Template iterations and maximal cofinitary groups

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- $\text{cofin}(S_\infty)$ is the set of cofinitary permutations in S_∞, i.e. permutations $\sigma \in S_\infty$ which have finitely many fixed points.
- A mapping $\rho : A \to S_\infty$ *induces* a cofinitary representation of $\mathbb{F}_A$ if the canonical extension of ρ to a homomorphism $\hat{\rho} : \mathbb{F}_A \to S_\infty$ is such that $\text{im}(\hat{\rho}) \subseteq \{I\} \cup \text{cofin}(S_\infty)$.

Evaluations

Let A be a set, $s \subseteq A \times \omega \times \omega$. For $a \in A$, let $s_a = \{(n, m) \in \omega \times \omega : (a, n, m) \in s\}$. For a word $w \in W_A$, define $e_w[s] \subseteq \omega \times \omega$ recursively as follows:

- if $w = a$ then $(n, m) \in e_w[s]$ iff $(n, m) \in s_a$,
- if $w = a^{-1}$ then $(n, m) \in e_w[s]$ iff $(m, n) \in s_a$, and
- if $w = a^i u$ for some $u \in W_A$ and $i \in \{1, -1\}$ without cancelation then

$$(n, m) \in e_w[s] \iff (\exists k) e_{a^i}[s](k, m) \wedge e_u[s](n, k).$$

- If s_a is a partial injection for all a, then $e_w[s]$ is a partial injection.
- We refer to $e_w[s]$ as the *evaluation* of w given s.
- By definition we let $e_\emptyset[s, \rho]$ be the identity in S_∞.

Let A, X be disjoint and let $\rho : X \to S_\infty$ be a function. For a word $w \in W_{A \cup X}$ and $s \subseteq A \times \omega \times \omega$, define

$$(n, m) \in e_w[s, \rho] \text{ iff } (n, m) \in e_w[s \cup \{(x, k, l) : \rho(x)(k) = l\}].$$

If s_a is a partial injection for a, then $e_w[s, \rho]$ is also a partial injection, referred to as the *evaluation* of w given s and ρ.

Forcing M.c.g.'s

Let A, X be disjoint non-empty sets and let $\rho : X \to S_\infty$ induce a cofinitary representation. Then $\mathbb{Q}_{A,\rho}$ is the poset of all (s, F) where $s \subseteq A \times \omega \times \omega$ is finite, s_a is a finite injection for all a and $F \subseteq \widehat{W}_{A \cup X}$ is finite. Define $(s, F) \leq_{\mathbb{Q}_{A,\rho}} (t, E)$ iff

- $s \supseteq t$, $F \supseteq E$ and,
- for all $n \in \omega$ and $w \in E$, if $e_w[s, \rho](n) = n$ then already $e_w[t, \rho](n)\downarrow$ and $e_w[t, \rho](n) = n$.

If $X = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$. If A is clear from the context we just write $\mathbb{Q}$.

- $\mathbb{Q}_{A,\rho}$ is Knaster.
- Let G be $\mathbb{Q}_{A,\rho}$ generic and let $\rho_G : A \cup X \to S_\infty$ be a mapping extending ρ and such that for all $a \in A$

$$\rho_G(a) = \bigcup \{s_a : (\exists F \in \widehat{W}_{A\cup X})\ (s, F) \in G\}.$$

Then ρ_G induces a cofinitary representation of $A \cup X$ extending ρ.

Lemma: No new fixed points

Let A and B be disjoint set and $\rho : B \to S_\infty$ a function inducing a cofinitary representation of $\mathbb{F}_B$. Then

- ("Domain extension") For any $(s, F) \in \mathbb{Q}_{A,\rho}$, $a \in A$ and $n \in \omega$ such that $n \notin \operatorname{dom}(s_a)$ there are cofinitely many $m \in \omega$ s.t. $(s \cup \{(a, n, m)\}, F) \leq (s, F)$.
- ("Range extension") For any $(s, F) \in \mathbb{Q}_{A,\rho}$, $a \in A$ and $m \in \omega$ such that $m \notin \operatorname{ran}(s_a)$ there are cofinitely many $n \in \omega$ s.t. $(s \cup \{(a, n, m)\}, F) \leq (s, F)$.

Definition: a-good words

Let $a \in A$ and $j \geq 1$. A word $w \in W_{A \cup X}$ is called *a-good* of *rank j* if it has the form

$$w = a^{k_j} u_j a^{k_{j-1}} u_{j-1} \cdots a^{k_1} u_1 \tag{1}$$

where $u_i \in W_{A \setminus \{a\} \cup X} \setminus \{\emptyset\}$ and $k_i \in \mathbb{Z} \setminus \{0\}$, for $1 \leq i \leq j$.

Lemma: Evaluations again

Let $w \in \widehat{W}_{A\cup B}$ and $(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_G](n) = m$ for some $n, m \in \omega$. Then $e_w[s, \rho](n)\downarrow$ and $e_w[s, \rho](n) = m$.

Proof:

By induction on $|\text{oc}(w) \cap A|$. If no letter from A occurs, the statement is true by definition of ρ_G. So suppose the claim holds for words with at most k letters from A and let w be such that $|oc(w) \cap A| = k + 1$. For a contradiction, assume $e_w[s, \rho](n)\uparrow$, but $(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_G](n) = m$. The word w can be written $w = w_1 w_0$ without cancelation where w_0 is a-good and a does not occur in w_1.

