THE DISTRIBUTIVITY SPECTRUM OF $\mathcal{P}(\omega)/\mathbf{fin}$

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Abstract. We construct a model in which there exists a distributivity matrix of regular height λ larger than $\mathfrak{h}$; both $\lambda = \mathfrak{c}$ and $\lambda < \mathfrak{c}$ are possible. A distributivity matrix is a refining system of mad families without common refinement. Of particular interest in our proof is the preservation of $\mathcal{B}$-Canjarness.

In our model, all branches through the underlying tree of the distributivity matrix are cofinal. We show that there can never be a base matrix tree of regular height larger than $\mathfrak{h}$ all of whose branches are cofinal. We also discuss the concept of distributivity matrices for $\mathcal{P}(\kappa)/{<}\kappa$ in place of $\mathcal{P}(\omega)/\mathrm{fin}$, where κ is a regular uncountable cardinal.

1. Introduction

The Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ has attracted a lot of attention in the last decades. One of the characteristics of a partial order is its distributivity. The distributivity of $\mathcal{P}(\omega)/\mathrm{fin}$ is the well-known cardinal characteristic $\mathfrak{h}$ which has been defined in [2], where the famous base matrix theorem is proved (see Theorem 2.5), and is tightly connected to many other structural properties of $\mathcal{P}(\omega)/\mathrm{fin}$ involving towers and mad families. In this paper, we define a distributivity spectrum for $\mathcal{P}(\omega)/fin$, i.e., a spectrum for $\mathfrak{h}$. Spectra have been considered for several cardinal characteristics, but not for $\mathfrak{h}$. For example, spectra for the tower number $\mathfrak{t}$ have been investigated in [25] and [14], spectra for the almost disjointness number $\mathfrak{a}$ in [25], [8], and [37], spectra for the bounding number $\mathfrak{b}$ in [14], spectra for the ultrafilter number $\mathfrak{u}$ in [34], [35], and [21], and spectra for the independence number $\mathfrak{i}$ in [18]. Furthermore, [5] develops a framework for dealing with several spectra.

Let $\mathfrak{h}(\mathbb{P})$ be the distributivity of the partial order $\mathbb{P}$, i.e., the least cardinal λ such that it is not λ-distributive; in other words, it is the least λ such that there is a system of λ many maximal antichains without common refinement, or, equivalently, such that $\mathbb{P}$ adds a new function from λ into the ordinals. We consider two possibilities to define a distributivity spectrum: the combinatorial distributivity spectrum (see Section 2.1), and the fresh function spectrum (see Section 2.2); in particular, we compute the fresh function spectrum of $\mathcal{P}(\omega)/\mathrm{fin}$ which turns out to depend only on $\mathfrak{h}$ and the size of the continuum. In [16], the fresh function spectrum is studied in greater generality.

The main focus of the paper is the combinatorial distributivity spectrum of $\mathcal{P}(\omega)/\mathrm{fin}$: this is the set of regular cardinals λ such that there exists a distributivity matrix of height λ for $\mathcal{P}(\omega)/\mathrm{fin}$. Such distributivity matrices are refining systems of mad families without common refinement and have an underlying tree

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structure. Some distributivity matrices have cofinal branches, and in some distributivity matrices all maximal branches are shorter than the height of the matrix (see [13] and [15]). This is strongly connected to the tower number $\mathfrak{t}$ and its spectrum (see [14]). We give a detailed analysis of these questions in Section 3. In particular, we show that a base matrix of regular height larger than $\mathfrak{h}$ necessarily has many branches which are dying out (see Theorem 3.7).

There are always distributivity matrices of height $\mathfrak{h}$. The main result of this paper (see Main Theorem 4.1) shows that it is consistent that there exists a distributivity matrix of regular height λ larger than $\mathfrak{h}$.

Main Theorem. *In a model of ZFC+GCH, let $\omega_1 < \lambda \leq \mu$ be cardinals such that λ is regular and* $\mathrm{cf}(\mu) > \omega$. *Then there is a cofinality preserving extension with $\mu = \mathfrak{c}$ such that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ and there exists a distributivity matrix of height λ.*

The proof strategy is as follows. We define a forcing iteration (see Section 4.1) which adds a distributivity matrix of height λ. Building on ideas from [25], we use c.c.c. iterands which approximate the distributivity matrix by finite conditions. To show that the generic object is actually a distributivity matrix (see Section 5), we consider certain complete subforcings to capture new subsets of ω. To show that $\mathfrak{h}$ is ω_1, we show that $\mathfrak{b}$ is ω_1 (and use that $\mathfrak{h} \leq \mathfrak{b}$). In fact, we show that the ground model reals $\mathcal{B}$ remain unbounded. For that, we represent our iteration as a finer iteration of Mathias forcings with respect to filters and use a characterization from [24] to show that these filters are $\mathcal{B}$-Canjar, i.e., that the corresponding Mathias forcings preserve the unboundedness of $\mathcal{B}$ (see Section 7). In [17], the same is done for Hechler's original forcings [25] to add a tower or to add a mad family.

We can use a genericity argument to show that the filters are $\mathcal{B}$-Canjar at the stage where they appear, but we need the $\mathcal{B}$-Canjarness in later stages of the iteration. Since the notion of $\mathcal{B}$-Canjarness of a filter is not absolute, we have to develop a method how to guarantee that the $\mathcal{B}$-Canjarness of a filter is not destroyed by Mathias forcings with respect to certain other filters (see Section 6).

At the end, we consider generalizations to $\mathcal{P}(\kappa)/{<\kappa}$ for regular uncountable κ. Building on [3] and [33], we compute the fresh function spectrum of $\mathcal{P}(\kappa)/{<\kappa}$ (see Section 8.2); then we discuss what we know about distributivity matrices for $\mathcal{P}(\kappa)/{<\kappa}$ (see Section 8.3). These considerations are also an attempt towards defining a κ-analogue of $\mathfrak{h}$ (this problem has been considered in [20] as well). A slightly different approach to generalize $\mathfrak{h}$ has been taken in [12]. We conclude the paper with some open questions (see Section 9).

2. Distributivity spectra

In this section, we introduce two notions of a distributivity spectrum and discuss their basic properties: the combinatorial distributivity spectrum (see Section 2.1) and the fresh function spectrum (see Section 2.2). Moreover, we provide a game characterization of being λ-distributive (see Section 2.3).

Recall the following basic definitions concerning forcing. Let[1] $(\mathbb{P}, \leq)$ be any forcing notion, i.e., a nonempty set $\mathbb{P}$ together with a partial order (or pre-order) $\leq$. A set $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there exists $q \in D$ such that $q \leq p$. Two conditions p and q in $\mathbb{P}$ are *incompatible* if there exists no $r \in \mathbb{P}$ with $r \leq p$ and $r \leq q$; otherwise they are *compatible*. A set $A \subseteq \mathbb{P}$ is an *antichain* if p and q are incompatible for all distinct p and q in A. An antichain A is *maximal* if for each $p \in \mathbb{P}$, there is $q \in A$ which is compatible with p. A forcing notion $\mathbb{P}$ has the *χ-c.c.* if $\mathbb{P}$ has no antichain of size χ. For (maximal) antichains A and

B in $\mathbb{P}$, we say that B *refines* A if for each $q \in B$ there is a $p \in A$ such that $q \leq p$. A forcing is *separative* if for each $q \nleq p$, there is $r \leq q$ which is incompatible with p. For an introduction to the theory of forcing, see, e.g., [29] or [27].

2.1. **Combinatorial distributivity spectrum.** As mentioned in the introduction, the *distributivity* of a forcing notion $\mathbb{P}$, denoted by $\mathfrak{h}(\mathbb{P})$, can be defined as the least λ such that there is a system of λ many maximal antichains without common refinement. As usual, we sometimes say that $\mathbb{P}$ is *δ-distributive* if $\delta < \mathfrak{h}(\mathbb{P})$.

Clearly, also for any δ larger than λ, there is a system of δ many maximal antichains without common refinement (so this does not yield a sensible definition of spectrum). However, this is not necessarily the case if the system is in addition required to be refining; note that $\mathfrak{h}(\mathbb{P})$ is actually the least λ such that there is a system of λ many *refining* maximal antichains without common refinement, which justifies the following combinatorial definition of a distributivity spectrum:

Definition 2.1. We say that $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ is a *distributivity matrix of height λ for* $\mathbb{P}$ if

(1) A_ξ is a maximal antichain in $\mathbb{P}$, for each $\xi < \lambda$,
(2) A_η refines A_ξ whenever $\eta \geq \xi$, and
(3) there is no common refinement, i.e., there is no maximal antichain B which refines every A_ξ.

The *combinatorial distributivity spectrum of* $\mathbb{P}$ (denoted by $\mathsf{COM}(\mathbb{P})$) is the set of regular cardinals λ such that there exists a distributivity matrix of height λ for $\mathbb{P}$.

We say that q *intersects* a distributivity matrix $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ if for each $\xi < \lambda$ there is an $a \in A_\xi$ with $q \leq a$. Note that Definition 2.1(3) is equivalent to

(3') $\{q \in \mathbb{P} \mid q \text{ intersects } \mathcal{A}\}$ is not dense in $\mathbb{P}$;

in particular, (3') holds if there is *no* q intersecting $\mathcal{A}$.

It is straightforward to check that the existence of distributivity matrices is only a matter of cofinality: if δ is a singular cardinal with $\mathrm{cf}(\delta) = \lambda$, then there exists a distributivity matrix of height δ for $\mathbb{P}$ if and only if there exists one of height λ (i.e., $\lambda \in \mathsf{COM}(\mathbb{P})$). Therefore, the restriction of the definition of $\mathsf{COM}(\mathbb{P})$ to regular cardinals makes sense.

Let us fix the following notation: for $\alpha, \beta \in \mathrm{Ord}$, let

$$[\alpha, \beta]_{Reg} := \{\lambda \mid \lambda \text{ is a regular cardinal with } \alpha \leq \lambda \leq \beta\}.$$

As discussed above, the least element of $\mathsf{COM}(\mathbb{P})$ is just the distributivity of $\mathbb{P}$:

$$\mathfrak{h}(\mathbb{P}) = \min(\mathsf{COM}(\mathbb{P})). \tag{1}$$

On the other hand, it is quite easy to get an upper bound for the elements of the combinatorial distributivity spectrum:

Proposition 2.2. *$\mathsf{COM}(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]_{Reg}$.*

[1] We use the "Boolean" (i.e., "downwards") notation: $q \leq p$ means "q is stronger than p" or "q has more information than p". We employ the "alphabet convention" (i.e., variables for stronger conditions come lexicographically later). The forcing will typically be non-atomic, i.e., for each condition, there are two stronger conditions which are incompatible.

Proof. We have to prove that no regular $\lambda > |\mathbb{P}|$ belongs to $\mathsf{COM}(\mathbb{P})$. So assume towards contradiction that $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ is a distributivity matrix of regular height $\lambda > |\mathbb{P}|$ for $\mathbb{P}$. We will show that item (3') fails, i.e., $\{q \in \mathbb{P} \mid q$ intersects $\mathcal{A}\}$ is dense in $\mathbb{P}$.

Fix $p \in \mathbb{P}$. For each $\xi < \lambda$, pick $a_\xi \in A_\xi$ which is compatible with p (this is possible since each A_ξ is a maximal antichain by item (1)). Since $|\mathbb{P}| < \lambda$ and λ is regular, there exists a condition $a^* \in \mathbb{P}$ such that $a_\xi = a^*$ for cofinally many $\xi < \lambda$. Let $q \in \mathbb{P}$ with $q \leq p$ and $q \leq a^*$. Since the matrix is refining by item (2), it is straightforward to check that q intersects $\mathcal{A}$, as desired. $\square$

The classical cardinal characteristic $\mathfrak{h}$ is defined as $\mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$. Since $\mathcal{P}(\omega)/\text{fin}$ is of size continuum, (1) and the above proposition immediately yield the following:

Corollary 2.3. $\{\mathfrak{h}\} \subseteq \mathit{COM}(\mathcal{P}(\omega)/\mathit{fin}) \subseteq [\mathfrak{h}, \mathfrak{c}]_{Reg}$.

In Main Theorem 4.1, we show that it is consistent that $\mathsf{COM}(\mathcal{P}(\omega)/\text{fin})$ contains more than one element, i.e., consistently, there exists a regular $\lambda > \mathfrak{h}$ with $\lambda \in \mathsf{COM}(\mathcal{P}(\omega)/\text{fin})$; in fact, both $\lambda = \mathfrak{c}$ and $\lambda < \mathfrak{c}$ are possible.

A special sort of distributivity matrices have been considered in the seminal paper [2] where $\mathfrak{h}$ has been introduced:

Definition 2.4. A distributivity matrix $\{A_\xi \mid \xi < \lambda\}$ for $\mathbb{P}$ is a *base matrix* if $\bigcup_{\xi<\lambda} A_\xi$ is dense in $\mathbb{P}$, i.e., for each $p \in \mathbb{P}$ there is $\xi < \lambda$ and $a \in A_\xi$ such that $a \leq p$.

It is straightforward to check that for a base matrix $\mathcal{A}$ the stronger version of (3') holds: there is *no* $q \in \mathbb{P}$ intersecting $\mathcal{A}$.

Recall the base matrix theorem (for[2] $\mathcal{P}(\omega)/\text{fin}$) from [2]:

Theorem 2.5. *There exists a base matrix of height* $\mathfrak{h}$ *for* $\mathcal{P}(\omega)/\mathit{fin}$.

2.2. **Fresh function spectrum.** We now turn to another version of distributivity spectrum of a forcing notion $\mathbb{P}$.

Definition 2.6. We say that f is a *fresh function on* λ *(over* V*)* if $f\colon \lambda \to \text{Ord}$ and $f \notin V$, but $f \restriction \gamma \in V$ for every $\gamma < \lambda$. The *fresh function spectrum of* $\mathbb{P}$ (denoted by $\mathsf{FRESH}(\mathbb{P})$) is the set of regular cardinals λ such that, in some extension by $\mathbb{P}$, there exists a fresh function on λ over V.

As an example, consider λ-Cohen forcing for regular λ, which adds a λ-Cohen real: this is a fresh function on λ (since the forcing is $<\lambda$-closed).

Analogous to the situation for $\mathsf{COM}(\mathbb{P})$ discussed above, it is straightforward to check that the existence of fresh functions is only a matter of cofinality: if δ is an ordinal with $\text{cf}(\delta) = \lambda$, then (in any extension by $\mathbb{P}$) there exists a fresh function on δ if and only if there exists one on λ. Therefore, the restriction of the definition of $\mathsf{FRESH}(\mathbb{P})$ to regular cardinals makes sense.

It is well-known and easy to check that the minimum of the fresh function spectrum $\mathsf{FRESH}(\mathbb{P})$ is equal to $\mathfrak{h}(\mathbb{P})$; in particular, the distributivity number $\mathfrak{h}$ equals the minimum of $\mathsf{FRESH}(\mathcal{P}(\omega)/fin)$.

It is easy to see that no regular cardinal strictly above the size of $\mathbb{P}$ belongs to $\mathsf{FRESH}(\mathbb{P})$: if there were such a function, then all its initial segments are in the ground model, so they can be decided by conditions

[2] A more general version for a wider class of forcings has been given in [1, Theorem 2.1].

in $\mathbb{P}$; by cardinality, one condition appears cofinally often, hence forces the entire function to be in the ground model. Together with the fact that $\mathfrak{h}(\mathbb{P})$ is the minimum of $\mathsf{FRESH}(\mathbb{P})$, this yields the following basic result:

Proposition 2.7. *$\mathsf{FRESH}(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]_{Reg}$.*

As an example, let λ be regular, and let $\mathbb{P}$ be λ-Cohen forcing. Since $\mathbb{P}$ is $<\lambda$-closed, $\mathfrak{h}(\mathbb{P})$ is λ, and, assuming $2^{<\lambda} = \lambda$, the size of $\mathbb{P}$ is λ as well, and so $\mathsf{FRESH}(\mathbb{P}) = \{\lambda\}$.

The following strengthening of the above proposition generalizes a well-known fact about branches of certain trees, which was essentially proved in [30, Lemma 3.8] (see also [31]). We will make use of this theorem in the proof of Main Theorem 4.1 (namely in Claim 7.4), as well as in the proof of Proposition 8.3. The fresh function spectrum is studied in more detail in [16]; for the convenience of the reader, we include a proof here.

Theorem 2.8. *If $\mathbb{P} \times \mathbb{P}$ has the χ-c.c. and $\delta \geq \chi$, then $\delta \notin \mathsf{FRESH}(\mathbb{P})$.*

Proof. Assume $\delta \in \mathsf{FRESH}(\mathbb{P})$, i.e., there exists $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ such that p forces $\dot{f}\colon \delta \to \mathrm{Ord}$ is not in V and $\dot{f} \restriction \gamma \in V$ for each $\gamma < \delta$. Therefore, we can, by induction on $i < \chi$, construct $\alpha_i < \delta$, $p_i \leq p$, and $q_i \leq p$ such that p_i and q_i decide $\dot{f}$ up to α_i, and α_i is the first point about which p_i and q_i disagree; more precisely, there is $s_i\colon \alpha_i + 1 \to \mathrm{Ord}$ and $t_i\colon \alpha_i + 1 \to \mathrm{Ord}$ such that

(1) $\alpha_j < \alpha_i$ for each $j < i$,
(2) $p_i \Vdash \dot{f} \restriction (\alpha_i + 1) = s_i$,
(3) $q_i \Vdash \dot{f} \restriction (\alpha_i + 1) = t_i$,
(4) $s_i \neq t_i$, and $s_i \restriction \alpha_i = t_i \restriction \alpha_i$.

Consider $\langle (p_i, q_i) \mid i < \chi \rangle$ and use that $\mathbb{P} \times \mathbb{P}$ has the χ-c.c. to obtain $i_0 < i_1$ such that (p_{i_0}, q_{i_0}) and (p_{i_1}, q_{i_1}) are compatible, and fix $(\bar{p}, \bar{q})$ with $(\bar{p}, \bar{q}) \leq (p_{i_0}, q_{i_0})$ and $(\bar{p}, \bar{q}) \leq (p_{i_1}, q_{i_1})$. It follows that both $\bar{p}$ and $\bar{q}$ force that $\dot{f} \restriction \alpha_{i_1} = s_{i_1} \restriction \alpha_{i_1}$. Moreover, $\bar{p} \Vdash \dot{f} \restriction (\alpha_{i_0} + 1) = s_{i_0}$ and $\bar{q} \Vdash \dot{f} \restriction (\alpha_{i_0} + 1) = t_{i_0}$, but $s_{i_0} \neq t_{i_0}$, which easily yields (using $\alpha_{i_0} < \alpha_{i_1}$) a contradiction. □

Standard arguments show (for a detailed proof, see [16]) that the combinatorial distributivity spectrum is a subset of the fresh function spectrum:

Proposition 2.9. *$\mathsf{COM}(\mathbb{P}) \subseteq \mathsf{FRESH}(\mathbb{P})$.*

In case $\mathbb{P}$ is a *complete* Boolean algebra, one can show that even equality holds.

We are now going to compute the fresh function spectrum of $\mathcal{P}(\omega)/\mathrm{fin}$. Let us recall from [2] that $\mathcal{P}(\omega)/\mathrm{fin}$ collapses $\mathfrak{c}$ to $\mathfrak{h}$, which follows from the existence of a base matrix of height $\mathfrak{h}$ (see Theorem 2.5) and the fact that there exists an antichain of size $\mathfrak{c}$ below each condition.

Using the fact that $\mathfrak{h}(\mathbb{P})$ is the distributivity of $\mathbb{P}$, any surjection from $\mathfrak{h}(\mathbb{P})$ to a regular cardinal λ yields a strictly increasing, cofinal map from $\mathfrak{h}(\mathbb{P})$ to λ; it is straightforward to check that the characteristic function of its range is a fresh function on λ, which gives the following lemma (for a more detailed proof, see [16]):

Lemma 2.10. *Let λ be a regular cardinal and $\mathbb{P}$ a forcing which collapses λ to $\mathfrak{h}(\mathbb{P})$. Then $\lambda \in \mathsf{FRESH}(\mathbb{P})$.*

Under the hypothesis of the above lemma, we actually have (since each regular cardinal between $\mathfrak{h}(\mathbb{P})$ and λ is collapsed as well)

$$[\mathfrak{h}(\mathbb{P}), \lambda]_{Reg} \subseteq \mathsf{FRESH}(\mathbb{P}).$$

Note that the above conclusion holds even if λ is singular and $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$. From this, and Proposition 2.7 we immediately obtain the fresh function spectrum of $\mathcal{P}(\omega)/\text{fin}$:

Proposition 2.11. *$\mathsf{FRESH}(\mathcal{P}(\omega)/fin) = [\mathfrak{h}, \mathfrak{c}]_{Reg}$.*

Note that the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ is *not* complete, so the above proposition does not imply that $\mathsf{COM}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]_{Reg}$.

Also note that, for singular $\mathfrak{c}$, it is consistent that $\mathcal{P}(\omega)/\text{fin}$ does not add a fresh function on $\mathfrak{c}$. Indeed, it is consistent that $\text{cf}(\mathfrak{c}) < \mathfrak{t}$ (see, e.g., [36, Theorem 3.7] for $\kappa = \omega$); thus, using the fact that $\mathfrak{t} \leq \mathfrak{h}$ (see (2) in Section 3.1), it is consistent that $\text{cf}(\mathfrak{c}) < \mathfrak{h}$. Therefore there is no fresh function on $\mathfrak{c}$, because there is no fresh function on $\text{cf}(\mathfrak{c})$.

