Projective Maximal Families of Orthogonal Measures with Large Continuum

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November 28, 2011
The results which we are to consider, concern the definability of certain combinatorial objects on the real line and in particular the question of how low in the projective hierarchy such objects exist.

- (Mathias) There is no $\Sigma^1_1$ mad family in $[\omega]^{\omega}$. 
- (Miller) If $V = L$, then there is a $\Pi^1_1$ mad family in $[\omega]^{\omega}$.
Theorem (L. Harrington)

The existence of $\Delta^1_3$-definable wellorder of the reals is consistent with $\mathfrak{c}$ being as large as desired and MA.

Theorem (S. D. Friedman)

The existence of $\Delta^1_3$-definable wellorder of the reals is consistent with $\mathfrak{c} = \omega_2$ and MA.
Note that $\Delta^1_3$ wellorder is optimal for models of $\mathfrak{c} > \aleph_1$, since by a result of Mansfield if there is a $\Sigma^1_2$ definable w.o. on $\mathbb{R}$, then all reals are constructible.
The existence of a $\Delta^1_3$-definable w.o. on the reals is consistent with each of the following:

- (F., Friedman) $\mathfrak{d} < c = \omega_2$; $b < g = c = \omega_2$; $b < a = s = c = \omega_2$;
- (Friedman, Zdomskyy) the existence of a $\Pi^1_2$ definable $\omega$-mad family on $[\omega]^\omega$ together with $b = c = \omega_2$;
- (F., Friedman, Zdomskyy) the existence of a $\Pi^1_2$ definable $\omega$-mad family on $[\omega]^\omega$ together with $b = c = \omega_3$;
- (F., Friedman, Zdomskyy) Martin’s Axiom and $c = \omega_3$. 
Let $X$ be a Polish space, and let $P(X)$ be the Polish space of Borel probability measures on $X$.

- If $\mu, \nu \in P(X)$ then $\mu$ and $\nu$ are said to be **orthogonal**, written $\mu \perp \nu$, if there is a Borel set $B \subseteq X$ such that $\mu(B) = 0$ and $\nu(X \setminus B) = 0$.

- A set of measures $\mathcal{A} \subseteq P(X)$ is said to be **orthogonal** if whenever $\mu, \nu \in \mathcal{A}$ and $\mu \neq \nu$ then $\mu \perp \nu$.

- A **maximal orthogonal family**, or m.o. family, is an orthogonal family $\mathcal{A} \subseteq P(X)$ which is maximal under inclusion.
Theorem (Preiss, Rataj, 1985)

There are no analytic m.o. families.

Theorem (F., Törnquist, 2009)

If $V = L$ then there is a $\Pi^1_1$ m.o. family.
We study $\Pi^1_2$ m.o. families in the context of $c \geq \omega_2$, with the additional requirement that there is a $\Delta^1_3$-definable wellorder of $\mathbb{R}$.

**Theorem (F., Freidman, Törnquist, 2011)**

*It is consistent with $c = b = \omega_3$ that there is a $\Delta^1_3$-definable well order of the reals, a $\Pi^1_2$ definable maximal orthogonal family of measures and there are no $\Sigma^1_2$-definable maximal sets of orthogonal measures.*

There is nothing special about $c = \omega_3$. In fact the same result can be obtained for any reasonable value of $c$. 
Theorem (F., Freidman, Törnquist, 2011)

It is consistent with $\mathfrak{b} = \omega_1$, $\mathfrak{c} = \omega_2$ that there is a $\Delta^1_3$-definable wellorder of the reals, a $\Pi^1_2$ definable maximal orthogonal family of measures and there are no $\Sigma^1_2$-definable maximal sets of orthogonal measures.
Let $X$ be a Polish space. Recall that if $\mu, \nu \in P(X)$ then $\mu$ is absolutely continuous with respect to $\nu$, written $\mu \ll \nu$, if for all Borel subsets of $X$ we have that $\nu(B) = 0$ implies that $\mu(B) = 0$. Two measures $\mu, \nu \in P(2^\omega)$ are called absolutely equivalent, written $\mu \approx \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$. 
For $s \in 2^{<\omega}$, let $N_s = \{ x \in 2^{\omega} : s \subseteq x \}$. Let $p(2^{\omega})$ be the set of all $f : 2^{<\omega} \rightarrow [0, 1]$ such that

$$f(\emptyset) = 1 \land (\forall s \in 2^{<\omega}) f(s) = f(s\upharpoonright 0) + f(s\upharpoonright 1).$$