We can find $s_1 \subseteq \{a\} \times \omega \times \omega$ finite such that $(s \cup s_1, F) \leq (s, F)$ and such that $n_1 = e_{w_0}[s \cup s_1, \rho](n) \neq e_{w_1}[s, \rho]^{-1}(m)$ if it is defined. Since

$$(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_{\dot{G}}](n) = m$$

and $(s \cup s_1, F) \Vdash e_{w_0}[\rho_{\dot{G}}](n) = n_1$ we must have

$$(s \cup s_1, F) \Vdash e_{w_1}[\rho_{\dot{G}}](n_1) = m.$$

By inductive hypothesis $e_{w_1}[s \cup s_1, \rho](n_1) = m$. Since $a \notin \mathrm{oc}(w_1)$ we have $e_{w_1}[s, \rho](n_1) = m$, contradicting the choice of n_1. □

Proposition

Let G be $\mathbb{Q}_{A,\rho}$-generic. Then $\rho_G : A \cup B \to S_\infty$ induces a cofinitary representation $\hat{\rho}_G : \mathbb{F}_{A\cup B} \to S_\infty$ such that $\hat{\rho}_G \restriction \mathbb{F}_B = \hat{\rho}$.

Proof:

For each $a \in A$, $n \in \omega$, let
$D_{a,n} = \{(s, F) \in \mathbb{Q}_{A,\rho} : (\exists m)(a, n, m) \in s\}$ and
$R_{a,n} = \{(s, F) \in \mathbb{Q}_{A,\rho} : (\exists m)(a, m, n) \in s\}$. For $w \in \widehat{W}_{A\cup B}$, let
$D_w = \{(s, F) \in \mathbb{Q}_{A,\rho} : w \in F\}$. Then D_w, $D_{a,n}$ and $R_{a,n}$ are dense and so $\rho_G : A \cup B \to S_\infty$ is a function as promised.

It remains to see that ρ_G induces a cofinitary representation. Let $w \in W_{A \cup B}$. There are $w' \in \widehat{W}_{A \cup B}$, $u \in W_{A \cup B}$ such that $w = u^{-1} w' u$. Since $D_{w'}$ is dense $\exists (s, F) \in G$ such that $w' \in F$. Suppose then $e_{w'}[\rho_G](n) = n$. Then there is some $(t, E) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ in G forcing this. But then $e_{w'}[t, \rho](n) = n$ and so by definition $e_{w'}[s, \rho](n) = n$. Thus

$$\mathrm{fix}(e_{w'}[\rho_G]) = \mathrm{fix}(e_{w'}[s, \rho]),$$

which is finite. Finally, $\mathrm{fix}(e_w[\rho_G]) = e_u[\rho_G]^{-1}(\mathrm{fix}(e_{w'}[\rho_G]))$, so $\mathrm{fix}(e_w[\rho_G])$ is finite. $\square$

Notation:

For $s \subseteq A \times \omega \times \omega$ and $A_0 \subseteq A$, write $s{\restriction}A_0$ for $s \cap A_0 \times \omega \times \omega$. For a condition $p = (s, F) \in \mathbb{Q}_{A,\rho}$ we will write $p{\restriction}A_0$ for $(s{\restriction}A_0, F)$, and $p \Vert\!\restriction A_0$ ("strong restriction") for $(s{\restriction}A_0, F \cap \widehat{W}_{A_0 \cup B})$.

Lemma: Strong embeddings

Let $A_0 \subset A$, $A_1 = A \backslash A_0$, $p = (s, F) \in \mathbb{Q}_{A,\rho}$. Then there is $t_0 \subseteq (\mathrm{oc}(s) \cap A_0) \times \omega \times \omega$ extending $s{\restriction}A_0$ such that

- $(t_0, F \cap \widehat{W}_{A_0 \cup B}) \leq_{\mathbb{Q}_{A_0,\rho}} p \Vert\!\restriction A_0$ and
- whenever $(t, E) \leq_{\mathbb{Q}_{A_0,\rho}} (t_0, F \cap \widehat{W}_{A_0 \cup B})$ then $(s \cup t, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$, and so $(s \cup t, F \cup E)$ is a common extension of (t, E) and (s, F).

Proof:

Let $\{w_1, \dots, w_n\} = F \setminus W_{A_0 \cup B}$. Then $w_i = u_{i,k_i} v_{i,k_i} \cdots u_{i,1} v_{i,1} u_{i,0}$ where $u_{i,j} \in W_{A_0 \cup B}$ and $v_{i,j} \in W_{A_1 \cup B}$ are non-$\emptyset$ except possibly u_{i,k_i}, $u_{i,0}$, each $v_{i,j}$ starts and ends with a letter from A_1. There is $t \subseteq A_0 \times \omega \times \omega$ such that $t_0 \supseteq s \restriction A_0$ and

- $\mathrm{dom}(e_{u_{i,j}}[s \cup t, \rho]) \supseteq \mathrm{ran}(e_{v_{i,j}}[s, \rho])$,
- $\mathrm{ran}(e_{u_{i,j}}[s \cup t_0, \rho] \supseteq \mathrm{dom}(e_{v_{i,j+1}}[s, \rho])$, and
- $(s \cup t_0, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$.

Let $(t, E) \leq_{\mathbb{Q}_{A_0,\rho}} (t_0, F \cap \widehat{W}_{A_0 \cup B})$. If $e_{w_i}[s \cup t, \rho](n)\downarrow$, then by definition of t_0 we have $e_{w_i}[s \cup t_0, \rho](n)\downarrow$. Therefore if $e_{w_i}[s \cup t, \rho](n) = n$ we have $e_{w_i}[s \cup t_0, \rho](n) = n$, and so since $(s \cup t_0, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ it follows $e_{w_i}[s, \rho](n) = n$. Thus $(s \cup t, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ as required. $\square$

Lemma: Strong Embedding

Let $B, C \subseteq D$, $B \cap C = A$ be given set and $p \in \mathbb{Q}_{B,\rho}$. Then there is a condition $p_0 \in \mathbb{Q}_{A,\rho}$ such that whenever $q_0 \leq_{\mathbb{Q}_{C,\rho}} p_0$, then q_0 is compatible in $\mathbb{Q}_{D,\rho}$ with p.

We say that $\mathbb{Q}_{B,\rho}$ has the strong embedding property and q_0 is called a strong reduction of p. If $C = A$, $B = D$ then the above gives in particular that $\mathbb{Q}_{A,\rho}$ is a complete suborder of $\mathbb{Q}_{B,\rho}$.

