Projective wellorders and mad families with large continuum

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Abstract

We show that $\mathfrak{b} = \mathfrak{c} = \omega_3$ is consistent with the existence of a Δ^1_3-definable wellorder of the reals and a Π^1_2-definable ω-mad subfamily of $[\omega]^\omega$ (resp. ω^ω).

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1. Introduction

The existence of a projective, in fact Δ^1_3-definable wellorder of the reals in the presence of large continuum, i.e. $\mathfrak{c} \geq \omega_3$, was established by Harrington in (8). In the present paper, we develop an iteration technique which allows one not only to obtain the consistency of the existence of a Δ^1_3-definable wellorder of the reals with large continuum (see Theorem 1), but in addition the existence of a Π^1_2-definable ω-mad family with $\mathfrak{b} = \mathfrak{c} = \omega_3$ (see Theorem 2). The method is a natural generalization to models with large continuum of the iteration technique developed in (5). We expect that an application of Jensen's coding techniques will lead to the same result with essentially arbitrary values for $\mathfrak{c}$.

For a more detailed introduction to the subject of projective wellorders of the reals and projective mad families, see (5) and (7). Recall that a family $\mathcal{A}$ of infinite subsets of ω is almost disjoint if any two of its elements have finite intersection. An infinite almost disjoint family $\mathcal{A}$ is maximal (abbreviated mad family), if for every infinite subset b of ω, there is an element $a \in \mathcal{A}$ such that $|a \cap b| = \omega$. If $\mathcal{A}$ is an almost disjoint family, let $\mathcal{L}(\mathcal{A}) = \{b \in [\omega]^\omega : b \text{ is not covered by finitely many elements of } \mathcal{A}\}$. A mad family $\mathcal{A}$

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is ω-mad if for every $B \in [\mathcal{L}(\mathcal{A})]^\omega$, there is $a \in \mathcal{A}$ such that $|a \cap b| = \omega$ for all $b \in B$. For the definition of $\mathfrak{b}$, as well as an introduction to the subject of cardinal characteristics of the continuum we refer the reader to (1).

2. Projective Wellorders with Large Continuum

Throughout the paper we work over the constructible universe L, thus unless otherwise specified $V = L$. Let $\langle G_\alpha : \alpha \in \omega_2 \cap \mathrm{cof}(\omega_1)\rangle$ be a $\Diamond_{\omega_2}(\mathrm{cof}(\omega_1))$ sequence which is Σ_1 definable over L_{ω_2}. For every $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 coding the ordinal α. Let $\vec{S} = \langle S_\alpha : \alpha < \omega_3\rangle$ be the sequence of stationary subsets of ω_2 defined as follows: $S_\alpha = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset\}$. In particular, the sets S_α are stationary subsets of $\mathrm{cof}(\omega_1) \cap \omega_2$ which are mutually almost disjoint (that is, for all $\alpha,\beta < \omega_3$, $\alpha \neq \beta$, we have that $S_\alpha \cap S_\beta$ is bounded). Let $S_{-1} = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = \emptyset\}$. Note that S_{-1} is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ disjoint from S_α for all $\alpha < \omega_3$.

Say that a transitive ZF^- model $\mathcal{M}$ is *suitable* if $\omega_3^{\mathcal{M}}$ exists and $\omega_3^{\mathcal{M}} = \omega_3^{L^{\mathcal{M}}}$. From this it follows, of course, that $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$ and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Step 0. For every $\alpha < \omega_3$ shoot a closed unbounded set C_α disjoint from S_α via a poset $\mathbb{P}^0_\alpha$. The poset $\mathbb{P}^0_\alpha$ consists of all bounded, closed subsets of ω_2, which are disjoint from S_α. The extension relation is end-extension. Note that $\mathbb{P}^0_\alpha$ is countably closed and $\aleph_2$-distributive (see (3)).

Let $\mathbb{P}_0 = \prod_{\alpha<\omega_3} \mathbb{P}^0_\alpha$ be the direct product of the $\mathbb{P}^0_\alpha$'s with supports of size ω_1. Then $\mathbb{P}_0$ is countably closed and by the Δ-system Lemma, also ω_3-c.c. And its ω_2-distributivity is easily established using the stationary set $S_{-1} \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$.

Step 1. In the following we treat 0 as a limit ordinal. Let $D_\alpha \subset \omega_2$ be a set coding the tuple $\langle C_\alpha, W_\alpha, W_\gamma\rangle$, where γ is the largest limit ordinal $\leq \alpha$. (More precisely, for a set X of ordinals denote by $0(X)$, $I(X)$, and $II(X)$ the sets $\{\eta : 3\eta \in X\}$, $\{\eta : 3\eta + 1 \in X\}$ and $\{\eta : 3\eta + 2 \in X\}$, respectively. Let D_α be such that $0(D_\alpha)$, $I(D_\alpha)$, and $II(D_\alpha)$ equal C_α, W_α, and W_γ, respectively.) Now let E_α be the club in ω_2 of intersections with ω_2 of elementary submodels of $L_{\alpha+\omega_2+1}[D_\alpha]$ which contain $\omega_1 \cup \{D_\alpha\}$ as a subset. (These elementary submodels form an ω_2-chain.) For a set X of ordinals, let $Even(X)$ be the subset of all even ordinals in X. Now choose Z_α to be a subset of ω_2 such that $Even(Z_\alpha) = D_\alpha$, and if $\beta < \omega_2$ is $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_\alpha \cap \beta \in \mathcal{M}$, then β belongs to E_α. (This is easily done by placing a code for a bijection $\phi : \beta_1 \to \omega_1$ on the interval $(\beta_0, \beta_0 + \omega_1)$ for each adjacent pair $\beta_0 < \beta_1$ from E_α.) Then we have:

$(*)_\alpha$: If $\beta < \omega_2$ and $\mathcal{M}$ is any suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \models \psi(\omega_2, Z_\alpha \cap \beta)$, where $\psi(\omega_2, X)$ is the formula "*Even*(X) codes a tuple

$\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\bar{\alpha}}}$".

Indeed, given a suitable model $\mathcal{M}$ with $\omega_2^{\mathcal{M}} = \beta$ and $Z_\alpha \cap \beta \in \mathcal{M}$, note that $\beta \in E_\alpha$ by the construction of Z_α and also that $D_\alpha \cap \beta \in \mathcal{M}$. Let $\mathcal{N}$ be an elementary submodel of $L_{\alpha+\omega_2+1}[D_\alpha]$ such that $\omega_1 \cup \{D_\alpha\} \subset \mathcal{N}$ and $\mathcal{N} \cap \omega_2 = \beta$. Denote by $\bar{\mathcal{N}}$ the transitive collapse of $\mathcal{N}$. Then $\bar{\mathcal{N}} = L_\xi[D_\alpha]$ for some $\omega_2 > \xi > \beta$ and $\omega_2^{\bar{\mathcal{N}}} = \omega_2^{\mathcal{M}} = \beta$. Therefore $\bar{\mathcal{N}} \subset \mathcal{M}$. Let $Z'_\alpha \subset \omega_2$ be such that $Even(Z'_\alpha) = Odd(Z'_\alpha) = D_\alpha$. By the definition of D_α, $L_{\alpha+\omega_2+1}[D_\alpha] \vDash \psi(\omega_2, Z'_\alpha)$. By elementarity, $\bar{\mathcal{N}} \vDash \psi(\omega_2, Z'_\alpha \cap \beta)$. Since the formula ψ is Σ_1, $\omega_2^{\bar{\mathcal{N}}} = \omega_2^{\mathcal{M}}$, we conclude that $\mathcal{M} \vDash \psi(\omega_2, Z'_\alpha \cap \beta)$. Since $Z_\alpha \cap \beta \in \mathcal{M}$ and $Even(Z'_\alpha) = Even(Z_\alpha)$, we have $\mathcal{M} \vDash \psi(\omega_2, Z_\alpha \cap \beta)$, which finishes the proof of $(*)_\alpha$.

Now similarly to $\vec{S}$ we can define a sequence $\vec{A} = \langle A_\xi : \xi < \omega_2 \rangle$ of stationary subsets of ω_1 using the "standard" $\Diamond$-sequence. Then in particular this sequence is nicely definable over L_{ω_1} and almost disjoint. Now we code Z_α by a subset X_α of ω_1 with the forcing $\mathbb{P}^1_\alpha$ consisting of all tuples $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$ and $t_0 \backslash s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$. Then X_α obviously satisfies the following condition:

$(**)_\alpha$: If $\omega_1 < \beta \leq \omega_2$ and $\mathcal{M}$ is a suitable model such that $\omega_2^{\mathcal{M}} = \beta$ and $\{X_\alpha\} \cup \omega_1 \subset \mathcal{M}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_\alpha)$, where $\phi(\omega_1, \omega_2, X)$ is the formula: " Using the sequence $\vec{A}$, X almost disjointly codes a subset $\bar{Z}$ of ω_2, whose even part $Even(\bar{Z})$ codes a tuple $\langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle$, where $\bar{W}$ and $\bar{\bar{W}}$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\bar{\alpha}}}$".