2.3. **A game characterization of distributivity.** We will now provide a game characterization of being λ-distributive, which we will make use of in the proof of Theorem 3.7, as well as to show that $\mathcal{P}(\kappa)/{<}\kappa$ is not ω-distributive (see Proposition 8.1). It generalizes the game characterization for being ω-distributive which can be found in [27, Lemma 30.23].

Definition 2.12. Let $\mathbb{P}$ be a forcing notion. Let $G_\lambda(\mathbb{P})$ denote the *λ-distributivity game* (which has length λ):

$$\begin{array}{c|cccccccccc}
\text{I} & a_0 & & a_1 & & \dots & & a_{\mu+1} & & \dots & \\
\hline
\text{II} & & b_0 & & b_1 & & \dots\ b_\mu & & b_{\mu+1} & & \dots
\end{array}$$

The players alternately pick conditions in $\mathbb{P}$ such that the resulting sequence is decreasing, i.e., $b_j \leq a_i$ and $a_{i+1} \leq b_i$ for every $i \leq j < \lambda$. Player I starts the game, and at limits μ, Player II has to play. If Player II cannot play at limits (because the sequence played till then has no lower bound), the game ends and Player I wins immediately. If the game continuous for λ many steps, Player II wins if and only if there exists a $b \in \mathbb{P}$ with $b \leq a_i$ for every successor $i < \lambda$.

Recall that, by definition, a forcing $\mathbb{P}$ is *$\leq\lambda$-strategically closed* if Player II has a winning strategy in $G_\lambda(\mathbb{P})$. A slightly weaker property turns out to be equivalent to being λ-distributive:

Proposition 2.13. *The following are equivalent:*

(1) Player I has no winning strategy in the game $G_\lambda(\mathbb{P})$.
(2) $\mathbb{P}$ is λ-distributive.

Proof. Assume Player I has a winning strategy σ. Let $a_0 := \sigma(\langle\rangle)$. We are going to construct a refining system of maximal antichains $\{A_\alpha \mid \alpha < \delta\}$ below a_0 by induction. First, let $A_0 := \{a_0\}$. Clearly, the set $\{\sigma(\langle a_0, b_0\rangle) \mid b_0 \leq a_0\}$ is dense below a_0. Pick an antichain A_1 in this set which is maximal below a_0.

More generally, for the successor step, assume that A_α has been defined. For a moment, fix $a \in A_\alpha$. Since $\{A_i \mid i \leq \alpha\}$ is a refining system of antichains by induction, for every $i \leq \alpha$ there is a unique $a_i \in A_i$ with $a \leq a_i$. Note that there is a sequence $\{b_i \mid i < \alpha\}$ such that these two sequences yield[3] a run

[3] For limits i, the a_i is not part of the run. Note that $a = a_\alpha$, which is the very last entry of s in case α is a successor ordinal, whereas for α limit, s has no last entry (and a does not occur in s).

$s := \langle a_0, b_0, \dots \rangle$ of the game where Player I plays according to σ. Clearly, the set $\{\sigma(s^\frown b) \mid b \leq a\}$ is dense below a. Pick an antichain $A^a_{\alpha+1}$ in this set which is maximal below a. Now let $A_{\alpha+1} := \bigcup\{A^a_{\alpha+1} \mid a \in A_\alpha\}$. Note that $A_{\alpha+1}$ refines A_α and is maximal below a_0.

Now assume that $\mu < \lambda$ is a limit ordinal and A_α has been defined for every $\alpha < \mu$. If there exists a refining antichain which is maximal below a_0, let A_μ be such an antichain. If no such antichain exists, we stop the construction here.

Assume A_α has been defined for every $\alpha < \lambda$. Any b intersecting this system of antichains would give a run of the game following σ which Player II wins. Since σ is a winning strategy for Player I, no such intersecting b exists. This shows that the construction ends after at most λ many steps.

Let $\delta \leq \lambda$ be maximal such that A_α is defined for every $\alpha < \delta$, and let B be an antichain such that $\{a_0\} \cup B$ is a maximal antichain in $\mathbb{P}$. It is easy to see that $\{A_\alpha \cup B \mid \alpha < \delta\}$ is a distributivity matrix in $\mathbb{P}$ of height δ. Hence $\mathbb{P}$ is not δ-distributive and therefore not λ-distributive.

For the other direction, assume $\mathbb{P}$ is not λ-distributive. So $\mathfrak{h}(\mathbb{P}) \leq \lambda$. Fix a[4] distributivity matrix $\{A_\alpha \mid \alpha < \mathfrak{h}(\mathbb{P})\}$ of height $\mathfrak{h}(\mathbb{P})$. This means that the conditions intersecting this matrix are not dense in $\mathbb{P}$. Let $a_0 \in \mathbb{P}$ such that no intersecting condition is stronger than a_0. Let us describe a winning strategy σ for Player I. Let $\sigma(\langle\rangle) := a_0$. Assume Player II played b_α in the αth round of the game (for $\alpha < \mathfrak{h}(\mathbb{P})$). Since A_α is a maximal antichain, there exists $a \in A_\alpha$ which is compatible with b_α. Let $a_{\alpha+1}$ be a witness for the compatibility and let $\sigma(\langle a_0, b_0, \dots, b_\alpha\rangle) := a_{\alpha+1}$.

If Player I follows the strategy σ, the game stops after at most $\mathfrak{h}(\mathbb{P})$ many rounds and Player I wins. Indeed, if there exists a run of the game of length $\mathfrak{h}(\mathbb{P})$ where Player I followed σ and has not won the game yet, then there exists a $b \in \mathbb{P}$ such that $b \leq a_{\alpha+1}$ for every $\alpha < \mathfrak{h}(\mathbb{P})$, which implies that b intersects the matrix and $b \leq a_0$, a contradiction. $\square$

3. Distributivity matrices for $\mathcal{P}(\omega)/\text{fin}$

In this section, we will discuss what distributivity matrices for $\mathcal{P}(\omega)/\text{fin}$ look like. After giving several basic definitions (see Section 3.1), we discuss (see Section 3.2) the tree structure of distributivity matrices, with the levels of the tree being mad families, and the branches (which can be cofinal in the tree or dying out) being $\subseteq^*$-decreasing sequences (typically towers). In Section 3.3, we discuss distributivity matrices of height $\mathfrak{h}$, and Dordal's model in which no such matrix has a cofinal branch. In Section 3.4, we discuss distributivity matrices of larger height, in the context of Main Theorem 4.1 and the Cohen model. In Section 3.5, we show that base matrices of regular height larger than $\mathfrak{h}$ (if they exist at all) always have maximal branches which are not cofinal. At the end, we summarize the possible existence of different types of distributivity matrices in various models of ZFC (see Section 3.6). Some of the question left open can be found in Section 9.

From now on, the main focus is on $\mathcal{P}(\omega)/\text{fin}$. To make the notation shorter, we use the following abbreviations:

Notation. We write COM instead of COM($\mathcal{P}(\omega)/\text{fin}$), and FRESH instead of FRESH($\mathcal{P}(\omega)/\text{fin}$).

[4]In fact, it is not necessary to assume that the system of maximal antichains is refining. One could just take any λ many maximal antichains without common refinement (even in case $\lambda > \mathfrak{h}(\mathbb{P})$) and perform the analogous proof.

3.1. **Basic definitions.** The forcing notion $([\omega]^\omega, \subseteq)$ is a non-separative pre-order whose separative quotient is the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ (see Remark 3.1). Alternatively, one can consider the separative pre-order $([\omega]^\omega, \subseteq^*)$, where $\subseteq^*$ denotes almost-inclusion: $b \subseteq^* a$ if $b \setminus a$ is finite. We write $a =^* b$ if $a \subseteq^* b$ and $b \subseteq^* a$. We say that a and b are *almost disjoint* if $a \cap b$ is finite (i.e., if they are incompatible). Moreover, we say that $A \subseteq [\omega]^\omega$ is an *almost disjoint family* if a and a' are almost disjoint whenever $a, a' \in A$ with $a \neq a'$ (i.e., if A is an antichain). An almost disjoint family A is *maximal* (called *mad family*) if for each $b \in [\omega]^\omega$ there exists $a \in A$ such that $|b \cap a| = \aleph_0$. For a sequence $\langle a_\xi \mid \xi < \delta \rangle \subseteq [\omega]^\omega$, we say that $b \in [\omega]^\omega$ is a *pseudo-intersection* of $\langle a_\xi \mid \xi < \delta \rangle$ if $b \subseteq^* a_\xi$ for each $\xi < \delta$. We say that $\langle a_\xi \mid \xi < \delta \rangle$ is a *tower of length* δ if $a_\eta \subseteq^* a_\xi$ for any $\eta > \xi$, and it does not have an infinite pseudo-intersection (i.e., if it is a decreasing sequence without lower bound).

Remark 3.1. As mentioned above, $\mathcal{P}(\omega)/\mathrm{fin}$ is[5] the separative quotient of $[\omega]^\omega$. More explicitly, it is the quotient of $[\omega]^\omega$ with respect to the equivalence relation $=^*$, with the order given by $\subseteq^*$ on the representatives. We will usually work with $([\omega]^\omega, \subseteq^*)$ instead of $\mathcal{P}(\omega)/\mathrm{fin}$ (i.e., we will work with representatives instead of equivalence classes). It is easy to see that all combinatorial objects we are interested in directly translate between the structures. Therefore, it does not matter which representation is chosen. Given, e.g., a maximal antichain in $\mathcal{P}(\omega)/\mathrm{fin}$, one can take arbitrary representatives of its elements to obtain a corresponding mad family. Also, $\mathsf{COM}(\mathcal{P}(\omega)/\mathrm{fin})$ equals $\mathsf{COM}([\omega]^\omega)$, which we call COM.

Let

$$spec(\mathfrak{t}) := \{\delta \mid \delta \text{ is regular and there is a tower of length } \delta\}$$

be the *tower spectrum*, and let $\mathfrak{t} := \min(spec(\mathfrak{t}))$ be the *tower number*. Note that whenever $\langle a_\xi \mid \xi < \delta \rangle$ is a tower, then there is a (sub)tower of length $\mathrm{cf}(\delta)$. On the other hand, each tower of length $\mathrm{cf}(\delta)$ can be expanded to one of length δ (by repeating elements). Therefore the restriction to regular cardinals in the definition of the tower spectrum makes sense. Moreover, let

$$spec(\mathfrak{a}) := \{\mu \mid \mu \text{ is an infinite cardinal and there is a mad family of size } \mu\}$$

be the *mad spectrum on* ω, and let $\mathfrak{a} := \min(spec(\mathfrak{a}))$ be the *almost disjointness number*. It is well-known and easy to see that there are always mad families of size $\mathfrak{c}$, i.e., $\mathfrak{c} \in spec(\mathfrak{a})$. Indeed, by identifying $2^{<\omega}$ with ω and taking the set of branches through the tree $2^{<\omega}$, we get an almost disjoint family of size $\mathfrak{c}$, which can be extended to a mad family (using the axiom of choice).

In Hechler's paper [25], it was shown that it is consistent that the continuum is large and $spec(\mathfrak{t})$ and $spec(\mathfrak{a})$ contain all uncountable regular cardinals up to the continuum. In [14], it was shown how to prevent certain cardinals from being in $spec(\mathfrak{t})$, and the same was done for $spec(\mathfrak{a})$ in [37]. A general framework for dealing with the spectra of nicely definable cardinal characteristics has been given in [5].

In the proof of Main Theorem 4.1, we will use the bounding number $\mathfrak{b}$. For $f, g \in \omega^\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We say that $\mathcal{B} \subseteq \omega^\omega$ is an *unbounded family*, if there exists no $g \in \omega^\omega$ with $f \leq^* g$ for all $f \in \mathcal{B}$. The *bounding number* $\mathfrak{b}$ is the smallest size of an unbounded family

[5] As usual in the context of forcing, one has to exclude the zero element of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ (i.e., the class of finite sets).

in ω^ω. The following inequalities between the cardinal characteristics are well-known and not too hard to prove:

$$\omega_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{a}. \tag{2}$$

In fact, it is easy to see (by diagonalization) that there are no towers of length ω (in other words, $\mathcal{P}(\omega)/\mathrm{fin}$ is σ-closed), i.e., $\omega \notin \mathit{spec}(\mathfrak{t})$, so the first inequality $\omega_1 \leq \mathfrak{t}$ holds true. We will give a proof of the inequality $\mathfrak{t} \leq \mathfrak{h}$ later (see Proposition 3.3). For general information on cardinal characteristics of the continuum, we refer the reader to Blass' paper [6]. In particular, the proof of the inequality $\mathfrak{b} \leq \mathfrak{a}$ can be found in [6, Proposition 8.4]. Since the fact that $\mathfrak{h} \leq \mathfrak{b}$ will be crucial in the proof of Main Theorem 4.1 (see Section 7), we give a sketch of the proof.

Proposition 3.2. $\mathfrak{h} \leq \mathfrak{b}$.

Proof. We will construct $\mathfrak{b}$ many mad families without common refinement. Let $\{f_i \mid i < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family of size $\mathfrak{b}$. For $b \in [\omega]^\omega$, let $\varphi_b \in \omega^\omega$ be its enumerating function, i.e., the unique strictly increasing function such that $b = \{\varphi_b(n) \mid n \in \omega\}$.

For $f \in \omega^\omega$ and $k \in \omega$, let $f^{+k} \in \omega^\omega$ be such that $f^{+k}(n) = f(n+k)$ for each $n \in \omega$. We say that $b \in [\omega]^\omega$ is *fast with respect to f* if $f^{+k} \leq^* \varphi_b$ for each $k \in \omega$. It is straightforward to check that, for each $f \in \omega^\omega$, the collection of sets which are fast with respect to f is dense in $[\omega]^\omega$ (fix $a \in [\omega]^\omega$; let $g \in \omega^\omega$ be such that $f^{+k} \leq^* g$ for each $k \in \omega$, and let $b \subseteq a$ be infinite such that $g \leq^* \varphi_b$).

Now fix, for each $i < \mathfrak{b}$, a mad family A_i within the dense collection of sets which are fast with respect to f_i. Assume towards contradiction that A is a mad family which refines A_i for each $i < \mathfrak{b}$. Fix $b \in A$. It is straightforward to check that $f_i \leq^* \varphi_b$ for every $i < \mathfrak{b}$ (fix $a \in A_i$ such that $b \subseteq^* a$, and observe that a and hence b is fast with respect to f_i), contradicting the unboundedness of $\{f_i \mid i < \mathfrak{b}\}$. □

3.2. **Tree structure of distributivity matrices.** A distributivity matrix $\{A_\xi \mid \xi < \lambda\}$ can be viewed as a tree (which we think of growing downwards): for each $\xi < \lambda$, the elements of the mad family A_ξ form the level ξ of the tree, and for $b \in A_\eta$ and $a \in A_\xi$ with $\eta > \xi$, the element b is below the element a in the tree if and only if $b \subseteq^* a$. Due to the refining structure of the distributivity matrix (see Definition 2.1(2)), each element of A_η is below exactly one element of A_ξ. Note that this tree is necessarily splitting[6]at some limit levels: this is because there always appear $\subseteq^*$-decreasing sequences of limit length which have no weakest lower bound, and so no single element below such a sequence can be enough to get maximality of the next level.

Let us say that $\langle a_\xi \mid \xi < \delta\rangle$ is a *branch through the distributivity matrix* $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ if $a_\xi \in A_\xi$ for each $\xi < \delta$, and $a_\eta \subseteq^* a_\xi$ for each $\xi \leq \eta < \delta$. We say that the branch is *maximal* if there is no branch through $\mathcal{A}$ strictly extending it. Note that each element $b \in A_\eta$ determines the branch $\langle a_\xi \mid \xi < \eta\rangle$ of its predecessors (which is not maximal).

For the nature of a maximal branch $\langle a_\xi \mid \xi < \delta\rangle$, there are several possibilities:

(1) The branch is not cofinal in the underlying tree (i.e., $\delta < \lambda$); in other words, it corresponds to a branch which is dying out (for an example, see[7]Theorem 3.5 and the discussion there).

[6]See also the discussion in Section 4.1 about the generic distributivity matrix of Main Theorem 4.1, whose underlying tree is splitting everywhere.

[7]For an example of a distributivity matrix for $\mathcal{P}(\kappa)/{<}\kappa$ in which all branches are dying out, see Section 8.3.

Let us argue that in this case the sequence $\langle a_\xi \mid \xi < \delta\rangle$ is necessarily a tower. If not, take any infinite $b \subseteq \omega$ which is a pseudo-intersection of the sequence; since A_δ refines A_ξ for each $\xi < \delta$, it is easy to see that every element of A_δ is either a pseudo-intersection of $\langle a_\xi \mid \xi < \delta\rangle$ (which is impossible since the branch is maximal), or it is almost disjoint from some a_ξ (in fact, from all but boundedly many). Therefore, b is almost disjoint from every element of A_δ, contradicting the fact that the almost disjoint family A_δ is maximal.

(2) The branch is cofinal in the underlying tree (i.e., $\delta = \lambda$).
Here, it might be the case that
 (a) the sequence $\langle a_\xi \mid \xi < \lambda\rangle$ is a tower;
 (b) the sequence $\langle a_\xi \mid \xi < \lambda\rangle$ is not a tower (i.e., has a pseudo-intersection), and
  (i) is eventually constant (i.e., there is $\xi < \lambda$ such that $a_\eta =^* a_\xi$ for each $\eta \geq \xi$), or
  (ii) is not eventually constant.

Recall that a base matrix (see Definition 2.4) is not intersected by any $b \in [\omega]^\omega$. In particular, it cannot have any branch of the above type 2(b); in other words, all maximal branches of base matrices are towers. In fact, it is easy to see that maximal branches which are not towers (i.e., branches of the above type 2(b)) do not play a crucial role in the context of distributivity matrices for $\mathcal{P}(\omega)/\mathrm{fin}$ in general;[8]this is due to the following argument.

Given a distributivity matrix $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$, the set

$$\{b \in [\omega]^\omega \mid b \text{ intersects } \mathcal{A}\}$$

is not dense (see Definition 2.1(3')). So we can fix $b \in [\omega]^\omega$ such that no infinite $b' \subseteq b$ is intersecting $\mathcal{A}$. For each mad family A, let

$$A \restriction b := \{a \cap b \mid a \in A \ \wedge \ |a \cap b| = \aleph_0\}$$

denote the "relativization" of A to b. It is straightforward to check that the relativized matrix $\{A_\xi \restriction b \mid \xi < \lambda\}$ is actually a distributivity matrix of height λ on b (in place of ω) with the additional property that it is not intersected by any infinite subset of b. "Transferring" the matrix from b to ω, we obtain a distributivity matrix with the property that there is no infinite set intersecting it; since any pseudo-intersection of a maximal branch which is cofinal would intersect the matrix, every maximal branch is a tower.

Convention. From now on, we will always tacitly assume that each maximal branch $\langle a_\xi \mid \xi < \delta\rangle$ of a distributivity matrix is a tower (whether cofinal or not).

It follows that the regular cardinal $\mathrm{cf}(\delta)$ which correspond to the length of the branch belongs to the tower spectrum $spec(\mathfrak{t})$ (and hence $\mathfrak{t} \leq \delta$). In particular, this yields the following well-known fact:

Proposition 3.3. $\mathfrak{t} \leq \mathfrak{h}$.

[8]On the other hand, it is always possible to manipulate a given distributivity matrix in such a way that it has branches of type 2(b). Indeed, one can, e.g., "transfer" the matrix from ω to the set of even numbers, and add the set of odd numbers to each level of the matrix to regain maximality in each level (either by letting the odd numbers be an additional element of each level, yielding a branch of type 2(b)(i), or – taking a cofinal tower through the original part of the matrix – adding the odd numbers to each element along the tower, yielding a branch of type 2(b)(ii)).

Proof. Since $\mathfrak{h} = \min(\mathsf{COM})$, we can fix a distributivity matrix $\{A_\xi \mid \xi < \mathfrak{h}\}$ of height $\mathfrak{h}$ for $\mathcal{P}(\omega)/\text{fin}$. Fix any maximal branch $\langle a_\xi \mid \xi < \delta\rangle$ through this matrix (which is a tower of length δ) and observe that $\mathfrak{t} = \min(spec(\mathfrak{t})) \leq \text{cf}(\delta) \leq \mathfrak{h}$. □

Observe that $\mathfrak{t} \leq \mathfrak{h}$ would directly follow from $\mathsf{COM} \subseteq spec(\mathfrak{t})$; however, $\mathsf{COM} \subseteq spec(\mathfrak{t})$ does not hold true in general: in fact, it is consistent that COM and $spec(\mathfrak{t})$ are disjoint (see Theorem 3.5 below).

3.3. **Branches through distributivity matrices of height $\mathfrak{h}$.** In case $\mathfrak{t} = \mathfrak{h}$ (so in particular under $\mathfrak{h} = \omega_1$), there are no towers of length strictly less than $\mathfrak{h}$, hence all maximal branches of a distributivity matrix of height $\mathfrak{h}$ are cofinal.