Lemma: Quotients

Let $A_0 \cap A_1 = \emptyset$, $A = A_0 \cup A_1$. Let G be $\mathbb{Q}_{A,\rho}$-generic, $H = G \cap \mathbb{Q}_{A_0,\rho}$. Then $K = \{(s\restriction A_1, F) : (s, F) \in G\}$ is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$ and $\rho_G = (\rho_H)_K$.

Proof:

Let $D \subseteq \mathbb{Q}_{A_1,\rho_H}$ be dense, $D \in V[H]$. Define $D' = \{p \in \mathbb{Q}_{A,\rho} : p \| \restriction A_0 \Vdash_{\mathbb{Q}_{A_0,\rho}} p \restriction A_1 \in \dot{D}\}$ and let $p_0 \in H$ forces "D is dense". We claim that D' is dense below p_0 (in $\mathbb{Q}_{A,\rho}$.) Let $(s,F) = p \leq_{\mathbb{Q}_{A,\rho}} p_0$. There is $p_0 \leq_{\mathbb{Q}_{A_0,\rho}} p \| \restriction A_0$ such that for any $p_1 \leq_{\mathbb{Q}_{A_0,\rho}} p_0$, p_1 is compatible with p. Thus we can find $q = (s_0, F_0) \in \mathbb{Q}_{A_1,\rho_H}$ and $(t,E) \leq_{\mathbb{Q}_{A_0,\rho}} p_0$ such that

$$(t,E) \Vdash_{\mathbb{Q}_{A_0,\rho}} \dot{q} \in \dot{D} \wedge \dot{q} \leq_{\mathbb{Q}_{A_1,\rho_{\dot{H}}}} \dot{p} \restriction A_1.$$

But then $(s_0 \cup t, F_0) \leq_{\mathbb{Q}_{A,\rho}} (s \restriction A_1 \cup t, F)$, and so $(s_0 \cup t, F_0 \cup E) \leq_{\mathbb{Q}_{A,\rho}} (s,F)$. Since clearly $(s_0 \cup t, F_0 \cup E) \in D'$, this shows that D' is dense below p_0. Now, since $p_0 \in G$ it follows that there is $q' \in D' \cap G$. In $V[H]$ it then holds that $q' \restriction A_1 \in D$, which shows that $K \cap D \neq \emptyset$. □

Theorem

Let $|A| > \aleph_0$ and G be a $\mathbb{Q}_{A,\rho}$-generic over V. Then $\mathrm{im}(\rho_G)$ is a maximal cofinitary group in $V[G]$.

Proof

Let $z \notin X \cup A$, where $\rho : X \to S_\infty$. Suppose there in $V[G]$ there is $\sigma \in \mathrm{cofin}(S_\infty)$ such that $\rho'_G : A \cup X \cup \{z\} \to S_\infty$ defined by $\rho'_G \restriction X \cup A = \rho_G$, $\rho'_G(z) = \sigma$ induces a cofinitary representation. Let $\dot\sigma$ be a name for σ. Then there is $A_0 \subseteq A$ countable so that $\dot\sigma$ is a $\mathbb{Q}_{A_0,\rho}$-name and so $\sigma \in V[H]$, where $H = G \cap \mathbb{Q}_{A_0,\rho}$.

Let $a_1 \in A \setminus A_0$ and let K be defined as in the previous Lemma. Note that for every $N \in \omega$

$$D_{\sigma,N} = \{(s,F) \in \mathbb{Q}_{A_1,\rho_H} : (\exists n \geq N) s_{a_1}(n) = \sigma(n)\}$$

is dense in $\mathbb{Q}_{A_1,\rho_H}$ and so in $V[H][K]$

$$\exists^\infty n((\rho_H)_K(a_1)(n) = \sigma(n)).$$

However $(\rho_H)_K = \rho_G$, which contradicts that ρ'_G induces a cofinitary representation. □

Definition: $\mathbb{L}$

$\mathbb{L}$ consists of pairs (σ, ϕ) such that $\sigma \in {}^{<\omega}({}^{<\omega}[\omega])$, $\phi \in {}^{\omega}({}^{<\omega}[\omega])$ such that $\sigma \subseteq \phi$, $\forall i < |\sigma|(|\sigma(i)| = i)$ and $\forall i \in \omega(|\phi(i)| \leq |\sigma|)$. The extension relation is defined as follows: $(\sigma, \phi) \leq (\tau, \psi)$ if and only if σ end-extends τ and $\forall i \in \omega\ (\psi(i) \subseteq \phi(i))$.

- A slalom is a function $\phi : \omega \to [\omega]^{<\omega}$ such that $\forall n \in \omega(|\phi(n)| \leq n)$. A slalom localizes a real $f \in {}^{\omega}\omega$ if there is $m \in \omega$ such that $\forall n \geq m(f(n) \in \phi(n))$.
- $\mathbb{L}$ adds a slalom which localizes all ground model reals.

- add($\mathcal{N}$) is the least cardinality of a family $F \subseteq \omega^\omega$ such that no slalom localizes all members of F
- cof($\mathcal{N}$) is the least cardinality of a family Φ of slaloms such that every real is localized by some $\phi \in \Phi$.
- $\mathfrak{a}_g \geq \text{non}(\mathcal{M})$.

In our intended forcing construction cofinally often we will force with the partial order $\mathbb{L}$, which using the above characterization will provide a lower bound for $\mathfrak{a}_g$.

Definition: σ-Suslin

Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion, $\mathbb{S} \subseteq {}^{<\omega}\omega \times {}^{\omega}\omega$. We say that $\mathbb{S}$ is *n-Suslin* if whenever $(s, f) \leq_{\mathbb{S}} (t, g)$ and (t, h) is a condition in $\mathbb{S}$ such that

$$h \restriction n \cdot |s| = g \restriction n \cdot |s|$$

then (s, f) and (t, h) are compatible. A forcing notion is called *σ-Suslin*, if it is n-Suslin for some n.

