# DEFINABLE MAD FAMILIES AND FORCING AXIOMS

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ABSTRACT. We show that ZFC+ BPFA (i.e., the Bounded Proper Forcing Axiom) + "there are no $\mathbf{\Pi}^1_2$ infinite MAD families" implies that $\omega_1$ is a remarkable cardinal in $\mathbf{L}$. In other words, under BPFA and an anti-large cardinal assumption there is a $\mathbf{\Pi}^1_2$ infinite MAD family. It follows that the consistency strength of ZFC+ BPFA + "there are no projective infinite MAD families" is exactly a $\Sigma_1$-reflecting cardinal above a remarkable cardinal. In contrast, if every real has a sharp—and thus under BMM—there are no $\mathbf{\Sigma}^1_3$ infinite MAD families.

## 1. INTRODUCTION

A. By a MAD family, we mean a collection $\mathcal{A}$ with the following two properties: Firstly, $\mathcal{A}$ is an almost disjoint (short: a.d.) family, that is, $\mathcal{A}$ consists of infinite subsets of $\omega$ and any two distinct $a, a' \in \mathcal{A}$ are almost disjoint, i.e., $a \cap a'$ is finite. Secondly, for any infinite set $b \subseteq \omega$ there is $a \in \mathcal{A}$ such that $|a \cap b| = \aleph_0$; that is, $\mathcal{A}$ is *maximal* among a.d. families under inclusion.

While finite MAD families exist trivially, infinite MAD families can be constructed using the Axiom of Choice. This makes them an example of an irregular set somewhat analogous to a set without the Baire property or a non-measurable set.

As is well-known, Mathias [16, 15] proved that no infinite MAD family can be analytic. On the other hand, Arnie W. Miller in [17] constructed a co-analytic infinite MAD family under the assumption that $\mathbf{V} = \mathbf{L}$, showing that Mathias' result is optimal.

Mathias [16] also produced a model of ZF + DC in which there are no infinite MAD families, starting from the assumption of a Mahlo cardinal. Much later, Törnquist showed that there are no infinite MAD families in Solovay's model [28], and Horowitz and Shelah produced a model of ZF + "there are no infinite MAD families" without making any large cardinal assumption [9].

The definability of MAD families has been investigated under many natural extensions of the axiomatic system ZFC. It was shown recently by Neeman and Norwood and independently by Bakke-Haga, Törnquist and the second author, that under the Axiom of Determinacy (AD) no infinite MAD family can be an element of $\mathbf{L}(\mathbb{R})$; and under the Axiom of Projective Determinacy and the Axiom of Dependent Choice (DC) there is no projective infinite MAD family [19, 2, 25]. In fact, as Neeman and Norwood were first to show, under $\mathsf{AD}^+$ (a technical strengthening of AD introduced by Woodin) there are no infinite MAD families.

Another natural family of extensions of ZFC are *forcing axioms*. Definability properties of irregular sets of reals under such axioms have long been investigated. An early example is work of Martin and Solovay [14] showing that Martin's Axiom for sets of size $\aleph_1$ ($\mathsf{MA}_{\aleph_1}$)

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implies that all $\mathbf{\Sigma}^1_2$ sets are measurable and have the Baire property. It was shown by Törnquist [28] that similarly $\mathsf{MA}_{\aleph_1}$ rules out the existence of infinite MAD families which are $\mathbf{\Sigma}^1_2$ in terms of definitional complexity.

As is well-known, $\mathsf{MA}_{\aleph_1}$ is equiconsistent with $\mathsf{ZFC}$; adding assumptions about regularity of definable sets of reals can increase this consistency strength. For instance it was shown by Harrington and Shelah [8] that $\mathsf{MA}_{\aleph_1}$ together with "all projective sets are measurable and have the Baire property" is equiconsistent with $\mathsf{ZFC}+$ there is a weakly compact cardinal (in fact, it suffices to add "all $\mathbf{\Delta}^1_3$ sets have the Baire property" or "all $\mathbf{\Delta}^1_3$ sets are measurable" to $\mathsf{MA}_{\aleph_1}$ to drive up its consistency strength).[1]

On the other hand, the theory $\mathsf{MA}_{\aleph_1}$ + "there is no infinite projective MAD family" (and even $\mathsf{MA}_{\aleph_1}$ + "there is no infinite MAD family which is definable from a parameter in $\mathrm{On}^\omega$") is equiconsistent with $\mathsf{ZFC}$; this is a by-product of Horowitz and Shelah's construction in [9] of a model of $\mathsf{ZF}$ + "there are no infinite MAD families" (since as part of this construction they force $\mathsf{MA}_{\aleph_1}$ to hold).

Stronger forcing axioms such as the Proper Forcing Axiom ($\mathsf{PFA}$) imply that $\mathsf{AD}$ holds in $\mathbf{L}(\mathbb{R})$, as was shown by Steel [26]. Thus under $\mathsf{PFA}$, just as under $\mathsf{AD}$, all sets of reals in $\mathbf{L}(\mathbb{R})$ are regular: They are measurable, have the Baire property, and no infinite MAD family can be found among them. As we shall show, this is not the case for the so-called *bounded* version of this axiom, the Bounded Proper Forcing Axiom ($\mathsf{BPFA}$). This axiom was introduced by Goldstern and Shelah in [7] and later recognized to be equivalent to a principle of generic absoluteness by Bagaria [1]; it is quite a bit stronger than $\mathsf{MA}_{\aleph_1}$ but much weaker than $\mathsf{PFA}$.

In the present paper, we show that the assumption that there are no infinite MAD families with a simple definition drives up the consistency strength of $\mathsf{BPFA}$.

**Theorem 1.1.** $\mathsf{ZFC}+\mathsf{BPFA}+$ *"there is no infinite $\mathbf{\Pi}^1_2$ MAD family" implies that $\omega_1$ is a remarkable cardinal in $\mathbf{L}$.*

This is indeed remarkable, since $\mathsf{ZFC}+\mathsf{BPFA}$ alone is known to have consistency strength of a $\Sigma_1$-reflecting cardinal, which is weaker than a remarkable cardinal. As we have already mentioned, it is known that $\mathsf{BPFA}$ (in fact, just $\mathsf{MA}_{\aleph_1}$) rules out the existence of $\mathbf{\Sigma}^1_2$ (in fact, of $\omega_1$-Suslin) infinite MAD families [28], so the complexity of $\mathbf{\Pi}^1_2$ in the above theorem cannot be improved.

By results of Schindler (see [22]) the consistency strength of $\mathsf{ZFC}+\mathsf{BPFA}+$ "every set in $\mathbf{L}(\mathbb{R})$ is measurable" is exactly a $\Sigma_1$-reflecting cardinal above a remarkable cardinal. With Theorem 1.1 and Törnquist's result from [28] at our disposal, we can adapt Schindler's argument to gauge the exact consistency strength of $\mathsf{ZFC}+\mathsf{BPFA}+$ "there is no infinite $\mathbf{\Pi}^1_2$ MAD family".[2]

**Corollary 1.2.** *The following are equiconsistent:*

*(1)* $\mathsf{ZFC}$ + *"there exists a $\Sigma_1$-reflecting cardinal above a remarkable cardinal."*
*(2)* $\mathsf{ZFC}+\mathsf{BPFA}$ + *"there is no infinite $\mathbf{\Pi}^1_2$ MAD family."*
*(3)* $\mathsf{ZFC}+\mathsf{BPFA}+$ *"there is no infinite MAD family in $\mathbf{L}(\mathbb{R})$."*

[1]The reader can find a wealth of further results in [3, Chapter 9].

[2]We thank the anonymous referee for calling Corollary 1.2 to our attention.

*Proof.* Clearly, Item (2) implies that the theory in Item (1) is consistent: It is well-known that BPFA implies that $\omega_2$ is $\Sigma_1$-reflecting in $\mathbf{L}$; by Theorem 1.1 $\omega_1$ is remarkable in $\mathbf{L}$.

It suffices to show that the theory in Item (3) is consistent assuming Item (1). So suppose $\kappa < \lambda$, $\kappa$ is remarkable, and $\lambda$ is $\Sigma_1$-reflecting in $V$. Let $G$ be $\mathrm{Coll}(\omega, <\kappa)$-generic over $V$, and let $H$ be generic for a proper forcing over $V[G]$ such that $V[G][H] \models \mathsf{BPFA}$. Since $\kappa$ is remarkable, it follows as in [22, Lemma 24] that for any $x \in \mathcal{P}(\omega) \cap V[G][H]$ there is $\xi < \kappa$ and some generic $G'$ for $\mathrm{Coll}(\omega, \xi)$ over $V$ such that $x \in V[G']$. It follows from this as in Lemma 8 of [22] (described as "folklore" there) that the first-order theory of $\mathbf{L}(\mathbb{R})^{V[G][H]}$ is the same as the theory of $\mathbf{L}(\mathbb{R})$ after forcing with $\mathrm{Coll}(\omega, <\kappa)$. But in the latter model, there is no infinite MAD family by [28]—or because universal Ramsey regularity and Ramsey-positive uniformization hold in this model, and by [25]. □

We can also view Theorem 1.1 from a different perspective. We have seen that forcing axioms which are strong enough to imply $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, rule out the existence of definable infinite MAD families. Our result shows that under an *anti-large cardinal assumption*, forcing axioms can lead to the opposite result: They imply the existence of infinite MAD families at a rather low level of the projective hierarchy.