Let $\mathbb{P}^1$ be the product of the $\mathbb{P}^1_\alpha$'s with countable support. The poset $\mathbb{P}^1$ is easily seen to be countably closed. Moreover, it has the ω_2-c.c. by a standard Δ-system argument.

Step 2. Now we shall force a localization of the X_α's. Fix ϕ as in $(**)_\alpha$.

Definition 1. Let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$ and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. We denote by $\mathcal{L}(X, X')$ the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$

3. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \upharpoonright \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \models \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The extension relation is end-extension.

Set $\mathbb{P}^2_{\alpha+m} = \mathcal{L}(X_{\alpha+m}, X_\alpha)$ for every $\alpha \in Lim(\omega_3)$ and $m \in \omega$. Let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. By the Δ-system Lemma in $L^{\mathbb{P}^0 * \mathbb{P}^1}$ the poset $\mathbb{P}^2$ has the ω_2-c.c.

Observe that the poset $\mathbb{P}^2_{\alpha+m}$ produces a generic function from ω_1 (of $L^{\mathbb{P}^0 * \mathbb{P}^1}$) into 2, which is the characteristic function of a subset $Y_{\alpha+m}$ of ω_1 with the following property:

$(***)_\alpha$: For every $\beta < \omega_1$ and any suitable $\mathcal{M}$ such that $\omega_1^{\mathcal{M}} = \beta$ and $Y_{\alpha+m} \cap \beta$ belongs to $\mathcal{M}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \phi(\omega_1, \omega_2, X_\alpha \cap \beta)$.

Lemma 1. The poset $\mathbb{P}_0 := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is ω-distributive.

Proof. Given a condition $p_0 \in \mathbb{P}_0$ and a collection $\{O_n\}_{n \in \omega}$ of open dense subsets of $\mathbb{P}_0$, choose the least countable elementary submodel $\mathcal{N}$ of some large L_θ (θ regular) such that $\{p_0\} \cup \{\mathbb{P}_0\} \cup \{O_n\}_{n \in \omega} \subset \mathcal{N}$. Build a subfilter g of $\mathbb{P}_0 \cap \mathcal{N}$, below p_0, which hits all dense subsets of $\mathbb{P}_0$ which belong to $\mathcal{N}$. Write g as $g(0) * g(1) * g(2)$. Now $g(0) * g(1)$ has a greatest lower bound $p(0) * p(1)$ because the forcing $\mathbb{P}^0 * \mathbb{P}^1$ is ω-closed. The condition $(p(0), p(1))$ is obviously $(\mathcal{N}, \mathbb{P}^0 * \mathbb{P}^1)$-generic.

On each component $\alpha+m \in \mathcal{N} \cap \omega_3$, where $\alpha \in Lim(\omega_3)$, $m \in \omega$, define $p(2)(\alpha+m) = \bigcup g(2)(\alpha+m)$. It suffices to verify that $p(2)(\alpha+m)$ is a condition in $\mathbb{P}^2_{\alpha+m}$, for this will give us a condition $p(2)$ so that $p(0) * p(1) * p(2)$ meets each of the O_n's.

As $(p(0)(\alpha), p(0)(\alpha+m), p(1)(\alpha), p(1)(\alpha+m))$ is a $(\mathcal{N}, \mathbb{P}^0_\alpha * \mathbb{P}^0_{\alpha+m} * \mathbb{P}^1_\alpha * \mathbb{P}^1_{\alpha+m})$-generic condition, if

$$G := G(0)(\alpha) * G(0)(\alpha+m) * G(1)(\alpha) * G(1)(\alpha+m)$$

is a $\mathbb{P}^0_\alpha * \mathbb{P}^0_{\alpha+m} * \mathbb{P}^1_\alpha * \mathbb{P}^1_{\alpha+m}$-generic filter over L containing it, then the isomorphism π of the transitive collapse $\bar{\mathcal{N}}$ of $\mathcal{N}$, onto $\mathcal{N}$ extends to an elementary embedding from

$$\bar{\mathcal{N}}_0 := \bar{\mathcal{N}}[\overline{g(0)}(\bar{\alpha}) * \overline{g(0)}(\bar{\alpha}+m) * \overline{g(1)}(\bar{\alpha}) * \overline{g(1)}(\bar{\alpha}+m)]$$

into $L_\theta[G]$. Here $\overline{g(i)} = \pi^{-1}(g(i))$, $i \in 2$, and $\bar{\xi} = \pi^{-1}(\xi)$ for all $\xi \in \mathcal{N} \cap \mathrm{Ord}$. By the genericity of G we know that, letting $X_\alpha = \bigcup G(1)(\alpha)$, $X_{\alpha+m} = \bigcup G(1)(\alpha+m)$, properties $(**)_\alpha$ and $(**)_{\alpha+m}$ hold. By elementarity, $\bar{\mathcal{N}}_0$ is a suitable model and $\bar{\mathcal{N}}_0 \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$, where $x_{\bar{\alpha}} = \bigcup g(1)(\alpha) = \bigcup \overline{g(1)}(\bar{\alpha})$ and $x_{\bar{\alpha}+m} = \bigcup g(1)(\alpha+m) =$

$\bigcup \overline{g(1)}(\bar{\alpha} + m)$. By the construction of $\mathbb{P}_0$, $\bar{\mathcal{N}}_0 = \bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$ and hence $\bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}] \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$.

Let ξ be such that $\bar{\mathcal{N}} = L_\xi$ and let $\mathcal{M}$ be any suitable model containing $p(2)(\alpha)$, $p(2)(\alpha + m)$, and such that $\omega_1^{\mathcal{M}} = \omega_1 \cap \mathcal{N}$. We have to show that $\mathcal{M} \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$. Set $\eta = \mathcal{M} \cap \mathrm{Ord}$ and consider the chain $\mathcal{M}_2 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ of suitable models, where $\mathcal{M}_2 = L_\eta[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$ and $\mathcal{M}_1 = L_\eta[p(2)(\alpha), p(2)(\alpha + m)]$. Three cases are possible.

Case a). $\eta > \xi$. Since $\mathcal{N}$ was chosen to be the least countable elementary submodel of L_θ containing the initial condition, the poset and the sequence of dense sets, it follows that ξ (and therefore also δ) is collapsed to ω in $L_{\xi+2}$, and hence this case cannot happen.

Case b). $\eta = \xi$. In this case $\mathcal{M}_2 \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$. (Indeed, $\mathcal{M}_2 = L_\eta[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}] = \bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$.) Since ϕ is a Σ_1-formula, $\omega_1^{\mathcal{M}_2} = \omega_1^{\mathcal{M}}$ and $\omega_2^{\mathcal{M}_2} = \omega_2^{\mathcal{M}}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$.

Case c). $\eta < \xi$. In this case $\mathcal{M}_2$ is an element of $\bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$. Since $L_\theta[G]$ satisfies $(**)_\alpha$ and $(**)_{\alpha+m}$, by elementarity so does the model $\bar{\mathcal{N}}[x_{\bar{\alpha}}, x_{\bar{\alpha}+m}]$ with X_α replaced by $x_{\bar{\alpha}}$ and $X_{\alpha+m}$ replaced by $x_{\bar{\alpha}+m}$. In particular, $\mathcal{M}_2 \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$. Since ϕ is a Σ_1-formula, $\omega_1^{\mathcal{M}_2} = \omega_1^{\mathcal{M}}$, and $\omega_2^{\mathcal{M}_2} = \omega_2^{\mathcal{M}}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, x_{\bar{\alpha}}) \wedge \phi(\omega_1, \omega_2, x_{\bar{\alpha}+m})$, which finishes our proof. $\square$

Set $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$. Let us fix $\xi \in \omega_3$ and denote by $\mathbb{P}^{0,\neq\xi}, \mathbb{P}^{1,\neq\xi}, \mathbb{P}^{2,\neq\xi}$ the following posets in $L, L^{\mathbb{P}^{0,\neq\xi}}$, and $L^{\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi}}$, respectively:

$$\textstyle\prod_{\alpha \in \omega_3 \setminus \{\xi\}} \mathbb{P}^0_\alpha \text{ with supports of size } \omega_1;$$

$$\textstyle\prod_{\alpha \in \omega_3 \setminus \{\xi\}} \mathbb{P}^1_\alpha \text{ with countable supports; and}$$

$$\textstyle\prod_{\alpha \in \omega_3 \setminus \{\xi\}} \mathbb{P}^2_\alpha \text{ with countable supports.}$$

Observe that $\tilde{\mathbb{P}}_0^{\neq\xi} := \mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi} * \mathbb{P}^{2,\neq\xi} <_c \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2 = \mathbb{P}_0$, where for posets $\mathbb{P} \subseteq \mathbb{Q}$ the notation $\mathbb{P} <_c \mathbb{Q}$ means that the identity embedding from $\mathbb{P}$ to $\mathbb{Q}$ is complete.[2] Let $\tilde{\mathbb{R}}$ be the quotient poset $\mathbb{P}_0 / \tilde{\mathbb{P}}_0^{\neq\xi}$. Thus $\tilde{\mathbb{P}}_0^{\neq\xi} * \tilde{\mathbb{R}} = \mathbb{P}_0$.