On the other hand, it is possible to have a distributivity matrix of height $\mathfrak{h}$ which has no cofinal branches. In fact, it was shown by Dow that this is the case in the Mathias model:

Theorem 3.4. *Assume CH. Let $\mathbb{P}$ be the countable support iteration of Mathias forcing of length ω_2. Then, in $V[\mathbb{P}]$, there exists a base matrix of height $\mathfrak{h}$ without cofinal branches (and $\omega_1 = \mathfrak{t} < \mathfrak{h} = \mathfrak{c} = \omega_2$).*

Proof. See [15, Lemma 2.17]. □

It is actually consistent that *no* distributivity matrix of height $\mathfrak{h}$ has cofinal branches. This was proved by Dordal by constructing a model in which $\mathfrak{h}$ does not belong to the tower spectrum:

Theorem 3.5. *It is consistent with ZFC that $spec(\mathfrak{t}) = \{\omega_1\}$ and $\mathfrak{h} = \omega_2 = \mathfrak{c}$.*

Proof. See [13] for Dordal's original model, or[9] [14, Corollary 2.6], where Dordal gives a more general result (which is, interestingly enough, easier to prove). The instance $A = \{\omega_1\}$ and $\lambda = \omega_2$ of this result yields the desired constellation. □

By Theorem 2.5, there are always base matrices of height $\mathfrak{h}$. In Dordal's model, base matrices do not have cofinal branches. We do not know whether the tower spectrum in the Mathias model contains ω_2 or not. In particular, we do not know whether there exist distributivity matrices of height $\mathfrak{h}$ with cofinal branches in the Mathias model. Note that the Mathias model would serve the same purpose as Dordal's models if $\omega_2 \notin spec(\mathfrak{t})$ holds true in the Mathias model.

3.4. **Distributivity matrices of height strictly above $\mathfrak{h}$.** There are always distributivity matrices of height $\mathfrak{h}$; in other words, $\mathfrak{h}$ always belongs – as its minimal element – to the combinatorial distributivity spectrum COM. Moreover, recall Corollary 2.3 and Proposition 2.11, which yield

$$\{\mathfrak{h}\} \subseteq \mathsf{COM} \subseteq [\mathfrak{h}, \mathfrak{c}]_{Reg} = \mathsf{FRESH}. \tag{3}$$

If $\mathfrak{h} = \mathfrak{c}$, then clearly equality holds everywhere in the above equation.

Let us now discuss the possible existence (and nature) of distributivity matrices of regular height strictly above $\mathfrak{h}$ (for that we have look at models[10]of $\mathfrak{h} < \mathfrak{c}$). The main result of this paper (see Main Theorem 4.1) demonstrates that the existence of distributivity matrices of regular height strictly above $\mathfrak{h}$ is consistent: in fact, we construct a model in which $\mathfrak{h}$ is ω_1, and there exists a distributivity matrix of regular height $\lambda > \omega_1$ (e.g., of height $\omega_2 = \mathfrak{c}$). This distributivity matrix of height λ is generically added by forcing, and all its

[9]In fact, this corollary in [14] also works for $\mathfrak{h} = \mathfrak{c}$ larger than ω_2, and for certain tower spectra which are more complicated than $\{\omega_1\}$.

maximal branches are cofinal. This shows that the tower spectrum in our model contains both $\omega_1 = \mathfrak{t} = \mathfrak{h}$ and λ.

In the Cohen model, the situation is somewhat different. Again, there are distributivity matrices of height ω_1 (which clearly have only cofinal branches), due to $\omega_1 = \mathfrak{t} = \mathfrak{h}$. However, we do not know whether there exist distributivity matrices of any larger regular height (compare with Question 9.2 and Question 9.3). In any case, there is a difference to the model of our main theorem: if, for some regular $\lambda > \omega_1$, there is a distributivity matrix of height λ, then it does not have any cofinal branch, due to the following fact.

Proposition 3.6. *Assume CH. Let $\mathbb{C}_\mu$ be the Cohen forcing which adds μ many ω-Cohen reals, and let $\lambda > \omega_1$ be a regular cardinal. In $V[\mathbb{C}_\mu]$, let $\langle a_\alpha \mid \alpha < \lambda \rangle$ be a $\subseteq^*$-decreasing sequence. Then there exists an $\alpha_0 < \lambda$ such that $a_{\alpha_0} =^* a_\beta$ for every $\beta \geq \alpha_0$. In particular, $spec(\mathfrak{t}) = \{\omega_1\}$ holds true in $V[\mathbb{C}_\mu]$.*

Proof. This proof is somewhat similar to the proof of [8, Proposition 3.1]. First note that it is enough to show the case $\lambda = \omega_2$. Indeed, if there exists a $\subseteq^*$-decreasing sequence $\langle a_\alpha \mid \alpha < \lambda \rangle$ for some regular $\lambda \geq \omega_2$ which is not eventually $=^*$-constant, then there exists a subsequence of length λ which is strictly $\subseteq^*$-decreasing. Then its initial segment of length ω_2 is $\subseteq^*$-decreasing and not eventually $=^*$-constant.

Recall that $\mathbb{C}_\mu$ consists of finite partial functions from μ to 2, ordered by reverse inclusion. For every $\alpha < \omega_2$, let $\dot{a}_\alpha$ be a $\mathbb{C}_\mu$-name such that it is forced (without loss of generality by the empty condition) that $\dot{a}_\beta \subseteq^* \dot{a}_\alpha$ for each $\alpha \leq \beta < \omega_2$, yet there is no $\alpha_0 < \omega_2$ as in the conclusion of the proposition.

For every $\alpha < \omega_2$, for every $n \in \omega$ find countable maximal antichains $\{p^\alpha_{n,i} \mid i \in \omega\} \subseteq \mathbb{C}_\mu$ which decide whether $n \in \dot{a}_\alpha$. Let $B_\alpha := \bigcup_{i,n\in\omega} \mathrm{dom}(p^\alpha_{n,i})$, and note that B_α is countable. Since CH holds, we can use the Δ-system lemma (see [29, Ch. II, Theorem 1.6]) to find $X \subseteq \omega_2$ of size ω_2 such that $\{B_\alpha \mid \alpha \in X\}$ is a Δ-system with root R. Furthermore, we can assume without loss of generality that $|B_\alpha \setminus R|$ is the same for every $\alpha \in X$. For $\alpha, \beta \in X$, let $\varphi_{\alpha,\beta}$ be a bijection between B_α and B_β such that $\varphi_{\alpha,\beta}$ is the identity on R. For $B \subseteq \mu$, let $\mathbb{C}_B$ be the subforcing of $\mathbb{C}_\mu$ consisting of those conditions whose domains are subsets of B. Clearly $\varphi_{\alpha,\beta}$ induces an isomorphism between $\mathbb{C}_{B_\alpha}$ and $\mathbb{C}_{B_\beta}$. This naturally extends to an isomorphism $\psi_{\alpha,\beta}$ between $\mathbb{C}_{B_\alpha}$-names and $\mathbb{C}_{B_\beta}$-names. This isomorphism can also be extended to an isomorphism on $\mathbb{C}_\mu$ which we again call $\psi_{\alpha,\beta}$.

Note that there are only $2^{\aleph_0} = \aleph_1$ many isomorphism types of names, so we may also assume without loss of generality that $\psi_{\alpha,\beta}(\dot{a}_\alpha) = \dot{a}_\beta$ and $\psi_{\alpha,\beta}(\dot{a}_\beta) = \dot{a}_\alpha$ for all $\alpha, \beta \in X$.

Let $\alpha, \beta \in X$ and $\beta > \alpha$. Recall that by assumption $\Vdash_{\mathbb{C}_\mu} \dot{a}_\beta \subseteq^* \dot{a}_\alpha$. Using the isomorphism, we get $\Vdash_{\psi_{\alpha,\beta}(\mathbb{C}_\mu)} \psi_{\alpha,\beta}(\dot{a}_\beta) \subseteq^* \psi_{\alpha,\beta}(\dot{a}_\alpha)$, which yields $\Vdash_{\mathbb{C}_\mu} \dot{a}_\alpha \subseteq^* \dot{a}_\beta$. So $\Vdash_{\mathbb{C}_\mu} \dot{a}_\alpha =^* \dot{a}_\beta$. Thus, if $\alpha_0 := \min(X)$, for every $\beta \in X$ we have $\Vdash_{\mathbb{C}_\mu} \dot{a}_{\alpha_0} =^* \dot{a}_\beta$. Since X is cofinal in ω_2 it follows that $a_{\alpha_0} =^* a_\beta$ for every $\beta \geq \alpha_0$. □

3.5. **Base matrices of height strictly above $\mathfrak{h}$.** We now show that a base matrix tree of regular height larger than $\mathfrak{h}$ necessarily has (below every node) branches which are dying out.

Theorem 3.7. *Let $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ be a base matrix of regular height[11] $\lambda > \mathfrak{h}$. Then below every $a \in \bigcup_{\xi<\lambda} A_\xi$ there is a maximal branch through $\mathcal{A}$ which is not cofinal.*

[10]Note that it is not clear at all why there should exist distributivity matrices of (regular) height $\mathfrak{c}$. As opposed to this, $spec(\mathfrak{a})$ always contains $\mathfrak{c}$ (this is not the case for $spec(\mathfrak{t})$).

Proof. Assume towards contradiction that the conclusion of the theorem fails. Fix a^* in the matrix (i.e., $a^* \in \bigcup_{\xi<\lambda} A_\xi$) such that every maximal branch below a^* is cofinal. By definition of $\mathfrak{h}$, the partial order $\mathcal{P}(\omega)/\mathrm{fin}$ is *not* $\mathfrak{h}$-distributive. Let $\mathbb{P} := \{b \mid b \subseteq^* a^*\}$ be the partial order below a^*. Recall that $\mathcal{P}(\omega)/\mathrm{fin}$ is homogenous, hence $\mathbb{P}$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}$; in particular, $\mathbb{P}$ is not $\mathfrak{h}$-distributive. Therefore, using the game characterization of distributivity (see Proposition 2.13), Player I has a winning strategy σ in the game $G_\mathfrak{h}(\mathbb{P})$.

Consider the following run of the game of (full) length $\mathfrak{h}$ (note that $b_0 \subseteq^* a^*$):

$$\begin{array}{c|cccccccccc} \mathrm{I} & b_0 & & b_1 & & \dots & & b_{\mu+1} & & \dots & \\ \hline \mathrm{II} & & a_0 & & a_1 & & \dots\ a_\mu & & a_{\mu+1} & & \dots \end{array}$$

where Player I plays according to σ (i.e., $b_0 = \sigma(\langle\rangle)$ and $b_{i+1} = \sigma(\langle b_0, a_0, \dots, a_i\rangle)$ for each $i < \mathfrak{h}$), and Player II plays as follows (where the a_i are going to be in the matrix for each $i < \mathfrak{h}$). For successors i (and for $i = 0$), let $a_i \subseteq^* b_i$ with a_i in the matrix; this is possible, because it is a base matrix. For limit $\mu \le \mathfrak{h}$, the following holds by induction: $\langle a_i \mid i < \mu\rangle$ is a $\subseteq^*$-decreasing sequence below a^* such that a_i is in the matrix for each $i < \mu$. So (for $\mu < \mathfrak{h}$) Player II can play a pseudo-intersection a_μ in the matrix, by the following claim.

Claim. *The sequence $\langle a_i \mid i < \mu\rangle$ has a pseudo-intersection in the matrix.*

Proof. We can assume that the sequence is not eventually $=^*$-constant. Moreover, we can assume that it is strictly $\subseteq^*$-decreasing. It is easy to check that there is a strictly increasing sequence $\langle \xi_i \mid i < \mu\rangle \subseteq \lambda$ with $a_i \in A_{\xi_i}$ for each $i < \mu$. Then $\sup(\{\xi_i \mid i < \mu\}) < \lambda$, because $\mu \le \mathfrak{h} < \lambda$ and λ is regular. So the corresponding branch is not cofinal in the matrix, hence it is not maximal by assumption. Consequently, there exists an a in the matrix such that $a \subseteq^* a_i$ for each $i < \mu$. □

Finally, for $\mu = \mathfrak{h}$, the claim yields a pseudo-intersection of $\langle a_i \mid i < \mathfrak{h}\rangle$, witnessing that Player II wins this run of the game. This contradicts that σ is a winning strategy for Player I. □

We do not know whether the existence of distributivity matrices of regular height strictly above $\mathfrak{h}$ with non-cofinal maximal branches is consistent; so, in particular, we do not know whether it is consistent at all that there are base matrices of any regular height strictly above $\mathfrak{h}$ (see also Question 9.4).

Since in the model of our main theorem all maximal branches of the generically added distributivity matrix of regular height $\lambda > \mathfrak{h}$ are cofinal, the above theorem in particular shows that it is not a base matrix; this can also be shown directly (see Remark 4.12).

3.6. **Summary.** Let COM^{cof} denote the set of regular cardinals λ such that there exists a distributivity matrix of height λ for $\mathcal{P}(\omega)/\mathrm{fin}$ in which all maximal branches are cofinal. In Table 1, we give an overview of the values of the cardinal characteristics $\mathfrak{t}$ and $\mathfrak{h}$ as well as the related spectra in various models of ZFC + $\mathfrak{c} = \omega_2$. It summarizes our discussion from Section 3.3 and Section 3.4 (see, in particular, Proposition 3.6, Main Theorem 4.1, and Theorem 3.5).

[11] In fact, $\lambda > \mathfrak{h}$ can be replaced by the weaker assumption that $\mathcal{P}(\omega)/\mathrm{fin}$ is, for some $\nu < \lambda$, not $\le\nu$-strategically closed. In the proof, one can turn (using a well-order on the base matrix tree) the description of the moves of Player II into a strategy for Player II, which is then a winning strategy.

	$\mathfrak{t}$	$spec(\mathfrak{t})$	$\mathfrak{h}$	COM^{cof}	COM	FRESH
Cohen model	ω_1	$\{\omega_1\}$	ω_1	$\{\omega_1\}$	?	$\{\omega_1, \omega_2\}$
Main Theorem	ω_1	$\{\omega_1, \omega_2\}$	ω_1	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_2\}$
Dordal model	ω_1	$\{\omega_1\}$	ω_2	$\emptyset$	$\{\omega_2\}$	$\{\omega_2\}$
Mathias model	ω_1	?	ω_2	?	$\{\omega_2\}$	$\{\omega_2\}$
MA	ω_2	$\{\omega_2\}$	ω_2	$\{\omega_2\}$	$\{\omega_2\}$	$\{\omega_2\}$

Table 1. Different spectra in various models of $\mathfrak{c} = \omega_2$

Finally, let us summarize our discussion about base matrices from Section 3.5. ZFC proves (see Theorem 3.7, Theorem 2.5, Proposition 2.9, and Proposition 2.11)

$$\mathsf{COM}^{base+cof} \subseteq \{\mathfrak{h}\} \subseteq \mathsf{COM}^{base} \subseteq \mathsf{COM} \subseteq \mathsf{FRESH} = [\mathfrak{h}, \mathfrak{c}]_{Reg},$$

where COM^{base} denotes the set of regular cardinals λ such that there exists a base matrix of height λ for $\mathcal{P}(\omega)/\text{fin}$, whereas $\mathsf{COM}^{base+cof}$ denotes the set of regular cardinals λ such that there exists a base matrix of height λ for $\mathcal{P}(\omega)/\text{fin}$ in which all maximal branches are cofinal.

4. Forcing a distributivity matrix

This section (as well as Section 5, Section 6, and Section 7) is devoted to the proof of our main result:

Main Theorem 4.1. *Let V_0 be a model of ZFC which satisfies GCH. In V_0, let*

$$\omega_1 < \lambda \leq \mu$$

be cardinals such that λ is regular and $\mathrm{cf}(\mu) > \omega$. *Then there is a c.c.c. (and hence cofinality preserving) extension W of V_0 such that*

$$W \models \omega_1 = \mathfrak{h} = \mathfrak{b} \ \wedge \ \lambda \in \mathsf{COM} \ \wedge \ \mu = \mathfrak{c}.$$

Note that, since $\mathfrak{h} \in \mathsf{COM}$, the above model W satisfies $\{\omega_1, \lambda\} \subseteq \mathsf{COM}$ (see also Question 9.1).

Letting $\lambda = \mu = \omega_2$, we immediately obtain the following (recall (3) from Section 3.4):

Corollary 4.2. *It is consistent with ZFC that*

$$\{\omega_1, \omega_2\} = \mathsf{COM} = [\mathfrak{h}, \mathfrak{c}]_{Reg} = \mathsf{FRESH}.$$

We construct our model W as follows. We start with V_0 and first go to the Cohen extension in which $\mathfrak{c} = \mu$. In this model V, we define a forcing iteration (see Section 4.1) which adds a distributivity matrix of height λ for $\mathcal{P}(\omega)/fin$, yielding $\lambda \in \mathsf{COM}$ in the final model W. Building on ideas from [25], we use c.c.c. iterands which approximate the distributivity matrix by finite conditions; we have to use an iteration, because after a single step of the forcing, new reals are added, which prevents the generically added antichains in $\mathcal{P}(\omega)/\text{fin}$ from being maximal. We show that the generic object is actually a distributivity matrix: in particular, the branches are towers (see Section 5.3) and the levels are mad families (see

Section 5.4); for that, we use complete subforcings to capture new subsets of ω (see Lemma 4.22 and Section 5.2).

To show that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ (and hence $\omega_1 \in \mathsf{COM}$), we show that $\mathfrak{b} = \omega_1$, and use the fact that $\mathfrak{h} \leq \mathfrak{b}$ holds in ZFC. In fact, we show that the ground model reals $\mathcal{B} := \omega^\omega \cap V_0$ remain unbounded. For that, we represent our iteration as a finer iteration of Mathias forcings with respect to filters (see Section 7.1). We use a characterization from [24] to show that these filters are $\mathcal{B}$-Canjar (see Section 6 and Section 7.2). In this context, it is problematic that $\mathcal{B}$-Canjarness is not absolute; however, we develop a method to overcome this issue (see Section 6.3 and the discussion before Lemma 7.3).

Remark 4.3. Let us remark that it is possible to derive a bit more from the proof of Main Theorem 4.1 than what is stated in the theorem. Actually, our forcing construction is based on the tree $\lambda^{<\lambda}$ (see Definition 4.5) and therefore results in a specific kind of distributivity matrix of height λ: first, all its maximal branches are cofinal, and second, the underlying tree has λ-splitting, i.e., each node has exactly λ many immediate successors. From the latter property, it immediately follows that $\lambda \in spec(\mathfrak{a})$ (in particular, $\mathfrak{a} \leq \lambda$).

We can modify the construction (by changing the underlying tree) to obtain different kinds of distributivity matrices of height λ. In fact, the following generalization of Main Theorem 4.1 holds true: if $\omega_1 \leq \lambda \leq \mathrm{cf}(\theta)$ and $\theta \leq \mu$ with λ regular and $\mathrm{cf}(\mu) > \omega$, then (using $\theta^{<\lambda}$ as the underlying tree) there is an extension such that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ and $\mu = \mathfrak{c}$, and there exists a distributivity matrix of height λ with θ-splitting (hence, in particular, $\theta \in spec(\mathfrak{a})$). The reason why we have to require $\lambda \leq \mathrm{cf}(\theta)$ is Lemma 5.14: otherwise, the proof would break down there (see Remark 5.16). It would even be possible to have different splitting at different nodes, provided that all the splitting sizes have cofinality at least λ. This way, we can get more values into $spec(\mathfrak{a})$ (similar as in Hechler's paper [25], where he constructs a model in which all uncountable regular cardinals up to $\mathfrak{c}$ are in $spec(\mathfrak{a})$).

Note that even $\lambda = \omega_1$ is possible in our forcing construction. It is true that it does not yield any additional information about COM (because $\mathfrak{h} = \omega_1$ holds true in our model which implies $\omega_1 \in \mathsf{COM}$ anyway), but we can obtain distributivity matrices of height ω_1 with additional features (e.g., by choosing[12] $\theta = \lambda = \omega_1$, resulting in a matrix with ω_1-splitting). Observe that it is always possible to turn a distributivity matrix with θ-splitting into a distributivity matrix with $\mathfrak{c}$-splitting (of the same height), by just taking every ωth level (and deleting all other levels). It is not clear whether it is possible to do it the other way round, i.e., to get a distributivity matrix with θ-splitting (for $\theta < \mathfrak{c}$) from a distributivity matrix with $\mathfrak{c}$-splitting (even if θ happens to be in $spec(\mathfrak{a})$). Therefore, we decided to state and prove Main Theorem 4.1 for $\theta = \lambda$ (i.e., small splitting), and not for $\theta = \mathfrak{c}$.

As a matter of fact, the Cohen model satisfies $spec(\mathfrak{a}) = \{\omega_1, \mathfrak{c}\}$ (see, e.g., [8, Proposition 3.1]). Thus, if $\omega_1 < \theta < \mathfrak{c}$, there are no mad families of size θ, and hence no distributivity matrix with θ-splitting in the Cohen model. If we choose, e.g., $\lambda = \omega_1$, $\theta = \omega_2$ and $\mu = \omega_3$ in the generalization of our main theorem described in the above remark, the generic matrix cannot exist in the Cohen model with $\mathfrak{c} = \omega_3$. On the other hand, our forcing construction with $\theta = \lambda = \omega_1$ and $\omega_1 < \mu$ actually results in the Cohen model

[12]Note that in this case our forcing iteration is equivalent to an iteration of Cohen forcing. This can be seen by representing the iteration as an iteration of Mathias forcings with respect to filters, as described in Section 7.1. If $\theta = \lambda = \omega_1$, all the filters are countably generated, therefore the respective Mathias forcings are equivalent to Cohen forcing.

with $\mathfrak{c} = \mu$ (see footnote 12), hence we can in particular derive the following from our proof of Main Theorem 4.1:

Observation 4.4. *Let $\mu > \omega_1$. Then, in the Cohen model with $\mathfrak{c} = \mu$ (i.e., in the extension of a GCH model by $\mathbb{C}_\mu$), there exists a distributivity matrix of height ω_1 which is ω_1-splitting[13]everywhere.*

4.1. **Definition of the forcing iteration.** In this section we will define a forcing for adding a distributivity matrix. The definition was motivated by the forcing for adding towers and mad families from Hechler's paper [25]. We proceed as follows. In V_0, let $\mathbb{C}_\mu$ be the usual forcing for adding μ many ω-Cohen reals, and let V be the extension by $\mathbb{C}_\mu$. In V, we perform our main forcing iteration of length λ which is going to add a distributivity matrix of height λ for $\mathcal{P}(\omega)/\text{fin}$. The iteration is going to be a finite support iteration whose iterands have the countable chain condition (see Lemma 4.7) and are of size continuum; in particular, the size of the continuum stays the same during the whole iteration, and hence $\mathfrak{c} = \mu$ holds true in the final model (see Lemma 4.8).