**Theorem 1.3.** *Suppose* BPFA *holds and that* $\omega_1$ *is not remarkable in* $\mathbf{L}$. *Then there is an infinite* $\mathbf{\Pi}^1_2$ *MAD family.*

We take this as evidence that under certain forcing axioms and anti-large cardinal assumptions, the universe behaves somewhat like $\mathbf{L}$ (as in $\mathbf{L}$ there are infinite $\Pi^1_1$ MAD families). This idea is also corroborated by the proof of the above theorem.

An obvious question is to which degree the above theorem can be generalized in the sense of replacing "almost-disjointness" by other relations. To state this question precisely, let us introduce some terminology: Let $E$ be a binary, symmetric, and anti-reflexive relation on a Polish space $X$. We view $G = \langle X, R\rangle$ as a simple graph with vertex set $X$ and edge relation $E$. To say such $G$ is Borel, $\mathbf{\Sigma}^1_1$, ...means that $E$ is Borel, $\mathbf{\Sigma}^1_1$, ...as a subset of $X^2$ (a Polish space, with the product topology). A set $D \subseteq X$ is called *$G$-discrete* if no two of its elements are $E$-related, and *maximal discrete* if it is $G$-discrete and maximal with respect to $\subseteq$ among $G$-discrete subsets of $X$.

We can now ask: Under the same hypothesis as in Theorem 1.3, which Borel graphs on Polish spaces have an infinite $\mathbf{\Pi}^1_2$ maximal discrete set? What about $\mathbf{\Sigma}^1_3$ graphs?

That the answer is not "all of them" is obvious from the fact that maximal discrete sets for the relation $xEy \iff (x \neq y \wedge |x \Delta y| < \aleph_0)$ on $\mathcal{P}(\omega)$ cannot be measurable and hence not $\mathbf{\Pi}^1_1$. Likewise, under $\mathsf{MA}_{\aleph_1}$ no $\mathbf{\Pi}^1_2$ maximal discrete set for this relation can exists, since under $\mathsf{MA}_{\aleph_1}$ all $\mathbf{\Sigma}^1_2$ sets are measurable.

The obstacle to generalizing our construction to arbitrary Borel (or $\mathbf{\Sigma}^1_3$) graphs is the coding mechanism in Fact 4.5 which relies heavily on the combinatorics of our specific graph. Vidnyánszky [29] has found a large class of graphs which admit a co-analytic maximal discrete set if $\mathcal{P}(\omega) \subseteq \mathbf{L}$: For instance, this holds for Borel graphs $G = \langle X, E\rangle$ with the property that for each countable $D \subseteq X$ and $d \in X$ such that $D \cup \{d\}$ is $G$-discrete, the set

$$\{d' \in X \mid D \cup \{d'\} \text{ is } G\text{-discrete and } D \cup \{d,' d\} \text{ is not}\}$$

is cofinal in the hyperarithmetic degrees. Vidnyánszky's construction uses properties specific to $\Pi^1_1$ sets, and it is not clear how to carry out this argument at the level of $\Pi^1_2$ sets (under BPFA).

Inspired by a question put to us by the anonymous referee, for which we are thankful, we proved the following theorem. Recall that Bounded Martin's Maximum (BMM) is the bounded forcing axiom for stationary set preserving forcing.

**Theorem 1.4.** $\mathsf{ZFC} + \mathsf{BMM}$ *implies that there are no infinite* $\boldsymbol{\Sigma}^1_3$ *MAD families.*

However, as is not hard to see, under just BPFA and thus also under BMM, there is an infinite MAD family which is $\Delta_1$ allowing a parameter from $\mathcal{P}(\omega_1)$ (using the well-ordering of $\mathcal{P}(\omega)$ from Theorem 2.1 below). The consistency strength of BMM is not known; the best upper bound, to our knowledge, is $(\omega+1)$-many Woodin cardinals (an unpublished result due to Woodin); the current lower bound is a strong cardinal [24].

B. Our work has some precursors in the literature: In [4] it is shown that under BPFA, if $\omega_1$ is not remarkable in $\mathbf{L}$ every predicate on $\mathcal{P}(\omega)$ which has a $\boldsymbol{\Sigma}_1$ definition in $\mathbf{H}(\omega_2)$ also has a $\boldsymbol{\Sigma}^1_3$ definition.

It was shown by Asger Törnquist in [27] that if there is an infinite $\boldsymbol{\Sigma}^1_2$ MAD family, there is an infinite $\boldsymbol{\Pi}^1_1$ MAD family. Unfortunately, the latter proof does not lift to show that there exists a $\boldsymbol{\Pi}^1_2$ infinite MAD under $\mathsf{BPFA}+\omega_1$ is not remarkable in $\mathbf{L}$. The reason for this is that Törnquist's proof relies on properties of $\Sigma^1_2$ and $\Pi^1_1$ sets which do not hold for $\Sigma^1_3$ and $\Pi^1_2$ sets.

C. The paper is organized as follows. In section §2 we discuss a result of Caicedo and Velickovic which can be summed up as follows: BPFA implies that there is a well-ordering of $\mathcal{P}(\omega)$ of length $\omega_2$ with definable initial segments. In §3 we discuss the role of the anti-large cardinal assumption, referring to work of Schindler, and discuss a technique of localization which we have used before (e.g., [6]) and which takes a particularly simple form under $\mathsf{BPFA} + \omega_1$ is not remarkable in $\mathbf{L}$. In §4 we prove Theorem 1.3, and in the short §5 we prove Theorem 1.4 We close with open questions in §6.

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## 2. A well-ordering with definable initial segments

It was shown by Moore [18] that under BPFA there is a well-ordering of $\mathcal{P}(\omega)$ of order-type $\omega_2$. Improving Moore's result, Caicedo and Velickovic [5] obtained, under BPFA, such a well-ordering which is definable by a $\Sigma_1$ formula with a parameter from $\mathcal{P}(\omega_1)$.

Their well-ordering has the following property which will be crucial to our argument.

**Theorem 2.1.** *Under* BPFA *there is a well-ordering* $\prec$ *of* $\mathcal{P}(\omega)$ *such that for some* $\Sigma_1$ *formula* $\Phi_\prec(u,v,w)$ *and some parameter* $c_\prec \subseteq \omega_1$,

$$\big(\forall x \in \mathcal{P}(\omega)\big)(\forall I)\ \Big(\Phi_\prec(x, I, c_\prec) \iff I = \{y \in \mathcal{P}(\omega) : y \prec x\}\Big)$$

Such a well-ordering is obviously very useful when one is interested in devising a recursive definition of optimal complexity. For convenience, we give a name to this type of well-order:

**Definition 2.2.** We say a well-order $\prec$ of $\mathcal{P}(\omega)$ with the property from Theorem 2.1 has $(\Sigma_1, \mathcal{P}(\omega_1))$-*definable good initial segments.*

In fact, what Caicedo and Velickovic show in their article [5] is that $\mathcal{P}(\omega)$ carries a well-order $\prec$ with the properties (i) and (ii) in the fact below. Of course, this is equivalent to having $(\Sigma_1, \mathcal{P}(\omega_1))$-definable initial segments.

**Fact 2.3.** That a well-ordering of $\mathcal{P}(\omega)$ has $(\Sigma_1, \mathcal{P}(\omega_1))$-definable initial segments (i.e., has the property from Theorem 2.1) is equivalent to the conjunction of the following:

(i) $\prec$ is $\Sigma_1$ with a parameter $c_\prec \subseteq \omega_1$,

(ii) There is a formula $\Phi_{\mathrm{is}}(u)$ such that for any transitive model $M$ with $c_\prec \in M$, $M \vDash \Phi_{\mathrm{is}}(c_\prec)$ if and only if $\prec$ is absolute for $M$ and $M \cap \mathcal{P}(\omega)$ is an initial segment of $\prec$.

*Remark* 2.4. The requirement in (ii) above that $M \vDash \Phi_{\mathrm{is}}(c_\prec)$ implies that $\prec$ is absolute for $M$ is redundant; it follows from Requirement (i) if we replace $\Phi_{\mathrm{is}}(c_\prec)$ by its conjunction with $\big(\forall x, y \in \mathcal{P}(\omega)\big)\ x \prec y \vee y \prec x$.

*Proof.* To see that (i)$\wedge$(ii) implies that $\prec$ is a well-ordering with $(\Sigma_1, \mathcal{P}(\omega_1))$-definable initial segments let $\Phi_\prec(x, I, c_\prec)$ be the formula

$$(\exists M)\ M \text{ is a transitive } \in\text{-model with } \{x, c_\prec, I\} \subseteq M \text{ and } M \vDash \text{“}\Phi_{\mathrm{is}}(c_\prec) \wedge I = \{y \in \mathcal{P}(\omega) \mid y \prec x\}\text{”}$$

and observe $I = \{y \in \mathcal{P}(\omega) : y \prec x\} \iff \Phi_\prec(x, I, c_\prec)$.