The spaces $p(2^{\omega})$ and $P(2^{\omega})$ are isomorphic via the $f \mapsto \mu_f$ where $\mu_f \in P(2^{\omega})$ is the measure uniquely determined by $\mu_f(N_s) = f(s)$ for all $s \in 2^{<\omega}$. The unique real $f \in p(2^{\omega})$ such that $\mu = \mu_f$ is called the code for $\mu$. 
Let $P_c(2^\omega)$ be the set of all non-atomic measures and $p_c(2^\omega)$ the set of all codes for non-atomic measures. We describe now a way of coding a given real $z \in 2^\omega$ into a measure $\mu \in P_c(2^\omega)$.

1. Let $\mu \in P_c(2^\omega)$, $s \in 2^{<\omega}$. Then let $t(s, \mu)$ be the lexicographically least $t \in 2^{<\omega}$ such that $s \subseteq t$, $\mu(N_t \cap 0) > 0$ and $\mu(N_t \cap 1) > 0$, if it exists and otherwise we let $t(s, \mu) = \emptyset$.

2. Define recursively $t^\mu_n \in 2^{<\omega}$ by letting $t^\mu_0 = \emptyset$ and $t^\mu_{n+1} = t(t^\mu_n \cap 0, \mu)$. Since $\mu$ is non-atomic, we have $\text{lh}(t^\mu_{n+1}) > \text{lh}(t^\mu_n)$. Let $t^\mu_\infty = \bigcup_{n=0}^\infty t^\mu_n$.

3. For $f \in p_c(2^\omega)$ and $n \in \omega \cup \{\infty\}$ we will write $t^f_n$ for $t^\mu_{n^f}$. Clearly the sequence $(t^f_n : n \in \omega)$ is recursive in $f$. 
Define the relation $R \subseteq p_c(2^\omega) \times 2^\omega$ as follows: $R(f, z)$ holds iff for all $n \in \omega$ we have

$$
(z(n) = 1 \iff f(t^f_n \uparrow 0) = \frac{2}{3} f(t^f_n) \land f(t^f_n \uparrow 1) = \frac{1}{3} f(t_n)) \land
$$

$$
(z(n) = 0 \iff f(t^f_n \uparrow 0) = \frac{1}{3} f(t^f_n) \land f(t^f_n \uparrow 1) = \frac{2}{3} f(t^f_n)).
$$

Whenever $(f, z) \in R$ we say that $f$ codes $z$. 
Lemma (Coding Lemma)

There is a recursive function $G : p_c(2^\omega) \times 2^\omega \rightarrow p_c(2^\omega)$ such that for all $f \in p_c(2^\omega)$ and $z \in 2^\omega$ we have:

- $\mu_{G(f, z)} \approx \mu_f$, and
- $R(G(f, z), z)$
Proposition

Let \( a \in \mathbb{R} \) and suppose that there either is a Cohen real over \( L[a] \) or there is a random real over \( L[a] \). Then there is no \( \Sigma^1_2(a) \) m.o. family.
We proceed with the construction of a generic extension of $L$ in which there is a $\Delta^1_3$ definable well order of the reals, there is a $\Pi^1_2$-definable m.o. family, there are no $\Sigma^1_2$-definable m.o. families and $c = \aleph_3$.

A transitive $ZF^- \text{ model is suitable if } \omega_3^M \text{ exists and } \omega_3^M = \omega_3^{L^M}$.
If $\mathcal{M}$ is suitable then also $\omega_1^\mathcal{M} = \omega_1^{L^\mathcal{M}}$ and $\omega_2^\mathcal{M} = \omega_2^{L^\mathcal{M}}$. 
Fix a $\diamondsuit_{\omega_2}(\text{cof}(\omega_1))$ sequence $\langle G_\xi : \xi \in \omega_2 \cap \text{cof}(\omega_1) \rangle$ which is $\Sigma_1$-definable over $L_{\omega_2}$.