- If $\mathbb{S}$ is n-Suslin and $m \geq n$, than $\mathbb{S}$ is also m-Suslin.
- Every σ-Suslin forcing notion is σ-linked and so has the Knaster property.
- Hechler forcing $\mathbb{H}$ is 1-Suslin, localization $\mathbb{L}$ is 2-Suslin.

Definition: Nice name for a real

Let $\mathbb{B}$ be a partial order and $y \in \mathbb{B}$. For each $n \geq 1$ let $\mathcal{B}_n$ be a maximal antichain below y. We will say that the set $\{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ is *a nice name for a real below y* if

1. whenever $n \geq 1$, $b \in \mathcal{B}_n$ then $s(b) \in {}^n\omega$
2. whenever $m > n \geq 1$, $b \in \mathcal{B}_n$, $b' \in \mathcal{B}_m$ and b, b' are compatible, then $s(b)$ is an initial segment of $s(b')$.

We can assume that all names for reals are nice and abusing notation we will write $\dot{f} = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \in \omega}$.

Lemma: Canonical Projection of a name for a real

Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$, $y \in \mathbb{B}$ and x a reduction of y to $\mathbb{A}$. Let $\dot{f} = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ be a nice name for a real below y. Then there is $\dot{g} = \{(a, s(a))\}_{a \in \mathcal{A}_n, n \geq 1}$, a $\mathbb{A}$-nice name for a real below x, such that for all $n \geq 1$, for all $a \in \mathcal{A}_n$, there is $b \in \mathcal{B}_n$ such that a is a reduction of b and $s(a) = s(b)$.

Whenever $\dot{f}, \dot{g}$ are as above, we will say that $\dot{g}$ is a canonical projection of $\dot{f}$ below x.

Definition: Good Suslin

Let $\mathbb{S}$ be a Suslin forcing notion, $\mathbb{S} \subseteq {}^{<\omega}\omega \times {}^{\omega}\omega$. Then $\mathbb{S}$ is said to be *good* if whenever $\mathbb{A}$ is a complete suborder of $\mathbb{B}$, $x \in \mathbb{A}$ is a reduction of $y \in \mathbb{B}$ and $\dot{f}$ is a nice name for a real below y such that $y \Vdash_{\mathbb{B}} (\check{s}, \dot{f}) \in \dot{\mathbb{S}}$ for some $s \in {}^{<\omega}\omega$, there is a canonical projection $\dot{g}$ of $\dot{f}$ below x such that $x \Vdash (\check{s}, \dot{g}) \in \dot{\mathbb{S}}$.

$\mathbb{D}$ and $\mathbb{L}$ are good σ-Suslin forcing notions.

- Let $(L, \leq)$ be a linearly ordered set, $x \in L$. Then $L_x := \{y \in L : y < x\}$.
- If $L_0 \subseteq L$ and $A \subseteq L$, then define the L_0-*closure* of A as follows:

$$\mathrm{cl}_{L_0}(A) = A \cup \bigcup_{x \in A} L_x \cap L_0.$$

Definition: Template

A *template* is a tuple $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ where $L = L_0 \cup L_1$, $L_0 \cap L_1 = \emptyset$, $(L, \leq)$ is a linear order, $\mathcal{I} \subseteq \mathcal{P}(L)$, such that

- $\mathcal{I}$ is closed under finite intersections and unions, $\emptyset, L \in \mathcal{I}$.
- If $x, y \in L$, $y \in L_1$ and $x < y$ then $\exists A \in \mathcal{I}(A \subseteq L_y \wedge x \in A)$.
- If $A \in \mathcal{I}$, $x \in L_1 \backslash A$, then $A \cap L_x \in \mathcal{I}$.
- $\{A \cap L_1 : A \in \mathcal{I}\}$ is well-founded when ordered by inclusion.
- All $A \in \mathcal{I}$ are L_0-closed.

- Define $\mathrm{Dp} : \mathcal{I} \to \mathbb{ON}$ by letting $\mathrm{Dp}(A) = 0$ for $A \subseteq L_0$ and

$$\mathrm{Dp}(A) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathcal{I} \wedge B \cap L_1 \subset A \cap L_1\}.$$

  Let $\mathrm{Rk}(\mathcal{T}) = \mathrm{Dp}(L)$.
- For $A \subseteq L$ let

$$\mathcal{T}_A = ((A, \leq), \mathcal{I}{\restriction}A, L_0 \cap A, L_1 \cap A),$$

  where $\mathcal{I}{\restriction}A = \{A \cap B : B \in \mathcal{I}\}$. If $A \in \mathcal{I}$ then $\mathrm{Rk}(\mathcal{T}_A) = \mathrm{Dp}(A)$.
- For $x \in L$ let $\mathcal{I}_x = \{B \in \mathcal{I} : B \subseteq L_x\}$.

Definition: Iterating good σ-Suslin posets along a template and adding m.c.g.

Let $\mathbb{Q} = \mathbb{Q}_{L_0}$ the poset adding a m.c.g. with L_0-generators, $\mathbb{S}$ good σ-Suslin. $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is defined recursively:

If $\mathrm{Rk}(\mathcal{T}) = 0$, then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S}) = \mathbb{Q}_{L_0}$. Let $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ be defined for all templates of rank $< \kappa$. Let $\mathrm{Rk}(\mathcal{T}) = \kappa$ and for all $B \in \mathcal{I}(\mathrm{Dp}(B) < \kappa)$ let $\mathbb{P}_B = \mathbb{P}(\mathcal{T}_B, \mathbb{Q}, \mathbb{S})$. Then

- $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ consists of all $P = (p, F^p)$ where p is a finite partial function with $\mathrm{dom}(p) \subseteq L$, $(p \restriction L_0, F^p) \in \mathbb{Q}$ and if $x_p \stackrel{\text{def}}{=} \max\{\mathrm{dom}(p) \cap L_1\}$ is defined then $\exists B \in \mathcal{I}_{x_p}$ such that $P \parallel\!\restriction L_{x_p} = (p \restriction L_{x_p}, F^p \cap \widehat{W}_{L_{x_p} \cap L_0}) \in \mathbb{P}_B$, $p(x_p) = (\check{s}^p_x, \dot{f}^p_x)$, where $s^p_x \in {}^{<\omega}\omega$, $\dot{f}^p_x$ is a $\mathbb{P}_B$ name for a real and $(P \parallel\!\restriction L_{x_p}, p(x_p)) \in \mathbb{P}_B * \dot{\mathbb{S}}$.

