A CO-ANALYTIC SACKS INDESTRUCTIBLE MAXIMAL INDEPENDENT FAMILY

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Abstract. We show that there is a co-analyitc maximal independent family, which is Sacks indestructible. Thus the consistency of $\mathfrak{i} < \mathfrak{c}$ can be witnessed by a co-analytic set.

1. Introduction

The study of the definability properties of various sets of real has been of increased interest in the past few decades. Mathias showed that there are no analytic maximal almost disjoint families, while Miller constructed a co-analytic maximal almost disjoint family in L. These results initiated a long list of theorems regarding the existence of various nicely definable combinatorial sets of reals: maximal families of orthogonal measure ([4]), maximal cofinitary groups (whose definability properties have been of particular interest) and more recently definable maximal towers ([3]).

In this paper we turn our attention towards maximal independent families. Recall that a family $\mathcal{A}$ of infinite subsets of ω is said to be independent if whenever F, G are disjoint finite subfamilies of $\mathcal{A}$, the intersection $(\cap F)\backslash(\cup G)$ is infinite. Many interesting results about independent families can be found in [1] and [7]. An independent family is said to be maximal if it is not properly contained in another independent family. The minimal size of a maximal independent family is denoted $\mathfrak{i}$ and is referred to as the *the independence number*. Even though the consistency of $\mathfrak{i} = \aleph_1 < \mathfrak{c} = \aleph_2$, as well as the existence of a Sacks indestructible maximal independent family (a result attributed to Eisworth and Shelah, see [1]) have been well-known, it is only recently that a written proof of the latter appeared in the literature ([5]). Indeed, the construction of a Sacks indestructible maximal independent family, implicitly appears in Shelah's proof of the consistency of $\mathfrak{i} < \mathfrak{u} = \aleph_2$, where $\mathfrak{u}$ denotes the minimal size of an ultrafilter base. In the following, we will show that the construction of a Sacks indestructible maximal independent family which originates in [10] can be naturally modified to produce a Σ^1_2-definable, and so by a recent result of Brendle and Khomkii, see [2], a co-analytic maximal independent family, which remains maximal after long products of Sacks forcing, as well as after the countable support iteration of the Sacks poset. As a corollary we obtain that the consistency of $\mathfrak{i} = \aleph_1 < \mathfrak{c} = \lambda$, where λ is an arbitrary regular uncountable cardinal, can be witnessed by a Π^1_1-set. Moreover, our construction shows

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that Shelah's witness to $\mathfrak{i} = \aleph_1$ in the model of $\mathfrak{i} = \aleph_1 < \mathfrak{u} = \aleph_2$ from [10] can be chosen to be co-analytic.

2. Sacks Indestructibility

A convenient way to describe the independence of a family $\mathcal{A} \subseteq [\omega]^\omega$ is to consider finite partial functions $h : \mathcal{A} \to \{0,1\}$ and define $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \omega \setminus A$. Then $\mathcal{A}$ is independent if for every $f \in \mathrm{FF}(\mathcal{A}) := \{h : h \colon \mathcal{A} \to \{0,1\}, |\mathrm{dom}(h)| < \omega\}$, the set $\mathcal{A}^h$ is infinite. In [5] the authors introduce a class of maximal independent families, *densely maximal independent families*, and give a characterization of their Sacks indestructibility in terms of properties of the following ideal referred to as *density ideal:*

$$\mathrm{id}(\mathcal{A}) := \{X \subseteq \omega : \forall h \in \mathrm{FF}(\mathcal{A}) \exists h' \in \mathrm{FF}(\mathcal{A}) \text{ s.t. } h' \supseteq h \text{ and } \mathcal{A}^{h'} \cap X =^* \emptyset\}.$$

In particular, if $\mathcal{A}$ is densely maximal and the dual filter $\mathrm{fil}(\mathcal{A})$ of $\mathrm{id}(\mathcal{A})$ is generated by a Ramsey filter and the co-finite sets, then $\mathcal{A}$ is Sacks indestructible ([5, Corollary 37]).

