{k(i(x)(n)+1)}(p)$. Thus we have $q(s) \Vdash \check{y}_x(n) \in \dot{Y}$. On the other hand, $s = x \restriction \mathrm{split}_{k(i(x)(n)+1)}(p) \subseteq x \restriction \mathrm{split}_{k(j_n+1)}(p) = x \restriction \mathrm{split}_n(q) = t$. □

Our last part of the proof resembles the proof of previous claim. Let x_s for $s \in \mathrm{split}(p)$ be the branch in X extending s such that if $s \subseteq t \subseteq x_s$ then $x_t = x_s$. The corresponding y_{x_s} is denoted y_s. The set y_s belongs to $\langle \mathrm{fil}(\mathcal{A}) \cap V\rangle_{\mathrm{up}}$. By Claim 1 for $\mathcal{G}_n$ being the family of all y_t's with $t \in \mathrm{split}_{\leq n+2}(p)$, we obtain $\{l(n) \colon n \in \omega\} \in \mathrm{fil}(\mathcal{A})$ such that

$$l(n+1) \in \bigcap\{y_t \colon t \in \mathrm{split}_{\leq l(n)+2}(p)\}.$$

Let us denote $C = \{l(n+1)\colon n \in \omega\}$. We shall construct a condition $q \le p$ such that $q \Vdash \check{C} \subseteq \dot{Y}$. Then $q \Vdash \dot{X} \in \mathrm{fil}(\mathcal{A})$ which is a contradiction.

We build a fusion sequence $\{(p_n, X_n)\}_{n\in\omega}$. Let $p_0 = p$, $X_0 = \{x_t\}$ for $t \in \mathrm{split}_0(p)$, and suppose we have defined p_n. For each $t \in \mathrm{split}_n(p_n)$ and each $i \in \{0,1\}$ take $w^*(t,i) \in \mathrm{split}_{l(n)+1}(p)$ such that $w^*(t,i)$ end-extends $t^\frown i$. Then

$$l(n+1) \in \bigcap\{y_{w^*(t,i)}\colon t \in \mathrm{split}_n(p_n), i \in \{0,1\}\}$$

and so for each t, i we take $w(t,i) = x_{w^*(t,i)} \restriction \mathrm{split}_{l(n+1)}(p)$. Note that by Claim 3 and the fact that $l(n+1) \ge j$ for $l(n+1) = y_{w^*(t,i)}(j)$ we obtain

$$p(w(t,i)) \Vdash \check{l}(n+1) \in \dot{Y}.$$

Take $p_{n+1} = \bigcup\{p(w(t,i))\colon t \in \mathrm{split}_n(p_n), i \in \{0,1\}\}$ and $X_{n+1} = \{x_{w(t,i)}\colon t \in \mathrm{split}_n(p_n), i \in \{0,1\}\}$. □

Theorem 24. *The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves* P*-points and Ramsey ultrafilters.*

Proof. We prove just first part. The second claim follows from the first one and the fact that the forcing notion $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding, see [7, Lemma 21.12]. Note that a family $\mathcal{G}$ generates an ultrafilter on ω if and only if $\mathcal{P}(\omega) = \langle\mathcal{G}\rangle_{\mathrm{up}} \cup \langle\mathcal{G}^*\rangle_{\mathrm{dn}}$.

Let $\mathcal{U}$ be an ultrafilter in V. We shall prove that the family $\mathcal{U}$ generates an ultrafilter in $V^{\mathbb{Q}(\mathcal{C})}$, i.e., $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{P}(\omega) = \langle\mathcal{U}\rangle_{\mathrm{up}} \cup \langle\mathcal{U}^*\rangle_{\mathrm{dn}}$. In $V^{\mathbb{Q}(\mathcal{C})}$, take any set in $\mathcal{P}(\omega)$. We fix $p \in \mathbb{Q}(\mathcal{C})$ and a $\mathbb{Q}(\mathcal{C})$-name $\dot{Y}$ such that $p \Vdash \dot{Y} \subseteq \omega$. By Lemma 13 we can assume that for all $m \in \omega$, for all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that

$$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t.$$

Note that the latter property remains true for any stronger condition q, since t in the m-th level of q is an extension of some s in the m-th level of p. Let $\{x_t\colon t \in p\} \subseteq [p]$ be a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$). We set $Y_t = \bigcup\{u_s\colon s \subseteq x_t\}$.

Claim. We can assume that the set $\mathcal{Y}_0 = \{Y_s\colon s \in p\}$ is in $\mathcal{U}$ or the set $\mathcal{Y}_1 = \{\omega \setminus Y_s\colon s \in p\}$ is in $\mathcal{U}$.