Define $Q \leq_{\mathbb{P}} P$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, $(q{\restriction}L_0, F^q) \leq_{\mathbb{Q}} (p{\restriction}L_0, F^p)$, and if x_p is defined then either

- $x_p < x_q$ and $\exists B \in \mathcal{I}_{x_q}$ such that $P \|{\restriction} L_{x_q}, Q \|{\restriction} L_{x_q} \in \mathbb{P}_B$ and $Q \|{\restriction} L_{x_q} \leq_{\mathbb{P}_B} P \|{\restriction} L_{x_q}$, or
- $x_p = x_q$ and $\exists B \in \mathcal{I}_{x_q}$ witnessing $P, Q \in \mathbb{P}$, and such that

$$(Q \|{\restriction} L_{x_q}, q(x_q)) \leq_{\mathbb{P}_B * \dot{\mathbb{S}}} (P \|{\restriction} L_{x_p}, p(x_p)).$$

Completeness of Embeddings Lemma

Let $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$, let $\mathbb{Q} = \mathbb{Q}_{L_0}$ be the poset for adding m.c.g. with L_0-generators, $\mathbb{S}$ be good σ-Suslin.

Let $B \in \mathcal{I}$, $A \subset B$ be closed. Then $\mathbb{P}_B$ is a poset, $\mathbb{P}_A \subset \mathbb{P}_B$, every $P = (p, F^p) \in \mathbb{P}_B$ has a canonical reduction $P_0 = (p_0, F^{p_0}) \in \mathbb{P}_A$ such that

- $\mathrm{dom}(p_0) = \mathrm{dom}(p) \cap A$, $F^{p_0} = F^p$,
- $s_x^{p_0} = s_x^p$ for all $x \in \mathrm{dom}(p_0) \cap L_1$
- $(p_0 \restriction L_0, F^{p_0})$ is a strong $\mathbb{Q}_A$-reduction of $(p \restriction L_0, F^p)$

and whenever $D \in \mathcal{I}$, $B, C \subseteq D$, C is closed, $C \cap B = A$ and $Q_0 \leq_{\mathbb{P}_C} P_0$, then Q_0 and P are compatible in $\mathbb{P}_D$.

If $A = C$, $D = B$ then $\mathbb{P}_A$ is a complete suborder of $\mathbb{P}_B$.

Transitivity:

If $\mathrm{Rk}(\mathcal{T})=0$, then since $\mathbb{P}=\mathbb{Q}_{L_0}$ clear. So assume the Lemma for all templates of rank $<\alpha$, and let $\mathrm{Rk}(\mathcal{T})=\alpha$. Fix $P_0, P_1, P_2 \in \mathbb{P}$ such that $P_1 \leq_{\mathbb{P}} P_0$ and $P_2 \leq_{\mathbb{P}} P_1$, and assume that x_{p_0} is defined. Fix witnesses $B_1 \in \mathcal{I}_{x_{p_1}}$ and $B_2 \in \mathcal{I}_{x_{p_2}}$ to $P_1 \leq_{\mathbb{P}} P_0$ and $P_2 \leq_{\mathbb{P}} P_1$. Since $\mathrm{Dp}(B_1 \cup B_2) < \alpha$, by inductive hypothesis

$$\mathbb{P}_{B_1}, \mathbb{P}_{B_2} \lessdot \mathbb{P}_{B_1 \cup B_2},$$

and so we have $P_i \parallel\!\upharpoonright L_{x_{p_2}} \in \mathbb{P}_{B_1 \cup B_2}$ for $0 \leq i \leq 2$, and

$$P_2 \parallel\!\upharpoonright L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_1 \parallel\!\upharpoonright L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_0 \parallel\!\upharpoonright L_{x_{p_2}}.$$

Thus by inductive hypothesis $P_2 \parallel\!\upharpoonright L_{x_{p_2}} \leq_{\mathbb{P}_{B_1 \cup B_2}} P_0 \parallel\!\upharpoonright L_{x_{p_2}}$.

If $x_{p_0} < x_{p_2}$ then by definition $P_2 \leq_{\mathbb{P}} P_0$. So assume that $x_{p_0} = x_{p_2}$. Then $p_i(x_{p_2})$ is a $\mathbb{P}_{B_1 \cup B_2}$-name for $0 \leq i \leq 2$. Since $\mathbb{P}_{B_1}, \mathbb{P}_{B_2} \lessdot \mathbb{P}_{B_1 \cup B_2}$ we must have that

- $P_1 \upharpoonright\upharpoonright L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1 \cup B_2}} p_1(x_{p_2}) \leq_{\dot{\mathbb{S}}} p_0(x_{p_2})$, and
- $P_2 \upharpoonright\upharpoonright L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1 \cup B_2}} p_2(x_{p_2}) \leq_{\dot{\mathbb{S}}} p_1(x_{p_2})$ and so
- $P_2 \upharpoonright\upharpoonright L_{x_{p_2}} \Vdash_{\mathbb{P}_{B_1 \cup B_2}} p_2(x_{p_2}) \leq_{\dot{\mathbb{S}}} p_0(x_{p_2})$.

Thus $(P_2 \upharpoonright\upharpoonright L_{x_{p_2}}, p_2(x_{p_2})) \leq_{\mathbb{P}_{B_1 \cup B_2} * \dot{\mathbb{S}}} (P_0 \upharpoonright L_{x_{p_2}}, p_0(x_{p_2}))$ as required.