Step 3. Fix a nicely definable sequence $\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega \rangle$ of almost disjoint subsets of ω. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$ such that $\mathbb{P}_0$ is as above, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset, in $L^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable wellorder of the reals and $\mathfrak{c} = \mathfrak{b} = \aleph_3$. Every $\mathbb{Q}_\alpha$ is going to add a generic real whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{u}_\alpha$ and we shall prove that $L[G_\alpha] \cap \omega^\omega = L[\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha \rangle] \cap \omega^\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α (see Lemma 2). This gives us a canonical wellorder of

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[2] It might seem unclear why we denote $\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi} * \mathbb{P}^{2,\neq\xi}$ by $\tilde{\mathbb{P}}_0^{\neq\xi}$ and not simply by $\mathbb{P}_0^{\neq\xi}$. It is to reserve the notation $\mathbb{P}_0^{\neq\xi}$ for a certain restriction of $\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi} * \mathbb{P}^{2,\neq\xi}$ appearing naturally in the proof of Lemma 3.

the reals in $L[G_\alpha]$, which depends only on the sequence $\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha \rangle$, whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{<}_\alpha$. We can additionally arrange that for $\alpha < \beta$ we have that $1_{\mathbb{P}_\beta}$ forces $\dot{<}_\alpha$ to be an initial segment of $\dot{<}_\beta$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L, $<^G = \bigcup\{\dot{<}_\alpha^G : \alpha < \omega_3\}$ will be the desired wellorder of the reals.

We proceed with the recursive construction of $\mathbb{P}_{\omega_3}$. Along this construction we shall also define a sequence $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3) \rangle$, where $\dot{A}_\alpha$ is a $\mathbb{P}_\alpha$-name for a subset of $[\alpha, \alpha + \omega)$. For every $\omega_2 \leq \nu < \omega_3$ fix a bijection $i_\nu : \{\langle \zeta, \xi \rangle : \zeta < \xi < \nu\} \to Lim(\omega_2)$. If G_α is $\mathbb{P}_\alpha$-generic over L, $<_\alpha = \dot{<}_\alpha^{G_\alpha}$ and x, y are reals in $L[G_\alpha]$ such that $x <_\alpha y$, let $x * y = \{2n : n \in x\} \cup \{2n+1 : n \in y\}$ and $\Delta(x * y) = \{2n+2 : n \in x * y\} \cup \{2n+1 : n \notin x * y\}$.

Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

Case 1. Suppose α is a limit ordinal and write it in the form $\omega_2 \cdot \alpha' + \xi$, where $\xi < \omega_2$. If $\alpha' > 0$, let $i = i_{o.t.(\dot{<}_{\omega_2 \cdot \alpha'}^{G_\alpha})}$ and $\langle \xi_0, \xi_1 \rangle = i^{-1}(\xi)$. Let $A_\alpha := \dot{A}_\alpha^{G_\alpha}$ be the set $\alpha + (\omega \setminus \Delta(x_{\xi_0} * x_{\xi_1}))$, where x_ζ is the ζ-th real in $L[G_{\omega_2 \cdot \alpha'}] \cap [\omega]^\omega$ according to the wellorder $\dot{<}_{\omega_2 \cdot \alpha'}^{G_\alpha}$ (here $G_{\omega_2 \cdot \alpha'} = G_\alpha \cap \mathbb{P}_{\omega_2 \cdot \alpha'}$). Let also

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1 \rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m \in \Delta(x_{\xi_0} * x_{\xi_1})} Y_{\alpha+m} \times \{m\}]^{<\omega}\},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ if and only if $s_1 \subset t_1$, s_0 is an initial segment of t_0 and $(t_0 \setminus s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m \rangle \in s_1$.

Case 2. If α is not of the form above, i.e. α is a successor or $\alpha < \omega_2$, then $\dot{A}_\alpha$ is a name for the empty set and $\dot{\mathbb{Q}}_\alpha$ is a name for the following poset adding a dominating real:

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1 \rangle : s_0 \in \omega^{<\omega}, s_1 \in [o.t.(\dot{<}_\alpha^{G_\alpha})]^{<\omega}\},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ if and only if s_0 is an initial segment of t_0, $s_1 \subset t_1$, and $t_0(n) > x_\xi(n)$ for all $n \in \mathrm{dom}(t_0) \setminus \mathrm{dom}(s_0)$ and $\xi \in s_1$, where x_ξ is the ξ-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $\dot{<}_\alpha^{G_\alpha}$.

In both cases $\mathbb{Q}_\alpha$ adds the generic real[3] $u_\alpha = \bigcup\{s_0 : \exists s_1 \langle s_0, s_1 \rangle \in g_\alpha\}$, where g_α is $\mathbb{Q}_\alpha$-generic over $V[G_\alpha]$ and $L[G_\alpha][u_\alpha] = L[G_\alpha][g_\alpha]$.

With this the definitions of $\mathbb{P} = \mathbb{P}_{\omega_3}$ and $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3) \rangle$ are complete.

Remark 1. Note that if the first case in the definition of $\dot{\mathbb{Q}}_\alpha$ above takes place, then in $L^{\mathbb{P}_\alpha}$ the poset $\dot{\mathbb{Q}}_\alpha$ produces a real r_α, which for certain reals x, y codes $Y_{\alpha+m}$ for all $m \in \Delta(x * y)$.

Let $\mathbb{H}$ be a poset. An $\mathbb{H}$-name $\dot{f}$ is called a *nice name for a real* if $\dot{f} = \bigcup_{i \in \omega}\{\langle \langle i, j_p^i \rangle, p \rangle : p \in \mathcal{A}_i(\dot{f})\}$ where for all $i \in \omega$, $\mathcal{A}_i(\dot{f})$ is a maximal antichain in $\mathbb{H}$, $j_p^i \in \omega$ and for all $p \in \mathcal{A}_i(\dot{f})$, $p \Vdash \dot{f}(i) = j_p^i$. From now on we will assume that all names for reals are nice.

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[3] $u_\alpha \in [\omega]^\omega$ in the first case and $u_\alpha \in \omega^\omega$ in the second case.

Using the fact that for every $p \in \mathbb{P}$ and $\alpha > 0$ the coordinate $p(\alpha)$ is a $\mathbb{P}_\alpha$-name for a finite set of ordinals, one can show that the set $\mathcal{D}$ of conditions p fulfilling the following properties is dense in $\mathbb{P}$:

- For every $\alpha > 0$ in the support of p, $p(\alpha) = \langle s_0, \check{s}_1 \rangle$ for some $s_1 \in [\mathrm{Ord}]^{<\omega}$ and $s_0 \in [\omega]^{<\omega}$ or $s_0 \in \omega^{<\omega}$ depending on $\dot{\mathbb{Q}}_\alpha$.

Lemma 2. Let $\gamma \leq \omega_3$ and let G_γ be a $\mathbb{P}_\gamma$-generic filter over L. Then $L[G_\gamma] \cap \omega^\omega = L[\langle \dot{u}_\delta^{G_\gamma} : \delta < \gamma \rangle] \cap \omega^\omega$.

Proof. Let $\dot{f} = \bigcup_{i \in \omega} \{ \langle \langle i, j_p^i \rangle, p \rangle : p \in \mathcal{A}_i(\dot{f}) \}$ be a nice $\mathbb{P}_\gamma$-name for a real such that $\bigcup_{i \in \omega} \mathcal{A}_i(\dot{f}) \subset \mathcal{D}$, $f = \dot{f}^{G_\gamma}$ and let p_i be the unique element of $\mathcal{A}_i(\dot{f}) \cap G_\gamma$. Set $u_\xi = \dot{u}_\xi^{G_\gamma}$ for all $\xi < \gamma$. Since $\mathbb{P}_0$ is countably distributive, there exists $q \in \mathbb{P}_0 \cap G_\gamma$ such that $q \leq p_i(0)$ for all $i \in \omega$.