Consider the poset of all pairs $(\mathcal{A}, A)$ where $\mathcal{A}$ is a countable independent family, $A \in [\omega]^\omega$ and for all $h \in \mathrm{FF}(\mathcal{A})(|\mathcal{A}^h \cap A| = \omega)$ with extension relation defined as follows: $(\mathcal{A}_2, A_2) \leq (\mathcal{A}_1, A_1)$ iff $\mathcal{A}_2 \supseteq \mathcal{A}_1$ and $A_2 \subseteq^* A_1$. Then $\mathbb{P}$ is countably closed ([10, 5]) and for each $X \in \mathrm{id}(\mathcal{A})$ and $(\mathcal{A}, A) \in \mathbb{P}$, the pair $(\mathcal{A}, A \backslash X)$ is still a condition in $\mathbb{P}$ extending $(\mathcal{A}, A)$. Following the notation of [5], whenever $\mathcal{E}$ is a partition of ω and $A \in [\omega]^\omega$, we will say that $\chi(\mathcal{E}, A)$ holds, if either there is $E \in \mathcal{E}$ such that $A \subseteq E$, or for all $E \in \mathcal{E}$ the set $E \cap A$ is of cardinality no greater than 1. The following statement can be found as Claim 2.1 in [10]. For a detailed proof see [5, Lemmas 17,18; Corollary 19].

Lemma 1 (Claim 2.1 in [10])**.**
1. If $(\mathcal{A}, A) \in \mathbb{P}$, there is $B \notin \mathcal{A}$, $B \subseteq A$ such that $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$.
2. If $(\mathcal{A}, A) \in \mathbb{P}$, $\mathcal{E}$ is a partition of ω and $h^0 \in \mathrm{FF}(\mathcal{A})$, then there exist $h^1 \supseteq h^0$, $B \subseteq A$ such that $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and $\chi(\mathcal{E}, \mathcal{A}^{h_1} \cap B)$.[1]
3. If $(\mathcal{A}, A) \in \mathbb{P}$ and $\mathcal{E}$ is a partition of ω, each element of which is finite, then there is $B \subseteq A$ such that $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and B is a semiselector for $\mathcal{E}$.

As an immediate corollary we obtain:

Lemma 2. Assume CH. Then, there is a strictly decreasing sequence $\langle (\mathcal{A}_\alpha, A_\alpha) : \alpha < \omega_1 \rangle$ of conditions in $\mathbb{P}$ such that if $\mathcal{A} = \bigcup_{\alpha \in \omega_1} \mathcal{A}_\alpha$ then:
1. for every partition $\mathcal{E}$ of ω and every $h \in \mathrm{FF}(\mathcal{A})$, there is $h' \in \mathrm{FF}(\mathcal{A})$ such that $h' \supseteq h$, and $\chi(\mathcal{E}, \mathcal{A}^{h'})$;
2. for every partition of $\mathcal{E}$ of ω into finite sets, there is $i \in \omega_1$ such that A_i is a semiselector for $\mathcal{E}$;
3. for each $\alpha \in \omega_1$ there is $A \subseteq A_\alpha$ such that $A \in \mathcal{A}_{\alpha+2}$;
4. for each $X \in \mathrm{id}(\mathcal{A})$ there are unboundedly many $\alpha \in \omega_1$ such that $X \subseteq \omega \backslash A_\alpha$.

[1] If $|E| < \omega$ for each $E \in \mathcal{E}$, then since $\mathcal{A}^{h^1} \cap B$ is a semiselector for $\mathcal{E}$.

Proof. Let $\{\mathcal{E}_\alpha : \alpha \in \mathrm{Succ}(\omega_1), \alpha \equiv 1 \mod 2\}$ be an enumeration of all partitions of ω such that each partition occurs cofinally often, let $\{X_\alpha : \alpha \in \mathrm{Succ}(\omega_1), \alpha \equiv 0 \mod 2\}$ be an enumeration of $[\omega]^\omega$ such that each set occurs cofinally often and let $(\mathcal{A}_0, A_0)$ be an arbitrary condition in $\mathbb{P}$. Suppose we have constructed $\{(\mathcal{A}_\beta, A_\beta) : \beta < \alpha\}$ for some $\alpha \in \omega_1$.