$\mathbb{P}_A \subset \mathbb{P}_B$:

Let $\mathcal{I}$ be of rank α. Let $A \subset B$ be closed, $B \in \mathcal{I}$. Let $R \in \mathbb{P}_A$ and let $x = x_r$. By definition there is $\bar{A} \in (\mathcal{I} \restriction A)_x$ such that

$$R \parallel\restriction L_x \in \mathbb{P}_{\bar{A}} \text{ and } \dot{f}_x^r \text{ is a } \mathbb{P}_{\bar{A}}\text{-name.}$$

By the properties of $\mathcal{I}$ there is $\bar{B} \in \mathcal{I}_{B,x}$ such that $\bar{A} = \bar{B} \cap A$. Then $\mathrm{Rk}(\mathcal{T}_{\bar{B}}) < \alpha$ and so by inductive hypothesis $\mathbb{P}_{\bar{A}} \lessdot \mathbb{P}_{\bar{B}}$. Therefore

$$R \parallel\restriction L_x \in \mathbb{P}_{\bar{B}} \text{ and } \dot{f}_x^r \text{ is a } \mathbb{P}_{\bar{B}}\text{-name.}$$

That is $R \in \mathbb{P}_B$.

Definition of $p_0(P, A, B)$

Let $\mathcal{I}$ be of rank α. Let $A \subset B$ be closed, $B \in \mathcal{I}$. Let $P = (p, F^p) \in \mathbb{P}_B$. We have to construct $P_0 = p_0(P, A, B)$. By definition there is $\bar{B} \in \mathcal{I}_{B,x}$ such that $\bar{P} = P \| \restriction L_x = (p \restriction L_x, F^p \cap \widehat{W}_{L_x \cap L_0}) \in \mathbb{P}_B$. Let $\bar{A} = \bar{B} \cap A$. Then by inductive hypothesis there is $\bar{P}_0 = p_0(\bar{P}, \bar{A}, \bar{B}) = (\bar{p}_0, F^{\bar{p}_0})$. Define $P_0 = (p_0, F^{p_0})$ as follows:

- $p_0 \restriction L_x = \bar{p}_0$, $p_0 \restriction L \backslash L_x = p \restriction L \backslash L_x$,
- If $x \notin A$ let $p_0(x) = p(x)$, and
- if $x \in A$ let $p_0(x)$ be a canonical projection of $p(x)$ below $\bar{P}_0$ (since $\mathbb{S}$ is a good Suslin, such projection exists).
- $F^{p_0} = F^p \cap \widehat{W}_A$.

Strong embedding of $\mathbb{P}$

Let $D \in \mathcal{I}$, C closed such that $C \cap B = A$, $C \cup B \subseteq D$. Let $Q_0 = (q_0, F^{q_0}) \leq_{\mathbb{P}_C} P_0$. We have to show that Q_0 is compatible with P (in $\mathbb{P}_D$).

Case $x \notin A$:

Suppose $x \notin A$. Then $x \notin C$. Using the properties of $\mathcal{I}$ find $\bar{C} \in (\mathcal{I}\restriction C)_x$, $\bar{D} \in \mathcal{I}_x$ such that $\bar{A} = \bar{B} \cap \bar{C}$, $\bar{B} \cup \bar{C} \subset \bar{D}$ and

$$\bar{Q}_0 := Q_0 \parallel\!\restriction L_x \leq_{\mathbb{P}_{\bar{C}}} P_0 \parallel\!\restriction L_x = \bar{P}_0.$$

Passing to an extension if necessary we can assume that $\bar{Q}_0\restriction L_0$ is a strong $\mathbb{Q}_{\bar{C}}$ reduction of $\mathbb{Q}_0\restriction L_0$. Since $\bar{P}_0$ is a canonical reduction of $\bar{P} = P \parallel\!\restriction L_x$, there is $\bar{Q} = (\bar{q}, F^{\bar{q}})$ which is a common extension of $\bar{Q}_0$ and $\bar{P}$ in $\mathbb{P}_{\bar{D}}$.

Define the common extension $Q = (q, F^q)$ as follows:

- $q \restriction L_x = \bar{q}$
- $q(x) = p(x)$
- $q \restriction \mathrm{dom}(q_0) \setminus L_x = q_0 \setminus L_x$
- $q \restriction \mathrm{dom}(p) \setminus (\mathrm{dom}(q_0) \cup L_x^{=}) = p \restriction \mathrm{dom}(p) \setminus (\mathrm{dom}(q_0) \cup L_x^{=})$
- $F^q = F^{q_0} \cup F^p$

Case $x \in A$:

Assume $x \in A$. Then $x \in C$. By the properties of $\mathcal{I}$ find $\bar{C} \in (\mathcal{I} \restriction C)_x$ and $\bar{D} \in \mathcal{I}_x$ such that $\bar{A} = \bar{C} \cap \bar{D}$, $\bar{C} \cup \bar{B} \subseteq \bar{D}$ and $\bar{C}$ is a witness to $Q_0 \| \restriction L_x^= \leq_{\mathbb{P}_C} P_0 \| \restriction L_x^=$. Thus in particular

- $\bar{Q}_0 = Q_0 \| \restriction L_x \leq_{\mathbb{P}_{\bar{C}}} \bar{P}_0 = P_0 \| \restriction L_x$, and
- $\bar{Q}_0 \Vdash_{\mathbb{P}_{\bar{C}}} q_0(x) \leq p_0(x) = p(x)$.

Passing to an extension if necessary we can assume that $\bar{Q}_0 \restriction L_0$ is a strong $\mathbb{Q}_{\bar{C}}$-reduction of $\bar{Q}_0 \restriction L_0$.