Observe that $\langle i, j \rangle \in f$ if and only if there exists $p \in \mathcal{A}_i(\dot{f})$ such that $p(0) \geq q$ and for every α in the support of p the following holds:

If $p \restriction \alpha$ forces $\dot{\mathbb{Q}}_\alpha$ to be an almost disjoint coding, i.e. $\alpha = \omega_2 \cdot \alpha' + i(\beta_0, \beta_1)$ for some $\alpha' > 0$ and $\beta_0 < \beta_1 < o.t.(\dot{<}_{\omega_2 \cdot \alpha'}^{G_\gamma})$ and $\mathbb{Q}_\alpha$ produces a real coding a stationary kill of $S_{\alpha+m}$ for all $m \in \Delta(x_{\beta_0} * x_{\beta_1})$, where x_δ is the δ-th real in $L[\langle u_\xi : \xi < \omega_2 \cdot \alpha' \rangle]$, ***then*** $p(\alpha)_0$ is an initial segment of u_α and $u_\alpha \setminus p(\alpha)_0$ is disjoint from $B_{\zeta,m}$ for all $\langle \zeta, m \rangle \in p(\alpha)_1$; and

If $p \restriction \alpha$ forces $\dot{\mathbb{Q}}_\alpha$ to be a poset adding a dominating function, i.e. $\mathbb{Q}_\alpha$ produces a real u_α dominating all reals in $L[\langle u_\xi : \xi < \alpha \rangle]$, ***then*** $p(\alpha)_0$ is an initial segment of u_α and $u_\alpha(n) > x_\xi(n)$ for all $\xi \in p(\alpha)_1$ and $n \geq \mathrm{dom}(p(\alpha)_0)$, where x_ξ is the ξ-th real in $L[\langle u_\zeta : \zeta < \alpha \rangle]$ according to the wellorder $\dot{<}_\alpha^{G_\gamma}$.

Since $\dot{<}_\beta^{G_\gamma}$ depends only on the sequence $\langle u_\zeta : \zeta < \beta \rangle$ for all $\beta < \gamma$, the definition of f above implies that $f \in L[\langle u_\zeta : \zeta < \gamma \rangle]$, which finishes our proof. □

Lemma 3. Let G be a $\mathbb{P}$-generic filter over L. Then for $\xi \in \bigcup_{\alpha \in Lim(\omega_3)} \dot{A}_\alpha^G$ there is no real coding a stationary kill of S_ξ (i.e., there is no closed unbounded set disjoint from S_ξ which is constructible from a real).

Proof. Let $p \in G$ be a condition forcing

$$\xi \in \bigcup_{\alpha \in Lim(\omega_3)} \dot{A}_\alpha^G.$$

Suppose that $\xi = \beta + 2n - 1$ for some limit β and $n \in \omega$. Without loss of generality, $p \in \mathbb{P}_\beta \cap \mathcal{D}$.

We define a finite support iteration iteration of a countably distributive poset followed by c.c.c. posets $\langle \bar{\mathbb{P}}_\alpha, \dot{\bar{\mathbb{Q}}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3 \rangle$, where $\bar{\mathbb{P}}_0 = \mathbb{P}_0 \restriction p(0)$ and in $L^{\bar{\mathbb{P}}_\alpha}$

we have $\bar{\mathbb{Q}}_\alpha = \mathbb{Q}_\alpha \restriction p(\alpha)$. Such an iteration is just another way of thinking of the poset $\mathbb{P} \restriction p$ which will appear useful for further considerations.

Let $p_0^{\neq\xi}, p_0^{\xi}$ be such that $p_0^{\neq\xi} \in \tilde{\mathbb{P}}_0^{\neq\xi}$, $p_0^{\neq\xi} \Vdash p_0^{\xi} \in \tilde{\mathbb{R}}$ and $\langle p_0^{\neq\xi}, p_0^{\xi}\rangle = p(0)$, where $\tilde{\mathbb{R}}$ is the quotient poset $\mathbb{P}_0/\tilde{\mathbb{P}}_0^{\neq\xi}$. Denote by $\mathbb{P}_0^{\neq\xi}$ the restriction $\tilde{\mathbb{P}}_0^{\neq\xi} \restriction p_0^{\neq\xi}$ and let $\mathbb{R}$ be the $\mathbb{P}_0^{\neq\xi}$-name for $\tilde{\mathbb{R}} \restriction p_0^{\xi}$. Note that $\mathbb{P}_0^{\neq\xi} * \mathbb{R} = \bar{\mathbb{P}}_0$[4].

Now we define a finite support iteration $\langle \mathbb{P}_\alpha^{\neq\xi}, \dot{\mathbb{Q}}_\gamma^{\neq\xi} : \alpha \leq \omega_3, \gamma < \omega_3\rangle$, where $\mathbb{P}_0^{\neq\xi}$ is as above and $\dot{\mathbb{Q}}_\gamma^{\neq\xi}$ is a name for a σ-centered poset. Also we define a sequence $\langle \dot{A}_\alpha^{\neq\xi} : \alpha \in Lim(\omega_3)\rangle$, where $\dot{A}_\alpha^{\neq\xi}$ is a $\mathbb{P}_\alpha^{\neq\xi}$-name for a subset of $[\alpha, \alpha+\omega)$. The intention is to show that in $\bar{\mathbb{P}} = \bar{\mathbb{P}}_{\omega_3}$ the components $\mathbb{P}_\xi^0$, $\mathbb{P}_\xi^1$, $\mathbb{P}_\xi^2$ of $\mathbb{P}^0$, $\mathbb{P}^1$, $\mathbb{P}^2$, respectively, can be left out in a certain sense. Thus the iteration $\langle \mathbb{P}_\alpha^{\neq\xi}, \dot{\mathbb{Q}}_\gamma^{\neq\xi} : \alpha \leq \omega_3, \gamma < \omega_3\rangle$ will be introduced along the lines of the definition of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3\rangle$. In particular, every $\mathbb{Q}_\alpha^{\neq\xi}$ will add a generic real with $\mathbb{P}_\alpha^{\neq\xi} * \mathbb{Q}_\alpha^{\neq\xi}$-name $\dot{u}_\alpha^{\neq\xi}$. Given a $\mathbb{P}_\alpha^{\neq\xi}$-generic filter $G = G_\alpha^{\neq\xi}$, this gives us a canonical wellorder of the reals in $L[\langle \dot{u}_\zeta^{\neq\xi^G} : \zeta < \alpha\rangle]$ which depends only on the sequence $\langle \dot{u}_\zeta^{\neq\xi^G} : \zeta < \alpha\rangle$, whose $\mathbb{P}_\alpha^{\neq\xi}$-name will be denoted by $\dot{<}_\alpha^{\neq\xi}$. We can additionally arrange that for $\alpha < \beta$ we have that $1_{\mathbb{P}_\beta^{\neq\xi}}$ forces $\dot{<}_\alpha^{\neq\xi}$ to be an initial segment of $\dot{<}_\beta^{\neq\xi}$. Along the recursive construction for every $\gamma < \omega_3$ we will establish the following properties:

1. $\mathbb{P}_\gamma^{\neq\xi} <_c \bar{\mathbb{P}}_\gamma$;
2. $\dot{u}_\gamma^{\neq\xi^{H_\gamma^{\neq\xi}}} = \dot{u}_\gamma^{H_\gamma}$, $\dot{<}_\gamma^{\neq\xi^{H_\gamma^{\neq\xi}}} = \dot{<}_\gamma^{H_\gamma}$ and $\dot{A}_\gamma^{H_\gamma} = \dot{A}_\gamma^{\neq\xi^{H_\gamma^{\neq\xi}}}$ for limit γ, where $H_\gamma^{\neq\xi} \subseteq \mathbb{P}_\gamma^{\neq\xi}$ is the preimage of the $\bar{\mathbb{P}}_\gamma$-generic filter H_γ under the complete embedding from (1);
3. Let $\mathbb{P}_{[1,\gamma)}^{\neq\xi}$, $\bar{\mathbb{P}}_{[1,\gamma)}$ be the quotient posets $\mathbb{P}_\gamma^{\neq\xi}/\mathbb{P}_0^{\neq\xi}$ and $\bar{\mathbb{P}}_\gamma/\bar{\mathbb{P}}_0$ respectively. Then $\Vdash_{\bar{\mathbb{P}}_0} \mathbb{P}_{[1,\gamma)}^{\neq\xi} = \bar{\mathbb{P}}_{[1,\gamma)}$; and
4. $L[H_\gamma] \cap [\mathrm{Ord}]^\omega = L[H_\gamma^{\neq\xi}] \cap [\mathrm{Ord}]^\omega$ where H_γ, $H_\gamma^{\neq\xi}$ are as in (2).

For $\gamma = 0$ the properties above follow from the corresponding definitions. Suppose that (1)-(4) are established for all $\eta < \gamma$.

Case 1. If γ is a limit, there is nothing to prove except for (4) (To see that $\mathbb{P}_\gamma^{\neq\xi}$ is completely embedded in $\bar{\mathbb{P}}_\gamma$ refer to the inductive hypothesis and (2, Lemma 10)). Let $H_0^{\neq\xi} = H_\gamma^{\neq\xi} \cap \mathbb{P}_0^{\neq\xi}$, $H_0 = H_\gamma \cap \mathbb{P}_0$ and let K be an $\mathbb{R}$-generic filter over $L[H_0^{\neq\xi}]$ such that $L[H_0] = L[H_0^{\neq\xi}][K]$. Let $\mathbb{E}$ be the poset $(\mathbb{P}_{[1,\gamma)}^{\neq\xi})^{H_0^{\neq\xi}} = \bar{\mathbb{P}}_{[1,\gamma)}^{H_0} \in L[H_0^{\neq\xi}]$ (the latter equality follows from (3)). Then $H_{[1,\gamma)} (= H_\gamma/H_0)$ is $\mathbb{E}$-generic over $L[H_0^{\neq\xi}][K]$. Therefore $L[H_0^{\neq\xi}][K][H_{[1,\gamma)}] = L[H_0^{\neq\xi}][H_{[1,\gamma)}][K]$.