Proof. We set $U_0 = \{s \in p\colon Y_s \in \mathcal{U}\}$ and $U_1 = \{s \in p\colon (\omega \setminus Y_s) \in \mathcal{U}\}$. The sets U_0, U_1 are disjoint and their union is p. We may distinguish two cases:

(i) There is $s \in p$ such that $p(s) \subseteq U_0$. In this case, just take $p(s)$.

(ii) For each $s \in p$ there is $t \in p(s)$ such that $t \in U_1$. We build a fusion sequence $\{(p_n, X_n)\}_{n\in\omega}$ such that the fusion has the required properties. Taking $s \in \mathrm{split}_0(p)$ there is $t \in p(s)$ such that $t \in U_1$. We set $p_0 = p(t)$ and $X_0 = \{x_t\}$.

Let $0 \le n < \omega$. Suppose we have defined $p_n \in \mathbb{Q}_\alpha$ and finite $X_n \subseteq [p_n]$. Let $s \in \mathrm{split}_n(p_n)$. Take node $r = x_s \restriction \mathrm{split}_{n+1}(p_n)$, and number $i = x_s(|s|)$ in $\{0,1\}$. We set $p(s,i) = p_n(r)$. Let $t \supseteq s^\frown\langle 1-i\rangle$ be splitting such that $t \in U_1$. We set $p(s, 1-i) = p(t)$.

Finally, let

$$p_{n+1} = \bigcup\{p(s,i)\colon s \in \operatorname{split}_n(p_n), i \in \{0,1\}\}.$$

and let X_{n+1} be the set of all x_t's for $t \in \operatorname{split}_{n+1}(p_{n+1})$. One can verify that the sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses.

□

We assume that $\mathcal{Y}_0 \in \mathcal{U}$, the other case may be handled analogously. We take a pseudointersection Z of $\mathcal{Y}_0$ in $\mathcal{U}$, with $Z \subseteq Y_\emptyset$. We shall simultaneously build two fusion sequences with witnesses, namely $\{(p_n^0, X_n^0)\}_{n\in\omega}$, $\{(p_n^1, X_n^1)\}_{n\in\omega}$, and a partition of Z into two sets Z_0, Z_1 such that for their respective fusions $q_0, q_1 \leq p$ we obtain $q_0 \Vdash \check{Z}_0 \subseteq \dot{Y}$ and $q_1 \Vdash \check{Z}_1 \subseteq \dot{Y}$.

Let $p^0 = p^1 = p$, $X_0^0 = X_0^1 = \{Y_\emptyset\}$, and $k_0 = 0$, $k_1 = 2$. We assume that p_n^0, p_n^1, k_{2n}, and k_{2n+1} are constructed. Let $t \in \operatorname{split}_{k_{2n}}(p) \cap \operatorname{split}(p_n^0)$, and set $w^*(t) = x_t \restriction \operatorname{split}_{k_{2n+1}}(p)$. For each $i \in \{0,1\}$, we take $w^*(t,i) \in \operatorname{split}_{k_{2n+1}+1}(p)$ extending $w^*(t)^\frown i$. There is $k_{2n+2} > k_{2n+1}+1$ such that

$$Z \setminus k_{2n+2} \subseteq \bigcap\{Y_{w^*(t,i)}\colon t \in \operatorname{split}_{k_{2n}}(p) \cap \operatorname{split}(p_n^0), i \in \{0,1\}\}.$$

We set $w(t,i) = x_{w^*(t,i)} \restriction \operatorname{split}_{k_{2n+2}}(p)$. Take $p_{n+1} = \bigcup\{p(w(t,i))\colon t \in \operatorname{split}_{k_{2n}}(p) \cap \operatorname{split}(p_n^0), i \in \{0,1\}\}$ and $X_{n+1} = \{x_{w(t,i)}\colon t \in \operatorname{split}_{k_{2n}}(p) \cap \operatorname{split}(p_n^0), i \in \{0,1\}\}$. One can see that $p_n^0 \Vdash \check{Z} \cap [k_{2n}, k_{2n+1}) \subseteq \dot{Y}$. The construction of condition p_n^1 and the choice of number k_{2n+3} are done similarly, and leads to $p_n^1 \Vdash \check{Z} \cap [k_{2n+1}, k_{2n+2}) \subseteq \dot{Y}$. Finally, we define

$$Z_0 = Z \cap \bigcup\{[k_{2n}, k_{2n+1})\colon n \in \omega\} \text{ and } Z_1 = Z \cap \bigcup\{[k_{2n+1}, k_{2n+2})\colon n \in \omega\}.$$

□

One of the most interesting open questions regarding the independence number is the consistency of $\mathfrak{i} < \mathfrak{a}$. Relying on the above preservation theorem, we obtain the consistency of $\mathfrak{i} < \mathfrak{a}_T$ where $\mathfrak{a}_T$ is the least cardinality of a maximal almost disjoint family of finitely branching subtrees of $2^{<\omega}$. Miller [9] showed that $\mathfrak{a}_T$ is the least cardinality of a partition of ω^ω into compact sets. Let us recall that in the $\mathbb{Q}(\mathcal{C})$-generic extension, $\mathcal{C}$ is no longer a partition of 2^ω.