Using the facts that $\mathbb{S}$ is good n-Suslin poset and that $p_0(x)$ is a canonical projection of $p(x)$ below $\bar{P}_0$ find $T = (t, F^t)$ extending $\bar{Q}_0$ and $\bar{P}$ in $\mathbb{P}_{\bar{D}}$ such that for some nice name $t(x)$ for a condition in $\mathbb{S}$ below T,

$$T = (t, F^t) \Vdash_{\mathbb{P}_{\bar{D}}} t(x) \leq_{\dot{\mathbb{S}}} q_0(x), p(x).$$

Define the common extension $Q = (q, F^q)$ of Q_0 and P as follows:

- $q \restriction L_x = t$,
- $q(x) = t(x)$
- $q \restriction \mathrm{dom}(q_0) \backslash L_x = q_0 \backslash L_x$
- $q \restriction \mathrm{dom}(p) \backslash (\mathrm{dom}(q_0) \cup L_x^{=}) = p \restriction \mathrm{dom}(p) \backslash (\mathrm{dom}(q_0) \cup L_x^{=})$
- $F^q = F^{q_0} \cup F^p$

□

Lemma

- $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is Knaster.
- Let $x \in L_1$, $A \in \mathcal{I}_x$. Then the two-step iteration $\mathbb{P}_A * \mathbb{S}$ completely embeds into $\mathbb{P}$.
- For any $p \in \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ there is countable $A \subseteq L$ such that $p \in \mathbb{P}_{\mathrm{cl}(A)}$. If τ is a $\mathbb{P}$-name for a real then there is a countable $A \subseteq L$ such that τ is a $\mathbb{P}_{\mathrm{cl}(A)}$-name.

Lemma

Let $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$ and let λ_0 be a regular uncountable cardinal such that $\lambda_0 \subseteq L_1$ (as an order), λ_0 is cofinal in L, and $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$. Then in $V^{\mathbb{P}}$, $\text{non}(\mathcal{M}) = \lambda_0$ and so $\mathfrak{a}_g \geq \lambda_0$.

Proof

Let G be $\mathbb{P}$-generic and let ϕ_α be the slalom added in coordinate $\alpha < \lambda_0$. Since λ_0 is regular, uncountable and is cofinal in L, the family $\langle \phi_\alpha : \alpha < \mu \rangle$ localizes all reals $V[G]$ (indeed any real must appear in some $V[G \cap \mathbb{P}_{L_\alpha}]$ for some $\alpha < \lambda_0$.) Thus $\mathrm{cof}(\mathcal{N}) \leq \lambda_0$. On the other hand, if $F \subseteq \omega^\omega$ is a family of size $< \lambda_0$ in $V[G]$, then there must be some $\alpha < \lambda_0$ such that all reals of F already are in $V[G \cap \mathbb{P}_{L_\alpha}]$, and so ϕ_α localizes all reals in F. Thus $\mathrm{add}(\mathcal{N}) \geq \lambda_0$. Therefore $\mathrm{non}(M) = \lambda_0$ and so $\mathfrak{a}_g \geq \mu$. □

Lemma

Let $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$, L of uncountable cofinality, L_0 cofinal in L. Then $\mathbb{P}$ adds a maximal cofinitary group of size $|L_0|$.

Proof:

Let G be $\mathbb{P}$-generic, $\rho_G : L_0 \to S_\infty$ be defined as follows: for $x \in L_0$ let $\rho_G(x) = \bigcup\{s^p_x : p \in G \wedge p \restriction L_0 = (s^p, F^p)\}$. Note that $\rho_G = \rho_{G_0}$ where $G_0 = G \cap \mathbb{P}_{L_0}$ and so it induced a cofinitary representation of $\mathbb{F}_{L_0}$. We claim that $\mathrm{im}(\rho_G)$ is a m.c.g.

Otherwise, there are $\sigma \in \mathrm{cofin}(S_\infty)$ and $b_0 \notin L_0$ such that $\rho'_G : L_0 \cup \{b_0\} \to S_\infty$, where $\rho'_G \restriction L_0 = \rho_G$ and $\rho'_G(b_0) = \sigma$, induces a cofinitary representation. Let $\dot{\sigma}$ be a $\mathbb{P}$-name for σ. Then for some countable $A \subseteq L$, $\dot{\sigma}$ is a $\mathbb{P}_{\mathrm{cl}(A)}$-name. Since L_0 is cofinal in L and L has uncountable cofinality, there is $x \in L_0$ such that $\mathrm{cl}(A) \subseteq L_x$ and so $\mathbb{P}_{\mathrm{cl}(A)} \lessdot \mathbb{P}_{L_x}$. Let $H = G \cap \mathbb{P}_{L_x}$.

Claim

In $V[H]$ the set $D_{\sigma,N}$ consisting of all $p \in \mathbb{P}/H$ such that for some $n \geq N(s_x^p(n) = \sigma(n))$ where $p \restriction L_0 = (s^p, F^p)\}$ is dense.

Proof:

Let $p_0 \in \mathbb{P}/H$. Thus $p \restriction L_0 \cap L_x \in H_0 := G \cap \mathbb{P}_{L_0 \cap L_x}$. The set $D^0_{\sigma,N,x} = \{p \in (\mathbb{Q}_{L_0}/\mathbb{Q}_{L_x \cap L_0}) : (\exists n \geq N) s_x^p(n) = \sigma(n)\}$ is dense in $V[H_0]$ and so $\exists (t, E) \leq (s^{p_0} \restriction L_0 \backslash L_x, F^{p_0})$ such that $(t, E) \in D^0_{\sigma,N,x}$ i.e. $t_x(n) = \sigma(n)$ for some $n \geq N$. Define $p_1 \in \mathbb{P}/H$ as follows: $p_1 \restriction L_x = p_0 \restriction L_x$, $p_1 \restriction (L_0 \backslash L_x) = (t, E)$, $p_1 \restriction L_1 \backslash L_x = p_0 \restriction L_1 \backslash L_x$. Then in $V[H]$, $p_1 \leq p_0$ and $p_1 \in D_{\sigma,n}$. □

Then in $V[G]$ there are infinitely many n such that $\sigma(n) = \sigma_x(n)$, contradicting the fact that ρ'_G induces a cofinitary representation. □

Assume CH. Let $\lambda = \bigcup_n \lambda_n$, where λ_n is a regular cardinal, $\{\lambda_n\}_{n\in\omega}$ increasing and $\lambda_0 \geq \aleph_2$. Consider a template $\mathcal{T} = (L, \mathcal{I})$ such that

- $\lambda_0 \subseteq L_1$, λ_0 is cofinal in L, $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$.
- L has uncountable cofinality, L_0 is cofinal in L.