As discussed in Section 3.2, a distributivity matrix can be viewed as a tree, where each node is equipped with an element of $\mathcal{P}(\omega)/\text{fin}$. In fact, our generic distributivity matrix $\{A_{\xi+1} \mid \xi < \lambda\}$ will be based on the tree $\lambda^{<\lambda}$: each node $\sigma \in \lambda^{<\lambda}$ of successor length will carry an infinite set $a_\sigma \subseteq \omega$ such that for each $\xi < \lambda$,

$$A_{\xi+1} = \{a_\sigma \mid \sigma \in \lambda^{\xi+1}\}$$

is a mad family, and $a_\sigma \subseteq^* a_\tau$ if σ extends τ. In particular, all maximal branches of our distributivity matrix will be cofinal.

We write $\tau \trianglelefteq \sigma$ if $\tau \subseteq \sigma$ (i.e., if σ extends τ); we write $\tau \triangleleft \sigma$ if $\tau \trianglelefteq \sigma$ and $\tau \neq \sigma$. The length of σ is denoted by $|\sigma|$. We think of the tree $\lambda^{<\lambda}$ as "growing downwards", i.e., we say that σ is below τ if $\tau \trianglelefteq \sigma$; moreover, we say that $\sigma^\frown j$ is to the left of $\sigma^\frown i$ whenever $j < i$.

Note that our mad families $A_{\xi+1}$ are indexed by successor ordinals only, for the following reason. Since there are $\subseteq^*$-decreasing sequences of limit length which do not have weakest lower bounds, and the mad family on the level directly below such a sequence has to be "maximal below the sequence" (i.e., each pseudo-intersection of the sequence is compatible with some element of the mad family), it is necessary that the underlying tree "splits" at such limit levels. However, $\lambda^{<\lambda}$ does not split at limit levels, so it is convenient to equip only nodes σ of successor length with infinite sets a_σ, and use nodes ρ of limit length to talk about the branch $\langle a_\sigma \mid \sigma \triangleleft \rho\rangle$.

Before giving the precise definition of our forcing iteration, let us describe the idea informally. We start with the tree $\lambda^{<\lambda}$ in V, and generically add a set $a_\sigma \subseteq \omega$ for every $\sigma \in \lambda^{<\lambda}$ (of successor length) in such a way that $a_\tau \supseteq^* a_\sigma$ if $\tau \trianglelefteq \sigma$, and $a_\sigma \cap a_\tau =^* \emptyset$ if $|\sigma| = |\tau|$. This results in a refining system of antichains in $\mathcal{P}(\omega)/\text{fin}$. But these antichains are not maximal, which can be seen as follows. The forcing adds new reals (any a_τ is a new infinite subsets of ω), so there are new branches through $\lambda^{<\omega}$. Let ρ be such a new branch of length ω; then $\langle a_{\rho\restriction n} \mid n < \omega\rangle$ is a $\subseteq^*$-decreasing sequence of length ω (in the extension), so (since $\omega < \mathfrak{t}$) it has an infinite pseudo-intersection b. It is easy to see that b is incompatible with all a_σ with $\sigma \in \lambda^{\omega+1} \cap V$, so the antichain $\{a_\sigma \mid |\sigma| = \omega + 1\}$ is not maximal.

[13]In particular, it is ω_1-splitting at limit levels (which is the non-trivial task).

To solve this problem, we use a finite support iteration

$$\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\}$$

of length λ. At each step $\mathbb{Q}_\alpha$ of the iteration, a set a_σ is added for every new node σ (of successor length) of the tree $\lambda^{<\lambda}$. In the definition below, we will use T_α to denote these new nodes (the nodes σ for which no sets a_σ have been added yet), and we will use T'_α to denote the old nodes (the nodes σ for which there has already been added a set a_σ at an earlier stage $\beta < \alpha$ of the iteration). The sets a_σ with $\sigma \in T'_\alpha$ will be used in the definition of the iterand $\mathbb{Q}_\alpha$. In the definition of the very first forcing $\mathbb{Q}_0$ of the iteration, the set T_0 will be the collection of all nodes in $\lambda^{<\lambda}$ (of successor length), and T'_0 will be empty (since no sets a_σ have been defined yet). After λ many steps, we are finished, because no new nodes appear at stage λ (see Lemma 4.9).

Definition 4.5. Let $\alpha < \lambda$, and assume that $\mathbb{P}_\alpha$ has already been defined. For every $\beta \leq \alpha$, let G_β be generic for $\mathbb{P}_\beta$. We work in $V[G_\alpha]$ to define our iterand[14] $\mathbb{Q}_\alpha$. First (letting succ denote the sequences of ordinals of successor length), let

$$T'_\alpha := \bigcup_{\beta<\alpha} (\lambda^{<\lambda} \cap \mathrm{succ})^{V[G_\beta]},$$

and let

$$T_\alpha := (\lambda^{<\lambda} \cap \mathrm{succ})^{V[G_\alpha]} \setminus T'_\alpha.$$

Note that for each $\sigma \in T'_\alpha$, there exists a minimal $\beta < \alpha$ such that $\sigma \in V[G_\beta]$, and hence, by induction, a_σ has been added by $\mathbb{Q}_\beta$. For each $\sigma \in T_\alpha$, the set a_σ is not defined yet, and will be added by $\mathbb{Q}_\alpha$ (see below, at the end of the definition).

Now $\mathbb{Q}_\alpha$ is defined[15] as follows: $p \in \mathbb{Q}_\alpha$ if p is a function with finite domain, $\mathrm{dom}(p) \subseteq T_\alpha$, and for each $\sigma \in \mathrm{dom}(p)$, we have

$$p(\sigma) = (s^p_\sigma, f^p_\sigma, h^p_\sigma) = (s_\sigma, f_\sigma, h_\sigma),$$

where[16]

(1) $s_\sigma \in 2^{<\omega}$,
(2) for each $\tau \in \mathrm{dom}(p)$ with $\tau \lhd \sigma$, $|s_\tau| \geq |s_\sigma|$,
(3) $\mathrm{dom}(f_\sigma) \subseteq (\mathrm{dom}(p) \cup T'_\alpha) \cap \{\tau \in T_\alpha \cup T'_\alpha \mid \tau \lhd \sigma\}$, finite,[17]
(4) $f_\sigma \colon \mathrm{dom}(f_\sigma) \to \omega$,
(5) whenever $\tau \in \mathrm{dom}(f_\sigma) \cap T_\alpha$, and $n \in \omega$ with[18] $n \in \mathrm{dom}(s_\tau) \cap \mathrm{dom}(s_\sigma)$ and $n \geq f_\sigma(\tau)$, we have

$$s_\tau(n) = 0 \to s_\sigma(n) = 0,$$

[14]We will often write $\mathbb{Q}$ to denote one of the iterands $\mathbb{Q}_\alpha$; see also Remark 4.10.

[15]The presentation of our forcing here is somewhat different from the presentation of Hechler's forcings to add towers or mad families in [25]. In [17], we represent these forcings in a form which is analogous to our definition of $\mathbb{Q}_\alpha$.

[16]The paragraph after the definition gives a short intuitive explanation of the roles of s_σ, f_σ, and h_σ.

[17]Note that in (6), it automatically follows that $\mathrm{dom}(h_\sigma)$ is finite because $\mathrm{dom}(h_\sigma) \subseteq \mathrm{dom}(p)$, but not here because $\mathrm{dom}(f_\sigma) \subseteq \mathrm{dom}(p) \cup T'_\alpha$.

[18]By (2), here it is actually sufficient to require $n \in \mathrm{dom}(s_\sigma)$.

and whenever $\tau \in \mathrm{dom}(f_\sigma) \cap T'_\alpha$ and $n \in \omega$ with $n \in \mathrm{dom}(s_\sigma)$ and $n \geq f_\sigma(\tau)$, we have[19]

$$a_\tau(n) = 0 \to s_\sigma(n) = 0,$$

(6) $\mathrm{dom}(h_\sigma) \subseteq \mathrm{dom}(p) \cap \{\rho^\frown j \mid j < i\}$ (where $\rho \in \lambda^{<\lambda}$ and $i \in \lambda$ such that $\sigma = \rho^\frown i$),
(7) $h_\sigma \colon \mathrm{dom}(h_\sigma) \to \omega$,
(8) whenever $\tau \in \mathrm{dom}(h_\sigma)$, and $n \in \omega$ with $n \in \mathrm{dom}(s_\tau) \cap \mathrm{dom}(s_\sigma)$ and $n \geq h_\sigma(\tau)$, we have

$$s_\tau(n) = 0 \vee s_\sigma(n) = 0.$$

The order on $\mathbb{Q}_\alpha$ is defined as follows: $q \leq p$ ("q is stronger than p") if
(i) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
(ii) and for each $\sigma \in \mathrm{dom}(p)$, we have
  (a) $s^p_\sigma \trianglelefteq s^q_\sigma$,
  (b) $\mathrm{dom}(f^p_\sigma) \subseteq \mathrm{dom}(f^q_\sigma)$ and $f^p_\sigma(\tau) \geq f^q_\sigma(\tau)$ for each $\tau \in \mathrm{dom}(f^p_\sigma)$,
  (c) $\mathrm{dom}(h^p_\sigma) \subseteq \mathrm{dom}(h^q_\sigma)$ and $h^p_\sigma(\tau) \geq h^q_\sigma(\tau)$ for each $\tau \in \mathrm{dom}(h^p_\sigma)$.

Given a generic filter G for $\mathbb{Q}_\alpha$, we define, for each $\sigma \in T_\alpha$,

$$a_\sigma := \bigcup \{s^p_\sigma \mid p \in G \wedge \sigma \in \mathrm{dom}(p)\}.$$

This completes the definition of the forcing.

Let us describe the role of the parts of a condition: s_σ is a finite approximation of the set a_σ assigned to σ, whereas the functions f_σ and h_σ are promises for guaranteeing that the branches through the generic matrix are $\subseteq^*$-decreasing and the levels are almost disjoint families, respectively. More precisely, f_σ promises that $a_\sigma \setminus f_\sigma(\tau) \subseteq a_\tau$ for each $\tau \in \mathrm{dom}(f_\sigma)$ (see Lemma 4.14), and h_σ promises that $a_\tau \cap a_\sigma \subseteq h_\sigma(\tau)$ for each $\tau \in \mathrm{dom}(h_\sigma)$ (see Lemma 4.15).

Remark 4.6. Note that $\mathbb{Q}_\alpha$ is not separative. As an example, we can take p and q as follows: $\mathrm{dom}(p) = \mathrm{dom}(q) = \{\sigma, \tau\}$ (where σ is to the left of τ within the same block), $p(\tau) = q(\tau) = (\langle 1 \rangle, \emptyset, h)$ where $h(\sigma) = 0$ and $p(\sigma) = (\langle\rangle, \emptyset, \emptyset)$ and $q(\sigma) = (\langle 0 \rangle, \emptyset, \emptyset)$. It is easy to see that $p \nleq q$, but $p \leq^* q$, i.e., any condition stronger than p is compatible with q. Therefore, we later need to provide certain iteration lemmas for the general case of non-separative forcings (see Lemma 5.3 and footnote 30).

4.2. **Countable chain condition and some implications.** We are now going to show that our iterands $\mathbb{Q}_\alpha$ have the c.c.c.; it immediately follows that their finite support iteration $\mathbb{P}_\lambda$ has the c.c.c. as well, and therefore it does not change cofinalities or cardinalities.

Lemma 4.7. *$\mathbb{Q}_\alpha$ has the[20] c.c.c. for every $\alpha < \lambda$.*

[19] In many of our proofs, we will not deal with the case involving a_τ, but just discuss the case involving s_τ, but it should always be clear how to handle the a_τ-case in an analogous way (see also Remark 4.10).

[20] In fact, $\mathbb{Q}_\alpha$ is even σ-centered: in Section 7, we are going to show that each $\mathbb{Q}_\alpha$ can be represented as a finite support iteration of length strictly less than $\mathfrak{c}^+$ of Mathias forcings with respect to certain filters; since filtered Mathias forcings are always σ-centered (see Definition 6.1 and the subsequent remark), and σ-centeredness is preserved under finite support iterations of length strictly less than $\mathfrak{c}^+$, it follows that $\mathbb{Q}_\alpha$ is σ-centered (see also Corollary 7.2).

Proof. Let $\{p_i \mid i < \omega_1\} \subseteq \mathbb{Q}_\alpha$. We want to show that the set cannot be an antichain. First note that it is possible to extend[21] all s^p_σ (with $\sigma \in \text{dom}(p)$) of a condition $p \in \mathbb{Q}_\alpha$ to the same length $N_p \in \omega$, by just adding 0's at the end. Therefore we can assume without loss of generality that there exists N such that[22] $|s^{p_i}_\sigma| = N$ for each $i \in \omega_1$ and each $\sigma \in \text{dom}(p_i)$. Since $\text{dom}(p_i) \subseteq T_\alpha \subseteq \lambda^{<\lambda}$ is finite for every i, we can apply the Δ-system lemma to find a subset $X \subseteq \omega_1$ of size ω_1 such that $\{\text{dom}(p_i) \mid i \in X\}$ is a Δ-system with root $R \subseteq T_\alpha$. Also, $\text{dom}(f^{p_i}_\sigma) \cap T'_\alpha$ is finite for each $i \in X$ and each $\sigma \in \text{dom}(p_i)$, so we can repeatedly apply[23] the Δ-system lemma, for each $\sigma \in R$ (hence finitely many times), to find a subset $Y \subseteq X$ of size ω_1 such that $\{\text{dom}(f^{p_i}_\sigma) \cap T'_\alpha \mid i \in Y\}$ is a Δ-system with root A_σ for each $\sigma \in R$. Moreover, we can assume without loss of generality that for each $\sigma \in R$, there are s^*_σ, f^*_σ, and h^*_σ such that for all $i \in Y$, we have $s^{p_i}_\sigma = s^*_\sigma$, $f^{p_i}_\sigma \restriction (R \cup A_\sigma) = f^*_\sigma$, and $h^{p_i}_\sigma \restriction R = h^*_\sigma$. Now it is straightforward to check that any two conditions from $\{p_i \mid i \in Y\}$ are compatible (in fact, any finitely many of them have a common lower bound; hence $\mathbb{Q}_\alpha$ is even precaliber ω_1). $\square$

We now show that the size of the continuum in the final model is as desired; in fact, the following holds:

Lemma 4.8. *Let $\alpha \leq \lambda$. Then, in $V[\mathbb{P}_\alpha]$, we have $\mathfrak{c} = \mu$.*

Proof. First note that $V \models \mathfrak{c} = \mu \wedge \mu^{<\mu} = \mu$, because it is the extension after adding μ many ω-Cohen reals over a model which satisfies GCH. We show simultaneously by induction on $\alpha \leq \lambda$ that[24]

(1) $|\mathbb{P}_\alpha| \leq \mu$ and
(2) $V[\mathbb{P}_\alpha] \models \mathfrak{c} = \mu \wedge |T_\alpha| \leq \lambda^{<\lambda} \leq \mu$.

Clearly (1) and (2) hold for $\mathbb{P}_0$ since $\mathbb{P}_0$ is the trivial forcing. Now assume that we have shown (1) and (2) for each $\alpha' < \alpha$.

To show (1), argue as follows. If α is a limit, then $|\mathbb{P}_\alpha| \leq \mu$, because we use finite support, each $\mathbb{P}_{\alpha'} \leq \mu$, and $\alpha \leq \mu$. If $\alpha = \alpha' + 1$ is a successor, $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'} * \mathbb{Q}_{\alpha'}$. By induction, $\mathbb{P}_{\alpha'} \Vdash |T_{\alpha'}| \leq \lambda^{<\lambda} \leq \mu$, and so it is easy to check that $|\mathbb{Q}_{\alpha'}| \leq \mu$, hence $|\mathbb{P}_{\alpha'} * \mathbb{Q}_{\alpha'}| \leq \mu$.

To show (2), we count nice names. For every real x in $V[\mathbb{P}_\alpha]$, there exists a nice name. Such a nice name consists of antichains X_n in $\mathbb{P}_\alpha$ for each entry $x(n)$. By the c.c.c., each X_n is countable, so the number of nice names for reals is $|\mathbb{P}_\alpha|^{\leq\omega} \leq \mu$, so there are only μ many reals in $V[\mathbb{P}_\alpha]$. Similarly, a nice name for an element of $\lambda^{<\lambda}$ consists of less than μ many countable antichains, and since $|\mathbb{P}_\alpha|^{<\mu} \leq \mu^{<\mu} = \mu$, there are at most μ many elements of $\lambda^{<\lambda}$ in $V[\mathbb{P}_\alpha]$. $\square$

The following lemma guarantees that, by the end of the iteration of length λ, a set a_σ has been added for every $\sigma \in \lambda^{<\lambda}$ (so T_λ would be empty, hence $\mathbb{Q}_\lambda$ would be the trivial forcing – if we would continue the iteration after λ many stages):

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[21]Here, we could also use the fact that the set of full conditions is dense (see Definition 4.18 and Lemma 4.19), but what we actually need here is much less.

[22]The reason why we want the $s^{p_i}_\sigma$ to have the same length, is to avoid trouble with Definition 4.5(2).

[23]In case $\alpha = 0$, this is not necessary, because $T'_0 = \emptyset$ (in the definition of $\mathbb{Q}_0$).

[24]For the more general version of Main Theorem 4.1 discussed in Remark 4.3, we would need $\theta^{<\lambda} \leq \mu$ instead of $\lambda^{<\lambda} \leq \mu$ in (2), which works as well.

Lemma 4.9. *Every node $\sigma \in \lambda^{<\lambda}$ from the final model $V[\mathbb{P}_\lambda]$ already appears in some intermediate model $V[\mathbb{P}_\alpha]$ with $\alpha < \lambda$.*

Proof. Let $\dot\sigma$ be a nice $\mathbb{P}_\lambda$-name for σ; more precisely, $\dot\sigma$ has the following form. First, $\dot\sigma$ contains an antichain which decides the length of $\dot\sigma$. Since $\mathbb{P}_\lambda$ has the c.c.c., this antichain is countable, so there are only countably many values possible for the length; let $\xi < \lambda$ be larger than all the possible values. Now, for all $\xi' < \xi$, there is an antichain deciding the entry of $\dot\sigma(\xi')$ (if ξ' is less than the length of $\dot\sigma$). Again, by c.c.c. all these antichains are countable. So there are ξ many countable antichains which are in $\dot\sigma$; the union of these antichains contains less than λ many elements. Since we use finite support, there exists an $\alpha < \lambda$ such that $\dot\sigma$ is a $\mathbb{P}_\alpha$-name, hence $\sigma \in V[\mathbb{P}_\alpha]$. □

4.3. **The generic distributivity matrix.** Let G be a generic filter for the iteration $\mathbb{P}_\lambda$. In the final model $V[G]$, we derive our "intended generic object" (which is going to be a distributivity matrix of height λ) from the generic filter G as follows. For each $\sigma \in \lambda^{<\lambda} \cap \mathrm{succ}$, we can fix the minimal $\alpha < \lambda$ such that $\sigma \in V[G_\alpha]$ (see Lemma 4.9). Then in $V[G_\alpha]$, the node σ belongs to T_α, and, letting $G(\alpha)$ be the corresponding filter for $\mathbb{Q}_\alpha$, the set

$$a_\sigma = \bigcup \{s^p_\sigma \mid p \in G(\alpha) \wedge \sigma \in \mathrm{dom}(p)\}$$

is added by $\mathbb{Q}_\alpha$. Back in the final model $V[G]$, we let, for each $\xi < \lambda$,

$$A_{\xi+1} := \{a_\sigma \mid |\sigma| = \xi + 1\}$$

(which is going to be a mad family). Here, we are going to show that the generic object $\{A_{\xi+1} \mid \xi < \lambda\}$ is a refining system of antichains in $\mathcal{P}(\omega)/\mathrm{fin}$.

Remark 4.10. In the rest of this section, we will give the proofs only for the first step $\mathbb{Q}_0$ of our iteration: in this case, no node σ has been equipped with a set a_σ yet. It is easy to see and left to the reader that the proofs can be adapted to the general case, i.e., $\mathbb{Q}_\alpha$ for $\alpha > 0$. In fact, the only difference in the proofs is that one also has to deal with the case (see Definition 4.5(5)) involving a_τ for some old node τ (i.e., $\tau \in T'_\alpha$), and not just the one involving s_τ for a new node τ (i.e., $\tau \in T_\alpha$; see also footnote 19).

In such cases, we will often write $\mathbb{Q}$ instead of $\mathbb{Q}_0$, to indicate that the lemmas actually hold for any $\mathbb{Q}_\alpha$; moreover, we will write T instead of T_0. In case $\mathbb{Q} = \mathbb{Q}_0$, we have $T = T_0 = \lambda^{<\lambda} \cap \mathrm{succ}$.