- For $\alpha < \omega_3$, let $W_\alpha$ be the $L$-least subset of $\omega_2$ coding $\alpha$ and let $S_\alpha = \{ \xi \in \omega_2 \cap \text{cof}(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset \}$.

Then $\vec{S} = \langle S_\alpha : 1 < \alpha < \omega_3 \rangle$ is a sequence of stationary subsets of $\omega_2 \cap \text{cof}(\omega_1)$, which are mutually almost disjoint.
For every $\alpha$ such that $\omega \leq \alpha < \omega_3$ shoot a club $C_\alpha$ disjoint from $S_\alpha$ via the poset $P^0_\alpha$, consisting of all closed subsets of $\omega_2$ which are disjoint from $S_\alpha$ with the extension relation being end-extension, and let $P^0 = \prod_{\alpha < \omega_3} P^0_\alpha$ be the direct product of the $P^0_\alpha$'s with supports of size $\omega_1$, where for $\alpha \in \omega$, $P^0_\alpha$ is the trivial poset. Then $P^0$ is countably closed, $\omega_2$-distributive and $\omega_3$-c.c.
For every $\alpha$ such that $\omega \leq \alpha < \omega_3$ let $D_\alpha \subseteq \omega_3$ be a set coding the triple $\langle C_\alpha, W_\alpha, W_\gamma \rangle$ where $\gamma$ is the largest limit ordinal $\leq \alpha$. Let

$$E_\alpha = \{M \cap \omega_2 : M < L_{\alpha+\omega_2+1}[D_\alpha], \omega_1 \cup \{D_\alpha\} \subseteq M\}.$$ 

Then $E_\alpha$ is a club on $\omega_2$. Choose $Z_\alpha \subseteq \omega_2$ such that:

- $\text{Even}(Z_\alpha) = D_\alpha$, where $\text{Even}(Z_\alpha) = \{\beta : 2 \cdot \beta \in Z_\alpha\}$, and
- if $\beta < \omega_2$ is the $\omega_2^M$ for some suitable model $M$ such that $Z_\alpha \cap \beta \in M$, then $\beta \in E_\alpha$. 

Then we have

\[(*)_\alpha: \text{ If } \beta < \omega_2, \mathcal{M} \text{ is a suitable model such that } \omega_1 \subset \mathcal{M},\]
\[\omega_2^\mathcal{M} = \beta, \text{ and } Z_\alpha \cap \beta \in \mathcal{M}, \text{ then } \mathcal{M} \models \psi(\omega_2, Z_\alpha \cap \beta), \text{ where}\]
\[\psi(\omega_2, X) \text{ is the formula } \text{“Even}(X) \text{ codes a triple } \langle \bar{C}, \bar{W}, \bar{\bar{W}} \rangle,\]
\[\text{where } \bar{W} \text{ and } \bar{\bar{W}} \text{ are the } L\text{-least codes of ordinals } \bar{\alpha}, \bar{\alpha} < \omega_3 \text{ such that } \bar{\alpha} \text{ is the largest limit ordinal not exceeding } \bar{\alpha}, \text{ and } \bar{C} \text{ is a club in } \omega_2 \text{ disjoint from } S_{\bar{\alpha}}”.\]
Similarly to $\vec{S}$ define a sequence $\vec{A} = \langle A_\xi : \xi < \omega_2 \rangle$ of stationary subsets of $\omega_1$ using the “standard” $\lozenge$-sequence. Code $Z_\alpha$ by a subset $X_\alpha$ of $\omega_1$ with the poset $P^1_\alpha$ consisting of all pairs $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff $s_0$ is an initial segment of $t_0$, $s_1 \subseteq t_1$ and $t_0 \setminus s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$. 
Then $X_\alpha$ satisfies the following condition:

\[(**)_\alpha: \text{ If } \omega_1 < \beta \leq \omega_2 \text{ and } \mathcal{M} \text{ is a suitable model such that } \omega_2^\mathcal{M} = \beta \text{ and } \{X_\alpha\} \cup \omega_1 \subset \mathcal{M}, \text{ then } \mathcal{M} \models \phi(\omega_1, \omega_2, X_\alpha), \]\n
where $\phi(\omega_1, \omega_2, X)$ is the formula: “Using the sequence $\vec{A}$, $X$ almost disjointly codes a subset $\vec{Z}$ of $\omega_2$, such that $\text{Even}(\vec{Z})$ codes a triple $\langle \vec{C}, \vec{W}, \vec{\bar{W}} \rangle$, where $\vec{W}$ and $\vec{\bar{W}}$ are the $L$-least codes of ordinals $\vec{\alpha}, \vec{\alpha} < \omega_3$ such that $\vec{\alpha}$ is the largest limit ordinal not exceeding $\vec{\alpha}$, and $\vec{C}$ is a club in $\omega_2$ disjoint from $S_{\vec{\alpha}}$.”
Let $\mathbb{P}^1 = \prod_{\alpha < \omega_3} \mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is the trivial poset for all $\alpha \in \omega$, with countable support. Then $\mathbb{P}^1$ is countably closed and has the $\omega_2$-c.c.
Now we shall force a localization of the $X_\alpha$'s. Fix $\phi$ as in $(**)_\alpha$.

**Definition**

Let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X), \phi(\omega_1, \omega_2, X')$ hold in any suitable $\mathcal{M}$ with $\omega_1^\mathcal{M} = \omega_1^L$, $X, X'$ in $\mathcal{M}$. Denote by $\mathcal{L}(X, X')$ the p.o. of all $r : |r| \to 2$, where $|r| \in \text{Lim}(\omega_1)$, such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$
3. if $\gamma \leq |r|$, $\mathcal{M}$ is countable, suitable, such that $r \upharpoonright \gamma \in \mathcal{M}$ and $\gamma = \omega_1^\mathcal{M}$, then $\mathcal{M} \models \phi(\omega_1, \omega_2, X \cap \gamma) \land \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The extension relation is end-extension.
Set $\mathbb{P}_\alpha^2 = \mathcal{L}(X_{\alpha+m}, X_\alpha)$ for every $\alpha \in \text{Lim}(\omega_3) \setminus \{0\}$ and $m \in \omega$. Let $\mathbb{P}_0^2$ be the trivial poset for every $m \in \omega$ and let

$$\mathbb{P}^2 = \prod_{\alpha \in \text{Lim}(\omega_3)} \prod_{m \in \omega} \mathbb{P}_{\alpha+m}^2$$

with countable supports. By the $\Delta$-system Lemma in $L^{\mathbb{P}^0 \ast \mathbb{P}^1}$ the poset $\mathbb{P}^2$ has the $\omega_2$-c.c.
Observe that the poset $\mathbb{P}^2_{\alpha+m}$, where $\alpha > 0$, produces a generic function from $\omega_1$ (of $L^{\mathbb{P}^0 \ast \mathbb{P}^1}$) into 2, which is the characteristic function of a subset $Y_{\alpha+m}$ of $\omega_1$ with the following property:

\[ (***)_{\alpha}: \text{For every } \beta < \omega_1 \text{ and any suitable } \mathcal{M} \text{ such that } \omega_1^\mathcal{M} = \beta \text{ and } Y_{\alpha+m} \cap \beta \text{ belongs to } \mathcal{M}, \text{ we have} \]

\[ \mathcal{M} \models \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \phi(\omega_1, \omega_2, X_\alpha \cap \beta). \]
Lemma

The poset $\mathbb{P}_0 := \mathbb{P}^0 \ast \mathbb{P}^1 \ast \mathbb{P}^2$ is $\omega$-distributive.
For $\alpha : 1 \leq \alpha < \omega_3$ we will say that there is a stationary kill of $S_\alpha$, if there is a closed unbounded set $C$ disjoint from $S_\alpha$. We will say that the stationary kill of $S_\alpha$ is coded by a real, if there is a closed unbounded set constructible from this real.
Let $\vec{B} = \langle B_\zeta, m : \zeta < \omega_1, m \in \omega \rangle \subseteq \omega$ be a nicely definable sequence of a. d. sets. We will define a f. s. iteration $\langle P_\alpha, Q_\beta : \alpha \leq \omega_3, \beta < \omega_3 \rangle$ such that $P_0 = P^0 \ast P^1 \ast P^2$, for every $\alpha < \omega_3$, $Q_\alpha$ is a $P_\alpha$-name for a $\sigma$-centered poset, in $L^{P_\omega_3}$ there is a $\Delta^1_3$-definable wellorder of the reals, a $\Pi^1_2$-definable m.o. family and there are no $\Sigma^1_2$-definable m.o. families.