If α is a limit, find a pseudointersection A_α of $\{A_\beta : \beta < \alpha\}$ such that $(\mathcal{A}_\alpha, A_\alpha)$ is a condition, where $\mathcal{A}_\alpha = \bigcup_{\beta\in\alpha} A_\beta$.

If α is a successor and $\alpha \equiv 1 \mod 2$, then in particular $\alpha = \beta + 1$ for some β. Fix an enumeration $\{h_n : n \in \omega\}$ of $\mathrm{FF}(\mathcal{A}_\beta)$ and inductively construct a decreasing sequence $\{(\mathcal{A}_{\beta,n}, A_{\beta,n}) : n \in \omega\}$ as follows. Let $\mathcal{A}_{\beta,0} = \mathcal{A}_\beta$, $A_{\beta,0} = A_\beta$. Suppose we have constructed $\{(\mathcal{A}_{\beta,i}, A_{\beta,i}) : i \leq n\}$. By Lemma 1.2, there is $h' \supseteq h_n$ and $B' \subseteq A_{\beta,n}$ such that $(\mathcal{A}_{\beta,n}, B') \leq (\mathcal{A}_{\beta,n}, A_{\beta,n})$ and $\chi(\mathcal{E}_\alpha, \mathcal{A}^{h'}_{\beta,n} \cap B')$. By Lemma 1.1, there is $B'' \subseteq B'$ such that $(\mathcal{A}_{\beta,n} \cup \{B''\}, B') \leq (\mathcal{A}_{\beta,n}, B')$. Then take $\mathcal{A}_{\beta,n+1} = \mathcal{A}_{\beta,n} \cup \{B''\}$, $A_{\beta,n+1} = B'$. Thus $(\mathcal{A}_{\beta,n+1}, A_{\beta,n+1}) \leq (\mathcal{A}_{\beta,n}, A_{\beta,n})$ and there is $h'' \in \mathrm{FF}(\mathcal{A}_{\beta,n+1})$ such that $\chi(\mathcal{E}_\alpha, \mathcal{A}^{h''}_{\beta,n+1})$, where $h'' = h' \cup \{(B'', 0)\}$. Finally, take $\mathcal{A}_\alpha = \bigcup_{i\in\omega} \mathcal{A}_{\beta,i}$ and let A'_α be a pseudointersection of $\{A_{\beta,n}\}_{n\in\omega}$ such that $(\mathcal{A}_\alpha, A'_\alpha)$ is a condition. If $\mathcal{E}_\alpha$ is a partition of ω into finite subsets, apply Lemma 1.3 to find $A_\alpha \subseteq A'_\alpha$ such that $(\mathcal{A}_\alpha, A_\alpha)$ is a condition and A_α is a semi-selector for $\mathcal{E}_\alpha$. Otherwise, take $A_\alpha = A'_\alpha$.

If α is a successor and $\alpha \equiv 0 \mod 2$, i.e. $\alpha = \beta + 2$ for some β then $(\mathcal{A}_{\beta+1}, A_{\beta+1}) \leq (\mathcal{A}_\beta, A_\beta)$ have already been defined. Thus $A_{\beta+1} \subseteq^* A_\beta$ and so for $K = A_{\beta+1} \backslash A_\beta$, we have that $(\mathcal{A}_{\beta+1}, A_{\beta+1} \backslash K) \leq (\mathcal{A}_{\beta+1}, A_{\beta+1})$. By Lemma 1.1 there is $B \subseteq A_{\beta+1} \backslash K \subseteq A_\beta$ such that $(\mathcal{A}_{\beta+1} \cup \{B\}, A_{\beta+1} \backslash K)$ is a condition. Define $\mathcal{A}_\alpha = \mathcal{A}_{\beta+2} = \mathcal{A}_{\beta+1} \cup \{B\}$, $A'_{\beta+2} = A_{\beta+1} \backslash K$. If $X_\alpha \in \mathrm{id}(\mathcal{A}_\alpha)$, then take $A_\alpha = A'_{\beta+2} \backslash X_\alpha$ and note that by a previous remark $(\mathcal{A}_\alpha, A_\alpha)$ is a condition. If $X_\alpha \notin \mathrm{id}(\mathcal{A}_\alpha)$, then let $A_\alpha = A'_{\beta+2}$.