Our first lemma guarantees that each a_σ is going to have infinitely many 1's:

Lemma 4.11. *For each $\sigma \in T$ and each $n \in \omega$, the set*

$$D_{\sigma,n} := \{q \in \mathbb{Q} \mid \sigma \in \mathrm{dom}(q) \text{ and}^{25} \exists m \geq n (s^q_\sigma(m) = 1)\}$$

is dense in $\mathbb{Q}$.

Proof. Let $\sigma \in T$, $n \in \omega$ and $p \in \mathbb{Q}$. If $\sigma \notin \mathrm{dom}(p)$ extend p to $p \cup \{(\sigma, (\langle\rangle, \emptyset, \emptyset))\}$. From now on, we assume that $\sigma \in \mathrm{dom}(p)$.

[25] Note that (in this and all future clauses of this kind) we actually mean "$m \in \mathrm{dom}(s) \wedge s(m) = 1$" whenever we write "$s(m) = 1$".

Let N be bigger than the maximal length of all the s^p_τ with $\tau \in \mathrm{dom}(p)$ and bigger than n. Now for every $\tau \in \mathrm{dom}(p)$, extend s^p_τ with 0's to length N. It is easy to see that this is a condition. Now we can extend s^p_τ to $s^p_\tau{}^\frown 1$ for every $\tau \trianglelefteq \sigma$ and $\tau \in \mathrm{dom}(p)$ (in particular, s^p_σ is extended in this way). This results in the desired condition q. □

In particular, it easily follows that each a_σ is a total function, i.e., $a_\sigma \in 2^\omega$ for each $\sigma \in T$; in fact, it follows from Lemma 4.11 that (we abuse notation and let a_σ also denote the respective subset of ω) a_σ is infinite, i.e., $a_\sigma \in [\omega]^\omega$.

Remark 4.12. It is easy to see that a slight generalization of the proof of Lemma 4.11 yields the following: for each infinite ground model set $b \subseteq \omega$, each a_σ has infinitely many 1's (and also infinitely many 0's) within b. If b is infinite and co-infinite, it follows that b splits a_σ. In particular, $a_\sigma \not\subseteq^* b$, so b witnesses that the generic matrix is not going to be a base matrix (see also the discussion in Section 3.5).

The next lemma shows that we can always assume that the domain of f^p_σ and the domain of h^p_σ is as large as possible. Parts of the lemma will be essential also later, for the notion of "full condition" (see Definition 4.18).

Lemma 4.13. *Let $p \in \mathbb{Q}$ and $\sigma \in \mathrm{dom}(p)$. Then there exists a $q \leq p$ such that $\mathrm{dom}(q) = \mathrm{dom}(p)$, and the following holds:*

(a) $\tau \in \mathrm{dom}(f^q_\sigma)$ for each $\tau \in \mathrm{dom}(q)$ with $\tau \vartriangleleft \sigma$, and

(b) (letting $\sigma = \rho^\frown i$) $\rho^\frown j \in \mathrm{dom}(h^q_\sigma)$ for each $j < i$ with $\rho^\frown j \in \mathrm{dom}(q)$.

In particular, the set

$$D := \{q \in \mathbb{Q} \mid \text{ (a) and (b) holds for each } \sigma \in \mathrm{dom}(q)\}$$

is dense in $\mathbb{Q}$.

Moreover, if $\mathbb{Q} = \mathbb{Q}_\alpha$ for $\alpha > 0$, then the following holds: whenever $\tau' \vartriangleleft \sigma$ with $\tau' \in T'_\alpha$ (i.e., $a_{\tau'}$ has already been added before), there exists $q \leq p$ such that $\mathrm{dom}(q) = \mathrm{dom}(p)$, $q \in D$, and $\tau' \in \mathrm{dom}(f^q_\sigma)$.

Proof. For every $\tau \in \mathrm{dom}(p) \setminus \mathrm{dom}(f^p_\sigma)$ with $\tau \vartriangleleft \sigma$, let $f^q_\sigma(\tau) := |s^p_\sigma|$. For every $\rho^\frown j \in \mathrm{dom}(p) \setminus \mathrm{dom}(h^p_\sigma)$ with $j < i$, let $h^q_\sigma(\rho^\frown j) := |s^p_\sigma|$. For the moreover part, let $f^q_\sigma(\tau') := |s^p_\sigma|$. If we (repeatedly) extend p in this way to q, it is clear that q is a condition with the properties we wanted. □

The next lemma will be used to show that, for $\tau \vartriangleleft \sigma$, the set of conditions which force $a_\sigma \subseteq^* a_\tau$ is dense:

Lemma 4.14. *Let $p \in \mathbb{Q}$, $\sigma \in \mathrm{dom}(p)$, and $\tau \in \mathrm{dom}(f^p_\sigma)$. Then*

$$p \Vdash a_\sigma \setminus f^p_\sigma(\tau) \subseteq a_\tau$$

(in particular, $p \Vdash a_\sigma \subseteq^* a_\tau$).

Proof. This follows easily from the definition of the forcing. □

Analogously to the previous lemma, the next lemma will be used to show that, for $\rho^\frown i$ and $\rho^\frown j$ with $i \neq j$, the set of conditions which force $a_{\rho^\frown i} \cap a_{\rho^\frown j} =^* \emptyset$ is dense:

Lemma 4.15. *Let $p \in \mathbb{Q}$, $\sigma \in \mathrm{dom}(p)$, and $\tau \in \mathrm{dom}(h^p_\sigma)$. Then*

$$p \Vdash a_\tau \cap a_\sigma \subseteq h^p_\sigma(\tau)$$

(in particular, $p \Vdash a_\tau \cap a_\sigma =^* \emptyset$).

Proof. This follows easily from the definition of the forcing. □

The next corollary shows that in the final model $V[\mathbb{P}_\lambda]$ the sets along branches of $\lambda^{<\lambda}$ are $\subseteq^*$-decreasing:

Corollary 4.16. *In $V[\mathbb{P}_\lambda]$, let $\tau, \sigma \in \lambda^{<\lambda} \cap \mathrm{succ}$ such that $\tau \lhd \sigma$. Then $a_\sigma \subseteq^* a_\tau$.*

Proof. Let $\eta < \lambda$ be minimal such that $\sigma \in (\lambda^{<\lambda})^{V[\mathbb{P}_\eta]}$. Lemma 4.11, Lemma 4.13, and Lemma 4.14 in particular imply that the set

$$\{q \in \mathbb{Q}_\eta \mid q \Vdash a_\sigma \subseteq^* a_\tau\}$$

is dense. Hence $V[\mathbb{P}_{\eta+1}] \models a_\sigma \subseteq^* a_\tau$, and this remains true in the final model. □

The next corollary shows that in the final model $V[\mathbb{P}_\lambda]$ the sets on one level of $\lambda^{<\lambda}$ are pairwise almost disjoint:

Corollary 4.17. *In $V[\mathbb{P}_\lambda]$, let $\rho \in \lambda^{<\lambda}$, and let $j < i < \lambda$. Then $a_{\rho^\frown j} \cap a_{\rho^\frown i} =^* \emptyset$.*

Indeed, the following holds. For each $\sigma, \sigma' \in \lambda^{<\lambda} \cap \mathrm{succ}$ satisfying $|\sigma| = |\sigma'|$ and $\sigma \neq \sigma'$, we have $a_\sigma \cap a_{\sigma'} =^* \emptyset$; in other words, for each $\xi < \lambda$,

$$A_{\xi+1} = \{a_\sigma \mid \sigma \in \lambda^{\xi+1}\}$$

is an almost disjoint family.

Proof. Let $\eta < \lambda$ be minimal such that $\rho \in (\lambda^{<\lambda})^{V[\mathbb{P}_\eta]}$. Lemma 4.11, Lemma 4.13, and Lemma 4.15 in particular imply that the set

$$\{q \in \mathbb{Q}_\eta \mid q \Vdash a_{\rho^\frown j} \cap a_{\rho^\frown i} =^* \emptyset\}$$

is dense; this proves the first assertion.

To prove the second assertion, find $\rho \in \lambda^{<\lambda}$ with $\rho \lhd \sigma, \sigma'$ and $i, j < \lambda$ with $j \neq i$ such that $\rho^\frown j \unlhd \sigma$ and $\rho^\frown i \unlhd \sigma'$, and apply the first assertion as well as Corollary 4.16. □

Altogether, we have proved that $\{A_{\xi+1} \mid \xi < \lambda\}$ is a refining system of ad families, i.e., for each $\xi < \lambda$, $A_{\xi+1}$ is an almost disjoint family, and for all $\xi < \xi' < \lambda$, $A_{\xi'+1}$ refines $A_{\xi+1}$.

To show that $\{A_{\xi+1} \mid \xi < \lambda\}$ is actually a distributivity matrix requires much more work. The proof will be completed in Section 5. After a lot of preparatory work, it will be shown in Section 5.4 that the sets $A_{\xi+1}$ are indeed maximal, and in Section 5.3 that the sets along branches are indeed towers, which implies that there is no set intersecting the whole family $\{A_{\xi+1} \mid \xi < \lambda\}$ (and hence there is no common refinement).

4.4. **Complete subforcings.** The goal of this section is to show that our forcing $\mathbb{Q}$ (see also Remark 4.10) has complete subforcings which use only part of $\lambda^{<\lambda} \cap \mathrm{succ}$ (see Lemma 4.22). In Section 5.2, this will be extended to the whole iteration (see Lemma 5.12), which will be an important ingredient of the proof that the generic object is a distributivity matrix (see Section 5.3 and Section 5.4). Moreover, we will show in Section 7 that each $\mathbb{Q}_\alpha$ (and hence our whole iteration) can be seen as an iteration of Mathias forcings with respect to certain filters; to show that these filters are $\mathcal{B}$-Canjar, we will again use Lemma 4.22. Let us start with a concept which is going to be very useful:

Definition 4.18. A condition $p \in \mathbb{Q}$ is called *full* if there exists an $N \in \omega$ such that for all $\sigma \in \mathrm{dom}(p)$

(1) $|s^p_\sigma| = N$,
(2) $N > \max(\mathrm{rng}(f^p_\sigma))$ and $N > \max(\mathrm{rng}(h^p_\sigma))$,
(3) $\tau \in \mathrm{dom}(f^p_\sigma)$ for each $\tau \in \mathrm{dom}(p)$ with $\tau \lhd \sigma$, and
(4) (letting $\sigma = \rho^\frown i$) $\rho^\frown j \in \mathrm{dom}(h^p_\sigma)$ for each $j < i$ with $\rho^\frown j \in \mathrm{dom}(p)$.

Moreover, $p \in \mathbb{P}_\lambda$ is full if[26] $p(0)$ is full.

The set of full conditions is dense:

Lemma 4.19. *For every condition $p \in \mathbb{Q}$ there exists a full condition q with $q \leq p$ and* $\mathrm{dom}(q) = \mathrm{dom}(p)$. *In particular, the set of full conditions is dense in $\mathbb{Q}$. Hence also the set of full conditions in $\mathbb{P}_\lambda$ is dense in $\mathbb{P}_\lambda$.*

Proof. We can assume that p belongs to the dense set D from Lemma 4.13, i.e., p fulfills (3) and (4) for each $\sigma \in \mathrm{dom}(p)$. Now let

$$N > \max(\mathrm{rng}(f^p_\sigma)), \max(\mathrm{rng}(h^p_\sigma)), |s^p_\sigma|$$

for every $\sigma \in \mathrm{dom}(p)$. Finally, for every $\sigma \in \mathrm{dom}(p)$, extend s^p_σ with 0's to length N. It is easy to see that this results in a condition q which is full. □

We now introduce a notation for the collection of conditions in $\mathbb{Q}$ whose domain is contained in a prescribed set of nodes:

Definition 4.20. Let $C \subseteq \lambda^{<\lambda}$. Define

$$\mathbb{Q}^C := \{p \in \mathbb{Q} \mid \mathrm{dom}(p) \subseteq C\}.$$

In our completeness lemma below we are going to show that $\mathbb{Q}^C$ is a complete subforcing of $\mathbb{Q}$ provided that C has a certain form.

Definition 4.21. Let $E \subseteq \lambda^{<\lambda}$. We call E *left-up-closed* if

- for each $\sigma \in E$ and each $\tau \lhd \sigma$ with $\tau \in T$, we have $\tau \in E$,
- for each ρ and i with $\rho^\frown i \in E$ and each $j < i$, we have $\rho^\frown j \in E$.

Now let $C = E \cup \bar{C}$, where $E \subseteq \lambda^{<\lambda}$ is left-up-closed, and either $\bar{C}$ is empty, or the following holds: $E \subseteq \lambda^{<\gamma}$ for some $\gamma < \lambda$ and $\bar{C} \subseteq \lambda^{\gamma'}$ for some $\gamma' \geq \gamma$ (with γ' successor), and,[27] for $\sigma, \sigma' \in \bar{C}$,

[26] Later, we will consider quotients $\mathbb{P}_\lambda / \mathbb{P}_\eta$ and therefore use a modification, where 0 is replaced by η, i.e., $p(\eta)$ is full; see Remark 5.10.

- either there exist ρ, i and j such that $\rho^\frown i = \sigma$ and $\rho^\frown j = \sigma'$ (i.e., σ and σ' are in the same block),
- or there exist two incomparable nodes $\tau, \tau' \in E$ with $\tau \vartriangleleft \sigma$ and $\tau' \vartriangleleft \sigma'$ (i.e., σ and σ' split within E).

So C consists of a left-up-closed part together with nodes from one later level.

For $p \in \mathbb{Q}$, let $p \upharpoonright\upharpoonright C$ be the condition p' with $\mathrm{dom}(p') = \mathrm{dom}(p) \cap C$, and $s^{p'}_\sigma = s^p_\sigma$, $f^{p'}_\sigma = f^p_\sigma \upharpoonright C$ and $h^{p'}_\sigma = h^p_\sigma \upharpoonright C$ for each $\sigma \in \mathrm{dom}(p')$. Clearly, p' is a condition in $\mathbb{Q}^C$. Note that if C is left-up-closed, then $p \upharpoonright\upharpoonright C = p \upharpoonright C$, because for every $\sigma \in \mathrm{dom}(p) \cap C$ clearly $f^p_\sigma \upharpoonright C = f^p_\sigma$ and $h^p_\sigma \upharpoonright C = h^p_\sigma$.

The following crucial completeness lemma is given in a quite general form. This way, it can be used in Section 5.2 as well as in Section 7.2. For Section 5.2, a somewhat easier version would be enough (see the proof of Lemma 5.12).

Lemma 4.22. *Let C be of the above form. Then $\mathbb{Q}^C$ is a complete subforcing of $\mathbb{Q}$. Moreover, if $p \in \mathbb{Q}$ is a full condition, then $\mathsf{RED}(p) = p \upharpoonright\upharpoonright C$ is a reduction of p to $\mathbb{Q}^C$.*

Note that the sets $\lambda^1 = \{\sigma \in \lambda^{<\lambda} \mid |\sigma| = 1\}$ and $1^{<\lambda} = \{\sigma \in \lambda^{<\lambda} \mid \sigma(\xi) = 0 \text{ for every } \xi\}$ are left-up-closed, hence the forcings $\mathbb{Q}_0^{(\lambda^1)}$ and $\mathbb{Q}_0^{(1^{<\lambda})}$ are complete subforcings of $\mathbb{Q}_0$ by the lemma. These forcings are isomorphic to the forcings introduced by Hechler [25] to add a mad family and a tower, respectively (compare with the respective definitions in [17]).

Proof of Lemma 4.22. We first show that $\mathbb{Q}^C \subseteq_{ic} \mathbb{Q}$. Let $p_0, p_1 \in \mathbb{Q}^C$ and $q \in \mathbb{Q}$ with $q \leq p_0, p_1$. We have to show that there exists a condition $q' \in \mathbb{Q}^C$ with $q' \leq p_0, p_1$. Let $q' := q \upharpoonright\upharpoonright C$. It is very easy to check that q' is as we wanted.

Let $p \in \mathbb{Q}$. We want to define a reduction $\mathsf{RED}(p) \in \mathbb{Q}^C$. Let $p' \leq p$ be a full condition with $\mathrm{dom}(p') = \mathrm{dom}(p)$ (see Lemma 4.19). Let $\mathsf{RED}(p) := p' \upharpoonright\upharpoonright C$. Let $q \leq \mathsf{RED}(p)$ with $q \in \mathbb{Q}^C$. We have to show that q is compatible with p. To show this, we define a witness r as follows. Let $\mathrm{dom}(r) := \mathrm{dom}(p') \cup \mathrm{dom}(q)$. For $\sigma \in \mathrm{dom}(q)$, let $s^r_\sigma := s^q_\sigma$, and for $\sigma \in \mathrm{dom}(q) \setminus \mathrm{dom}(p')$, let $f^r_\sigma := f^q_\sigma$ and $h^r_\sigma := h^q_\sigma$.

For $\sigma \in \mathrm{dom}(q) \cap \mathrm{dom}(p')$, let $\mathrm{dom}(f^r_\sigma) := \mathrm{dom}(f^q_\sigma) \cup \mathrm{dom}(f^{p'}_\sigma)$ and let $f^r_\sigma(\sigma') := \min(f^q_\sigma(\sigma'), f^{p'}_\sigma(\sigma'))$ for every $\sigma' \in \mathrm{dom}(f^q_\sigma) \cap \mathrm{dom}(f^{p'}_\sigma)$ and $f^r_\sigma(\sigma') := f^{p'}_\sigma(\sigma')$ for $\sigma' \in \mathrm{dom}(f^{p'}_\sigma) \setminus \mathrm{dom}(f^q_\sigma)$. Similarly, let $\mathrm{dom}(h^r_\sigma) := \mathrm{dom}(h^q_\sigma) \cup \mathrm{dom}(h^{p'}_\sigma)$ and let $h^r_\sigma(\sigma') := \min(h^q_\sigma(\sigma'), h^{p'}_\sigma(\sigma'))$ for every $\sigma' \in \mathrm{dom}(h^q_\sigma) \cap \mathrm{dom}(h^{p'}_\sigma)$ and $h^r_\sigma(\sigma') := h^{p'}_\sigma(\sigma')$ for $\sigma' \in \mathrm{dom}(h^{p'}_\sigma) \setminus \mathrm{dom}(h^q_\sigma)$.

For $\sigma \in \mathrm{dom}(p') \setminus \mathrm{dom}(q)$, make the following definition. Let $f^r_\sigma := f^{p'}_\sigma$ and $h^r_\sigma := h^{p'}_\sigma$. If there is no $\tau \in \mathrm{dom}(q)$ with $\sigma \trianglelefteq \tau$, let $s^r_\sigma := s^{p'}_\sigma$. If there exists $\tau \in \mathrm{dom}(q)$ with $\sigma \trianglelefteq \tau$, extend $s^{p'}_\sigma$ to the maximal length of the s^q_τ for $\tau \in \mathrm{dom}(q)$ with $\sigma \vartriangleleft \tau$ in the following way: if $n \geq |s^{p'}_\sigma|$ and there exists $\tau \in \mathrm{dom}(p')$ which extends σ with $s^q_\tau(n) = 1$, let $s^r_\sigma(n) = 1$, and let $s^r_\sigma(n) = 0$ otherwise. This makes sure that $s^r_\sigma(n) = 1$ whenever $s^r_\tau(n) = 1$ for $\sigma \vartriangleleft \tau$ and $\sigma \in \mathrm{dom}(f^{p'}_\tau)$.

Claim. *r is a condition.*

[27] Note that we have to impose some restrictions on $\sigma, \sigma' \in \bar{C}$ to ensure that after forcing with $\mathbb{Q}^C$, the two sets a_σ and $a_{\sigma'}$ are almost disjoint (which is necessary for $\mathbb{Q}^C$ being a complete subforcing); this is guaranteed by either item below: the two sets are forced to be almost disjoint either because it happens in the same block, or because they are almost contained in almost disjoint sets which are already added by $\mathbb{Q}^E$.

Proof. It is very easy to check that s^r_σ, f^r_σ and h^r_σ are well-defined with the right domains and ranges for all $\sigma \in \mathrm{dom}(r)$.

If $\sigma \trianglelefteq \tau$, then $|s^r_\sigma| \geq |s^r_\tau|$: if σ and τ are both in $\mathrm{dom}(q)$, so $s^r_\sigma = s^q_\sigma$ and $s^r_\tau = s^q_\tau$, so the length is ok, because they are both from q; if $\sigma \notin \mathrm{dom}(q)$, we lengthened s^r_σ to make it as long as all the s's of τ's which extend it.