Along the iteration $\forall \alpha < \omega_3$, in $V^{P_\alpha}$ we will define a set $O_\alpha \subseteq P_c(2^\omega)$ of orthogonal measures and for $\alpha \in Lim(\omega_3)$, a subset $A_\alpha \subseteq [\alpha, \alpha + \omega)$.
\( \mathbb{Q}_\alpha \) will add a generic real \( u_\alpha \). We will have that 
\[
L[G_\alpha] \cap \omega \omega = L[\langle \dot{u}_\xi^G : \xi < \alpha \rangle] \cap \omega \omega.
\]
This gives a canonical w.o. of the reals in \( L[G_\alpha] \) which depends only on \( \langle \dot{u}_\xi : \xi < \alpha \rangle \), whose 
\( p_\alpha \)-name will be denoted by \( \dot{<}_\alpha \). Additionally arrange that for 
\( \alpha < \beta \), \( <_\alpha \) is an initial segment of \( <_\beta \), where \( <_\alpha = \dot{<}_\alpha^G \) and
\( <_\beta = \dot{<}_\beta^G \). Then if \( G \) is a \( \mathbb{P}_{\omega_3} \)-generic filter:

1. \( \dot{<}^G = \bigcup \{ \dot{<}_\alpha^G : \alpha < \omega_3 \} \) will be is the desired w.o. of \( \mathbb{R} \) and,
2. \( O = \bigcup_{\alpha < \omega_3} O_\alpha \subseteq P_c(2^\omega) \) will be \( \Pi^1_2 \)-definable maximal family of orthogonal measures.
Recursively define $P_{\omega_3}$ as follows. For $\nu \in [\omega_2, \omega_3)$ let
\[ i_\nu : \nu \cup \{ \langle \xi, \eta \rangle : \xi < \eta < \nu \} \rightarrow Lim(\omega_3) \] be a fixed bijection. If $G_\alpha$ is a $P_\alpha$-generic, $<_\alpha = \dot{<_\alpha}^G_\alpha$ and $x, y \in L[G_\alpha] \cap \omega \omega$ such that $x <_\alpha y$, let $x \ast y = \{2n\}_{n \in x} \cup \{2n + 1\}_{n \in y}$ and
\[ \Delta(x \ast y) = \{2n + 2 : n \in x \ast y\} \cup \{2n + 1 : n \notin x \ast y\}. \]

Suppose $P_\alpha$ has been defined and let $G_\alpha$ be a $P_\alpha$-generic filter. If $\alpha = \omega_2 \cdot \alpha' + \xi$, where $\alpha' > 0$, $\xi \in Lim(\omega_2)$, let $\nu = o.t.(\dot{<_\omega}^{G_\alpha}_{\omega_2 \cdot \alpha'})$ and let $i = i_\nu$. 

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Coding the w.o.. If $i^{-1}(\xi) = \langle \xi_0, \xi_1 \rangle$ for some $\xi_0 < \xi_1 < \nu$, let $x_{\xi_0}, x_{\xi_1}$ be the $\xi_0$-th, $\xi_1$-th reals in $L[G_{\omega_2 \cdot \alpha'}]$ according to $\dot{G}_\alpha$. In $L^{p_\alpha}$ let

$$Q_\alpha = \{ \langle s_0, s_1 \rangle : s_0 \in [\omega]^{<\omega}, s_1 \in \bigcup_{m \in \Delta(x_{\xi_0} * x_{\xi_1})} Y_{\alpha + m} \times \{m\}]^{<\omega} \},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff $s_1 \subseteq t_1$, $s_0$ is an initial segment of $t_0$ and $(t_0 \setminus s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m \rangle \in s_1$. Let $u_\alpha$ be the generic real added by $Q_\alpha$, $A_\alpha = \alpha + \omega \setminus \Delta(x_{\xi_0} * x_{\xi_1})$ and $O_\alpha = \emptyset$. 
Coding the m.o. family. Let $i^{-1}(\xi) = \zeta \in \nu$. If the $\zeta$-th real $x_\zeta$ according to $\leq_{\omega_2 \cdot \alpha'}$ is not the code of a measure orthogonal to $O'_\alpha = \bigcup_{\gamma < \alpha} O_\gamma$, let $Q_\alpha$ be trivial, $A_\alpha = \emptyset$, $O_\alpha = \emptyset$. Otherwise, let