The following standard fact may be compared to (9, Lemma 15.19).

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[4] In fact, one can prove that $\Vdash_{\tilde{\mathbb{P}}_0^{\neq\xi}} \tilde{\mathbb{R}} = \mathbb{P}_0^{\xi} * \mathbb{P}_1^{\xi} * \mathbb{P}_2^{\xi}$, but this does not simplify the proof.

Claim. Suppose that $\mathbb{P}, \mathbb{Q}$ are in V, $\mathbb{P}$ is ω-distributive and $\mathbb{Q}$ is c.c.c. in $V^{\mathbb{P}}$. Then $\mathbb{P}$ is ω-distributive in $V^{\mathbb{Q}}$. In particular, if $\mathbb{P}$ is ω-distributive and $\mathbb{Q}$ is a finite support iteration of σ-centered posets, then $\mathbb{P}$ is ω-distributive in $V^{\mathbb{Q}}$.

Proof. Let $G \times H$ be $\mathbb{P} \times \mathbb{Q}$-generic. Let $f : \omega \to \mathrm{Ord}$ be in $V[H][G] = V[G][H]$ and σ be a $\mathbb{Q}$-name for f in $V[G]$. Without loss of generality, σ is a nice name which can be written as $\bigcup_{i\in\omega}\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{A}_i\}$, where j^i_p is an ordinal and $\mathcal{A}_i \in V[G]$ is a maximal antichain in $\mathbb{Q}$. As $\mathbb{Q}$ is c.c.c. in $V[G]$, each $\mathcal{A}_i$ is countable in $V[G]$, and hence σ is countable in $V[G]$. Therefore $\sigma \in V$ by the countable distributivity of $\mathbb{P}$. It follows that f belongs to $V[H]$. ☐

By the above Claim, $\mathbb{R}$ is countably distributive in $L[H^{\neq\xi}_0][H_{[1,\gamma)}] = L[H^{\neq\xi}_\gamma]$ and hence $L[H_\gamma] \cap [\mathrm{Ord}]^\omega = L[H^{\neq\xi}_\gamma] \cap [\mathrm{Ord}]^\omega$.

Case 2). $\gamma = \eta + 1$.

Let $H^{\neq\xi}_\eta$ be a $\mathbb{P}^{\neq\xi}_\eta$-generic filter over L and let K be a $\mathbb{R}$-generic filter over $L[H^{\neq\xi}_0]$, where $H^{\neq\xi}_0 = H^{\neq\xi}_\eta \cap \mathbb{P}^{\neq\xi}_0$. In $L[H^{\neq\xi}_0]$, the quotient poset $\mathbb{P}_{[1,\eta)} = \mathbb{P}_\eta/\mathbb{P}_0$ is a finite support iteration of σ-centered posets. Since $\mathbb{P}^{\neq\xi}_{[1,\eta)}$ has c.c.c. in $L[H^{\neq\xi}_0][K]$ and $\mathbb{R}$ is ω-distributive, $H^{\neq\xi}_{[1,\eta)}$ is $\mathbb{P}^{\neq\xi}_{[1,\eta)}$-generic over $L[H^{\neq\xi}_0][K]$. By (3), the equality $\mathbb{P}^{\neq\xi}_{[1,\eta)} = \bar{\mathbb{P}}_{[1,\eta)}$ holds in $L[H^{\neq\xi}_0][K]$. Therefore $H_\eta := H^{\neq\xi}_0 * K * H^{\neq\xi}_{[1,\eta)}$ is $\bar{\mathbb{P}}_\eta$-generic over L.

Since $p \in \mathcal{D}$, one of the following alternatives holds.

Case a). $\dot{\bar{\mathbb{Q}}}_\eta$ is a name for an almost disjoint coding below the condition $p(\eta) = \check{\langle s^\eta_0, s^\eta_1\rangle}$. Set $\bar{\mathbb{Q}}_\eta = \dot{\bar{\mathbb{Q}}}_\eta^{H_\eta}$, $u_\delta = \dot{u}_\delta^{H_\eta}$, $A_\delta = \dot{A}_\delta^{H_\eta}$, and $<_\delta = \dot{<}_\delta^{H_\eta}$ for all $\delta \leq \eta$.

It follows that:

- η is a limit ordinal that can be written in the form $\eta = \omega_2 \cdot \nu + \zeta$, where $\zeta = i(\zeta_0, \zeta_1)$ for some $\zeta_0, \zeta_1 < o.t.(<^{H_\eta}_{\omega_2\cdot\nu})$ and $i = i_{o.t.(<^{H_\eta}_{\omega_2\cdot\nu})}$;
- $A_\eta = \eta + (\omega \setminus \Delta(x_{\zeta_0} * x_{\zeta_1}))$, where x_ϵ is the ϵ-th real in $L[\langle u_\delta : \delta < \omega_2 \cdot \nu] \cap \omega^\omega$ according to the natural wellorder $<^{H_\eta}_{\omega_2\cdot\nu}$ of this set;
- $\bar{\mathbb{Q}}_\eta = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega}, s_1 \in \big[\bigcup_{m\in\Delta(x_{\zeta_0}*x_{\zeta_1})} Y_{\eta+m} \times \{m\}\big]^{<\omega}$, s_0 end-extends s^η_0, $s_1 \supseteq s^\eta_1$ and $s_0 \setminus s^\eta_0 \cap B_{\epsilon,m} = \emptyset$ for all $\langle\epsilon, m\rangle \in s^\eta_1\}$ ordered as before.

Our choice of p and the fact that the upwards closure of H_η in $\mathbb{P}_\eta$ is a $\mathbb{P}_\eta$-generic filter containing p imply that Y_ξ is not among the $Y_{\eta+m}$'s involved into the definition of $\bar{\mathbb{Q}}_\eta$. Thus $\bar{\mathbb{Q}}_\eta \in L[H^{\neq\xi}_\eta]$. Moreover, $\bar{\mathbb{Q}}_\eta$ is fully determined by the relevant $Y_{\eta+m}$'s and the sequence $\langle u_\delta : \delta < \eta\rangle$ which belongs to $L[H^{\neq\xi}_\eta]$ and does not depend on K by (2). Therefore $\bar{\mathbb{Q}}_\eta$ does not depend on K and hence we may set $\mathbb{Q}^{\neq\xi}_\eta := \bar{\mathbb{Q}}_\eta$, $A^{\neq\xi}_\eta := A_\eta$. Let $\dot{\mathbb{Q}}^{\neq\xi}_\eta$, $\dot{A}^{\neq\xi}_\eta$ be $\mathbb{P}^{\neq\xi}_\eta$-names for $\mathbb{Q}^{\neq\xi}_\eta$ and $A^{\neq\xi}_\eta$ respectively. By the definition, (3) and the third part of (2) hold true.

The equality $L[H_\eta] \cap [\mathrm{Ord}]^\omega = L[H_\eta^{\neq\xi}] \cap [\mathrm{Ord}]^\omega$ and the σ-centeredness of $\bar{\mathbb{Q}}_\eta$ imply that any $\mathbb{Q}_\eta^{\neq\xi}$-generic over $L[H_\eta^{\neq\xi}]$ is $\bar{\mathbb{Q}}_\eta$-generic over $L[H_\eta]$ and vice versa. Therefore $\mathbb{P}_{\eta+1}^{\neq\xi} <_c \bar{\mathbb{P}}_{\eta+1}$ (note that H_η may be thought of as being an arbitrary $\bar{\mathbb{P}}_\eta$-generic filter over L). This establishes (1).

Let h_η be a $\mathbb{Q}_\eta^{\neq\xi}$-generic over $L[H_\eta^{\neq\xi}]$ (or, equivalently, $\bar{\mathbb{Q}}_\eta$-generic filter over $L[H_\eta]$). Since a (nice) $\bar{\mathbb{Q}}_\eta$-name for a countable set of ordinals in $L[H_\eta]$ can be naturally identified with a countable set of ordinals, every $\bar{\mathbb{Q}}_\eta$-name $\sigma \in L[H_\eta]$ for a countable set of ordinals is in fact in $L[H_\eta^{\neq\xi}]$. Therefore $L[H_{\eta+1}] \cap [\mathrm{Ord}]^\omega = L[H_{\eta+1}^{\neq\xi}] \cap [\mathrm{Ord}]^\omega$, where $H_{\eta+1} = H_\eta * h_\eta$. This proves (4).