Let $\sigma, \tau \in \mathrm{dom}(r)$ with[28] $\tau \in \mathrm{dom}(f^r_\sigma)$ and $m \geq f^r_\sigma(\tau)$ and $s^r_\sigma(m) = 1$; we have to show that $s^r_\tau(m) = 1$. *Case 1:* σ and τ are both in $\mathrm{dom}(q)$. It follows that $\tau \in C \cap \mathrm{dom}(f^r_\sigma) = \mathrm{dom}(f^q_\sigma)$, $f^r_\sigma(\tau) = f^q_\sigma(\tau)$ and $s^r_\sigma = s^q_\sigma$ and $s^r_\tau = s^q_\tau$, so they fit together, because q is a condition. *Case 2:* $\sigma \in \mathrm{dom}(q)$, $\tau \notin \mathrm{dom}(q)$. Since $\tau \notin \mathrm{dom}(q)$ and $\tau \in \mathrm{dom}(f^r_\sigma)$, it follows that $\tau \in \mathrm{dom}(f^{p'}_\sigma)$. In particular $f^{p'}_\sigma$ is defined, so $\sigma \in \mathrm{dom}(p')$. If $m < |s^{p'}_\tau|$, it follows that $s^r_\tau(m) = s^{p'}_\tau(m)$ and $s^r_\sigma(m) = s^{p'}_\sigma(m)$, and $f^r_\sigma(\tau) = f^{p'}_\sigma(\tau)$. So $s^r_\tau(m) = 1$, because p' is a condition. If $m \geq |s^{p'}_\tau| = |s^{p'}_\sigma|$, then $s^r_\sigma(m) = 1$ implies that $s^q_\rho(m) = 1$ for some $\rho \trianglerighteq \sigma$, but then $\rho \trianglerighteq \tau$, and therefore also $s^r_\tau(m) = 1$. *Case 3:* $\sigma \notin \mathrm{dom}(q)$. So $f^r_\sigma = f^{p'}_\sigma$, and it follows that $\tau \in \mathrm{dom}(f^{p'}_\sigma) \subseteq \mathrm{dom}(p')$. If $m < |s^{p'}_\sigma|$, it follows that $s^{p'}_\tau(m) = s^r_\tau(m) = 1$, because p' is a condition and $\tau \in \mathrm{dom}(f^{p'}_\sigma)$. If $m \geq |s^{p'}_\sigma|$, then our definition implies that there exists a ρ with $\rho \triangleright \sigma$, $\rho \in \mathrm{dom}(p')$ and $s^q_\rho(m) = 1$. So both ρ and τ are in $\mathrm{dom}(p')$, $\tau \in \mathrm{dom}(f^{p'}_\rho)$ and $m \geq f^{p'}_\rho(\tau)$, hence $s^r_\rho(m) = 1$ implies that $s^r_\tau(m) = 1$ by definition of s^r_τ. This finishes the proof that s^r_σ and s^r_τ fit together (with respect to f^r_σ).

Assume $\rho, \rho' \in \mathrm{dom}(r)$ and $\rho' \in \mathrm{dom}(h^r_\rho)$, $m \geq h^r_\rho(\rho')$ and $s^r_\rho(m) = 1$; we have to show that $s^r_{\rho'}(m) = 0$, if it is defined. *Case 1:* $\rho, \rho' \in \mathrm{dom}(q)$. The requirement follows, because q is a condition. *Case 2:* $\rho, \rho' \in \mathrm{dom}(p') \setminus \mathrm{dom}(q)$. In this case the requirement holds, because p' is a condition. *Case 3:* One of them is in $\mathrm{dom}(q)$, the other one not. Since $\rho' \in \mathrm{dom}(h^r_\rho)$, both are in $\mathrm{dom}(p')$ and $h^r_\rho(\rho') = h^{p'}_\rho(\rho')$ (because q cannot provide an h-value for a pair of two nodes if not both of them are in $\mathrm{dom}(q)$). So for $m < |s^{p'}_\rho|$, the requirement holds, because it depends only on p'. The form of C implies that for at most one of the two nodes ρ and ρ' there exists a node in C extending it. Therefore, for $m \geq |s^{p'}_\rho|$, only one of $s^r_\rho(m)$ and $s^r_{\rho'}(m)$ is defined, and we have nothing to show. This finishes the proof that s^r_ρ and $s^r_{\rho'}$ fit together (with respect to h^r_ρ). □

It is straightforward to check that r extends both q and p' (and therefore p). □

5. No refinement, and madness of levels

This section is dedicated to the central part of the proof that the generic object added by our forcing iteration is a distributivity matrix of height λ: we will show that the levels are mad families and that there is no further refinement. This will be done in Section 5.4 and Section 5.3, respectively. Before that, we provide several preliminary lemmas and concepts.

5.1. **General forcing lemmas.** In this section, we give some lemmas about forcing iterations (and completeness) in general, i.e., they are not specific for our forcing from Definition 4.5. We will need them for our proofs. For a good source about forcing iteration, see [22]. Here $\mathbb{P}$, $\mathbb{Q}$, etc. are arbitrary forcing notions.

[28] Note that, for $\mathbb{Q}_\alpha$ with $\alpha > 0$, one also has to deal with the case where $\tau \in \mathrm{dom}(f^r_\sigma)$, but $\tau \notin \mathrm{dom}(r)$, which is analogous (see Remark 4.10).

For two forcing notions $\mathbb{P}$ and $\mathbb{P}'$, let $\mathbb{P}' \lessdot \mathbb{P}$ denote that $\mathbb{P}'$ is a complete subforcing of $\mathbb{P}$. Recall that $\mathbb{P}' \lessdot \mathbb{P}$ if and only if

(1) $\mathbb{P}' \subseteq_{ic} \mathbb{P}$, i.e., for each $q, q' \in \mathbb{P}'$, we have $q \perp_{\mathbb{P}'} q' \implies q \perp_{\mathbb{P}} q'$, and
(2) for each condition $p \in \mathbb{P}$, there is $q \in \mathbb{P}'$ such that q is a *reduction of p to $\mathbb{P}'$*, i.e., for each $r \in \mathbb{P}'$ with $r \leq q$, we have $r \not\perp_{\mathbb{P}} p$.

Let us first recall two easy facts:

Lemma 5.1. *Suppose that $\mathbb{P}_0 \lessdot \mathbb{P}$ and*[29] *$\mathbb{P}_1 \lessdot \mathbb{P}$ satisfying $\mathbb{P}_0 \subseteq \mathbb{P}_1$. Then $\mathbb{P}_0 \lessdot \mathbb{P}_1$. Moreover, if $q \in \mathbb{P}_0$ is a reduction of $p \in \mathbb{P}_1$ from $\mathbb{P}$ to $\mathbb{P}_0$, then q is also a reduction of p from $\mathbb{P}_1$ to $\mathbb{P}_0$.*

Proof. Let us first show that q is a reduction of p from $\mathbb{P}_1$ to $\mathbb{P}_0$. Fix $q' \in \mathbb{P}_0$ with $q' \leq q$. Since q is a reduction of p from $\mathbb{P}$ to $\mathbb{P}_0$, we know that q' and p are compatible within $\mathbb{P}$. But since $\mathbb{P}_1 \subseteq_{ic} \mathbb{P}$, it follows that q' and p are also compatible within $\mathbb{P}_1$, as desired.

It remains to show that $\mathbb{P}_0 \subseteq_{ic} \mathbb{P}_1$: two conditions in $\mathbb{P}_0$ which are compatible within $\mathbb{P}_1$ are clearly compatible within $\mathbb{P}$, hence (due to $\mathbb{P}_0 \subseteq_{ic} \mathbb{P}$) they are also compatible within $\mathbb{P}_0$, as desired. $\square$

Lemma 5.2. *Suppose that $\mathbb{P}' \lessdot \mathbb{P}$. Let φ be some formula, let $\dot{x}$, $\dot{y}$, etc. be $\mathbb{P}'$-names, and let $p \in \mathbb{P}$ such that*

$$p \Vdash_{\mathbb{P}} \varphi(\dot{x}, \dot{y}, \ldots).$$

Then for each $p' \in \mathbb{P}'$ which is a reduction of p,

$$p' \Vdash_{\mathbb{P}'} \varphi(\dot{x}, \dot{y}, \ldots).$$

Proof. If not, then there were $p'' \in \mathbb{P}'$ such that $p'' \leq p'$ and $p'' \Vdash_{\mathbb{P}'} \neg\varphi(\dot{x}, \dot{y}, \ldots)$, and hence $p'' \Vdash_{\mathbb{P}} \neg\varphi(\dot{x}, \dot{y}, \ldots)$; but since p' is a reduction of p, the conditions p'' and p are compatible in $\mathbb{P}$, which is a contradiction. $\square$

Let us now recall the following well-known fact (see, e.g., [11]): If $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\}$ and $\{\mathbb{P}'_\alpha, \dot{\mathbb{Q}}'_\alpha \mid \alpha < \delta\}$ are finite support iterations such that $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}'_\alpha \lessdot \dot{\mathbb{Q}}_\alpha$ for each $\alpha < \delta$, then $\mathbb{P}'_\delta$ is complete in $\mathbb{P}_\delta$. For the convenience of the reader, and since we will need a technical strengthening of the fact (see the moreover part of Lemma 5.4), we give detailed proofs here. We start with the successor step:

Lemma 5.3. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ and $\mathbb{P}' * \dot{\mathbb{Q}}'$ be two-step iterations satisfying $\mathbb{P}' \lessdot \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}' \lessdot \dot{\mathbb{Q}}$. Then $\mathbb{P}' * \dot{\mathbb{Q}}'$ is a complete subforcing of $\mathbb{P} * \dot{\mathbb{Q}}$.*

Moreover, the following holds. Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$, and let p' be a reduction of p to $\mathbb{P}'$. Then there exists a $\mathbb{P}'$-name $\dot{q}'$ such that $p \Vdash$ "$\dot{q}'$ is a reduction of $\dot{q}$ to $\dot{\mathbb{Q}}'$" and $(p', \dot{q}')$ is a reduction of $(p, \dot{q})$ to $\mathbb{P}' * \dot{\mathbb{Q}}'$, and, if $p \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}'$, then $p \Vdash_{\mathbb{P}} \dot{q}' = \dot{q}$, and, moreover, if $\dot{q}$ is a $\mathbb{P}'$-name with $p \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}'$, then $\dot{q}' = \dot{q}$.

Proof. We begin with the moreover part. Since $p \Vdash \dot{q} \in \dot{\mathbb{Q}} \wedge \dot{\mathbb{Q}}' \lessdot \dot{\mathbb{Q}}$, it is forced by p that there exists a condition in $\dot{\mathbb{Q}}'$ which is a reduction of $\dot{q}$. Therefore, we can fix a $\mathbb{P}'$-name $\dot{q}'$ such that

$$p \Vdash_{\mathbb{P}} \dot{q}' \in \dot{\mathbb{Q}}' \ \wedge\ \dot{q}' \text{ is a reduction of } \dot{q}. \tag{4}$$

Since each condition in the subforcing $\dot{\mathbb{Q}}'$ is a reduction of itself, we can assume that, in case $p \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}'$, we have chosen $\dot{q}'$ in such a way that $p \Vdash_{\mathbb{P}} \dot{q}' = \dot{q}$, and that, moreover, if $\dot{q}$ is in addition a $\mathbb{P}'$-name, we

[29] In fact, $\mathbb{P}_1 \subseteq_{ic} \mathbb{P}$ is sufficient for the proof to go through.

have chosen $\dot q'$ to be equal to $\dot q$. Note that $(p', \dot q') \in \mathbb{P}' * \dot{\mathbb{Q}}'$; indeed, p' is a reduction of p, hence (4) together with Lemma 5.2 shows that $p' \Vdash_{\mathbb{P}} \dot q' \in \dot{\mathbb{Q}}'$.

To show that $(p', \dot q')$ is a reduction of $(p, \dot q)$, fix $(p^*, \dot q^*) \in \mathbb{P}' * \dot{\mathbb{Q}}'$ such that $(p^*, \dot q^*) \leq (p', \dot q')$. Since p' is a reduction of p, we can fix $\bar p \in \mathbb{P}$ below both p^* and p. Since then

$$\bar p \Vdash_{\mathbb{P}} \dot q^* \in \dot{\mathbb{Q}}' \ \wedge\ \dot q^* \leq \dot q' \ \wedge\ \dot q' \text{ is a reduction of } \dot q,$$

we can fix a $\mathbb{P}$-name $\dot{\bar q}$ such that

$$\bar p \Vdash_{\mathbb{P}} \dot{\bar q} \in \dot{\mathbb{Q}} \ \wedge\ \dot{\bar q} \leq \dot q^* \ \wedge\ \dot{\bar q} \leq \dot q,$$

finishing the proof that $(\bar p, \dot{\bar q})$ is a condition in $\mathbb{P} * \dot{\mathbb{Q}}$ which witnesses the compatibility of $(p^*, \dot q^*)$ and $(p, \dot q)$.

To finish the proof that $\mathbb{P}' * \dot{\mathbb{Q}}'$ is a complete subforcing of $\mathbb{P} * \dot{\mathbb{Q}}$, it remains to show that $\mathbb{P}' * \dot{\mathbb{Q}}' \subseteq_{ic} \mathbb{P} * \dot{\mathbb{Q}}$. For that, fix two conditions $(p, \dot q)$ and $(p', \dot q')$ in $\mathbb{P}' * \dot{\mathbb{Q}}'$ which are compatible in $\mathbb{P} * \dot{\mathbb{Q}}$, and fix a witness $(\bar p, \dot{\bar q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ below both $(p, \dot q)$ and $(p', \dot q')$. Let $p^* \in \mathbb{P}'$ be a reduction of $\bar p$. Note that[30] $p^* \leq^* p$ (and, analogously, $p^* \leq^* p'$): if not, then, by definition of $\leq^*$, there is $r \in \mathbb{P}'$ such that $r \leq p^*$ and r is incompatible with p (in $\mathbb{P}'$, and therefore in $\mathbb{P}$ due to $\mathbb{P}' \subseteq_{ic} \mathbb{P}$); but, by definition of reduction, r is compatible with $\bar p$, a contradiction.

Now note that $\bar p \Vdash_{\mathbb{P}} \dot q \not\perp_{\dot{\mathbb{Q}}} \dot q'$. Since $\dot{\mathbb{Q}}' \subseteq_{ic} \dot{\mathbb{Q}}$ is forced, we also get $\bar p \Vdash_{\mathbb{P}} \dot q \not\perp_{\dot{\mathbb{Q}}'} \dot q'$, so Lemma 5.2 implies that $p^* \Vdash_{\mathbb{P}'} \dot q \not\perp_{\dot{\mathbb{Q}}'} \dot q'$. So we can fix a $\mathbb{P}'$-name $\dot q^*$ such that

$$p^* \Vdash_{\mathbb{P}'} \dot q^* \in \dot{\mathbb{Q}}' \ \wedge\ \dot q^* \leq \dot q \ \wedge\ \dot q^* \leq \dot q'.$$

It follows that $(p^*, \dot q^*)$ is a condition in $\mathbb{P}' * \dot{\mathbb{Q}}'$ satisfying $(p^*, \dot q^*) \leq^* (p, \dot q)$ and $(p^*, \dot q^*) \leq^* (p', \dot q')$; since compatibility with respect to $\leq^*$ is equivalent to compatibility with respect to $\leq$, it immediately follows that $(p, \dot q) \not\perp_{\mathbb{P}' * \dot{\mathbb{Q}}'} (p', \dot q')$, as desired. $\square$

We now provide the completeness lemma for arbitrary finite support iterations. Recall that, in a forcing iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\}$, the forcing $\mathbb{P}_0$ is the trivial forcing, and hence the ground model iterand $\mathbb{Q}_0$ can be viewed as a $\mathbb{P}_0$-name; $\mathbb{P}_1$ is basically the same as $\mathbb{Q}_0$, and $\mathbb{P}_2$ corresponds to $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$, and so on. In a finite support iteration, conditions can be either viewed as finite partial functions, or as (total) functions whose values are (forced to be) the weakest condition except for finitely many places. We will use both of these two alternative representations (and tacitly identify them); see also [22].

Lemma 5.4. *Let $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\}$ and $\{\mathbb{P}'_\alpha, \dot{\mathbb{Q}}'_\alpha \mid \alpha < \delta\}$ be[31]finite support iterations such that for each $\alpha < \delta$,*

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}'_\alpha \lessdot \dot{\mathbb{Q}}_\alpha.$$

Then $\mathbb{P}'_\delta$ is a complete subforcing of $\mathbb{P}_\delta$.

[30] Note that the involved forcings are not required to be separative; after all, we want to apply this lemma to our forcings $\mathbb{Q}_\alpha$ from Definition 4.5, which are not seperative (see Remark 4.6). Therefore, we have to speak about $\leq^*$ rather than $\leq$. Recall that $q \leq^* p$ if and only if $q \Vdash p \in G$.

[31] We do not seem to need here that our finite support iterations are c.c.c.; however, finite support iterations of non-c.c.c. iterands collapse cardinals. We will use the lemma for our forcings from Definition 4.5, so, in our application, everything will have the c.c.c. anyway.

Moreover, if $\mathsf{RED}\colon \mathbb{Q}_0 \to \mathbb{Q}_0'$ *is a map such that* $\mathsf{RED}(q)$ *is a reduction of* q *for each* $q \in \mathbb{Q}_0$*, then for each* $p \in \mathbb{P}_\delta$*, there is a* $p' \in \mathbb{P}_\delta'$ *such that* p' *is a reduction of* p*, and* $p'(0) = \mathsf{RED}(p(0))$*, and, if* $\alpha \geq 1$ *and* $p(\alpha)$ *is a* $\mathbb{P}_\alpha'$*-name with* $p \restriction \alpha \Vdash p(\alpha) \in \dot{\mathbb{Q}}_\alpha'$*, then* $p'(\alpha) = p(\alpha)$*.*

Proof. By induction on $\delta \geq 1$, we prove that $\mathbb{P}_\delta' \lessdot \mathbb{P}_\delta$, and also define a map $\mathsf{RED}_\delta\colon \mathbb{P}_\delta \to \mathbb{P}_\delta'$ such that for each $p \in \mathbb{P}_\delta$, the following properties hold:

(1) $\mathsf{RED}_\delta(p)$ is a reduction of p.
(2) $\mathsf{RED}_\delta(p)(0) = \mathsf{RED}(p(0))$.
(3) If $\alpha \geq 1$ and $p(\alpha)$ is a $\mathbb{P}_\alpha'$-name with $p \restriction \alpha \Vdash p(\alpha) \in \dot{\mathbb{Q}}_\alpha'$, then $\mathsf{RED}_\delta(p)(\alpha) = p(\alpha)$.
(4) If $1 \leq \alpha < \delta$ and $p \in \mathbb{P}_\alpha$, then $\mathsf{RED}_\delta(p) = \mathsf{RED}_\alpha(p)$.
(5) If $\delta = \alpha + 1 > 1$ is a successor ordinal, then $\mathsf{RED}_\delta(p) \restriction \alpha = \mathsf{RED}_\alpha(p \restriction \alpha)$.

Note that (4) $\wedge$ (5) basically says that the mappings RED_α (with $\alpha \leq \delta$) are coherent.

(Initial step $\delta = 1$) Since $\mathbb{Q}_0' \lessdot \mathbb{Q}_0$ by assumption, $\mathbb{P}_1' = \mathbb{Q}_0'$ is a complete subforcing of $\mathbb{P}_1 = \mathbb{Q}_0$. We let $\mathsf{RED}_1 := \mathsf{RED}$, which clearly satisfies the properties (1) to (5).

(Successor step $\delta = \alpha + 1$) Note that $\mathbb{P}_\alpha' \lessdot \mathbb{P}_\alpha$ by induction, and by assumption, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha' \lessdot \dot{\mathbb{Q}}_\alpha$. Given $(p, \dot{q}) \in \mathbb{P}_\delta = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, let $p' := \mathsf{RED}_\alpha(p)$ (which is a reduction of p by induction) and apply Lemma 5.3 to obtain the reduction $(p', \dot{q}')$. Let $\mathsf{RED}_\delta((p, \dot{q})) := (p', \dot{q}')$. It is straightforward to check that RED_δ is as desired. Moreover, it follows from Lemma 5.3 that $\mathbb{P}_\delta' \lessdot \mathbb{P}_\delta$ (i.e., in addition to the existence of the reduction map RED_δ, we have $\mathbb{P}_\delta' \subseteq_{ic} \mathbb{P}_\delta$).

(Limit step δ) Let us first define the reduction map $\mathsf{RED}_\delta\colon \mathbb{P}_\delta \to \mathbb{P}_\delta'$ as follows: given $p \in \mathbb{P}_\delta$, we can fix $\alpha < \delta$ such that $p \in \mathbb{P}_\alpha$ (since $\mathbb{P}_\delta$ is a finite support iteration, and hence $\mathbb{P}_\delta = \bigcup_{\alpha<\delta} \mathbb{P}_\alpha$); define $\mathsf{RED}_\delta(p) := \mathsf{RED}_\alpha(p)$. Note that this is well-defined: by the coherence property (4) of the maps RED_α (with $\alpha < \delta$), which holds by induction, it does not matter which $\alpha < \delta$ we picked. It is straightforward to check that RED_δ satisfies properties (2) to (5).

To show that RED_δ satisfies property (1), let $p \in \mathbb{P}_\delta$. Fix $p^* \in \mathbb{P}_\delta'$ with $p^* \leq \mathsf{RED}_\delta(p)$. Fix $\alpha < \delta$ such that both $p \in \mathbb{P}_\alpha$ and $p^* \in \mathbb{P}_\alpha'$. Note that, by (4), $\mathsf{RED}_\delta(p) = \mathsf{RED}_\alpha(p)$, hence $\mathsf{RED}_\delta(p)$ is a reduction of p to $\mathbb{P}_\alpha'$; it follows that there exists $\bar{p} \in \mathbb{P}_\alpha \subseteq \mathbb{P}_\delta$ with $\bar{p} \leq p^*$ and $\bar{p} \leq p$. Therefore, $\bar{p}$ witnesses that p^* and p are compatible in $\mathbb{P}_\delta$, as desired.