$$Q_\alpha = \{\langle s_0, s_1 \rangle: s_0 \in [\omega]^{<\omega}, s_1 \in \bigcup_{m \in \Delta(x_\zeta)} Y_{\alpha+m} \times \{m\}]^{<\omega}\},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff $s_1 \subseteq t_1$, $s_0$ is an initial segment of $t_0$ and $(t_0 \setminus s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m \rangle \in s_1$. Let $u_\alpha$ be the generic real added by $Q_\alpha$. In $L^{P_{\alpha+1}} = L^{P_{\alpha} \ast Q_\alpha}$ let $g_\alpha = G(x_\zeta, u_\alpha)$ be the code of a measure equivalent to $\mu_{x_\zeta}$ which codes $u_\alpha$, let $O_\alpha = \{\mu g_\alpha\}$ and let $A_\alpha = \alpha + \omega \setminus \Delta(u_\alpha)$. 
If $\alpha$ is not of the above form, i.e. $\alpha$ is a successor or $\alpha \in \omega_2$, let $Q_\alpha$ be the following poset for adding a dominating real:

$$Q_\alpha = \{ \langle s_0, s_1 \rangle : s_0 \in \omega^{<\omega}, s_1 \in \overline{\{o.t.(\dot{\mathcal{G}}_\alpha)^{<\omega}\}} \},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff $s_0$ is an initial segment of $t_0$, $s_1 \subseteq t_1$, and $t_0(n) > x_\xi(n)$ for all $n \in \text{dom}(t_0) \setminus \text{dom}(s_0)$ and $\xi \in s_1$, where $x_\xi$ is the $\xi$-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $\dot{\mathcal{G}}_\alpha$. Let $A_\alpha = \emptyset$, $O_\alpha = \emptyset$. 
With this the definition of $\mathbb{P}_{\omega_3}$ is complete. Let $O = \bigcup_{\alpha < \omega_3} O_{\alpha}$.

Note in particular, that if $\mu \in O$, then $f_\mu$ codes $u_\alpha$ for some $\alpha \in \omega_3$. By definition $u_\alpha$ codes the code $f_\nu$ for a measure $\nu$ equivalent to $\mu$ and the sequence $\langle Y_{\alpha+m} : m \in \Delta(f_\nu) \rangle$. We will write $f_\nu = r(\mu)$. 
Lemma A

Let $\gamma \leq \omega_3$ and let $G_\gamma$ be a $\mathbb{P}_\gamma$-generic filter over $L$. Then

$L[G_\gamma] \cap \omega^\omega = L[\langle \dot{u}_\delta^{G_\gamma} : \delta < \gamma \rangle] \cap \omega^\omega$. 
Lemma B

Let $G$ be a $\mathbb{P}$-generic filter over $L$. Then for $\xi \in \bigcup_{\alpha \in \text{Lim}(\omega_3)} \dot{A}^G_{\alpha}$ there is no real coding a stationary kill of $S_\xi$. 
Corollary A

Let $G$ be $\mathbb{P}$-generic over $L$ and let $x, y$ be reals in $L[G]$. Then

- $x <^G y$ iff there is $\alpha < \omega_3$ such that for all $m$, the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x \ast y)$.

- $\mu \in O$ iff there is $\alpha < \omega_3$ such that for all $m$, the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(r(\mu))$. 
Proof:
Let \( x <^G y \) and let \( \alpha' > 0 \) be minimal with \( x, y \in L[G_{\omega_2 \cdot \alpha'}] \), \( i = i_{o.t.}(\dot{\omega}_{\omega_2 \cdot \alpha'}) \). Find \( \xi \in Lim(\omega_2) \) such that \( i(\xi) = (\xi_x, \xi_y) \) where \( x, y \) are the \( \xi_x \)-th, \( \xi_y \)-th real resp. in \( L[G_{\omega_2 \cdot \alpha'}] \) according to \( \dot{\omega}_{\omega_2 \cdot \alpha'} \).