Let us denote by $u_\eta^{\neq\xi} \in [\omega]^\omega \cap L[H_{\eta+1}^{\neq\xi}]$ the union of the first coordinates of elements of h_η. By the maximality principle, this gives us a $\mathbb{P}_{\eta+1}^{\neq\xi}$-name $\dot{u}_\eta^{\neq\xi}$. By the definitions of $\dot{u}_\eta$ and $\dot{u}_\eta^{\neq\xi}$, $\dot{u}_\eta^{H_\eta * h_\eta} = \dot{u}_\eta^{\neq\xi^{H_\eta^{\neq\xi} * h_\eta}}$, which proves the first part of (2). By (4) and Lemma 2,

$$
\begin{aligned}
L[H_\eta^{\neq\xi} * h_\eta] \cap [\omega]^\omega &= (L[H_\eta^{\neq\xi} * h_\eta] \cap [\mathrm{Ord}]^\omega) \cap [\omega]^\omega = \\
&= (L[H_\eta * h_\eta] \cap [\mathrm{Ord}]^\omega) \cap [\omega]^\omega = L[H_\eta * h_\eta] \cap [\omega]^\omega = \\
= L[\langle \dot{u}_\delta^{H_\eta * h_\eta} : \delta \leq \eta\rangle] \cap [\omega]^\omega &= L[\langle \dot{u}_\delta^{\neq\xi^{H_\eta^{\neq\xi} * h_\eta}} : \delta \leq \eta\rangle] \cap [\omega]^\omega,
\end{aligned}
$$

which implies the second equality in (2) and thus concludes *Case a).*

Case b). $\dot{\bar{\mathbb{Q}}}_\eta$ is a name for a poset adjoining a dominating function restricted to the condition $p(\eta) = \langle s_0^{\eta}{}^{\vee}, s_1^\eta\rangle$. This case is analogous to, but easier than the *Case a)* (here we do not have to worry about Y_ξ) and we leave it to the reader.

This finishes our construction of $\langle \mathbb{P}_\alpha^{\neq\xi}, \dot{\mathbb{Q}}_\gamma^{\neq\xi} : \alpha \leq \omega_3, \gamma < \omega_3\rangle$. Observe that conditions (1)-(4) hold for $\gamma = \omega_3$. In particular, $L[G] \cap \omega^\omega = L[G^{\neq\xi}] \cap \omega^\omega$, where $G^{\neq\xi} \subset \mathbb{P}_{\omega_3}^{\neq\xi}$ is the preimage of the $\bar{\mathbb{P}}_{\omega_3}$-generic filter G under the complete embedding from (1). So it is sufficient to show that in $L[G^{\neq\xi}]$ there is no real coding a closed unbounded subset disjoint from S_ξ. Since $\mathbb{P}_{[1,\omega_3)}^{\neq\xi}$ is a $\mathbb{P}_0^{\neq\xi}$-name for a c.c.c poset and $\mathbb{P}^{2,\neq\xi}, \mathbb{P}^{1,\neq\xi}$ are $\mathbb{P}^{0,\neq\xi} * \mathbb{P}^{1,\neq\xi}, \mathbb{P}^{0,\neq\xi}$-names for ω_2-c.c. posets, respectively, every closed unbounded subset of ω_2 in $L[G^{\neq\xi}]$ contains a closed unbounded subset of ω_2 in $L[G^{0,\neq\xi}]$, see (9, Lemma 22.25). (Here $G^{0,\neq\xi} = G^{\neq\xi} \cap \mathbb{P}^{0,\neq\xi}$ is the $\mathbb{P}^{0,\neq\xi}$-generic filter over L induced by $G^{\neq\xi}$). Thus it suffices to verify that S_ξ is stationary in $L^{\mathbb{P}^{0,\neq\xi}}$. We shall use here an idea from (6).

Fix $p \in \mathbb{P}^{0,\neq\xi}$ and let $\dot{C}$ be a name for a club in ω_2. We would like to find $q \in \mathbb{P}^{0,\neq\xi}$ such that $q \leq p$ and $q \Vdash_{\mathbb{P}^{0,\neq\xi}} \dot{C} \cap S_\xi \neq \emptyset$. Let $\langle \mathcal{M}_i : i < \omega_2\rangle$ be a continuous chain of elementary submodels of some large L_θ such that $\mathcal{M}_0$ contains $p, \alpha, \dot{C}$, $\omega_1 + 1 \subset \mathcal{M}_0$, $\gamma_i := \mathcal{M}_i \cap \omega_2 \in \omega_2$, $\mathrm{cof}(\gamma_i) = \omega_1$, and $\mathcal{M}_i^{<\omega_1} \subset \mathcal{M}_i$ for all $i \in \omega_2$. Set $S_\xi^0 = \{i \in S_\xi : \gamma_i = i\}$ and note that S_ξ^0 is stationary.

Claim. There exists $i \in S_\xi^0$ such that $i \notin S_\alpha$ for all $\alpha \in \mathcal{M}_i \setminus \{\xi\}$.

Proof. Note that $\alpha \in \mathcal{M}_i$ is equivalent to $\alpha < \gamma_i$, and hence to $\alpha < i$ since $i \in S^0_\xi$. Suppose that for every $i \in S^0_\xi$ there exists $f(i) < i$ such that $i \in S_{f(i)}$ and $f(i) \neq \xi$. By Fodor's Lemma there exists $j \in \omega_2$ and a stationary $T \subset S^0_\xi$ such that $f(i) \equiv j$ for all $i \in T$. It follows that $T \subset S_j$, and hence $T \subset S_j \cap S_\xi$, a contradiction. □

Choose i as in the Claim above. We shall build an ω_1-sequence $p = p_0 \geq p_1 \geq \cdots$ with a lower bound forcing $i \in \dot{C}$. Let $\langle i_\alpha : \alpha < \omega_1 \rangle$ be an increasing continuous sequence of ordinals such that $\sup_{\alpha \in \omega_1} i_\alpha = i$. Given p_α, let $p_{\alpha+1} \leq p_\alpha$ be such a condition in $\mathbb{P}^{0,\neq\xi} \cap \mathcal{M}_i$ such that $p_{\alpha+1}$ forces some ordinal $j_{\alpha+1} \in [i_{\alpha+1}, i)$ to belong to $\dot{C}$. For limit α and $\zeta \in i \setminus \{\xi\}$ set

$$p_\alpha(\zeta) = \bigcup_{\beta<\alpha} p_\beta(\zeta) \cup \{\sup \bigcup_{\beta<\alpha} p_\beta(\zeta),\ i_\alpha\}.$$

Since S_ζ's consist of ordinals of cofinality ω_1 and $\mathcal{M}_i$ is closed under countable sequences of its elements, $p_\alpha \in \mathbb{P}^{0,\neq\xi} \cap \mathcal{M}_i$. This finishes our construction of the sequences $\langle p_\alpha : \alpha < \omega_1 \rangle \in \mathcal{M}_i^{\omega_1}$ and $\langle j_\alpha : \alpha < \omega_1 \rangle \in i^{\omega_1}$. Set $q(\zeta) = \bigcup_{\alpha \in \omega_1} p_\alpha(\zeta) \cup \{i\}$ for all $\zeta \in i \setminus \xi$. Since $i \notin S_\zeta$ for all $\zeta \in i \setminus \{\xi\}$, we conclude that $q(\zeta) \cap S_\zeta = \emptyset$ for all $\zeta \in i \setminus \{\xi\}$. From the above it follows that $q \in \mathbb{P}^{0,\neq\xi}$ and $q \Vdash_{\mathbb{P}^{0,\neq\xi}} i \in \dot{C}$, which finishes our proof. □

Corollary 1. Let G be a $\mathbb{P}$-generic filter over L and let x, y be reals in $L[G]$. Then $x <^G y$ if and only if there is $\alpha < \omega_3$ such that for all m, the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x * y)$.

Proof. Suppose that $x <^G y$. Let $\alpha' > 0$ be minimal such that $x, y \in L[G_{\omega_2 \cdot \alpha'}]$ and let $i = i_{o.t.(\dot{<}^G_{\omega_2 \cdot \alpha'})}$. Find $\xi \in Lim(\omega_2)$ such that $i(\xi) = (\xi_x, \xi_y)$ where x and y are the ξ_x-th and ξ_y-th real respectively in $L[G_{\omega_2 \cdot \alpha'}]$ according to the wellorder $\dot{<}^G_{\omega_2 \cdot \alpha'}$. (By Lemma 2 such a ξ exists). Let $\alpha = \omega_2 \cdot \alpha' + \xi$. Then $\mathbb{Q}_\alpha$ adds a real coding a stationary kill for $S_{\alpha+m}$ for all $m \in \Delta(x * y)$. On the other hand if $m \notin \Delta(x * y)$, then $\alpha + m \in \dot{A}^G_\alpha = \alpha + (\omega \setminus \Delta(x * y))$ and so by Lemma 3, there is no real in $L[G]$ coding the stationary kill of $S_{\alpha+m}$.