For the other direction, firstly observe that if $\prec$ has $(\Sigma_1, \mathcal{P}(\omega_1))$-definable initial segments then obviously $\prec$ is $\Sigma_1$ in the parameter $c_\prec$. Secondly, let $\Phi_{\mathrm{is}}(c_\prec)$ be the formula

$$\big(\forall x, y \in \mathcal{P}(\omega)\big)\ x \prec y \vee y \prec x \wedge \big(\forall x \in \mathcal{P}(\omega)\big)(\exists I)\ \Phi_\prec(x, I, c_\prec). \qquad \square$$

For a proof that under BPFA there is such a well-ordering of $\mathcal{P}(\omega)$ with $(\Sigma_1, \mathcal{P}(\omega_1))$-definable good initial segments, we refer the reader to the excellent exposition in [5].

## 3. Coding, reshaping, and localization

We start by recalling the following well-known fact.

**Fact 3.1.** Let $\mathcal{B} = \langle b_\xi : \xi < \omega_1 \rangle$ be an arbitrary sequence of pairwise almost disjoint infinite subsets of $\omega$. Under $\mathsf{MA}_{\aleph_1}$, for every subset of $S \subseteq \omega_1$ there is a $c \subseteq \omega$ such that

$$S = \{\xi < \omega_1 : c \cap b_\xi \text{ is infinite}\}. \tag{1}$$

The proof of this fact is equally well-known; it uses Solovay's almost disjoint coding (see [10] or, e.g., [12]).

We take the opportunity to introduce the following rather natural terminology:

**Definition 3.2.** We shall say that $c \subseteq \omega$ *almost disjointly via $\mathcal{B}$ codes* the set $S$ to mean precisely that (1) holds.

Our only use of the assumption that $\omega_1$ is not remarkable in $\mathbf{L}$ is in the following fact (this was shown by Ralf Schindler in [20, 21]).

**Fact 3.3.** Suppose $\omega_1$ is not remarkable in $\mathbf{L}$ and $\mathsf{BPFA}$ holds. Then there exists $r \in \mathcal{P}(\omega)$ such that $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$.

**Notation 3.4.**

*(1) For the rest of this article, let us suppose that $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ for some $r \in \mathcal{P}(\omega)$ which from now on shall remain fixed.*
*(2) Fix an almost disjoint family $\mathcal{F} = \langle f_\xi : \xi < \omega_1 \rangle$ which has a $\Sigma_1$ definition in $\mathbf{L}[r]$ and such that for any $\alpha < \omega_1$, $\langle f_\xi : \xi < (\omega_1)^{\mathbf{L}_\alpha[r]} \rangle$ is the set satisfying this definition in $\mathbf{L}_\alpha[r]$.*

It is a consequence of $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ and $\mathsf{MA}_{\aleph_1}$ that any predicate which is $\Sigma_1$ in $\mathbf{H}(\omega_2)$ (with a parameter) can be localized in a strong sense. A version of this result can, e.g., be found in [4].

To state the following localization lemma, let us make a definition which will be used throughout the paper.

**Definition 3.5** (Suitable models)**.** A *suitable model* is a countable transitive $\in$-model $N$ such that $r \in N$, $N \vDash \mathsf{ZF}^-$ and $N \vDash$"$\omega_1$ exists".

**Lemma 3.6** (A form of localization)**.** *Suppose $\mathsf{MA}_{\aleph_1}$ holds (and recall that we are working under the assumption that $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ made in 3.4). Let $\phi(y, \omega_1)$ be an arbitrary formula, where $y \in \mathcal{P}(\omega)$ and $\omega_1$ are parameters, and suppose that for some transitive $\in$-model $M$ with $\{\omega_1, y\} \in M$ it holds that $M \vDash \phi(y, \omega_1)$. Then there is $c \subseteq \omega$ such that the following holds:*

(2) *Given any suitable model $N$ with $\{c, y\} \subseteq N$ the following must hold in $N$: "There is a transitive $\in$-model $M^*$ such that $\{y, (\omega_1)^N\} \subseteq M^*$ and $M^* \vDash \phi\big(y, (\omega_1)^N\big)$".*

*Proof.* Fix a transitive model $M$ as in the lemma. We can assume $M$ to have size $\omega_1$. Find $S \subseteq \omega_1$ such that via Gödel pairing, $S$ gives rise to a well-founded binary relation $S^*$ on $\omega_1$ whose transitive collapse is $\langle M, \in \restriction M \rangle$. We can ask that $y$ and $\omega_1$ are mapped to specific points in $\langle \omega_1, S^* \rangle$ by the inverse of the collapsing map, say to 0 and 1.

Let

$$D = \{\beta \in \omega_1 : (\exists \mathcal{N}^*)\, \mathcal{N}^* \prec L_{\omega_2}[S^*, y], \{S^*, y\} \in \mathcal{N}^*, \beta = \omega_1 \cap \mathcal{N}^*\}. \tag{3}$$

For $Y \subseteq \mathrm{On}$, let $\mathrm{Even}(Y) = \{\xi : 2\xi \in Y\}$ and $\mathrm{Odd}(Y) = \{\xi : 2\xi + 1 \in Y\}$. Choose $Y$ to be any subset of $\omega_1$ such that $\mathrm{Even}(Y) = S^*$ and for each $\beta \in D$, the preimage under Gödel pairing of $\mathrm{Odd}(Y) \cap [\beta, \beta + \omega)$ is a well-founded binary relation of rank at least $\min\big(D \setminus (\beta + 1)\big)$.

**Claim 3.7.** *$Y \subseteq \omega_1$ satisfies the following:*

(4) *Given any suitable model $N$ with $\{Y \cap (\omega_1)^N, y\} \subseteq N$ the following must hold in $N$: "There is a transitive $\in$-model $M^*$ such that $\{y, (\omega_1)^N\} \subseteq M^*$ and $M^* \vDash \phi\big(y, (\omega_1)^N\big)$".*

*Proof.* To see that $Y$ indeed satisfies (4) let $N$ as in (4) be given. Letting $\beta = (\omega_1)^N$ it must hold that $\beta \in D$: For if $\beta' < \beta$, since $\beta \cap Y \in N$, this model contains a well-founded binary relation of length $\min\big(D \setminus (\beta' + 1)\big)$ as an element, and so $\min\big(D \setminus (\beta' + 1)\big) < \beta = (\omega_1)^N$ because $N$ is a model of $\mathsf{ZF}^-$. As $D$ is closed, $\beta \in D$. It also follows that $S^* \cap \beta \in N$.

By definition of $D$ we may pick $\mathcal{N}^*$ as in (3). Letting $\overline{\mathcal{N}}^*$ be the transitive collapse of $\mathcal{N}^*$ we obtain an elementary embedding $j : \overline{\mathcal{N}}^* \to L_{\omega_2}[S^*, y]$ with critical point $\beta = (\omega_1)^N$ (namely, the inverse of the collapsing map) such that $\{S^*, y, \omega_1\} \subseteq \operatorname{ran}(j)$. By elementarity $\overline{\mathcal{N}}^* \vDash$"The transitive collapse of $\langle \beta, S^* \restriction \beta \rangle$ is a transitive $\in$-model $M^*$ such that $M^* \vDash \Phi(y, \beta)$". But this transitive collapse is also an element of $N$, so by absoluteness of $\Delta_1$ formulas $N$ must satisfy the same sentence. Claim 3.7. □

Finally, we find $c \in \mathcal{P}(\omega)$ which almost disjointly via $\mathcal{F}$ codes the set $Y \subseteq \omega_1$ constructed above. To see that $c$ satisfies (2) let $N$ as in (2) be given. By choice of $\mathcal{F}$ (see Notation 3.4) $\langle f_\xi \colon \xi < (\omega_1)^N \rangle \in N$ and so since $N \vDash \mathsf{ZF}^-$ it holds that $Y \cap (\omega_1)^N \in N$. By (4) the sentence "there is a transitive $\in$-model $M^*$ such that $\{y, (\omega_1)^N\} \subseteq M^*$ and $M^* \vDash \phi\big(y, (\omega_1)^N\big)$" holds in $N$, verifying (2). Lemma 3.6. □

## 4. Proof of Theorem 1.3

In this section we prove Theorem 1.3 in the following, slightly more general form:

**Theorem 4.1.** *Suppose there is a well-ordering of $\mathcal{P}(\omega)$ of length $\omega_2$ with $(\Sigma_1, \mathcal{P}(\omega_1))$-definable initial segments, $\mathsf{MA}_{\aleph_1}$ holds, and $\omega_1 = (\omega_1)^{\mathbf{L}[r]}$ where $r \in \mathcal{P}(\omega)$. Then there is an infinite $\mathbf{\Pi}^1_2$ MAD family.*

It is clear by Theorem 2.1 and Fact 3.3 that $\mathsf{BPFA} + \omega_1$ is not remarkable in $\mathbf{L}$ implies the hypothesis, so proving the above theorem will indeed prove Theorem 1.3.