Then in $V^{\mathbb{P}}$ for $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$

- $\lambda_0 = \text{non}(\mathcal{M})$, and so $\lambda_0 \leq \mathfrak{a}_g$
- there is a mcg of size λ and so $\mathfrak{a}_g \leq \lambda$.

An isomorphism of names argument provides that in $V^{\mathbb{P}}$ there are no mcg of size $< \lambda$ and so $V^{\mathbb{P}} \vDash \mathfrak{a}_g = \lambda$.

Theorem (V.F., A. Törnquist)

It is consistent with the usual axioms of set theory that the minimal size of a maximal cofinitary group is of countable cofinality.

Thank you!

Good words

Let $\widehat{W_A}$ consist of all $w \in W_A$ such that either $w = a^n$ for some $a \in A$ and $n \in \mathbb{Z} \setminus \{0\}$, or w starts and ends with a different letter (i.e. there are $u \in W_A$, $a, b \in A$, $a \neq b$, and $i, j \in \{-1, 1\}$ such that $w = a^i u b^j$ *without cancelation*). Any $w \in W_A$ can be written as $w = u^{-1} w' u$ for some $w' \in \widehat{W_A}$ and $u \in W_A$.

Lemma

Let $s \subseteq A \times \omega \times \omega$ be finite such that s_a is a partial injection for all $a \in A$. Fix $a \in A$, and let $w \in W_{A \cup X}$ be a-good. Then for any $n \in \omega \setminus \mathrm{dom}(s_a)$ and $C \subseteq \omega$ finite there are cofinitely many $m \in \omega$ such that for all $l \in \omega$

$$e_w[s \cup \{(a, n, m)\}, \rho](l) \in C \text{ iff } e_w[s, \rho](l){\downarrow} \wedge e_w[s, \rho](l) \in C$$

Proof:

By induction on the rank j. Let w be an a-good word of rank 1,

$$w = a^{k_1} u_1.$$

Assume first $k_1 > 0$. Then pick $m \notin \mathrm{dom}(a)$ and $m \notin C$. Suppose $e_w[s \cup \{(a,n,m)\}, \rho](l) \in C$ but $e_w[s,\rho](l)\uparrow$. Then there is some $0 < i < k_1$ such that $e_{a^i u_1}[s,\rho](l) = n$. If $i < k_1 - 1$ then $e_{a^{i+2} u_1}[s \cup \{(a,n,m)\}, \rho](l)\uparrow$, so we must have $i = k_1 - 1$. But then $e_w[s \cup \{(a,n,m)\}, \rho](l) = m \notin C$, a contradiction.

The case $k_1 < 0$ is analogous.

Now let w be a-good of rank $j > 1$, $w = a^{k_j} u_j \bar{w}$, where $\bar{w}$ is a-good of rank $j-1$. Let $C' = e_{u_j^{-1} a^{-k_j}}[s,\rho](C)$. By IH there is $I_0 \subseteq \omega$ cofinite such that for all $m \in I_0$, all $l \in \omega$ we have that $e_{\bar{w}}[s \cup \{(a,n,m)\},\rho](l) \in C'$ iff $e_{\bar{w}}[s,\rho](l)\downarrow \wedge\ e_{\bar{w}}[s,\rho](l) \in C'$. Let $I_1 \subseteq \omega$ be cofinite such that for all $m \in I_1$, and all $l \in \omega$

$$\begin{aligned} & e_{a^{k_i} u_j}[s \cup \{(a,n,m)\},\rho](l) \in C \\ & \iff e_{a^{k_i} u_j}[s,\rho](l)\downarrow \wedge\ e_{a^{k_i} u_j}[s,\rho](l) \in C. \end{aligned}$$

Then let $m \in I_1 \cap I_0$, and suppose $e_w[s \cup \{(a,n,m)\},\rho](l) \in C$. Then $e_{\bar{w}}[s \cup \{(a,n,m)\},\rho](l) \in C'$ and so $e_{\bar{w}}[s,\rho](l) \in C'$. It follows that $e_{a^{k_j} u_j}[s \cup \{(a,n,m)\},\rho](e_{\bar{w}}[s,\rho](l)) \in C$ and so we have $e_{a^{k_j} u_j}[s,\rho](e_{\bar{w}}[s,\rho](l)) = e_w[s,\rho](l) \in C$, as required. □

Proof: No New Fixed Points, Domain Extension

Sufficient to consider the case $F = \{w\}$. We may assume that $a \in \mathrm{oc}(w)$ (otherwise - done). If w is a-good, then the statement follows from the previous lemma. If w is not a-good, then write $w = uva^k$ (without cancelation), where $u \in W_{A\setminus\{a\}\cup B}$, v is a-good, and $k \in \mathbb{Z}$. Let $\bar{w} = va^k u$. Then $\bar{w}$ is a-good, and so $\exists I \subseteq \omega$ cofinite such that

$$(\forall m \in I)(s \cup \{(a, n, m)\}, \{\bar{w}\}) \leq_{\mathbb{P}_{A,\rho}} (s, \{\bar{w}\}).$$

We claim that $(s \cup \{(a,n,m)\}, \{w\}) \leq (s, \{w\})$ when $m \in I$. Indeed, if $e_w[s \cup \{(a,n,m)\}, \rho](I) = I$ then it is not hard to check that

$$e_{\bar{w}}[s \cup \{(a,n,m)\}, \rho](e_{va^k}[s \cup \{(a,n,m)\}, \rho](I))$$
$$= e_{va^k}[s \cup \{(a,n,m)\}, \rho](I)$$

and so

$$e_{\bar{w}}[s, \rho](e_{va^k}[s \cup \{(a,n,m)\}, \rho](I)) = e_{va^k}[s \cup \{(a,n,m)\}, \rho](I),$$

which implies $e_w[s, \rho](I) = I$.