With this the inductive construction of $\{(\mathcal{A}_\alpha, A_\alpha) : \alpha \in \omega_1\}$ is completed. Let $\mathcal{A} = \bigcup_{\alpha\in\omega_1} \mathcal{A}_\alpha$. To verify property (1), take any partition $\mathcal{E}$ of ω and $h \in \mathrm{FF}(\mathcal{A})$. Let $\alpha < \omega_1$ be minimal such that $h \in \mathrm{FF}(\mathcal{A}_\alpha)$. Then there is $\beta \geq \alpha$ such that $\mathcal{E} = \mathcal{E}_\beta$ and so there is β such that $h \in \mathrm{FF}(\mathcal{A}_\beta)$, $\mathcal{E} = \mathcal{E}_\beta$. By construction of $\mathcal{A}_{\beta+1}$ there is $h' \in \mathrm{FF}(\mathcal{A}_{\beta+1}) \subseteq \mathrm{FF}(\mathcal{A})$ such that $\chi(\mathcal{E}, \mathcal{A}^{h'}_{\beta+1})$, and so, since $\mathcal{A}^{h'}_{\beta+1} = \mathcal{A}^{h'}$ we have $\chi(\mathcal{E}, \mathcal{A}^{h'})$. To verify part (2), consider any partition $\mathcal{E}$ of ω into finite sets, and fix α such that $\mathcal{E} = \mathcal{E}_\alpha$. Then $A_{\alpha+1}$ has been chosen to be a semiselector for $\mathcal{E}_\alpha$. Part (3) has been provided at even successor stages of the construction. For part (4) consider an arbitrary $X \in \mathrm{id}(\mathcal{A})$. Fix $\alpha \in \omega_1$ minimal such that $X \in \mathrm{id}(\mathcal{A}_\alpha)$ and note that for each $\beta \geq \alpha$, $X \in \mathcal{A}_\beta$. Find $\beta \geq \alpha$, such that X_β is defined and $X_\beta = X$. Then by construction $X_\beta \subseteq \omega \backslash A_\beta$. Since X occurs cofinally often in the sequence $\{X_\alpha : \alpha \in \omega_1\}$, there are unboundedly many α's for which $X \subseteq \omega \backslash A_\alpha$. □

Theorem 3. *Let $\{(\mathcal{A}_\alpha, A_\alpha) : \alpha \in \omega_1\}$ be a strictly decreasing sequence of conditions in $\mathbb{P}$ satisfying properties $(1) - (4)$ of Lemma 2. Let $\mathcal{A} = \bigcup_{\alpha\in\omega_1} \mathcal{A}_\alpha$. Then $\mathcal{A}$ is a maximal independent family, which remains maximal after the countable support product of Sacks forcing, as well as after the countable support iteration of Sacks forcing.*

Proof. To see that $\mathcal{A}$ is a maximal independent family, consider an arbitrary $X \in [\omega]^\omega \backslash \mathcal{A}$. Then $\mathcal{E}_X = \{X, \omega\backslash X\}$ is a finite partition of ω. If $h \in \mathrm{FF}(\mathcal{A})$ is arbitrary, then there is $h' \supseteq h$ such that $\chi(\mathcal{E}_X, \mathcal{A}^{h'})$ and so $\mathcal{A}^{h'} \subseteq X$, or $\mathcal{A}^{h'} \subseteq \omega\backslash X$. Thus $\mathcal{A} \cup \{X\}$ is not independent. To show that $\mathcal{A}$

is Sacks indestructible, it is sufficient to show that $\mathcal{A}$ is generated by $\{\omega\backslash A_\alpha : \alpha < \omega_1\}$. Indeed, if this is the case then fil($\mathcal{A}$) is generated by the tower $\{A_i : i \in \omega_1\}$, which is a Ramsey filter (by construction), and the co-finite sets. Thus by [5, Corollary 37] the family is Sacks indestructible.