**Notation 4.2.** *From now on, we suppress the parameter $r$ and assume $\omega_1 = (\omega_1)^{\mathbf{L}}$; our argument will relativize to $r$ trivially.*

*By Theorem 2.1 we can fix a well-ordering $\prec$ of $\mathcal{P}(\omega)$ with $(\Sigma_1, \mathcal{P}(\omega_1))$-definable initial segments, together with a parameter $c_\prec \subseteq \omega$ and a formula $\Phi_{\mathrm{is}}(c_\prec)$ as in Fact 2.3.*

We shall inductively construct a sequence $\langle a_\nu : \nu < \omega_2 \rangle$ such that $\mathcal{A} = \{a_\nu : \nu < \omega_2\}$ will be a $\mathbf{\Pi}^1_2$ MAD family.

The most straightforward formula defining a MAD family $\mathcal{A}$ would express that $a \in \mathcal{A}$ iff there is an initial segment $\langle a_\nu : \nu \leq \xi \rangle$ of the construction with $a = a_\xi$; that is, assuming we can find a formula expressing that $\langle a_\nu : \nu \leq \xi \rangle$ is an initial segment of this construction. But of course it is not clear how any projective formula should express such a fact about $\langle a_\nu : \nu < \xi \rangle$, this being an object of size $\omega_1$. A first step towards a solution is that $a_\xi$ should *code* certain sets of size $\omega_1$, including $\langle a_\nu : \nu < \xi \rangle$. Almost disjoint coding via $\mathcal{F}$ (see Fact 3.1) allows us to find a real coding these large sets. We then want to find a real 'localizing' this coding, i.e., ensuring that the property of coding an initial segment of the construction is expressible by a $\Pi^1_2$ formula. Using a variant of the coding from [17] we can then code these reals into $a_\xi$.

### 4.1. Coding into an almost disjoint family.

We call the following fact from Miller's article [17] to the reader's attention.

**Fact 4.3** (see [17, Lemma 8.24, p. 195]). Fix $z \in \mathcal{P}(\omega)$ and suppose $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ is a countable sequence of pairwise almost disjoint infinite sets. For any $d \in [\omega]^\omega$ which is almost disjoint from every element of $\vec{a}$ there is $a \in [\omega]^\omega$ such that

- $a \cap d$ is infinite,
- $a$ is almost disjoint from each $a_\nu$ for $\nu < \xi$,
- and $z$ is computable from $a$ and $\vec{a} \restriction \omega = \langle a_n : n < \omega \rangle$.

Using this fact, Miller succeeds in constructing a co-analytic MAD family in $\mathbf{L}$: he recursively constructs $\langle a_\nu : \nu < \omega_1 \rangle$ such that in the end, $\mathcal{A} = \{a_\nu : \nu < \omega_1\}$ turns out to be a $\mathbf{\Pi}^1_1$ MAD family. At some initial stage $\xi < \omega_1$ having constructed $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ he considers a counterexample $d$ to the maximality of the family $\{a_\nu : \nu < \xi\}$ constructed so far. Instead of adding this set $d$ to $\vec{a}$, he adds $a$ as in the fact above, which in addition codes some information $z$ so as to bring down the definitional complexity of $\mathcal{A}$.

Since we shall need a variant of this type of coding, let us repeat Miller's proof of the above fact.

*Proof of Fact 4.3.* Let $\vec{b} = \langle b_n : n \in \mathbb{N} \rangle$ enumerate $\{a_\nu : \omega \le \nu < \xi\}$. For each $n \in \omega$, choose a finite set $G_n \subseteq a_n \setminus \bigcup \big( \{ b_k : k < n\} \cup \{ a_k : k < n\} \big)$ so that $|G_n \cup (a_n \cap d)|$ is even if $n \in z$, and odd otherwise. Finally, let $a = d \cup \bigcup \{G_n : n \in \omega\}$. □

For our purposes the previous fact is useless, since as $2^\omega = \omega_2$ under $\mathsf{BPFA}$ we shall have to deal with uncountable sequences $\vec{a} = \langle a^\nu : \nu < \xi \rangle$. Interestingly, there is a variant of the above construction that allows us to deal with uncountable sequences.

Before we describe this variant let us commit, once and for all, to some sequence (to be used for coding purposes) as an initial segment of the MAD family we are about to construct.

**Notation 4.4.** *Let us fix, for the rest of this article, some sequence $\vec{a}_\omega = \langle a_n : n \in \omega \rangle$ of infinite sets any two of which are almost disjoint.*

We now state our variant of Miller's coding lemma. For this variant, we must make an additional assumption (the existence of $c$ below) which in our case will easily be seen to follow from $\mathsf{MA}_{\aleph_1}$ (see Remark 4.6 below)

**Fact 4.5.** Suppose $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ is a (possibly uncountable) sequence of pairwise almost disjoint infinite subsets of $\omega$ such that $\vec{a} \restriction \omega = \vec{a}_\omega$. Further suppose we have $c \in [\omega]^\omega$ satisfying the following:

- $c$ is almost disjoint from each $a_\nu$, for $\omega \le \nu < \xi$, and
- $c \cap a_n$ is infinite for each $n \in \omega$.

Then for any $z \in \mathcal{P}(\omega)$ and any $d \in [\omega]^\omega$ which is almost disjoint from every element of $\operatorname{ran}(\vec{a})$ there is $a \in [\omega]^\omega$ such that

- $a \cap d$ is infinite,
- $a$ is almost disjoint from each $a_\nu$ for $\nu < \xi$,
- and $z$ is computable from $a$ and $\vec{a} \restriction \omega = \langle a_n : n < \omega \rangle$.

In fact there are functions $\mathsf{dc}\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ and $\mathsf{cd}\colon \mathcal{P}(\omega)^3 \to \mathcal{P}(\omega)$, both of which are computable in $\vec{a}_\omega$, such that $a$ as above is given by $a = \mathsf{cd}(d, c, z)$ and $z$ can be recovered from $a$ as $z = \mathsf{dc}(a)$.

The name $\mathsf{dc}$ was chosen to remind us that this function will be used to 'decode' $z$ from $a$, and likewise, the name $\mathsf{cd}$ should remind us that the function produces a 'code' (for $z$).

*Remark* 4.6. We will use Fact 4.5 in the situation where $\vec{a} = \langle a_\nu : \nu < \xi \rangle$ is of length $\xi < \omega_2$ and $\mathsf{MA}_{\aleph_1}$ holds. Then it is easy to see $c$ as in Fact 4.5 exists: Just use Fact 3.1 to obtain $c$ so that $\{\nu < \xi \mid c \cap a_\nu \text{ is infinite }\} = \omega$.

*Proof of Fact 4.5.* We define $\mathsf{cd}\colon \mathcal{P}(\omega)^3 \to \mathcal{P}(\omega)$ as follows. Let $F_n$ be the shortest finite initial segment of

$$c \cap a_n \setminus \bigcup \{a_k : k < n\}$$

such that $|F_n \cup (d \cap a_n)|$ is even if $n \in z$ and odd otherwise. Clearly, $F_n$ can be found by a procedure which is computable in $\vec{a}_\omega$, $c$, $d$, and $z$. Now define the function $\mathsf{cd}$ by

$$\mathsf{cd}(d, c, z) = d \cup \bigcup \{F_n : n \in \omega\}.$$

Moreover, we define $\mathsf{dc}\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ as follows: Given $a \in [\omega]^\omega$ let

$$\mathsf{dc}(a) = \{n \in \omega : |a \cap a_n| \text{ is even}\}.$$

Clearly, these functions satisfy the conditions in the lemma. □

4.2. **Minimal local witnesses.** The functions $\mathsf{cd}$ and $\mathsf{dc}$ together with the almost disjoint coding into reals of subsets of $\omega_1$ via $\mathcal{F}$ will help us arrange that $a_\xi$ codes $\langle a_\nu : \nu < \xi \rangle$. But crucially, we need the fact that $a_\xi$ codes an initial segment of the construction (up to stage $\xi$, some ordinal below $\omega_2$) to be witnessed by a $\mathbf{\Pi}^1_2$ formula (the same formula for all $\xi < \omega_2$). This involves uniquely selecting a real $c_\xi \in \mathcal{P}(\omega)$ which we call a *minimal local witnesses* and whose task is to *localize* the coding to suitable countable models. Uniquely selecting such a real is a non-trivial task, and to tackle it we introduce some terminology.

**Notation 4.7.** *Let $F\colon \omega^2 \to \omega$ denote some fixed recursive bijection for the remainder of this article.*

**Definition 4.8.**

(1) Given $c \subseteq \omega$ and $n \in \omega$ we write $(c)_n$ for $\{m \in \omega : F(n, m) \in c\}$.
(2) Given $c \subseteq \omega$ we write $\mathrm{Seq}\,(c)$ for the sequence $\langle (c)_n : n \in \omega \rangle$.
(3) Let $G\colon \mathrm{On}^2 \to \mathrm{On}$ denote the Gödel pairing function. We say $c \subseteq \omega$ *almost disjointly via $\mathcal{F}$ codes the sequence $\vec{b}$* to mean that $c \subseteq \omega$ almost disjointly via $\mathcal{F}$ codes a set $S \subseteq \omega_1$ and $\vec{b} = \langle b_\nu : \nu < \xi \rangle$ where:
   - For $\theta < \omega_1$, letting
   $$S_\theta = \{\eta < \omega_1 \mid G(\theta, \eta) \in S\},$$
   $$S^*_\theta = \{(\zeta_0, \zeta_1) \in (\omega_1)^2 \mid \omega + G(\zeta_0, \zeta_1) \in S_\theta\},$$
   it holds that $\langle \omega_1, S^*_\theta \rangle$ is a well-ordering and $\xi = \{\mathrm{otp}\, S^*_\theta \mid \theta < \omega_1, S_\theta \neq \emptyset\}$;
   - For each $\nu < \xi$ there is exactly one $\theta$ such that $S_\theta \neq \emptyset$ and $\mathrm{otp}\, S^*_\theta = \nu$, and for this $\theta$ it holds that
   $$b_\nu = \omega \cap S_\theta.$$

The crucial definition for our proof of Theorem 4.1 (and thus, of Theorems 1.1 and 1.3) is that of *minimal local witness*.