Let \( \alpha = \omega_2 \cdot \alpha' + \xi \). Then \( \mathbb{Q}_\alpha \) adds a real coding a stationary kill for \( S_{\alpha+m} \) for all \( m \in \Delta(x \ast y) \). On the other hand if \( m \notin \Delta(x \ast y) \), then \( \alpha + m \in \dot{A}_\alpha^G = \alpha + (\omega \setminus \Delta(x \ast y)) \) and so by Lemma B, there is no real in \( L[G] \) coding the stationary kill of \( S_{\alpha+m} \).
Now suppose that there exists $\alpha$ such that the stationary kill of $S_{\alpha+m}$ is coded by a real iff $m \in \Delta(x \ast y)$. Since the stationary kill of some $\alpha + m$'s is coded by a real in $L[G]$, Lemma B implies that $\dot{Q}_\alpha^G$ introduced a real coding stationary kill for all $m \in \Delta(a \ast b)$ for some reals $a \lessdot^G \alpha b$, while there are no reals coding a stationary kill of $S_{\alpha+m}$ for $m \notin \Delta(a \ast b)$. Therefore $\Delta(a \ast b) = \Delta(x \ast y)$ and hence $a = x$, $b = y$, consequently $x \lessdot^G \alpha y$. 
Lemma

Let $G$ be a $\mathbb{P}$-generic real over $L$, $x, y \in \omega^\omega \cap L[G]$ and $\mu \in \mathcal{P}_c(2^\omega) \cap L[G]$. Then

1. $x < y$ iff there is a real $r$ such that for every countable suitable model $\mathcal{M}$ such that $r \in \mathcal{M}$, there is $\bar{\alpha} < \omega_3^\mathcal{M}$ such that for all $m \in \Delta(x \ast y)$, $(L[r])^\mathcal{M} \models S_{\bar{\alpha}+m}$ is not stationary.

2. $\mu \in O$ iff for every countable suitable model $\mathcal{M}$ such that $\mu \in \mathcal{M}$, there is $\bar{\alpha} < \omega_3^\mathcal{M}$ such that $S_{\bar{\alpha}+m}$ is nonstationary in $(L[r(\mu)])^\mathcal{M}$ for every $m \in \Delta(r(\mu))$. 

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By Corollary A, there exists $\alpha < \omega_3$ such that $\mathcal{Q}^G_\alpha$ adds a real $r$ coding a stationary kill of $S_{\alpha+m}$ for all $m \in \Delta(x \ast y)$. Let $\mathcal{M}$ be a countable suitable model containing $r$. It follows that $Y_{\alpha+m} \cap \omega_1^\mathcal{M} \in \mathcal{M}$ and hence $X_\alpha \cap \omega_1^\mathcal{M}$, $X_{\alpha+m} \cap \omega_1^\mathcal{M}$ also belong to $\mathcal{M}$. Observe that these sets are actually in $\mathcal{N} := (L[r])^\mathcal{M}$.

Note also that $\mathcal{N}$ is a countable suitable model and consequently by the definition of $\mathcal{L}(X_{\alpha+m}, X_\alpha)$ we have that for every $m \in \Delta(x \ast y)$, $\mathcal{N} \models$
“Using $\vec{A}$, $X_{\alpha+m} \cap \omega_1$ (resp. $X_\alpha \cap \omega_1$) a. d. codes a subset $\vec{Z}_m$ (resp. $\vec{Z}_0$) of $\omega_2$, such that $\text{Even}(\vec{Z}_m)$ (resp. $\text{Even}(\vec{Z}_0)$) codes $\langle \vec{C}, \vec{W}_m, \bar{W}_m \rangle$ (resp. $\langle \vec{C}, \vec{W}_0, \bar{W}_0 \rangle$), where $\vec{W}_m$, $\bar{W}_m$ are the $L$-least codes of ordinals $\vec{\alpha}_m$, $\bar{\alpha}_m < \omega_3$ (resp. $\vec{W}_0 = \bar{W}_0$ is the $L$-least code for a limit ordinal $\vec{\alpha}_0$) such that $\vec{\alpha}_m = \vec{\alpha}_0$ is the largest limit ordinal not exceeding $\vec{\alpha}_m$ and $\vec{C}$ is a club in $\omega_2$ disjoint from $S_{\vec{\alpha}_m}$.”