Claim 4. id($\mathcal{A}$) is generated by $\{\omega\backslash A_\alpha : \alpha \in \omega_1\}$.

Proof. Let $\mathcal{I}$ denote the ideal generated by $\{\omega\backslash A_i\}_{i\in\omega}$. By property (4), id$(\mathcal{A}) \subseteq \mathcal{I}$. To see that $\mathcal{I} \subseteq$ id$(\mathcal{A})$, consider an arbitrary $X \in \mathcal{I}$. Then, there are $i_1, \cdots, i_k$ (taken in increasing order) in ω_1 such that $X \subseteq \bigcup_{j=1}^k \omega\backslash A_{i_j} = \omega\backslash \bigcap_{j=1}^k A_{i_j}$. Since $\{A_i\}_{i\in\omega_1}$ is an almost decreasing sequence, there is a finite set K_0 such that $A_{i_k}\backslash K_0 = \bigcap_{j=1}^k A_{i_j}$. Thus $X \subseteq \omega\backslash A_{i_k} \cup K_0$. Now, fix any $h \in$ FF$(\mathcal{A})$ and consider the partition $\mathcal{E}_{K_0} = \{K_0, \omega\backslash K_0\}$. Then, by part (1) of the Lemma, there is $h_0 \supseteq h$ such that $\mathcal{A}^{h_0} \subseteq \omega\backslash K_0$. Pick any $j > \max(\{i : A_i \in \text{dom}(h_0)\} \cup \{i_k\})$ and by part (3) of the Lemma, a set $B \subseteq A_j$ such that $B \in \mathcal{A}$. Let $h_0' = h_0 \cup \{(B, 0)\}$. Then $\mathcal{A}^{h_0'} \subseteq B \subseteq A_j$. However $A_j \subseteq^* A_{i_k}$ and so there is a finite K_1, such that $A_j\backslash K_1 \subseteq A_{i_k}$. Now, consider $\mathcal{E}_{K_1} = \{K_1, \omega\backslash K_1\}$. By part (1) of the Lemma, there is $h_1 \supseteq h_0'$ such that $\mathcal{A}^{h_1} \subseteq \omega\backslash K_1$. But then $\mathcal{A}^{h_1} \subseteq \omega\backslash K_1 \cap A_j \cap \omega\backslash K_0 = A_j\backslash K_1 \cap \omega\backslash K_0 \subseteq A_{i_k} \cap \omega\backslash K_0$. However $X \subseteq (\omega\backslash A_{i_k}) \cup K_0 = \omega\backslash(A_{i_k} \cap \omega\backslash K_0)$ and so $\mathcal{A}^{h_1} \cap X = \emptyset$. □

□

3. Definable Approximations

Throughout the section we work over the constructible universe L. As usual $<_L$ denotes the natural wellorder on L. In this section we modify the definition of the Sacks indestructible maximal independent family from Theorem 3 to make it nicely definable.

Definition 5. We say that a condition $(\mathcal{A}, a)$ is *prepared* for a family of partitions $\{\mathcal{E}_\gamma\}_{\gamma\leq\beta}$ if for every $h \in$ FF$(\mathcal{A})$ and every $\gamma \leq \beta$, there is $h' \in$ FF$(\mathcal{A})$ such that $\chi(\mathcal{A}^{h'}, \mathcal{E}_\gamma)$.

Definition 6. A triple $\langle\{\mathcal{E}_\beta\}_{\beta<\alpha}, \{(\mathcal{A}_\beta, A_\beta)\}_{\beta<\alpha}, \{X_\beta\}_{\beta<\alpha}\rangle$ is said to be an *approximating sequence* if