*Remark* 4.9. In the end, our MAD family will be

$$\mathcal{A} = \{a_n \colon n \in \omega\} \cup \big\{a \in [\omega]^\omega : c = \mathsf{dc}(a) \text{ is a minimal local witness and } a = \mathsf{cd}\big((c)_0, (c)_1, c\big)\big\}.$$

We will show below that being a minimal local witness is expressible by a $\mathbf{\Pi}^1_2$ formula. Thus, $\mathcal{A}$ will be $\mathbf{\Pi}^1_2$. The low definitional complexity will be achieved through a careful recursive construction of $\mathcal{A}$. We will have $\mathcal{A} = \{a_\xi : \xi < \omega_2\}$ where letting $c_\xi = \mathsf{dc}(a_\xi)$, $(c_\xi)_2$ almost disjointly via $\mathcal{F}$ codes $\langle a_\nu : \nu < \xi\rangle$.

Before we introduce the notion of minimal local witnesses, we make another convenient definition, for which some motivation should be provided by the previous remark.

**Definition 4.10.** We shall say that a sequence $\vec{b} = \langle b_\nu : \nu < \xi\rangle$ is a *coherent candidate* if $\vec{a}_\omega \subseteq \vec{b}$ and moreover, for each $\nu < \xi$ it holds that $\big(\mathsf{dc}(b_\nu)\big)_2$ almost disjointly via $\mathcal{F}$ codes the sequence $\vec{b} \restriction \nu$.

We proceed towards the definition of minimal local witness, by defining the notions of $k$-witness, minimal $k$-witness and $k$-localizer, by induction on $k \in \omega$, $k \geq 3$.

**Definition 4.11.** We say $\bar{c} \in \mathcal{P}(\omega)^3$ *is a* 3*-witness* if and only if

$(*)_3$
- (a) $\bar{c}(2)$ almost disjointly via $\mathcal{F}$ codes a sequence $\vec{b} = \langle b_\nu : \nu < \xi\rangle$.
- (b) $\vec{b}$ is a coherent candidate.
- (c) $\bar{c}(1)$ is subset of $\omega$ such that $c \cap b_\nu$ is infinite if $\nu < \omega$ and finite for all other $\nu < \xi$.
- (d) $\bar{c}(0)$ is an element of $[\omega]^\omega$ which is almost disjoint from each $b_\nu$ for $\nu < \xi$;

*Remark* 4.12. Clearly, the sequence $\vec{b}$ from (a) is intended to be an initial segment of the MAD family under construction. We ask (b) as a first step towards ensuring that this is indeed the case. The reader will notice that in (c) we require that $\bar{c}(1)$ has the same properties as $c$ in Fact 4.5, and in (d) we require that $\bar{c}(0)$ has the same properties as $d$ in said fact. The reader may think of $\bar{c}(0)$ as a counterexample to maximality of $\vec{b}$ which we wish to eliminate at stage $\xi$ of our construction of $\mathcal{A}$ by adding a 'self-coding' element to our MAD family which has infinite intersection with $\bar{c}(0)$.

We continue with the definition of minimal 3-witness to a sequence $\vec{b}$ of subsets of $\omega$.

**Definition 4.13.** For any 3-witness $\bar{c} \in \mathcal{P}(\omega)^3$, we say $\bar{c}$ *is a witness to* $\vec{b}$ if $\vec{b}$ is the sequence coded by $\bar{c}(2)$ as in (a) above. We also write $\vec{b}\big(\bar{c}(2)\big)$ for this sequence. Write $\prec^3$ for the lexicographic ordering on $\mathcal{P}(\omega)^3$ induced by $\prec$. We call a 3-witness $\bar{c} \in \mathcal{P}(\omega)^3$ *minimal* if it is $\prec^3$-minimal among all 3-witnesses to the same sequence $\vec{b}$. This is the same as saying that $\bar{c}(2)$ is $\prec$-minimal satisfying (d) in $(*)_3$, $\bar{c}(1)$ is $\prec$-minimal satisfying (c), and $\bar{c}(0)$ is $\prec$-minimal satisfying (a).

As is not hard to see, the notion of minimal 3-witness is sufficiently absolute for our purposes:

**Lemma 4.14.** *The notion of* 3*-witness is absolute for transitive models $M$ of* $\mathsf{ZF}^-$ *such that $\omega_1 \in M$ and the notion of minimal* 3*-witness is absolute for such models if in addition* $M \models \Phi_{\text{is}}(c_\prec)$.

*Proof.* The statement $(*)_3$(a) that $\bar{c}(2)$ almost disjointly via $\mathcal{F}$ codes a sequence $\vec{b}$ is easily seen to be equivalent in $\mathsf{ZF}^-$ to a $\Sigma_1$ property of $\bar{c}(2)$, allowing $\omega_1$ as a parameter. Moreover, this $\Sigma_1$ statement is absolute for transitive models $M$ of $\mathsf{ZF}^-$ since a witness can be constructed inside $M$ using the Replacement Axiom, and as $\mathcal{F} \in M$ by choice of $\mathcal{F}$. That $\vec{b}$ is a coherent candidate is absolute for the same reasons. Statements (c) and (d) are obviously $\Delta_1$ in the parameters $\vec{b}$ and $\bar{c}$. This shows that the notion of 3-witness is absolute for transitive models $M$ of $\mathsf{ZF}^-$.

Minimality of 3-witnesses is now easily seen to be absolute provided that in addition $M \models \Phi_{\text{is}}(c_\prec)$ since $\prec$ is absolute for such $M$ and $\mathcal{P}(\omega) \cap M$ is an initial segment of $\prec$: If $\bar{c} \in M$ is a minimal 3-witness, $M \models$"$\bar{c}$ is a minimal 3-witness" by absoluteness of $\prec$ and of the notion of 3-witness. Vice versa, suppose $\bar{c} \in M$ and $M \models$"$\bar{c}$ is a minimal 3-witness". Then $\bar{c}(0)$ must minimal satisfying $(*)_3$(a) since if there were $c' \prec \bar{c}(0)$ satisfying $(a)$, it would have to be the case that $c' \in M$ since $\mathcal{P}(\omega) \cap M$ is a $\prec$-initial segment, contradicting $M \models$"$\bar{c}$ is a minimal 3-witness". Likewise for $\bar{c}(1)$ and $\bar{c}(2)$. □

We now give the crucial definition of a 3-*localizer*—a real which ensures that minimal 3-witnesses can be recognized from a local (i.e., a $\mathbf{\Pi}^1_2$) property.

**Definition 4.15.** Given $\bar{c} \in \mathcal{P}(\omega)^3$ (a putative 3-witness) we say $c \in \mathcal{P}(\omega)$ is a 3-*localizer for* $\bar{c}$ if and only if:

$(*)_4$ For any suitable model $N$ with $\{\bar{c}, c, \vec{a}_\omega\} \subseteq N$, the following holds in $N$: There is a transitive model $M$ of $\mathsf{ZF}^-$ such that $M \models \Phi_{\text{is}}(c_\prec)$, $\{\omega_1, \bar{c}, \vec{a}_\omega\} \subseteq M$, and

(a) $M \models$"$\bar{c}$ is a minimal 3-witness".

(b) Writing $\vec{b}\big(\bar{c}(2)\big)^M$ as $\langle b_\nu : \nu < \xi\rangle$ it holds that for each $\nu < \xi$, $M \models$"$\bar{c}^*_\nu \restriction 3$ is a minimal 3-witness", where $\bar{c}^*_\nu = \operatorname{Seq}(\mathsf{dc}(b_\nu))$.

*Remark* 4.16. Note that "$\bar{c}_\nu \restriction 3$ is a minimal 3-witness" is a statement which uses $\vec{a}_\omega$ as a parameter.