Theorem 25. *Assume* CH. *There is a cardinals preserving generic extension in which*

$$\operatorname{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \omega_1 < \mathfrak{a}_T = \omega_2.$$

Proof. Let V denote the ground model. We assume that $\mathcal{A}$ is a selective independent family in V, $\mathcal{U}$ is a P-point in V, and $\mathcal{E}$ is a tight `MAD` family in V (according to [6]). Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta\colon \alpha \leq \omega_2, \beta < \omega_2\rangle$ of posets such that for each α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}(\mathcal{C})$ for some uncountable partition of 2^ω into compact sets and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \omega_2$. $\mathbb{P}_{\omega_2}$ is ${}^\omega\omega$-bounding, and therefore $\operatorname{cof}(\mathcal{N}) = \omega_1$. By Theorem 23, the family $\mathcal{A}$ remains maximal independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. Similarly, $\mathcal{U}$ generates a P-point in $V^{\mathbb{P}_{\omega_2}}$, so $\mathfrak{u} = \omega_1$ as well. And finally, $\mathfrak{a} = \omega_1$ since $\mathcal{E}$ is a tight `MAD` family (see [6]). □

For a maximal ideal $\mathcal{I}$ on ω, the forcing notion $\mathbb{Q}_{\mathcal{I}}$ is the one used by S. Shelah [12]. He has shown that $\mathbb{Q}_{\mathcal{I}}$ is proper, ${}^{\omega}\omega$-bounding and even has a Sacks property. In the $\mathbb{Q}_{\mathcal{I}}$-generic extension, $\mathcal{I}$ is no longer a maximal ideal. The assumptions of the next theorem are satisfied in the constructible universe.

Theorem 26. *Assume $2^{\omega} = \omega_1$, $2^{\omega_1} = \omega_2$, and $\diamondsuit_{(\delta<\omega_2:\mathrm{cf}(\delta)=\omega_1)}$. There is a cardinals preserving generic extension in which*

$$\mathrm{cof}(\mathcal{N}) = \mathfrak{i} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2.$$

Proof. Let V denote the ground model. We assume that $\mathcal{A}$ is a selective independent family in V. Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2\rangle$ of posets such that for even α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}(\mathcal{C})$ for some uncountable partition of 2^ω into compact sets, for odd α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}_{\mathcal{I}}$ for some maximal ideal on ω, and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \mathfrak{u} = \omega_2$. $\mathbb{P}_{\omega_2}$ is ${}^{\omega}\omega$-bounding, and therefore $\mathrm{cof}(\mathcal{N}) = \omega_1$. By Theorem 23, the family $\mathcal{A}$ remains maximal independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. $\square$

7. Questions

The poset $\mathbb{Q}(\mathcal{C})$ satisfies Axiom A. We recall that it is ${}^{\omega}\omega$-bounding and has the Sacks property (see [9, 16]). By our Theorem 24, $\mathbb{Q}(\mathcal{C})$ preserves P-points and Ramsey ultrafilters. Furthermore, the countable support iteration of $\mathbb{Q}(\mathcal{C})$ preserves selective independent families, see our Theorem 23. Finally, it preserves tight `MAD` families (see [6]).

Question 27. Does the forcing notion $\mathbb{Q}(\mathcal{C})$ preserve Q-points?

We know the only **ZFC** bound for $\mathfrak{a}_T$, namely $\mathfrak{d} \leq \mathfrak{a}_T$ (and its consequences). It is known that $\mathfrak{a}_T = \omega_1$ in Sacks and random real models. Together with our result we obtain that $\mathfrak{a}_T$ is independent on $\mathfrak{i}$, $\mathfrak{u}$, $\mathfrak{r}$, $\mathrm{cof}(\mathcal{M})$, $\mathrm{cof}(\mathcal{N})$, $\mathrm{non}(\mathcal{M})$, and $\mathrm{cov}(\mathcal{N})$. However, we do not know about $\mathfrak{a}$ and $\mathrm{non}(\mathcal{N})$ (see [6] as well).

Question 28. Is any of the inequalities $\mathfrak{a} \leq \mathfrak{a}_T$ or $\mathrm{non}(\mathcal{N}) \leq \mathfrak{a}_T$ provable in **ZFC**?

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