1. for every $\beta < \alpha$, $\mathcal{E}_\beta$ is the $<_L$-least set such that $\mathcal{E}_\beta \notin \{\mathcal{E}_\gamma\}_{\gamma<\beta}$, which is a partition of ω;
2. for every $\beta < \alpha$, X_β is the $<_L$-least infinite subset of ω, such that $X_\beta \notin \{X_\gamma\}_{\gamma<\beta}$;
3. for every $\beta < \alpha$, $(\mathcal{A}_\beta, A_\beta)$ is the $<_L$-least set such that
   - (a) $(\mathcal{A}_\beta, A_\beta) \in \mathbb{P}$, $(\mathcal{A}_\beta, A_\beta) \leq_{\mathbb{P}} (\mathcal{A}_\gamma, A_\gamma)$ for all $\gamma < \beta$;
   - (b) $\mathcal{A}_\beta$ is prepared for $\{\mathcal{E}_\gamma\}_{\gamma<\beta}$;
   - (c) For each $X_\gamma \in$ id$(\mathcal{A}_\beta)$, $A_\beta \cap X_\gamma = \emptyset$;
   - (d) if $\beta = \alpha + 2$ for some α (i.e. β is even), then there is $A \in \mathcal{A}_\beta$ such that $A \subseteq A_\alpha$.

Definition 7. A *maximal approximating sequence* is a sequence

$$\langle\{\mathcal{E}_\alpha\}_{\alpha<\omega_1}, \{(\mathcal{A}_\alpha, A_\alpha)\}_{\alpha<\omega_1}, \{X_\alpha\}_{\alpha<\omega_1}\rangle$$

such that for each $\beta < \omega_1$ the sequence $\langle\{\mathcal{E}_\beta\}_{\beta<\alpha}, \{(\mathcal{A}_\beta, A_\beta)\}_{\beta<\alpha}, \{X_\beta\}_{\beta<\alpha}\rangle$ is approximating and $\{\mathcal{E}_\alpha\}_{\alpha<\omega_1}$ (resp. $\{X_\alpha\}_{\alpha<\omega_1}$) enumerate all partitions of ω (resp. all infinite subsets of ω).

Clearly, in the constructible universe L, one can inductively construct a maximal approximating sequence.

Theorem 8. *Assume $V = L$. Let $\tau = \langle\{\mathcal{E}_\beta\}_{\beta<\omega_1}, \{(\mathcal{A}_\beta, A_\beta)\}_{\beta<\omega_1}, \{X_\beta\}_{\beta<\omega_1}\rangle$ be a maximal approximating sequence. Then $\mathcal{A} = \bigcup_{\beta<\omega_1} \mathcal{A}_\beta$ is a maximal independent family, which remains maximal after the countable support iteration of Sacks forcing, as well as after the countable support product of Sacks forcing.*

Proof. It is sufficient to observe that $\{(\mathcal{A}_\alpha, A_\alpha)\}_{\alpha<\omega_1}$ is a decreasing sequence of conditions in $\mathbb{P}$ which satisfies conditions $(1)-(4)$ of Lemma 2. □

4. A co-analytic Sacks indestructible maximal independent family

Let $\mathcal{L}_\in$ denote the language of set theory. For a real $x \in 2^\omega$, consider the binary relation $\in_x \subseteq \omega\times\omega$ defined by $m \in_x n$ iff $x(\langle m, n\rangle) = 1$, where $\langle m, n\rangle$ is a standard Gödel pairing function. Thus every $x \in 2^\omega$ in a natural way determines a $\mathcal{L}_\in$-structure, $M_x = (\omega, \in_x)$. Whenever M_x is well-founded and extensional, denote by $\mathrm{tr}(M_x)$ its transitive collapse and by $\pi_x : M_x \to \mathrm{tr}(M_x)$ the corresponding transitive isomorphism. The following can be found in [8, 13.8].

Lemma 9.

(1) Let $\varphi(x_0, \cdots, x_{k-1})$ be an $\mathcal{L}_\in$-formula with all free variable shown. Then

$$\{(x, m_0, \cdots, m_{k-1}) \in 2^\omega \times \omega \times \cdots \times \omega : M_x \vDash \varphi[m_0, \cdots, m_{k-1}]\}$$

is arithmetical;

(2) If M_x is well-founded and extensional, where $x \in 2^\omega$, then for every $f \in \mathcal{N}$, the relation $\{(m, f) \in \omega \times \mathcal{N} : \pi_x(m) = f\}$ is arithmetical.