We need the following crucial lemmas:

**Lemma 4.17.** *Suppose $\bar{c} \in \mathcal{P}(\omega)^3$ is a minimal 3-witness, $\vec{b}\big(\bar{c}(2)\big) = \langle b_\nu \mid \nu < \xi\rangle$ and for each $\nu < \xi$ it holds that $\operatorname{Seq}(\mathsf{dc}(b_\nu)) \restriction 3$ is a minimal 3-witness. Then there exists a 3-localizer for $\bar{c}$.*

*Proof.* Suppose $\bar{c} \in \mathcal{P}(\omega)^3$ is as in the lemma. Fix a transitive model $M$ of $\mathsf{ZF}^-$ such that $\{\omega_1, \bar{c}\} \subseteq M$ and so that $M \models \Phi_{\text{is}}(c_\prec)$. By Lemma 4.14 the property of being a minimal 3-witness is absolute for $M$, so $M \models$"$\bar{c}$ is a minimal 3-witness" and $M \models$"$\bar{c}^*_\nu \restriction 3$ is a minimal 3-witness" where $\bar{c}^*_\nu = \operatorname{Seq}(\mathsf{dc}(b_\nu))$ for each $\nu < \xi$.

Now as in the proof of Lemma 3.6, find $c$ coding almost disjointly via $\mathcal{F}$ a subset of $\omega_1$ which is isomorphic to $\in \restriction M$ and such that for any suitable model $N$, if $c, \bar{c} \in N$ then it holds in $N$ that $c$ codes a model $M^*$ which witnesses the $\Sigma_1$ statement expressing $\bar{c}$ and each $\bar{c}^*_\nu \restriction 3$ are minimal 3-witnesses. Clearly, $c$ is a 3-localizer for $\bar{c}$. □

**Lemma 4.18.** *Suppose $\bar{c} \in \mathcal{P}(\omega)^3$. If there exists a 3-localizer for $\bar{c}$, then $\bar{c}$ is a minimal 3-witness, and letting $\vec{b}\big(\bar{c}(2)\big) = \langle b_\nu \mid \nu < \xi\rangle$ it holds for each $\nu < \xi$ that $\operatorname{Seq}(\mathsf{dc}(b_\nu)) \restriction 3$ is a minimal 3-witness.*

*Proof.* Suppose $c$ is a 3-localizer for $\bar{c}$. Let $\bar{N}$ be a countable elementary submodel of $\mathbf{L}_{\omega_2}[c, \bar{c}, \vec{a}_\omega]$ with $\{\omega_1, c, \bar{c}, \vec{a}_\omega\} \subseteq \bar{N}$ and let $N$ be the transitive collapse of $\bar{N}$. Then $N$ is suitable, and so by $(*)_4$ the following holds in $N$: There is a transitive model $M$ of $\mathsf{ZF}^-$ such that $M \models \Phi_{\text{is}}(c_\prec)$, $\{(\omega_1)^N, \bar{c}, \vec{a}_\omega\} \subseteq M$, and

(a) $M \models$"$\bar{c}$ is a minimal 3-witness".

(b) Writing $\vec{b}\big(\bar{c}(2)\big)^M$ as $\langle b_\nu : \nu < \xi\rangle$, for each $\nu < \xi$, $M \Vdash$"$\bar{c}^*_\nu \restriction 3$ is a minimal 3-witness", where $\bar{c}^*_\nu = \operatorname{Seq}(\mathsf{dc}(b_\nu))$.

By elementarity, there exists such a model $M$ in $\mathbf{L}_{\omega_2}[c, \bar{c}, \vec{a}_\omega]$ with all of the above properties, where $(\omega_1)^N$ is replaced by $\omega_1$. Since $\omega_1 \in M$ and $M \vDash \mathsf{ZF}^- \wedge \Phi_{\text{is}}(c_\prec)$, by Lemma 4.14 the property of being a minimal 3-witness is absolute for $M$. Hence $\bar{c}$ and each $\bar{c}^*_\nu \restriction 3$ for $\nu < \xi$ are a minimal 3-witnesses, finishing the proof. □

Thus we have shown, roughly, that $\bar{c}$ is a minimal 3-witness coding a coherent candidate consisting of minimal 3-witnesses if and only if there exists a 3-localizer for $\bar{c}$. Of course, there may be more than one 3-localizer for a given minimal 3-witness.

**Definition 4.19.** We say $\bar{c} \in \mathcal{P}(\omega)^4$ is a *minimal 4-witness* if and only if $\bar{c}(3)$ is the $\prec$-least localizer for $\bar{c} \restriction 3$.

We now continue the definition of $k$-localizer and minimal $k+1$-witness for elements of $\mathcal{P}(\omega)^k$ by induction on $k$, following the template given by the definition for $k = 3$.

**Definition 4.20.** Let $k \in \omega \setminus 4$ and suppose we have already defined what it means to be a minimal $k$-witness for elements of $\mathcal{P}(\omega)^k$. Given $\bar{c} \in \mathcal{P}(\omega)^k$ (a putative $k$-witness) we say $c \in \mathcal{P}(\omega)$ is *is a $k$-localizer for $\bar{c}$* if and only if the following holds:

$(*)_k$ For any suitable model $N$ with $\{\bar{c}, c, \vec{a}_\omega\} \subseteq N$, the following holds in $N$: There is a transitive model $M$ of $\mathsf{ZF}^-$ such that $M \vDash \Phi_{\text{is}}(c_\prec)$, $\{\omega_1, \bar{c}, \vec{a}_\omega\} \subseteq M$, and
(a) $M \vDash$"$\bar{c}$ is a minimal $k$-witness",
(b) Writing $\vec{b}(\bar{c}(2))^M$ as $\langle b_\nu : \nu < \xi\rangle$, for each $\nu < \xi$ it holds that $M \vDash$"$\bar{c}_\nu \restriction k$ is a minimal $k$-witness", where $\bar{c}_\nu = \mathrm{Seq}\,(\mathsf{dc}(b_\nu))$.

Moreover, we say $\bar{c} \in \mathcal{P}(\omega)^{k+1}$ is a *minimal $(k+1)$-witness* if and only if $\bar{c}(k)$ is the $\prec$-least $k$-localizer for $\bar{c} \restriction k$.

Finally, we say $\bar{c} \in \mathcal{P}(\omega)^\omega$ is a *minimal local witness* if and only if

$(**)$ for each $k \in \omega \setminus 3$, $\bar{c}(k)$ is a $k$-localizer for $\bar{c} \restriction k$

and we say $c \in \mathcal{P}(\omega)$ is a *minimal local witness* if and only if $\mathrm{Seq}\,(c)$ is a minimal local witness.

Given arbitrary $\bar{c} \in \mathcal{P}(\omega)^{\leq\omega}$ let us say $\bar{c}$ *codes* $\vec{b}$ if $\bar{c}(2)$ almost disjointly via $\mathcal{F}$ codes the sequence $\vec{b}$. In this case let us also write $\vec{b}(\bar{c})$ for $\vec{b}$. We shall also say $\bar{c}$ is a witness to $\vec{b}$ to mean that $\bar{c}$ is a $\mathrm{lh}(\bar{c})$-witness or, if $\mathrm{lh}(\bar{c}) = \omega$, a minimal local witness, and $\vec{b}(\bar{c}) = \vec{b}$.

Just as before for $k = 3$ we have the following crucial lemma:

**Lemma 4.21.** *Suppose $k \in \omega \setminus 4$ and $\bar{c} \in \mathcal{P}(\omega)^k$. There exists a $k$-localizer for $\bar{c}$ if and only if $\bar{c}$ is a minimal $k$-witness, and letting $\vec{b}(\bar{c}(2)) = \langle b_\nu \mid \nu < \xi\rangle$ it holds for each $\nu < \xi$ that* $\mathrm{Seq}\,(\mathsf{dc}(b_\nu)) \restriction k$ *is a minimal $k$-witness.*

*Proof.* This is shown precisely as Lemmas 4.18 and 4.17 above. □

In the next lemma, we verify for the reader's convenience that the minimal local witness to a sequence is uniquely determined by this sequence.

**Lemma 4.22.** *For each sequence $\vec{b} = \langle b_\xi : \xi < \nu\rangle$, there is at most one minimal local witness $\bar{c} \in \mathcal{P}(\omega)^\omega$ coding $\vec{b}$. Likewise, if two sequences $\bar{c}$ and $\bar{c}'$ are minimal local witnesses and $\bar{c}(2) = \bar{c}'(2)$, then $\bar{c} = \bar{c}'$.*

*Proof.* Suppose $\bar{c}$ and $\bar{c}'$ are minimal local witnesses coding $\vec{b}$. Since $\bar{c}(3)$ is a 3-localizer to $\bar{c} \restriction 3$, by Lemma 4.18 the latter is a minimal 3-witness to $\vec{b}$. The same holds for $\bar{c}'$. But obviously, there is only one minimal 3-witness to $\vec{b}$, so $\bar{c} \restriction 3 = \bar{c}' \restriction 3$. But since $\bar{c}(4)$ is a 4-localizer for $\bar{c} \restriction 4$, $\bar{c}(3)$ is the $\prec$-least 3-localizer by Lemma 4.21. Since the same holds for $\bar{c}'(4)$ we have $\bar{c}(3) = \bar{c}'(3)$. Continue by induction to obtain $\bar{c} = \bar{c}'$. The second statement follows, since if $\bar{c}(2) = \bar{c}'(2)$, also $\vec{b}(\bar{c}) = \vec{b}(\bar{c}')$. □

We are now ready to begin the proof.