Now suppose that there exists α such that the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x * y)$. Since the stationary kill of some $\alpha + m$'s is coded by a real in $L[G]$, Lemma 3 implies that $\dot{\mathbb{Q}}^G_\alpha$ introduced a real coding stationary kill for all $m \in \Delta(a * b)$ for some reals $a \dot{<}^G_\alpha b$, while there are no reals coding a stationary kill of $S_{\alpha+m}$ for $m \notin \Delta(a * b)$. Therefore $\Delta(a * b) = \Delta(x * y)$ and hence $a = x$ and $b = y$, and consequently $x \dot{<}^G_\alpha y$. □

Lemma 4. Let G be $\mathbb{P}$-generic over L and let x, y be reals in $L[G]$. If $x <^G y$, then there is a real r such that for every countable suitable model $\mathcal{M}$ such that $r \in \mathcal{M}$, there is $\bar{\bar{\alpha}} < \omega_3^{\mathcal{M}}$ such that for all $m \in \Delta(x * y)$,

$$(L[r])^{\mathcal{M}} \models S_{\bar{\bar{\alpha}}+m} \text{ is not stationary.}$$

Proof. By Corollary 1, there exists $\alpha < \omega_3$ such that $\dot{\mathbb{Q}}^G_\alpha$ adds a real r coding a stationary kill of $S_{\alpha+m}$ for all $m \in \Delta(x * y)$. Let $\mathcal{M}$ be a countable suitable model containing r. It follows that $Y_{\alpha+m} \cap \omega_1^{\mathcal{M}} \in \mathcal{M}$ and hence $X_\alpha \cap \omega_1^{\mathcal{M}}$, $X_{\alpha+m} \cap \omega_1^{\mathcal{M}}$ also belong to $\mathcal{M}$. Observe that these sets are actually in $\mathcal{N} := (L[r])^{\mathcal{M}}$. Note also that $\mathcal{N}$ is a countable suitable model and consequently by the definition of $\mathcal{L}(X_{\alpha+m}, X_\alpha)$ we have that for every $m \in \Delta(x * y)$, $\mathcal{N} \vDash$

> " Using the sequence $\vec{A}$, $X_{\alpha+m} \cap \omega_1$ (resp. $X_\alpha \cap \omega_1$) almost disjointly codes a subset $\bar{Z}_m$ (resp. $\tilde{Z}_0$) of ω_2, whose even part $Even(\bar{Z}_m)$ (resp. $Even(\tilde{Z}_0)$) codes a tuple $\langle \bar{C}, \bar{W}_m, \bar{\bar{W}}_m \rangle$ (resp. $\langle \tilde{C}, \tilde{W}_0, \tilde{\tilde{W}}_0 \rangle$), where $\bar{W}_m$ and $\bar{\bar{W}}_m$ are the L-least codes of ordinals $\bar{\alpha}_m, \bar{\bar{\alpha}}_m < \omega_3$ (resp. $\tilde{W}_0 = \tilde{\tilde{W}}_0$ is the L-least code for a limit ordinal $\tilde{\tilde{\alpha}}_0$) such that $\bar{\bar{\alpha}}_m = \tilde{\tilde{\alpha}}_0$ is the largest limit ordinal not exceeding $\bar{\alpha}_m$ and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}_m}$.[5]

Note that in particular for every $m \neq m'$ in $\Delta(x * y)$, $\bar{\bar{\alpha}}_m = \bar{\bar{\alpha}}_{m'}$. □

Lemma 5. Let G be $\mathbb{P}$-generic over L and let x, y be reals in $L[G]$. If there is a real r such that for every countable suitable model $\mathcal{M}$ containing r as an element, there is $\bar{\bar{\alpha}} < \omega_3^{\mathcal{M}}$ such that for every $m \in \Delta(x * y)$,

$$(L[r])^{\mathcal{M}} \vDash S_{\bar{\bar{\alpha}}+m} \text{ is not stationary,}$$

then $x <^G y$.

Proof. Suppose that there is such a real r. By the Löwenheim-Skolem theorem, it has the property described in the formulation with respect to *all* suitable models $\mathcal{M}$, in particular for $\mathbb{H}^{\mathbb{P}}_\Theta$, where Θ is sufficiently large. That is there is $\alpha < \omega_3$ such that for every $m \in \Delta(x * y)$

$$L_\Theta[r] \vDash S_{\alpha+m} \text{ is not stationary.}$$

Thus in particular the stationary kill of at least some $S_{\alpha+m}$ was coded by a real. Lemma 3 implies that $\dot{\mathbb{Q}}^G_\alpha$ introduced a real u_α (perhaps different from r) coding stationary kill for all $m \in \Delta(a * b)$ for some reals $a \dot{<}^G_\alpha b$, while there are no reals coding a stationary kill of $S_{\alpha+m}$ for $m \notin \Delta(a*b)$. Therefore $\Delta(a*b) \supset \Delta(x*y)$, which yields $\Delta(a*b) = \Delta(x*y)$. From the above, it follows that $a = x$, $b = y$ and hence $x \dot{<}^G_\alpha y$, which finishes our proof. □

Combining Lemmata 4,5 and the fact that we have added dominating reals cofinally often, we get the following result.

Theorem 1. It is consistent with $\mathfrak{c} = \mathfrak{b} = \aleph_3$, that there is a projective (indeed Δ^1_3-definable) wellorder of the reals.

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[5] In the above, $\vec{A}$, $S_{\bar{\alpha}_m}$, $S_{\bar{\bar{\alpha}}_m}$, ω_1, ω_2, ω_3 refer of course to their interpretations in the model $\mathcal{N}$.

3. Projective mad families

The main result of this section and of the whole paper is the following theorem which answers (7, Question 19) in the positive.

Theorem 2. It is consistent with $\mathfrak{c} = \mathfrak{b} = \aleph_3$, that there is a Δ^1_3-definable wellorder of the reals and a Π^1_2-definable ω-mad subfamily of $[\omega]^\omega$ (resp. ω^ω).

The proof is completely analogous to that of Theorem 2. Moreover, we believe that adding the argument responsible for ω-mad families would just make the proof in the previous section messier without introducing any new ideas besides those used in the proof of Theorem 1 and in (7). Therefore the proof of Theorem 2 is just sketched here. More precisely, we shall define the corresponding poset $\mathbb{P}_{\omega_3}$ and leave it to the reader to verify that the proof of Theorem 1 can be carried over.

Let $\mathcal{B} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega\rangle$ be as in the proof of Theorem 1. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma : \alpha \leq \omega_3, \gamma < \omega_3\rangle$, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset and in $L^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable wellorder of the reals, a Π^1_2-definable ω-mad subfamily of $[\omega]^\omega$ (the case of subfamilies of ω^ω is completely analogous, see (7)), and $\mathfrak{c} = \mathfrak{b} = \aleph_3$.

$\mathbb{P}_0$ is a three step iteration $\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$, where $\mathbb{P}^0$ and $\mathbb{P}^1$ are exactly the same as in the proof of Theorem 1. The poset $\mathbb{P}^2$ uses the following modification of Definition 1, where ϕ is as in $(**)_\alpha$ from the previous section.

Definition 2. Let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$ and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. Let also η be a countable limit ordinal. We denote by $\mathcal{L}_\eta(X, X')$ the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

1. $|r| \geq \eta$
2. if $\gamma < \eta$ then $r(\gamma) = 0$
3. if $\gamma < |r|$ then $\gamma \in X$ iff $r(\eta + 3\gamma) = 1$
4. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(\eta + 3\gamma + 1) = 1$
5. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \models \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$ holds in $\mathcal{M}$.

The extension relation is end-extension.

Set $\mathbb{P}^2_{\alpha+m} = \prod_{\eta \in Lim(\omega_1)} \mathcal{L}_\eta(X_{\alpha+m}, X_\alpha)$ and let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. By the Δ-system Lemma in $L^{\mathbb{P}^0*\mathbb{P}^1}$ the poset $\mathbb{P}^2$ has the ω_2-c.c. Analogously to Lemma 1 we conclude that $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is ω-distributive.

If α is limit and $m \in \omega$, we shall refer to the localizing set for $X_{\alpha+m}$ produced by $\mathcal{L}_\eta(X_{\alpha+m}, X_\alpha)$ as $Y_{\alpha+m,\eta}$. That is $Y_{\alpha+m,\eta} \subseteq \omega_1 \setminus \eta$ and $Y_{\alpha+m,\eta}$ codes both $X_{\alpha+m}$ and X_α.

Every $\mathbb{Q}_\alpha$ is going to add a generic real whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{u}_\alpha$ and similarly to the proof of Lemma 2 one can prove that $L[G_\alpha]\cap\omega^\omega = L[\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha\rangle]\cap\omega^\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α. This gives us a canonical wellorder of the reals in $L[G_\alpha]$ which depends only on the sequence $\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha\rangle$, whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{<}_\alpha$. We can additionally arrange that for $\alpha < \beta$ we have that $1_{\mathbb{P}_\beta}$ forces $\dot{<}_\alpha$ to be an initial segment of $\dot{<}_\beta$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L, $<^G = \bigcup\{\dot{<}_\alpha^G : \alpha < \omega_3\}$ will be the desired wellorder of the reals.