Note that in particular for every $m \neq m'$ in $\Delta(x \ast y)$, $\vec{\alpha}_m = \vec{\alpha}_{m'}$. 
Suppose there is such a real \( r \). By Löwenheim-Skolem, \( r \) has the property from the formulation with respect to all suitable \( M \), and so for \( H^P_\Theta \), where \( \Theta \) is sufficiently large. That is \( \exists \alpha < \omega_3 \) such that \( \forall m \in \Delta(x \ast y) \ L_\Theta[r] \models S_{\alpha+m} \) is not stationary. Then the stationary kill of at least some \( S_{\alpha+m} \) was coded by a real.

By Lemma B, \( Q^G_\alpha \) adds a real \( u_\alpha \) coding stationary kill for all \( m \in \Delta(a \ast b) \) for some reals \( a <^G_\alpha b \), while there are no reals coding a stationary kill of \( S_{\alpha+m} \) for \( m \not\in \Delta(a \ast b) \). Therefore \( \Delta(a \ast b) \supset \Delta(x \ast y) \), and so \( \Delta(a \ast b) = \Delta(x \ast y) \). Thus \( a = x, b = y \) and hence \( x <^G_\alpha y \).
Lemma

*The family $O$ is maximal in $P_c(2^\omega)$.***
Proof:
Suppose in $L^P\omega_3$ there is a code $x$ for a measure orthogonal to every measure in the family $O$. Choose $\alpha$ minimal such that $\alpha = \omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$, $\xi \in \text{Lim}(\omega_2)$ and $x \in L[G_{\omega_2 \cdot \alpha'}].$

Let $\nu = o.t.(G_{\omega_2 \cdot \alpha'})$ and let $i = i_\nu$. Then $x = x_\zeta$ is the $\zeta$-th real according to the wellorder $G_{\omega_2 \cdot \alpha'}$, where $\zeta \in \nu$ and so for some $\xi \in \text{Lim}(\omega_2)$, $i^{-1}(\xi) = \zeta$. But then $x_\zeta = x$ is the code of a measure orthogonal to $O'_\alpha = \bigcup_{\gamma < \alpha} O_\gamma$ and so by construction $O_\alpha$ contains a measure equivalent to $\mu_x$, which is a contradiction. \qed
To obtain a $\Pi^1_2$-definable m.o. family in $L^{P_{\omega_3}}$ consider the union of $O$ with the set of all point measures.

Since $P_{\omega_3}$ is a finite support iteration, we have added Cohen reals along the iteration cofinally often. Thus for every real $a$ in $L^{P_{\omega_3}}$ there is a Cohen real over $L[a]$ and so in $L^{P_{\omega_3}}$ there are no $\Sigma^1_2$ m.o. families. Also note that since cofinally often we have added dominating reals, $L^{P_{\omega_3}} \models b = \omega_3$. 
Theorem (F., Friedman, Törnquist)

*It is consistent with c = b = ω_3 that there is a Δ^1_3-definable wellorder of the reals, a Π^1_2-definable maximal orthogonal family of measures and there are no Σ^1_2-definable maximal sets of orthogonal measures.*
THANK YOU!
Let $X$ be a topological space and $\mu : \mathcal{B}(X) \to [0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(X) = 1$ and $\mu(\bigcup_{n \in \omega} A_n) = \sum_{n \in \omega} \mu(A_n)$ for every pairwise disjoint family $\{A_n\}_{n \in \omega} \subseteq \mathcal{B}(X)$. 
If $s_n$ enumerates $2^{<\omega}$ and $f_n : 2^\omega \to \mathbb{R}$ is defined as follows:

$$f_n(x) = \begin{cases} 1 & \text{if } s_n \subseteq x \\ 0 & \text{otherwise}, \end{cases}$$

then the metric on $P(2^\omega)$ defined by

$$\delta(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n-1} \left| \int f_n d\mu - \int f_n d\nu \right| \frac{\|f_n\|_\infty}{\|f_n\|_\infty}$$

makes the map $f \mapsto \mu_f$ an isometric bijection if we equip $p(2^\omega)$ with the metric

$$d(f, g) = \sum_{n=0}^{\infty} 2^{-n-1} |f(s_n) - g(s_n)|.$$