*Proof of Theorems 1.3 and 4.1.* As we have stated earlier, we shall inductively construct a sequence $\langle a_\nu : \nu < \omega_2 \rangle$ such that $\mathcal{A} = \{a_\nu : \nu < \omega_2\}$ will be a $\mathbf{\Pi}^1_2$ MAD family. For the first $\omega$ elements of $\langle a_\nu : \nu < \omega_2 \rangle$ take the sequence $\vec{a}_\omega = \langle a_k : k \in \omega \rangle$ fixed in 4.4 (since our coding functions $\mathsf{cd}$ and $\mathsf{dc}$ use $\vec{a}_\omega$). Fix $c_{\mathcal{A}} \in \mathcal{P}(\omega)$ from which both $\vec{a}_\omega$ and $c_\prec$ are computable; in the end $\mathcal{A}$ will be $\Pi^1_2(c_{\mathcal{A}})$.

Suppose we have already constructed $\langle a_\nu : \nu < \xi \rangle$ (where $\omega \leq \xi < \omega_2$) and assume as induction hypothesis that for each $\nu < \xi$, letting $c_\nu = \mathsf{dc}(a_\nu)$ and $\bar{c}_\nu = \mathrm{Seq}\,(c_\nu)$ we have that $a_\nu = \mathsf{cd}(\bar{c}_\nu(0), \bar{c}_\nu(1), c_\nu)$ and $\bar{c}_\nu$ (or equivalently, $c_\nu$) is a minimal local witness. Also, let us write $d_\nu = \bar{c}_\nu(0)$.

Write $\mathcal{A}_\xi = \{a_\nu : \nu < \xi\}$. We will now define $a_\xi$. First find $d_\xi$ such that

(5) $d_\xi$ is the $\prec$-least element of $[\omega]^\omega$ which is almost disjoint from every element of $\mathcal{A}_\xi$.

Such $d_\xi$ exists since $\mathsf{BPFA}$ implies that there is no MAD family of size less than $\omega_2$.

We now find a minimal local witness $\bar{c}_\xi \in \mathcal{P}(\omega)^\omega$ to $\langle a_\nu : \nu < \xi \rangle$ (see Definition 4.13).

- Of course, we let $\bar{c}_\xi(0) = d_\xi$.
- By Fact 3.1 (see also Remark 4.6) there exists $c \in [\omega]^\omega$ satisfying the requirement from Fact 4.5 that $\{\nu < \xi : |c \cap a_\nu| < \omega\} = \xi \setminus \omega$. We let $\bar{c}_\xi(1)$ be the $\prec$-least such $c$.
- Also by Fact 3.1, there exists a subset of $\omega$ which almost disjointly via $\mathcal{F}$ codes $\langle a_\nu : \nu < \xi \rangle$; let $\bar{c}_\xi(2)$ be the $\prec$-least such subset.

By construction $\bar{c}_\xi \restriction 3$ is a minimal 3-witness. Let $\bar{c}_\xi(3)$ be the $\prec$-least 3-localizer for $\bar{c}_\xi \restriction 3$, which exists by Lemma 4.18. Continue defining $\bar{c}_\xi \restriction k+1$ by recursion on $k$ for $k > 3$, letting $\bar{c}_\xi(k)$ be the $\prec$-least $k$-localizer for $\bar{c}_\xi \restriction k$, using Lemma 4.21, arriving at a minimal local witness $\bar{c}_\xi$ to $\langle a_\nu : \nu < \xi \rangle$ with $\bar{c}_\xi(0) = d_\xi$.

Finally, we write $c_\xi$ for the element of $\mathcal{P}(\omega)$ such that $\mathrm{Seq}\,(c_\xi) = \bar{c}_\xi$ and define

$$a_\xi = \mathsf{cd}(\bar{c}_\xi(0), \bar{c}_\xi(1), c_\xi),$$

finishing the recursive definition of $\langle a_\xi : \xi < \omega_2 \rangle$. Write $\mathcal{A} = \{a_\xi : \xi < \omega_2\}$. Clearly, by choice of $c_\xi(0) = d_\xi$ and $\bar{c}_\xi(1)$ and by the properties of the function $\mathsf{cd}$ from Fact 4.5, this is an almost disjoint family.

It is not hard see that $\mathcal{A}$ is maximal. We first point out the following simple observation:

**Claim 4.23.** *Whenever $\nu < \xi < \omega_2$, $d_\nu \prec d_\xi$.*

*Proof.* This is clear by the definition: Suppose otherwise that $d_\xi \preceq d_\nu$. Since $\mathcal{A}_\nu \subseteq \mathcal{A}_\xi$, $d_\xi$ is almost disjoint from every set in $\mathcal{A}_\nu$. So by minimality of $d_\nu$, we infer $d_\nu = d_\xi$. But then since $d_\nu \cap a_\nu$ is infinite by the properties of the function $\mathsf{cd}$ from Fact 4.5,

$d_\xi$ is not almost disjoint from every element of $\mathcal{A}_\xi$, contradicting how $d_\xi$ was chosen. Claim 4.23. □

**Claim 4.24.** *The set $\mathcal{A}$ is a maximal almost disjoint family.*

*Proof.* Suppose towards a contradiction that $d \in [\omega]^\omega \setminus \mathcal{A}$ and $\mathcal{A} \cup \{d\}$ is an almost disjoint family. Let $\xi < \omega_2$ be the least ordinal such that $d \preceq d_\xi$; such an ordinal exists since $\prec$ well-orders the reals in ordertype $\omega_2$ and so the sequence $\langle d_\xi : \xi < \omega_2 \rangle$ is $\prec$-cofinal in $\mathcal{P}(\omega)$. But since at stage $\xi$ in the construction of $\mathcal{A}$, $d_\xi$ was chosen to be the least element almost disjoint from every element of $\{a_\nu : \nu < \xi\}$, we have $d = d_\xi$. Then since $a_\xi = \mathsf{cd}(\bar{c}_\xi(0), \bar{c}_\xi(1), \bar{c}_\xi)$ and $\bar{c}_\xi(0) = d_\xi = d$, $a_\xi \cap d$ is infinite by the properties of the function $\mathsf{cd}$ from Fact 4.5, contradiction. Claim 4.24. □

We now show that $\mathcal{A}$ is $\Pi^1_2(c_\mathcal{A})$. We first show:

**Claim 4.25.** *There is a $\Pi^1_2(c_\mathcal{A})$ formula $\Theta(x)$ such that $\Theta(\bar{c})$ holds if and only if $\bar{c}$ is a minimal local witness.*

*Proof.* It is easily seen that for each $k \in \omega \setminus 3$ the set

$$\{(c, c') \in \mathcal{P}(\omega) \times \mathcal{P}(\omega)^k : c \text{ is a } k\text{-localizer for } c'\}$$

is definable by a $\Pi^1_2(c_\mathcal{A})$ formula $\Theta_k(x, y)$, namely, the formula obtained by expressing $(*)_k$ in the language of set theory. In fact, $\langle \Theta_k(x, y) : k \in \omega \rangle$ is a recursive sequence of formulas, and so using a universal definable $\Pi^1_2$ truth predicate we can find a $\Pi^1_2(c_\mathcal{A})$ formula $\Theta(\bar{c})$ equivalent to

$$(\forall k \in \omega)\ \Theta_k(\bar{c}(k+3), \bar{c} \restriction (k+3)).$$ □

Let now $\Psi(a)$ be defined as follows:

$$\Psi(a) \overset{\text{def}}{\iff} \big[(\exists n \in \omega)\ a = a_n\big] \vee \\ (\forall c \in (\mathcal{P}(\omega))\ \Big[c = \mathsf{dc}(a) \Rightarrow \Big(a = \mathsf{cd}\left((c)_0, (c)_1, c\right) \wedge \Theta\big(\operatorname{Seq}(c)\big)\Big)\Big].$$

Clearly this formula is $\Pi^1_2(c_\mathcal{A})$. We will show that $\Psi(a) \iff a \in \mathcal{A}$. The non-trivial direction is "$\Rightarrow$," which we show first.

**Lemma 4.26.** $(\forall a \in [\omega]^\omega)\ \Psi(a) \Rightarrow a \in \mathcal{A}$.

*Proof.* Suppose $\Psi(a)$ and to avoid trivialities let us suppose $a \notin \{a_n \mid n \in \omega\}$. Then $\bar{c} = \operatorname{Seq}(\mathsf{dc}(a))$ is a minimal local witness and so $\vec{b}(\bar{c})$ is defined, namely as the unique sequence coded by $\bar{c}(2)$ as in $(*)_3$(a). Let us write $\vec{b}(\bar{c}) = \langle b_\xi : \xi < \alpha \rangle$. We need the following claim:

**Claim 4.27.** *The sequence $\vec{b}(\bar{c}) = \langle b_\xi : \xi < \alpha \rangle$ is an initial segment of $\langle a_\nu : \nu < \omega_2 \rangle$.*

*Proof.* Suppose not. Let $\nu < \alpha$ be least such that $b_\nu \neq a_\nu$. Write $c^*_\nu = \mathsf{dc}(b_\nu)$ and $\bar{c}^*_\nu = \operatorname{Seq}(c^*_\nu)$. Since $\bar{c}$ is a minimal local witness, $\vec{b}(\bar{c})$ is a coherent candidate, and so $(c^*_\nu)_2$ codes almost disjointly via $\mathcal{F}$ the sequence $\vec{b}(\bar{c}) \restriction \nu$, which by assumption is $\langle a_\xi : \xi < \nu \rangle$.