(3) There is an $\mathcal{L}_\in$-formula σ_0 such that for every $x \in {}^\omega 2$ if $M_x \vDash \sigma_0$, M_x is well-founded and extensional, then $M_x \cong L_\delta$ for some limit ordinal $\delta < \omega_1$.

(4) There is an $\mathcal{L}_\in$-formula $\varphi_0(x_0, x_1)$ which defines the canonical well-ordering of $\mathcal{L}_\delta$ for all $\delta > \omega$;

(5) The set of $\delta < \omega_1$ such that $L_\delta \cong M_x$ for some x is unbounded in ω_1.

Theorem 10. *In the constructible universe L, there is a Σ^1_2 definable Sacks indestructible maximal independent family.*

Proof. Let $\tau = \langle\{\mathcal{E}_\alpha\}_{\alpha<\omega_1}, \{(\mathcal{A}_\alpha, A_\alpha)\}_{\alpha<\omega_1}, \{X_\alpha\}_{\alpha<\omega_1}\rangle$ be a maximal approximating sequence. Consider the relation $P(s, x) \subseteq \mathcal{N} \times 2^\omega$ where

(1) M_x is well-founded and extensional, $M_x \vDash \sigma_0$, $\pi_x(m) = s$

(2) $s \in 2^\omega$ recursively encodes an ordinal $\ln(s) \leq \omega$ and a sequence of reals $\{s(n) : n < \ln(s)\}$ such that for each n, $s(n)$ recurseively encodes an approximating sequence $\Delta(s(n))$

(3) $\bigcup_{n\in\omega} \Delta(s(n))$ is approximating.

Thus in particular, for each $1 \leq n < \ln(s)$, $\Delta(s(n))$ is the $<_L$-least approximating sequence extending $\Delta(s(n-1))$. Property (1) is Π^1_1, while (2) is arithmetical. The relation $P(s, x)$ holds if and only if $\langle \Delta(s(n)) : n < \mathrm{lh}(s) \rangle$ is an approximating sequence the construction of which is witnessed by M_x, where $M_x \cong L_\delta$ for some limit $\delta < \omega_1$. Then we have:

$$X \in \mathcal{A} \text{ iff } \exists x \exists s [P(s,x) \text{ and } \exists n \exists m A \in (\Delta(s(n))_{(2)(m)(1)})],$$

where $(z)_y$ is the projection onto the y-coordinate. Thus $\mathcal{A} = \bigcup_{\alpha \in \omega_1} \mathcal{A}_\alpha$ is Σ^1_2. □

Theorem 11. *In the constructible universe there is a co-analytic maximal independent family, which remains maximal after the countable support iteration of Sacks forcing, as well as after the countable support product of Sacks forcing. In particular, the consistency of* $\mathfrak{i} = \omega_1 < \mathfrak{c}$ *can be witnessed by a* Π^1_1 *set, while* $\mathfrak{c}$ *is arbitrarily large.*

Proof. Work over L and construct a Σ^1_2 definable Sacks indestructible maximal independent family $\mathcal{A}$. Given an arbitrary regular uncountable cardinal κ use a countable support product of Sacks forcing to increase the continuum to κ. In the resulting model $V[G]$, $\mathcal{A}$ is maximal. By a recent result of Brendle and Khomskii [2], in $V[G]$ the existence of a Σ^1_2 definable maximal independent family implies the existence of a co-analytic maximal independent family $\mathcal{A}_0$ of the same cardinality. Thus $\mathcal{A}_0$ is necessarily Sacks indestructible. Since $\mathcal{A}$ remains maximal also after the countable support iteration of Sacks forcing, an analogous argument shows the existence of a co-analytic maximal independent family, which remains maximal after the countable support iteration of Sacks forcing. □

5. Questions

The presence of Cohen reals prohibits that existence of projective maximal independent families (see [2]). Thus the results of the current paper suggest the following question:

Question: Is it consistent that there is a Π^1_2, or even a Σ^1_2 (and so a Π^1_1), maximal independent family in a model of $\mathfrak{i} = \mathfrak{c} = \aleph_2$?

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