To finish the proof, it remains to show that $\mathbb{P}_\delta' \subseteq_{ic} \mathbb{P}_\delta$. For that, fix two conditions p and p' in $\mathbb{P}_\delta'$ which are compatible in $\mathbb{P}_\delta$, and fix a witness $\bar{p} \in \mathbb{P}_\delta$ stronger than both p and p'. Now fix $\alpha < \delta$ such that p and p' are in $\mathbb{P}_\alpha'$, and $\bar{p}$ is in $\mathbb{P}_\alpha$. By induction, $\mathbb{P}_\alpha' \subseteq_{ic} \mathbb{P}_\alpha$, so p and p' are in fact compatible within $\mathbb{P}_\alpha'$, witnessed by some p^* in $\mathbb{P}_\alpha'$. This shows that p and p' are also compatible within $\mathbb{P}_\delta'$, as desired. ◻

The following concept has been introduced by Brendle (see, e.g., [9] and [10]):

Definition 5.5. A system of forcings $\mathbb{R}_0, \mathbb{R}_1 \lessdot \mathbb{R}$ with $\mathbb{R}_0 \cap \mathbb{R}_1 \lessdot \mathbb{R}_0, \mathbb{R}_1$ is *correct* if any two conditions $p_0 \in \mathbb{R}_0$ and $p_1 \in \mathbb{R}_1$ which have a common reduction in $\mathbb{R}_0 \cap \mathbb{R}_1$ are compatible in $\mathbb{R}$.

In the following lemma, we are considering a system where $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$, $\mathbb{R}_0 = \mathbb{P}$, and $\mathbb{R}_1 = \mathbb{P}' * \dot{\mathbb{Q}}'$. It is easy to check that, under the assumptions of the lemma, this is a correct system. We do not know, however, whether the conclusion of the lemma holds for every correct system.

Lemma 5.6. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ and $\mathbb{P}' * \dot{\mathbb{Q}}'$ be two-step iterations satisfying $\mathbb{P}' \lessdot \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}' \lessdot \dot{\mathbb{Q}}$. Then*

$$V[\mathbb{P}' * \dot{\mathbb{Q}}'] \cap V[\mathbb{P}] = V[\mathbb{P}'].$$

Proof. We will only show the special case which we will need later (it is straightforward to extend the proof to the general case): for any $\delta, \varepsilon \in \mathrm{Ord}$,

$$\delta^{\varepsilon} \cap V[\mathbb{P}' * \dot{\mathbb{Q}}'] \cap V[\mathbb{P}] \subseteq V[\mathbb{P}'].$$

Let G be a generic filter for $\mathbb{P}'$, and let $\dot{f}_0$ be a $\mathbb{P}$-name, and let $\dot{f}_1$ be a $\mathbb{P}' * \dot{\mathbb{Q}}'$-name. Work in $V[G]$. Assume towards contradiction that there is a condition $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ with $p \in \mathbb{P}/G$ such that

$$(p, \dot{q}) \Vdash \dot{f}_0 = \dot{f}_1 \in \mathrm{Ord}^{<\mathrm{Ord}} \ \wedge \ \dot{f}_0 \notin V[G]. \tag{5}$$

Let $p' \in G$ be a reduction of p to $\mathbb{P}'$. By Lemma 5.3, we can fix a $\mathbb{P}'$-name $\dot{q}'$ such that $p \Vdash$ "$\dot{q}'$ is a reduction of $\dot{q}$" and $(p', \dot{q}') \in \mathbb{P}' * \dot{\mathbb{Q}}'$.

Since p is reduction of $(p, \dot{q})$ to $\mathbb{P}$, it follows from (5) and Lemma 5.2 that $p \Vdash \dot{f}_0 \notin V[G]$. Therefore, we can fix $\gamma \in \varepsilon$ such that p does not decide $\dot{f}_0(\gamma)$ in $\mathbb{P}/G$. Let $(p_1, \dot{q}_1) \leq (p', \dot{q}')$ and $\xi_1 \in \delta$ such that $p_1 \in G$ and $(p_1, \dot{q}_1) \Vdash \dot{f}_1(\gamma) = \xi_1$. Since p does not decide $\dot{f}_0$ at γ, we can fix $p_0 \in \mathbb{P}/G$ with $p_0 \leq p$ and $\xi_0 \in \delta$ with $\xi_0 \neq \xi_1$ such that $p_0 \Vdash \dot{f}_0(\gamma) = \xi_0$. Now we want to find a condition $(p^*, \dot{q}^*)$ which is stronger than $(p, \dot{q}), (p_1, \dot{q}_1)$ and $(p_0, \mathbb{1})$.

First note that p_0 and p_1 are compatible, because $p_0 \in \mathbb{P}/G$ and $p_1 \in G$, and fix $p^* \leq p_0, p_1$. Since $p^* \leq p, p_1$ it follows that $p^* \Vdash$ "$\dot{q}'$ is a reduction of $\dot{q}$ and $\dot{q}_1 \leq \dot{q}'$" hence $p^* \Vdash \dot{q}_1 \not\perp \dot{q}$. Let $\dot{q}^*$ be a $\mathbb{P}$-name such that $p^* \Vdash \dot{q}^* \leq \dot{q}_1, \dot{q}$. It is easy to check that $(p^*, \dot{q}^*) \leq (p, \dot{q}), (p_1, \dot{q}_1), (p_0, \mathbb{1})$. Now $(p^*, \dot{q}^*) \Vdash \dot{f}_0 = \dot{f}_1 \wedge \dot{f}_0(\gamma) = \xi_0 \wedge \dot{f}_1(\gamma) = \xi_1$, but $\xi_0 \neq \xi_1$, a contradiction. □

We conclude with an easy observation we will need later on:

Lemma 5.7. *Suppose that $\mathbb{P}' \lessdot \mathbb{P}$, and $\dot{b}$ is a $\mathbb{P}'$-name, and $p \in \mathbb{P}$ is such that $p \Vdash \dot{b} \in [\omega]^{\omega}$. Then for each $N \in \omega$ there exists $r \in \mathbb{P}'$ and $m > N$ such that $r \Vdash m \in \dot{b}$ and r is compatible with p.*

Proof. Since $\mathbb{P}'$ is complete in $\mathbb{P}$, we can fix a reduction p' of p, and apply Lemma 5.2 to see that $p' \Vdash_{\mathbb{P}'} \dot{b} \in [\omega]^{\omega}$. So, given $N \in \omega$, there exists $r \in \mathbb{P}'$ with $r \leq p'$ and $m > N$ such that $r \Vdash m \in \dot{b}$. Since p' is a reduction of p and $r \leq p'$, it follows that r is compatible with p, as desired. □

5.2. **Complete subforcings: hereditarily below γ.** In this section, we give some technical definitions and lemmas as a preparation for the main proofs in Section 5.3 and Section 5.4. More precisely, we define, for each $\gamma < \lambda$, the subforcings of "hereditarily below γ" conditions of our iteration and show that they form complete subforcings (see Lemma 5.12). Furthermore, we show that each condition is hereditarily below γ for some $\gamma < \lambda$ (see Lemma 5.14).

Let us now provide the following recursive definition (we give the definition for the entire iteration but we will actually need it for tails of the iteration; see Remark 5.10):

Definition 5.8. Let $\gamma < \lambda$. By recursion on $\alpha \leq \lambda$ we define when a condition $p \in \mathbb{P}_\alpha$ is[32] *hereditarily below γ* (and introduce the notation ${}^{<\gamma}\mathbb{P}_\alpha$):

[32] In the more general situation described in Remark 4.3, i.e., if we work with the tree $\theta^{<\lambda}$ in place of $\lambda^{<\lambda}$, we would rather need a pair of ordinals (ε, δ) in place of γ, where $\varepsilon < \theta$ and $\delta < \lambda$, and $\gamma^{<\gamma}$ in the definition would be replaced by $\varepsilon^{<\delta}$ (see also Remark 5.16).

(1) $p \in \mathbb{Q}_0$ is hereditarily below γ, if $\mathrm{dom}(p) \subseteq \gamma^{<\gamma}$.
(2) Let ${}^{<\gamma}\mathbb{P}_\alpha := \{p \in \mathbb{P}_\alpha \mid p \text{ hereditarily below } \gamma\}$.
(3) $p \in \mathbb{P}_{\alpha+1}$ is hereditarily below γ, if $p \restriction \alpha$ is hereditarily below γ and $p(\alpha)$ is a ${}^{<\gamma}\mathbb{P}_\alpha$-name, and $p \restriction \alpha \Vdash \mathrm{dom}(p(\alpha)) \subseteq \gamma^{<\gamma}$.
(4) For α limit, $p \in \mathbb{P}_\alpha$ is hereditarily below γ, if $p \restriction \beta$ is hereditarily below γ, for every $\beta < \alpha$.

For $\alpha \leq \lambda$ a $\mathbb{P}_\alpha$-name $\dot{b}$ is *hereditarily below* γ, if for all $(\dot{x}, p) \in \dot{b}$, $p \in {}^{<\gamma}\mathbb{P}_\alpha$ and $\dot{x}$ is hereditarily below γ (this is by recursion).

Clearly, if $p \in \mathbb{P}_\alpha$ is hereditarily below γ and $\gamma' > \gamma$, then p is also hereditarily below γ'. The same holds for a $\mathbb{P}_\alpha$-name $\dot{b}$.

Definition 5.9. Let $\gamma < \lambda$ and $\tau \in \lambda^{<\lambda}$. By recursion on $\alpha \leq \lambda$ we define when a condition $p \in \mathbb{P}_\alpha$ is *almost hereditarily below γ except for τ* (and introduce the notation ${}^{<\gamma+\tau}\mathbb{P}_\alpha$):

(1) $p \in \mathbb{Q}_0$ is almost hereditarily below γ except for τ, if $\mathrm{dom}(p) \subseteq \gamma^{<\gamma} \cup \{\tau\}$.
(2) Let ${}^{<\gamma+\tau}\mathbb{P}_\alpha := \{p \in \mathbb{P}_\alpha \mid p \text{ almost hereditarily below } \gamma \text{ except for } \tau\}$.
(3) $p \in \mathbb{P}_{\alpha+1}$ is almost hereditarily below γ except for τ, if $p \restriction \alpha$ is almost hereditarily below γ except for τ and $p(\alpha)$ is a[33] ${}^{<\gamma}\mathbb{P}_\alpha$-name, and $p \restriction \alpha \Vdash \mathrm{dom}(p(\alpha)) \subseteq \gamma^{<\gamma}$.
(4) For α limit, $p \in \mathbb{P}_\alpha$ is almost hereditarily below γ except for τ, if $p \restriction \beta$ is almost hereditarily below γ except for τ, for every $\beta < \alpha$.

For $\alpha \leq \lambda$ a $\mathbb{P}_\alpha$-name $\dot{b}$ is *almost hereditarily below γ except for τ*, if for all $(\dot{x}, p) \in \dot{b}$, both p and $\dot{x}$ are almost hereditarily below γ except for τ. We will write *almost hereditarily below* γ and omit the τ if it is clear from the context which τ is meant.

Clearly, if $p \in \mathbb{P}_\alpha$ is almost hereditarily below γ and $\gamma' > \gamma$, then p is also almost hereditarily below γ' and if $p \in \mathbb{P}_\alpha$ is hereditarily below γ, then it is almost hereditarily below γ except for τ for every τ. The same holds for a $\mathbb{P}_\alpha$-name $\dot{b}$.

Remark 5.10. As mentioned above, we will need several of our concepts for tails of the iteration instead of the whole iteration. We will later have the following situation: $\eta < \lambda$ will be fixed, and we will work in $V[G_\eta]$ for a fixed generic filter $G_\eta \subseteq \mathbb{P}_\eta$. We will use variants of the above definitions and the subsequent lemmas for the tail iteration $\{\mathbb{P}_\alpha / G_\eta, \dot{\mathbb{Q}}_\alpha \mid \eta \leq \alpha < \lambda\}$. In the definitions and lemmas, $\mathbb{Q}_\eta$ plays the role of $\mathbb{Q}_0$ (see for example Lemma 5.18(3)). So, e.g., in the definition of almost hereditarily below γ except for τ (with $\tau \in (\lambda^{<\lambda})^{V[G_\eta]}$), we want $\mathrm{dom}(p(\eta)) \subseteq (\gamma^{<\gamma})^{V[G_\eta]} \cup \{\tau\}$.

Before proving completeness, let us recall that $\gamma^{<\gamma}$ is left-up-closed; actually, we will need a bit more:

Lemma 5.11. *Let $\mathbb{P}'$ be a complete subforcing of $\mathbb{P}$, and G generic for $\mathbb{P}$. Then in $V[G]$, the set $(\gamma^{<\gamma})^{V[G \cap \mathbb{P}']}$ is left-up-closed.*

Proof. Suppose σ and $\rho^\frown i$ belong to $(\gamma^{<\gamma})^{V[G\cap\mathbb{P}']}$. Note that the following holds in $V[G]$: $\sigma \restriction (\xi + 1) \in V[G\cap\mathbb{P}']$ for each $\xi < |\sigma|$, and $\rho^\frown j \in V[G\cap\mathbb{P}']$ for each $j < i$. Therefore $(\gamma^{<\gamma})^{V[G\cap\mathbb{P}']}$ is left-up-closed. $\square$

We can now show that the subforcing of conditions which are (almost) hereditarily below γ is a complete subforcing:

[33] This is not a typo: we really require $p(\alpha)$ to be a ${}^{<\gamma}\mathbb{P}_\alpha$-name, not just a ${}^{<\gamma+\tau}\mathbb{P}_\alpha$-name.

Lemma 5.12. *Let $\gamma < \lambda$. Then ${}^{<\gamma}\mathbb{P}_\lambda$ is a complete subforcing of $\mathbb{P}_\lambda$.*
Also, if $\tau \in \lambda^{<\lambda}$ is such that either

(1) $|\tau| \geq \gamma$, or
(2) τ is such that $\gamma^{<\gamma} \cup \{\tau\}$ is left-up-closed,

then ${}^{<\gamma+\tau}\mathbb{P}_\lambda$ is a complete subforcing of $\mathbb{P}_\lambda$.
Moreover, if p is full and almost hereditarily below γ except for τ, then

$$(p(0) \restriction \gamma^{<\gamma}, p(1), p(2), \dots)$$

is a reduction of p to ${}^{<\gamma}\mathbb{P}_\lambda$.

Proof. We show by induction on α that ${}^{<\gamma}\mathbb{P}_\alpha$ (as well as ${}^{<\gamma+\tau}\mathbb{P}_\alpha$) is a complete subforcing of $\mathbb{P}_\alpha$, for $1 \leq \alpha \leq \lambda$. In fact, we will define ${}^{<\gamma}\mathbb{P}_\alpha$-names $\dot{\mathbb{Q}}'_\alpha$ such that ${}^{<\gamma}\mathbb{P}_\delta$ (or ${}^{<\gamma+\tau}\mathbb{P}_\delta$, respectively) is the finite support iteration of the $\dot{\mathbb{Q}}'_\alpha$'s; the only difference of the two iterations will be the first iterand $\mathbb{Q}'_0$.

(Initial step $\alpha = 1$) Note that ${}^{<\gamma}\mathbb{P}_1 = \mathbb{Q}_0^{\gamma^{<\gamma}}$ is a complete subforcing of $\mathbb{P}_1 = \mathbb{Q}_0$: this is an easy instance of Lemma 4.22, letting $C = E = \gamma^{<\gamma}$ (which is left-up-closed). Similarly, ${}^{<\gamma+\tau}\mathbb{P}_1 = \mathbb{Q}_0^{\gamma^{<\gamma}\cup\{\tau\}}$ is a complete subforcing of $\mathbb{Q}_0$: in case (2) holds, we let $C = E = \gamma^{<\gamma} \cup \{\tau\}$ (which is left-up-closed by assumption); in case (1) holds, we let $C = E \cup \bar{C}$ with $E = \gamma^{<\gamma}$ and $\bar{C} = \{\tau\}$ (which is easily seen to be a possible instance of the assumption of Lemma 4.22). Take $\mathbb{Q}'_0 = \mathbb{Q}_0^{\gamma^{<\gamma}}$ in the iteration representing ${}^{<\gamma}\mathbb{P}_\lambda$, and take $\mathbb{Q}'_0 = \mathbb{Q}_0^{\gamma^{<\gamma}\cup\{\tau\}}$ in the iteration representing ${}^{<\gamma+\tau}\mathbb{P}_\lambda$.

(Successor step $\alpha + 1$) Assume that ${}^{<\gamma}\mathbb{P}_\alpha$ and ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ are complete subforcings of $\mathbb{P}_\alpha$. We show that ${}^{<\gamma}\mathbb{P}_{\alpha+1}$ and ${}^{<\gamma+\tau}\mathbb{P}_{\alpha+1}$ are complete subforcings of $\mathbb{P}_{\alpha+1}$. In $V[G]$, for G generic for $\mathbb{P}_\alpha$, let[34] $E := (\gamma^{<\gamma})^{V[G\cap {}^{<\gamma}\mathbb{P}_\alpha]}$; by Lemma 5.11, E is left-up-closed, so Lemma 4.22 implies (letting $C = E$) that in $V[G]$, $\mathbb{Q}_\alpha^E$ is a complete subforcing of $\mathbb{Q}_\alpha$. We use the following, which we will prove after finishing the proof of the lemma:

Claim 5.13. *$\mathbb{Q}_\alpha^E$ is an element of $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$.*

Using the claim, we can fix a ${}^{<\gamma}\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}'_\alpha$ for $\mathbb{Q}_\alpha^E$. Since ${}^{<\gamma}\mathbb{P}_\alpha \subseteq {}^{<\gamma+\tau}\mathbb{P}_\alpha$ and both are complete subforcings of $\mathbb{P}_\alpha$, Lemma 5.1 implies that ${}^{<\gamma}\mathbb{P}_\alpha$ is a complete subforcing of ${}^{<\gamma+\tau}\mathbb{P}_\alpha$, so the ${}^{<\gamma}\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}'_\alpha$ is also a ${}^{<\gamma+\tau}\mathbb{P}_\alpha$-name. So we can apply Lemma 5.4 to obtain that ${}^{<\gamma}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ and ${}^{<\gamma+\tau}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ are complete subforcings of $\mathbb{P}_{\alpha+1}$. By definition, ${}^{<\gamma}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ is equivalent to ${}^{<\gamma}\mathbb{P}_{\alpha+1}$, and ${}^{<\gamma+\tau}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ is equivalent to ${}^{<\gamma+\tau}\mathbb{P}_{\alpha+1}$, so the successor step is finished.

(Limit step α) It follows by Lemma 5.4 that the limit of the finite support iteration of the $\dot{\mathbb{Q}}'_{\alpha'}$ with $\alpha' < \alpha$ is a complete subforcing of $\mathbb{P}_\alpha$, and by definition, ${}^{<\gamma}\mathbb{P}_\alpha$ (or ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ in the other case) is equivalent to the limit of this finite support iteration.

Now let us show the moreover part. By the moreover part of Lemma 4.22, $p(0) \restriction \gamma^{<\gamma}$ is a reduction of $p(0)$ to ${}^{<\gamma}\mathbb{P}_1$ (which is $\mathbb{Q}'_0$ in the iteration representing ${}^{<\gamma}\mathbb{P}_\lambda$). Since $p \in {}^{<\gamma+\tau}\mathbb{P}_\lambda$, which is the iteration of the $\dot{\mathbb{Q}}'_\alpha$ (for $\alpha \geq 1$ the iterands of the two iterations coincide), so $p \restriction \alpha \Vdash p(\alpha) \in \mathbb{Q}'_\alpha$ for $\alpha \geq 1$, therefore Lemma 5.4 completes the proof. $\square$

[34]Note that E is really defined this way for both cases (see also footnote 33).

Proof of Claim 5.13. We work in $V[G]$. Let $G_\beta := G \cap \mathbb{P}_\beta$. Let $T'_\alpha = \bigcup_{\beta<\alpha}(\lambda^{<\lambda} \cap \mathrm{succ})^{V[G_\beta]}$ and $T_\alpha = (\lambda^{<\lambda} \cap \mathrm{succ})^{V[G]} \setminus T'_\alpha$, as in the definition of $\mathbb{Q}_\alpha$.

It is straightforward to check that $\mathbb{Q}^E_\alpha$ can be defined in $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$ provided that $E \cap T'_\alpha$ (and hence also $E \cap T_\alpha$) belongs to $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$. First note that

$$E = (\gamma^{<\gamma})^{V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]} = \gamma^{<\gamma} \cap V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$$

and

$$T'_\alpha = \bigcup_{\beta<\alpha}(\lambda^{<\lambda} \cap \mathrm{succ})^{V[G_\beta]} = \bigcup_{\beta<\alpha}(\lambda^{<\lambda} \cap \mathrm{succ} \cap V[G_\beta]).$$

So

$$E \cap T'_\alpha = \bigcup_{\beta<\alpha}(\gamma^{<\gamma} \cap \mathrm{succ} \cap V[G \cap {}^{<\gamma}\mathbb{P}_\alpha] \cap V[G_\beta]).$$

Apply Lemma 5.6 to $\mathbb{P}_\beta * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is the quotient $\mathbb{P}_\alpha \,/\, \mathbb{P}_\beta$, and ${}^{<\gamma}\mathbb{P}_\beta * \dot{\mathbb{Q}}'$, where $\dot{\mathbb{Q}}'$ is the quotient ${}^{<\gamma}\mathbb{P}_\alpha / {}^{<\gamma}\mathbb{P}_\beta$ (which is possible since ${}^{<\gamma}\mathbb{P}_\beta \lessdot \mathbb{P}_\beta$ by induction hypothesis, and $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}' \lessdot \dot{\mathbb{Q}}$ by Lemma 5.4 for the tail iterations) to obtain

$$\gamma^{<\gamma} \cap V[G \cap {}^{<\gamma}\mathbb{P}_\alpha] \cap V[G_\beta] = \gamma^{<\gamma} \cap V[G_\beta \cap {}^{<\gamma}\mathbb{P}_\alpha].$$

Therefore,

$$E \cap T'_\alpha = \bigcup_{\beta<\alpha}(\gamma^{<\gamma} \cap \mathrm{succ})^{V[G_\beta \cap {}^{<\gamma}\mathbb{P}_\alpha]},$$

which clearly belongs to $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$, as desired. □

The next lemma shows that every condition in $\mathbb{P}_\lambda$ is (essentially) hereditarily below γ for some $\gamma < \lambda$.