We proceed with the recursive construction of $\mathbb{P}_{\omega_3}$. Along this construction we shall also define a sequence $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3)\rangle$, where $\dot{A}_\alpha$ is a $\mathbb{P}_\alpha$-name for a subset of $[\alpha, \alpha+\omega)$. Let $i : \omega\times\omega \to \omega$ and

$$j_\nu : \nu \cup \{\langle\zeta,\xi\rangle : \zeta < \xi < \nu\} \to Lim(\omega_2)$$

be some bijections, where $\nu \in [\omega_2, \omega_3)$. Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

Case 1. α is a limit ordinal that can be written in the form $\omega_2\cdot\alpha' + \xi$ for some $\alpha' > 0$, $\xi < \omega_2$, and the preimage $j^{-1}(\xi)$ is a tuple $\langle\xi_0,\xi_1\rangle$ for some $\xi_0 \dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'} \xi_1$, where $j = j_{o.t.(\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'})}$. In this case the definition of $\dot{\mathbb{Q}}_\alpha$ is the same as in the proof of Theorem 1.

Case 2. α is a limit ordinal that can be written in the form $\omega_2\cdot\alpha' + \xi$ for some $\alpha' > 0$ and the preimage $j^{-1}(\xi)$ is an ordinal $\zeta \in o.t.(\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'})$, where $j = j_{o.t.(\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'})}$. In this case we use a simplified version of the poset from (7, Theorem 1). More precisely, ordinals fulfilling the condition above will be used for the construction of a Π^1_2 definable ω-mad family $\mathcal{A}$.

For a subset s of ω and $l \in |s|$ $(=\mathrm{card}(s) \le \omega)$ we denote by $s(l)$ the l-th element of s. In what follows we shall denote by $E(s)$ and $O(s)$ the sets $\{s(2i) : 2i \in |s|\}$ and $\{s(2i+1) : 2i+1 \in |s|\}$, respectively. Let $\mathcal{A}_\alpha$ be the approximation to $\mathcal{A}$ constructed thus far. Suppose also that

$$\forall \mathcal{D} \in [\mathcal{A}_\alpha]^{<\omega}\ \forall B \in \mathcal{B}\ (|E(B)\setminus \cup\mathcal{D}| = |O(B)\setminus\cup\mathcal{D}| = \omega). \tag{$*$}$$

Observe that equation $(*)$ yields $|E(B)\setminus\cup\mathcal{D}| = |O(B)\setminus\cup\mathcal{D}| = \omega$ for every $\mathcal{D} \in [\mathcal{B}\cup\mathcal{A}_\alpha]^{<\omega}$ and $B \in \mathcal{B}\setminus\mathcal{D}$. Let x_ζ be the ζ-th real in $L[G_{\omega_2\cdot\alpha'}]\cap[\omega]^\omega$ according to the wellorder $\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'}$. Set $C_n = \{z_\zeta(i(n,m)) : m \in \omega\} \in [\omega]^\omega$ and $C = \{C_n : n \in \omega\}$. Unless the following holds, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset: none of the C_n's is covered by a finite subfamily of $\mathcal{A}_\alpha$. In the latter case $\mathbb{Q}_\alpha := \dot{\mathbb{Q}}_\alpha^{G_\alpha}$ is defined as follows.

Let us fix a limit ordinal $\eta_\alpha \in \omega_1$ such that there are no finite subsets $J, \mathcal{E}$ of $(\omega_1 \setminus \eta_\alpha) \times \omega$, $\mathcal{A}_\alpha$, respectively and $n \in \omega$, such that $C_n \subset \bigcup_{\langle \eta, m\rangle \in J} B_{\eta,m} \cup \bigcup \mathcal{E}$. (The almost disjointness of the $B_{\eta,m}$'s imply that if $C_n \subset \bigcup \mathcal{B}' \cup \bigcup \mathcal{A}'$ for some $\mathcal{B}' \in [\mathcal{B}]^{<\omega}$ and $\mathcal{A}' \in [\mathcal{A}_\alpha]^{<\omega}$, then $C_n \setminus \bigcup \mathcal{A}'$ has finite intersection with all elements of $\mathcal{B} \setminus \mathcal{B}'$. This easily yields the existence of such an η_α.) Let Z_α be an infinite subset of ω coding a surjection from ω onto η_α. For a subset s of ω we denote by Δs the set $\{2k+1 : k \in (\sup s \setminus s)\} \cup \{2k+2 : k \in s\}$.

In $V[G_\alpha]$, $\mathbb{Q}_\alpha$ consists of pairs $\langle s, s^*\rangle$ such that $s \in [\omega]^{<\omega}$, $s^* \in [\{B_{\beta,m} : m \in \Delta(s), \beta \in Y_{\alpha+m,\eta_\alpha}\} \cup \mathcal{A}_\alpha]^{<\omega}$, and for every $2n \in |s \cap B_{0,0}|$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap B_{0,0})(2n) = B_{0,0}(2m)$. For conditions $p = \langle s, s^*\rangle$ and $q = \langle t, t^*\rangle$ in $\mathbb{Q}_\alpha$, we let $q \leq p$ if and only if t is an end-extension of s and $t \setminus s$ has empty intersection with all elements of s^*.

Let h_α be a $\mathbb{Q}_\alpha$-generic filter over $L[G_\alpha]$. Set $u_\alpha = \bigcup_{\langle s, s^*\rangle \in h_\alpha} s$, $A_\alpha = \alpha + (\omega \setminus \Delta(u_\alpha))$, and $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \cup \{u_\alpha\}$. As a consequence of the definition of $\mathbb{Q}_\alpha$ and the genericity of h_α we get[6]

(1) $u_\alpha \in [\omega]^\omega$, u_α is almost disjoint from all elements of $\mathcal{A}_\alpha$, and has infinite intersection with C_n for all $n \in \omega$;

(2) If $m \in \Delta(u_\alpha)$, then $|u_\alpha \cap B_{\beta,m}| < \omega$ if and only if $\beta \in Y_{\alpha+m,\eta_\alpha}$;

(3) For every $n \in \omega$, $n \in Z_\alpha$ if and only if there exists $m \in \omega$ such that $(u_\alpha \cap B_{0,0})(2n) = B_{0,0}(2m)$; and

(4) Equation $(*)$ holds for $\alpha + 1$, i.e. for every $B \in \mathcal{B}$ and a finite subfamily $\mathcal{A}'$ of $\mathcal{A}_{\alpha+1}$, $\mathcal{A}'$ covers neither a cofinite part of $E(B)$ nor of $O(B)$.

By (2) u_α codes $Y_{\alpha+m,\eta_\alpha}$ for all $m \in \Delta(u_\alpha)$.

Case 3. If α is not of the form above, i.e. α is a successor or $\alpha < \omega_2$, then $\dot{A}_\alpha$ is a name for the empty set and $\dot{\mathbb{Q}}_\alpha$ is a name for the poset adding a dominating real defined in *Case 2* of the proof of Theorem 1.

With this the definitions of $\mathbb{P} = \mathbb{P}_{\omega_3}$ and $\langle \dot{A}_\alpha : \alpha \in Lim(\omega_3)\rangle$ are complete. Let G be a $\mathbb{P}$-generic over L.

Just as in the proof of Theorem 1 one can verify that Lemmata 2 and 3 hold true. These were of crucial importance for the proof of Corollary 1, which in turn was used in the proofs of Lemmata 4 and 5. Again, a direct verification shows that all of these statements still hold and hence $<^G$ is a Δ^1_3-wellorder of the reals in $L[G]$.

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[6] See (7, Claim 11) for an analogous argument

Lemma 2 implies that the family $\mathcal{A}$ we construct in the instances of *Case 2* is an ω-mad subfamily of $[\omega]^\omega$. Condition (3) above yields $\eta_\alpha < \omega_1^{\mathcal{M}}$ for all countable suitable models $\mathcal{M}$ containing $\dot{u}_\alpha^G$ provided that at stage α, *Case 2* took place (i.e., there is a condition in G which forces this). Combining this with the ideas of the proofs of Lemmata 4 and 5 we get that *$a \in \mathcal{A}$ iff for every countable suitable model $\mathcal{M}$ of* ZF$^-$ *containing a as an element there exists $\bar{\alpha} < \omega_3^{\mathcal{M}}$ such that $S^{\mathcal{M}}_{\bar{\alpha}+k}$ is nonstationary in $(L[a])^{\mathcal{M}}$ for all $k \in \Delta(a)$.* This provides a Π^1_2 definition of $\mathcal{A}$, which finishes our proof of Theorem 2.

4. Questions

The consistency of the existence of a Δ^1_3-definable wellorder of the reals in the presence of $\mathfrak{c} \geq \aleph_3$ and MA, is still open. A second question naturally emerging from the developed techniques is the existence of a model in which a desired inequality betwen the cardinal characteristics of the real line holds, there is a Δ^1_3-definable wellorder of the reals and $\mathfrak{c} \geq \aleph_3$. Note that the bookkeeping argument which we have used in Theorems 1 and 2 allows only for handling of countable objects, which presents an additional difficulty in obtaining such models.

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