We verify that $\bar{c}^*_\nu$, too, is a minimal local witness: Firstly, $\bar{c}(3)$ is a 3-localizer for $\bar{c} \restriction 3$. Then by (b) in $(*)_4$ and by Lemma 4.18 it holds that $\bar{c}^*_\nu \restriction 3$ is a minimal 3-witness. More generally, since $\bar{c}(k)$ is a $k$-localizer for $c \restriction k$, by (b) in $(*)_k$ we see that $\bar{c}^*_\nu \restriction k$ is a minimal $k$-witness (cf. Lemma 4.21). Since this holds for each $k \in \omega$, $\bar{c}^*_\nu$ is a minimal local witness.

But then since $\bar{c}^*_\nu$ and $\bar{c}_\nu$ are both minimal local witnesses for the sequence $\langle a_\xi : \xi < \nu \rangle$, we must have $\bar{c}^*_\nu = \bar{c}_\nu$ by the definition of minimal local witness (see also Lemma 4.22). It follows that $a_\nu = \mathsf{cd}(\bar{c}_\nu(0), \bar{c}_\nu(1), c_\nu) = \mathsf{cd}(\bar{c}^*_\nu(0), \bar{c}^*_\nu(1), c^*_\nu) = b_\nu$, contradicting the choice of $\nu$. Claim 4.25. □

By the claim we can fix $\nu < \omega_2$ such that $\vec{b}(\bar{c}) = \langle a_\xi : \xi < \nu \rangle$. By the same argument as in the previous paragraph, $a = a_\nu$. Lemma 4.26. □

Finally, for any $\xi$ such that $\omega \leq \xi < \omega_2$ it is clear by construction that $a_\xi = \mathsf{cd}(\bar{c}_\xi)$ and $\bar{c}_\xi$ is a minimal local witness. Therefore $\Psi(a_\xi)$ holds. So $a \in \mathcal{A} \Rightarrow \Psi(a)$. Theorems 1.3 & 4.1. □

## 5. Infinite MAD families, sharps, and Bounded Martin's Maximum

In this section, we proove Theorem 1.4, i.e., that under $\mathsf{ZFC} + \mathsf{BMM}$ there are no infinite $\mathbf{\Sigma}^1_3$ MAD families. In fact, we show the following:

**Theorem 5.1.** *Suppose for every $a \in \mathcal{P}(\omega)$, $a^\sharp$ exists. Then there are no infinite $\mathbf{\Sigma}^1_3$ MAD families.*

*Proof.* Under the assumption of the theorem, any $\Sigma^1_3(a)$ set, where $a \in \mathcal{P}(\omega)$, is equal to $p[T]$ for some tree $T$ on $\omega \times \kappa$ (for some ordinal $\kappa$); in fact one can take $T \in \mathbf{L}[a^\sharp]$ (this is implicit in [13]; see [11, pp. 198–204] for a proof, where the result is credited to Martin). Since also $(a^\sharp)^\sharp$ exists, $\mathcal{P}(\mathcal{P}(\omega))^{\mathbf{L}[a^\sharp]}$ is countable. Now let us suppose that $p[T]$, for some such tree $T$, is an infinite almost disjoint family. Following [2] we show that $p[T]$ cannot be maximal: For let $r$ be generic over $\mathbf{L}[a^\sharp]$ for Mathias forcing relative to the ideal generated by $p[T]$, as computed in $\mathbf{L}[a^\sharp]$; then $r$ is almost disjoint from any element of $p[T]$ by [2, Main Proposition 3.6]. □

Theorem 1.4 follows by a result of Schindler:

*Proof of Theorem 1.4.* As Schindler showed in [23, Theorem 1.3], $\mathsf{BMM}$ implies that every set has a sharp. Now use the previous theorem. □

## 6. Questions

**Question 6.1.** *Can* BPFA *be replaced by the Bounded Forcing Axiom for Axiom A in Theorem 1.3?*

**Question 6.2.** *Can we assume a forcing axiom* stronger *than* BPFA *but still compatible with an appropriate,* weaker *anti-large cardinal assumption and derive a form of Theorem 1.3?*

## References

[1] Joan Bagaria, *Bounded forcing axioms as principles of generic absoluteness*, Arch. Math. Logic **39** (2000), no. 6, 393–401. MR 1773776

[2] K. Bakke Haga, D. Schrittesser, and A. Törnquist, *Mad families, determinacy, and forcing*, `arXiv:1810.03016[math.LO]`, October 2018.

[3] Tomek Bartoszyński and Haim Judah, *Set theory - on the structure of the real line*, A K Peters, Ltd., Wellesley, MA, 1995. MR 1350295

[4] Andrés Eduardo Caicedo and Sy-David Friedman, BPFA *and projective well-orderings of the reals*, J. Symbolic Logic **76** (2011), no. 4, 1126–1136. MR 2895389
[5] Andrés Eduardo Caicedo and Boban Veličković, *The Bounded Proper Forcing Axiom and well orderings of the reals*, Math. Res. Lett. **13** (2006), no. 2-3, 393–408. MR 2231126
[6] Vera Fischer, Sy Friedman, David Schrittesser, and Asger Törnquist, *Good projective witnesses*, preprint, 2018.
[7] Martin Goldstern and Saharon Shelah, *The Bounded Proper Forcing Axiom*, J. Symbolic Logic **60** (1995), no. 1, 58–73. MR 1324501
[8] Leo Harrington and Saharon Shelah, *Some exact equiconsistency results in set theory*, Notre Dame J. Formal Logic **26** (1985), no. 2, 178–188. MR 783595
[9] Haim Horowitz and Saharon Shelah, *Can you take Törnquist's inaccessible away?*, `arXiv:1605.02419[math.LO]`, May 2016.
[10] R. B. Jensen and R. M. Solovay, *Some applications of almost disjoint sets*, Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968), North-Holland, Amsterdam, 1970, pp. 84–104. MR 0289291
[11] Akihiro Kanamori, *The higher infinite*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2009, Large cardinals in set theory from their beginnings, Paperback reprint of the 2003 edition. MR 2731169
[12] K. Kunen, *Set theory. an introduction to independence proofs*, North Holland, 1980.
[13] D. A. Martin and R. M. Solovay, *A basis theorem for* $\Sigma^1_3$ *sets of reals*, Ann. of Math. (2) **89** (1969), 138–159. MR 255391
[14] ______, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), no. 2, 143–178. MR 270904
[15] A. R. D. Mathias, *On a generalization of Ramsey's Theorem*, Research Fellowship, Peterhouse, Cambridge 1969 and Ph.D. thesis Cambridge, 1970, available at the author's website.
[16] ______, *Happy families*, Ann. Math. Logic **12** (1977), no. 1, 59–111. MR 0491197
[17] Arnold W. Miller, *Infinite combinatorics and definability*, Ann. Pure Appl. Logic **41** (1989), no. 2, 179–203. MR 983001
[18] Justin Tatch Moore, *Set mapping reflection*, J. Math. Log. **5** (2005), no. 1, 87–97. MR 2151584
[19] Itay Neeman and Zach Norwood, *Happy and mad families in* $L(\mathbb{R})$, J. Symb. Log. **83** (2018), no. 2, 572–597. MR 3835078
[20] Ralf-Dieter Schindler, *Proper forcing and remarkable cardinals*, Bull. Symbolic Logic **6** (2000), no. 2, 176–184. MR 1765054
[21] ______, *Proper forcing and remarkable cardinals. II*, J. Symbolic Logic **66** (2001), no. 3, 1481–1492. MR 1856755
[22] ______, *Forcing axioms and projective sets of reals*, Classical and new paradigms of computation and their complexity hierarchies, Trends Log. Stud. Log. Libr., vol. 23, Kluwer Acad. Publ., Dordrecht, 2004, pp. 207–222. MR 2156740
[23] ______, *Semi-proper forcing, remarkable cardinals, and bounded Martin's maximum*, MLQ Math. Log. Q. **50** (2004), no. 6, 527–532. MR 2096166
[24] ______, *Bounded Martin's maximum and strong cardinals*, Set theory, Trends Math., Birkhäuser, Basel, 2006, pp. 401–406. MR 2267158
[25] David Schrittesser and Asger Törnquist, *The Ramsey property implies no mad families*, Proc. Natl. Acad. Sci. USA **116** (2019), no. 38, 18883–18887. MR 4012549
[26] John R. Steel, *PFA implies* $\mathrm{AD}^{L(\mathbb{R})}$, J. Symbolic Logic **70** (2005), no. 4, 1255–1296. MR 2194247
[27] Asger Törnquist, $\Sigma^1_2$ *and* $\Pi^1_1$ *mad families*, J. Symbolic Logic **78** (2013), no. 4, 1181–1182. MR 3156517
[28] ______, *Definability and almost disjoint families*, Adv. Math. **330** (2018), 61–73. MR 3787540
[29] Zoltán Vidnyánszky, *Transfinite inductions producing coanalytic sets*, Fund. Math. **224** (2014), no. 2, 155–174. MR 3180585

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