Lemma 5.14. *For every $p \in \mathbb{P}_\lambda$, there exists a $\gamma < \lambda$ and a condition $p' \in {}^{<\gamma}\mathbb{P}_\lambda$ which is equivalent to p.*

Proof. We will actually show by induction on α that for every $p \in \mathbb{P}_\alpha$, there exists a $\gamma < \lambda$ and a condition p' equivalent to p such that $p' \in {}^{<\gamma}\mathbb{P}_\alpha$.

(Initial step $\alpha = 1$) Given $p \in \mathbb{P}_1 = \mathbb{Q}_0$, note that $\mathrm{dom}(p) \subseteq \lambda^{<\lambda}$ is finite. So the maximum length of the nodes σ in the domain as well as the maximal entry of the nodes are bounded, i.e., there is $\gamma < \lambda$ such that $\mathrm{dom}(p) \subseteq \gamma^{<\gamma}$. So $p \in {}^{<\gamma}\mathbb{P}_1$.

(Limit step α) Let $p \in \mathbb{P}_\alpha$. By induction hypothesis, for each $\beta < \alpha$ there exists γ_β such that $p \restriction \beta \in {}^{<\gamma_\beta}\mathbb{P}_\beta$. Since we are using finite support, there exists $\beta^* < \alpha$ which is an upper bound of the support of p. Then $p \in {}^{<\gamma_{\beta^*}}\mathbb{P}_\alpha$.

(Successor step $\alpha + 1$) Let $(p, \dot{q}) \in \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. First, by the induction hypothesis, we can assume without loss of generality that there exists $\gamma_p < \lambda$ such that $p \in {}^{<\gamma_p}\mathbb{P}_\alpha$. We will describe a name $\dot{q}'$ which is equivalent to $\dot{q}$ (more precisely, $p \Vdash \dot{q} = \dot{q}'$) and analyze it, to find a $\gamma < \lambda$ such that $\dot{q}'$ is a ${}^{<\gamma}\mathbb{P}_\alpha$-name and $p \Vdash \mathrm{dom}(\dot{q}') \subseteq \gamma^{<\gamma}$.

Claim 5.15. *Let $\dot{\sigma}$ be a $\mathbb{P}_\alpha$-name of a sequence of ordinals of length less than λ; then there exists a $\gamma < \lambda$ such that there exists a ${}^{<\gamma}\mathbb{P}_\alpha$-name which is equivalent to $\dot{\sigma}$. The same holds true if $\dot{\sigma}$ is a name for a finite sequence of such sequences.*

Proof. Clearly, by c.c.c., there exists a $\delta < \lambda$ which is an upper bound for the length of $\dot\sigma$. Since $\mathbb{P}_\alpha$ has the c.c.c., for each $\xi < \delta$, $\dot\sigma(\xi)$ is represented by a countable antichain. So only $|\delta| \cdot \aleph_0$ many (hence less than λ many) conditions appear in $\dot\sigma$. By inductive hypothesis we can assume that each of these conditions belongs to ${}^{<\gamma}\mathbb{P}_\alpha$ for some $\gamma < \lambda$, so we can fix a $\gamma_{\dot\sigma} < \lambda$ which is an upper bound of all the appearing γ. So $\dot\sigma$ is actually a ${}^{<\gamma_{\dot\sigma}}\mathbb{P}_\alpha$-name. The statement about names for finite sequences of sequences follows easily. $\square$

Now, let $\dot N$ be a $\mathbb{P}_\alpha$-name such that $p \Vdash |\mathrm{dom}(\dot q)| = \dot N$; by (a simple instance of) Claim 5.15, we can fix $\gamma_{\dot N} < \lambda$ and assume that $\dot N$ is a ${}^{<\gamma_{\dot N}}\mathbb{P}_\alpha$-name. To represent $\dot q$, we provide ω-sequences $\langle \dot\sigma_k \mid k \in \omega\rangle$ (of names for potential members of $\mathrm{dom}(\dot q)$) and $\langle (\dot s_k, \dot f_k, \dot h_k) \mid k \in \omega\rangle$ such that

$$p \Vdash \mathrm{dom}(\dot q) = \{\dot\sigma_k \mid k \in \dot N\} \ \wedge\ \forall k \in \dot N\ (\dot q(\dot\sigma_k) = (\dot s_k, \dot f_k, \dot h_k)),$$

where $\dot\sigma_k$ is forced to be a sequence of ordinals of length less than λ, and $\dot f_k$ and $\dot h_k$ can be represented as finite sequences of such sequences, together with finite sequences of natural numbers, and $\dot s_k$ is forced to be an element of $2^{<\omega}$. Using Claim 5.15, we can find $\gamma' < \lambda$, larger than $\gamma_{\dot N}$, such that there exist ${}^{<\gamma'}\mathbb{P}_\alpha$-names $\dot\sigma'_k$, $\dot s'_k$, $\dot f'_k$, and $\dot h'_k$ which are equivalent to $\dot\sigma_k$, $\dot s_k$, $\dot f_k$, and $\dot h_k$, respectively. By replacing all $\dot\sigma_k$, $\dot s_k$, $\dot f_k$, and $\dot h_k$ in $\dot q$ by their respective equivalent names, we get a ${}^{<\gamma'}\mathbb{P}_\alpha$-name $\dot q'$ such that $p \Vdash \dot q = \dot q'$.

Again by the c.c.c., there exist $\varepsilon, \delta < \lambda$ such that $p \Vdash \dot\sigma_k \in \varepsilon^{<\delta}$ for every $k < \omega$. Let $\gamma := \max(\gamma_p, \gamma', \varepsilon, \delta) < \lambda$. Then $(p, \dot q') \in {}^{<\gamma}\mathbb{P}_{\alpha+1}$, and it is equivalent to $(p, \dot q)$, which finishes the proof. $\square$

Remark 5.16. In the more general situation described in Remark 4.3, i.e., if we work with the tree $\theta^{<\lambda}$ in place of $\lambda^{<\lambda}$ (see also footnote 32), we have to require that $\mathrm{cf}(\theta) \geq \lambda$. The reason is that $|\sigma_k|$ can be arbitrarily large below λ: if $\mathrm{cf}(\theta) < \lambda$, it could happen that there does not exist an $\varepsilon < \theta$ which is needed in the end of the generalization of the above proof.

Lemma 5.17. *Let G be $\mathbb{P}_\lambda$-generic and $V[G] \models b \subseteq \omega$. Then there exists a $\gamma < \lambda$ and a $\mathbb{P}_\lambda$-name $\dot b$ for b which is hereditarily below γ.*

Proof. For every condition $p \in \mathbb{P}_\lambda$, let $\gamma_p < \lambda$ be such that there exists a condition in ${}^{<\gamma_p}\mathbb{P}_\lambda$ which is equivalent to p (which is possible by Lemma 5.14). Let $\dot b'$ be a nice name for b, and let $\dot b$ be a name where every condition p appearing in $\dot b'$ is replaced by an equivalent condition in ${}^{<\gamma_p}\mathbb{P}_\lambda$. Since b is a countable set and $\mathbb{P}_\lambda$ has the c.c.c., the set B of conditions which appeared in $\dot b'$ is countable. Let $\gamma := \sup\{\gamma_p \mid p \in B\} < \lambda$; then $\dot b$ is a ${}^{<\gamma}\mathbb{P}_\lambda$-name. $\square$

We conclude with a technical lemma which will be crucial later on:

Lemma 5.18. *Suppose $\tau \in \lambda^{<\lambda} \setminus \gamma^{<\gamma}$. Let $p, r \in \mathbb{P}_\lambda$ such that p is a full condition which is almost hereditarily below γ except for τ, and r is hereditarily below γ, and p and r are compatible (in $\mathbb{P}_\lambda$). Then there exists a $p^* \in \mathbb{P}_\lambda$ such that*

(1) *p^* is almost hereditarily below γ except for τ,*
(2) *$p^* \leq p, r$, and*
(3) *$p^*(0)(\tau) = p(0)(\tau)$.*

Proof. Without loss of generality we can assume that $\mathrm{dom}(p(0)) \supsetneq \{\tau\}$. Since p is full and almost hereditarily below γ, by (the "moreover part" of) Lemma 5.12

$$\textsf{RED}(p) := (p(0) \restriction \gamma^{<\gamma}, p(1), p(2), \dots)$$

is a reduction of p to ${}^{<\gamma}\mathbb{P}_\lambda$. We show that $\textsf{RED}(p) \not\perp_{{}^{<\gamma}\mathbb{P}_\lambda} r$. Assume not. Since ${}^{<\gamma}\mathbb{P}_\lambda$ is a complete subforcing of $\mathbb{P}_\lambda$, it follows that $\textsf{RED}(p) \perp_{\mathbb{P}_\lambda} r$. But $p \leq \textsf{RED}(p)$, so $p \perp_{\mathbb{P}_\lambda} r$, which is a contradiction to the assumption of the lemma.

Let $q^* \in {}^{<\gamma}\mathbb{P}_\lambda$ be such that $q^* \leq \textsf{RED}(p), r$; without loss of generality, we can assume that $q^*(0)$ is full. Since $q^*(0) \leq \textsf{RED}(p)(0) = p(0) \restriction \gamma^{<\gamma}$ and $p(0) \restriction \gamma^{<\gamma}$ is a reduction of $p(0)$ by Lemma 4.22 (recall that $p(0) \restriction\restriction \gamma^{<\gamma} = p(0) \restriction \gamma^{<\gamma}$ because $\gamma^{<\gamma}$ is left-up-closed), it follows that $q^*(0)$ is compatible with $p(0)$. Let $\bar{q}(0)$ be a full witness for that. So $\bar{q}(0) \leq p(0), r(0), q^*(0)$.

Let $p^*(0) := \bar{q}(0) \restriction \gamma^{<\gamma} \cup \{(\tau, p(0)(\tau))\}$, and for $\alpha > 0$, let $p^*(\alpha) := q^*(\alpha)$.

Claim. *$p^*(0)$ is a condition.*

Proof. For $\sigma, \sigma' \in \mathrm{dom}(\bar{q}(0) \restriction \gamma^{<\gamma})$, it is clear that the requirements for being a condition are fulfilled, because $\bar{q}(0)$ is a condition.

Let $\sigma \trianglelefteq \tau$ and $\sigma \in \mathrm{dom}(p^*(0))$. Let $\sigma' \in \mathrm{dom}(p(0)) \setminus \{\tau\}$. Clearly, $s_\sigma^{p^*(0)} = s_\sigma^{\bar{q}(0)}$ and $s_\tau^{p^*(0)} = s_\tau^{p(0)}$. Since $\bar{q}(0)$ and p are full, it follows that $|s_\sigma^{\bar{q}(0)}| = |s_{\sigma'}^{\bar{q}(0)}|$ and hence $|s_\sigma^{p^*(0)}| = |s_{\sigma'}^{p^*(0)}| \geq |s_{\sigma'}^{p(0)}| = |s_\tau^{p(0)}| = |s_\tau^{p^*(0)}|$.

Let $\sigma \in \mathrm{dom}(f_\tau^{p^*(0)})$ and assume that $s_\tau^{p^*(0)}(m) = 1$ for some $m \geq f_\tau^{p^*(0)}(\sigma)$. We have to show that $s_\sigma^{p^*(0)}(m) = 1$. Since $\bar{q}(0)$ extends $p(0)$, we have $s_\tau^{\bar{q}(0)}(m) = 1$ and $\mathrm{dom}(f_\tau^{p^*(0)}) \subseteq \mathrm{dom}(f_\tau^{\bar{q}(0)})$, and for $\sigma \in \mathrm{dom}(f_\tau^{p^*(0)})$ it holds that $f_\tau^{p^*(0)}(\sigma) \geq f_\tau^{\bar{q}(0)}(\sigma)$, so $\sigma \in \mathrm{dom}(f_\tau^{\bar{q}(0)})$ and $m \geq f_\tau^{\bar{q}(0)}(\sigma)$. Since $\bar{q}(0)$ is a condition, it follows that $s_\sigma^{p^*(0)}(m) = s_\sigma^{\bar{q}(0)}(m) = 1$.

Let $\sigma \in \mathrm{dom}(h_\tau^{p^*(0)})$ and assume that $s_\tau^{p^*(0)}(m) = 1$ for some $m \geq h_\tau^{p^*(0)}(\sigma)$. We have to show that $s_\sigma^{p^*(0)}(m) = 0$. Since $\bar{q}(0)$ extends $p(0)$, we have $s_\tau^{\bar{q}(0)}(m) = 1$ and $\mathrm{dom}(h_\tau^{p^*(0)}) \subseteq \mathrm{dom}(h_\tau^{\bar{q}(0)})$, and for $\sigma \in \mathrm{dom}(h_\tau^{p^*(0)})$ it holds that $h_\tau^{p^*(0)}(\sigma) \geq h_\tau^{\bar{q}(0)}(\sigma)$, so $\sigma \in \mathrm{dom}(h_\tau^{\bar{q}(0)})$ and $m \geq h_\tau^{\bar{q}(0)}(\sigma)$. Since $\bar{q}(0)$ is a condition, it follows that $s_\sigma^{p^*(0)}(m) = s_\sigma^{\bar{q}(0)}(m) = 0$. □

Moreover, $p^*(0) \leq q^*(0)$, because $\bar{q}(0) \leq q^*(0)$ and q^* hereditarily below γ except for τ. So p^* is a condition. Clearly p^* is almost hereditarily below γ and $p^*(0)(\tau) = p(0)(\tau)$.

Since $r(0)$ is hereditarily below γ and $p(0)$ is almost hereditarily below γ, and $\bar{q}(0) \leq r(0), p(0)$, it is clear that $p^*(0)$ extends $r(0)$ and $p(0)$. So clearly $p^* \leq r, p$. □

5.3. **No refinement: branches are towers.** Now we are ready to prove that the generic matrix has no refinement. More precisely, we show that the sets along any branch in our tree have no pseudo-intersection, i.e., they form a tower.

Lemma 5.19. *In $V[\mathbb{P}_\lambda]$, the sequence $\langle a_{\sigma\restriction\xi} \mid \xi < \lambda\rangle$ is a tower for each $\sigma \in \lambda^\lambda$.*

Proof. Let G_λ be generic for $\mathbb{P}_\lambda$ and work in $V[G_\lambda]$. Fix $\sigma \in \lambda^\lambda$. By Corollary 4.16, $\langle a_{\sigma\restriction\xi} \mid \xi < \lambda\rangle$ is $\subseteq^*$-decreasing. Let us show that $\langle a_{\sigma\restriction\xi} \mid \xi < \lambda\rangle$ is actually a tower. Let $b \subseteq \omega$ be infinite, and assume towards a contradiction that $b \subseteq^* a_{\sigma\restriction\xi}$ for every $\xi < \lambda$.

Apply Lemma 5.17 to get $\gamma < \lambda$ and a $\mathbb{P}_\lambda$-name $\dot{b}$ for b which is hereditarily below γ. Without loss of generality we can assume that γ is a successor ordinal. Fix $\eta < \lambda$ minimal such that $\sigma \restriction \gamma \in V[G_\eta]$ (such an η exists by Lemma 4.9). From now on, we work in $V[G_\eta]$, and we consider[35] the tail forcing $\mathbb{P}_\lambda/G_\eta$. The $\mathbb{P}_\lambda$-name $\dot{b}$ can be understood as a $\mathbb{P}_\lambda/G_\eta$-name for b which is hereditarily below γ.

Since $b \subseteq^* a_{\sigma\restriction\gamma}$ holds in $V[G_\lambda]$, we can pick $n \in \omega$ and $p \in \mathbb{P}_\lambda/G_\eta$ such that

$$p \Vdash \dot{b} \setminus n \subseteq a_{\sigma\restriction\gamma}.$$

From now on, whenever we say "almost hereditarily below γ", we shall mean "almost hereditarily below γ except for $\sigma \restriction \gamma$". Note that (the canonical name for) $a_{\sigma\restriction\gamma}$ is almost hereditarily below γ; also $\dot{b}$ is almost hereditarily below γ (because $\dot{b}$ is hereditarily below γ).

By Lemma 5.12 and Lemma 5.2, we can fix p' which is almost hereditarily below γ such that

$$p' \Vdash \dot{b} \setminus n \subseteq a_{\sigma\restriction\gamma}.$$

Recall that η is minimal with $\sigma \restriction \gamma \in V[G_\eta]$, so $\mathbb{Q}_\eta$ will assign a set $a_{\sigma\restriction\gamma}$ to $\sigma \restriction \gamma$. Therefore we can assume without loss of generality that $\sigma \restriction \gamma \in \mathrm{dom}(p'(\eta))$, and we can assume that p' is a full[36] condition.

By Lemma 5.7 there is $r \in \mathbb{P}_\lambda/G_\eta$ hereditarily below γ and $m > n, |s^{p'(\eta)}_{\sigma\restriction\gamma}|$ such that r is compatible with p', and $r \Vdash m \in \dot{b}$. Apply Lemma 5.18 to obtain $p'' \leq p', r$ such that p'' is almost hereditarily below γ, and moreover

$$p''(\eta)(\sigma \restriction \gamma) = p'(\eta)(\sigma \restriction \gamma).$$

It follows that $p'' \Vdash m \in \dot{b}$. In particular $m > |s^{p''(\eta)}_{\sigma\restriction\gamma}|$, thus we can strengthen p'' to a condition q (only strengthening $p''(\eta)$) by extending $s^{p''(\eta)}_{\sigma\restriction\gamma}$ to length $> m$ with $s^{q(\eta)}_{\sigma\restriction\gamma}(m) = 0$. Then $q \Vdash m \in \dot{b} \wedge m \notin a_{\sigma\restriction\gamma}$, which is a contradiction to the fact that p' forces $\dot{b} \setminus n \subseteq a_{\sigma\restriction\gamma}$. □

5.4. **Levels are mad families.** Finally we want to show that the levels of the generic matrix form maximal antichains in $\mathcal{P}(\omega)/\mathrm{fin}$, i.e., mad families.

Lemma 5.20. *In $V[\mathbb{P}_\lambda]$, the family $A_{\xi+1} = \{a_\sigma \mid |\sigma| = \xi + 1\}$ is mad for each $\xi < \lambda$.*

Proof. Let G_λ be generic for $\mathbb{P}_\lambda$ and work in $V[G_\lambda]$. The main work lies in the following claim, which guarantees "local madness" below branches. We will prove it after finishing the proof of the lemma.

Claim 5.21. *Let $\rho \in \lambda^{<\lambda}$, and let $b \subseteq \omega$ be infinite such that $b \cap a_{\rho\restriction\zeta}$ is infinite for every successor $\zeta \leq |\rho|$. Then there exists an $i < \lambda$ such that $b \cap a_{\rho^\frown i}$ is infinite.*

Fix $\xi < \lambda$. By Corollary 4.17, $A_{\xi+1}$ is an almost disjoint family. Using the claim, we will show that $A_{\xi+1}$ is actually mad. Let $b \subseteq \omega$ be infinite. To find $\sigma \in \lambda^{\xi+1}$ such that $b \cap a_\sigma$ is infinite, we construct, by induction on ζ, a branch $\langle \rho_\zeta \mid \zeta \leq \xi + 1 \rangle$ with $|\rho_\zeta| = \zeta$ for each ζ, and $\rho_{\zeta'} \trianglelefteq \rho_\zeta$ for $\zeta' \leq \zeta$, such that $b \cap a_{\rho_\zeta}$ is infinite for every successor $\zeta \leq \xi + 1$.

Let $\rho_0 := \langle\rangle$. Now assume we have constructed $\langle \rho_{\zeta'} \mid \zeta' < \zeta \rangle$. If ζ is a limit, just let $\rho_\zeta := \bigcup\{\rho_{\zeta'} \mid \zeta' < \zeta\}$. If $\zeta = \zeta' + 1$ is a successor, $\rho_{\zeta'}$ fulfills the assumptions of the claim by induction. Let $i < \lambda$ be given by the claim, and let $\rho_\zeta := \rho_{\zeta'}{}^\frown i$. Then $b \cap a_{\rho_\zeta}$ is infinite, as required. Finally, $\sigma := \rho_{\xi+1}$ is as desired. □

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[35]Here we use our modifications discussed in Remark 5.10.

[36]Here we use the modification of Definition 4.18, where 0 is replaced by η, i.e., $p'(\eta)$ is full.

Proof of Claim 5.21. Assume towards contradiction that $b \cap a_{\rho\restriction\zeta}$ is infinite for every successor $\zeta \leq |\rho|$, but $b \cap a_{\rho^\frown i}$ is finite for every $i < \lambda$.

Let η be minimal with $\rho \in V[G_\eta]$ (such an η exists by Lemma 4.9). Thus $a_{\rho^\frown i}$ (for any i) is not defined in $V[G_\eta]$ but it will get defined in the next step of the forcing iteration. From now on, we work in $V[G_\eta]$, and we consider[37] the tail forcing $\mathbb{P}_\lambda / G_\eta$, and apply Lemma 5.17 to get a $\mathbb{P}_\lambda/G_\eta$-name $\dot{b}$ for b and $\gamma' < \lambda$ such that $\dot{b}$ is hereditarily below γ'. Let $\gamma < \lambda$ be any ordinal strictly above $|\rho| + 1$, $\sup(\mathrm{rng}(\rho))$, and γ'.

Note that we can pick $n \in \omega$ and $p \in \mathbb{P}_\lambda / G_\eta$ such that

(1) $p \Vdash \dot{b} \cap a_{\rho^\frown\gamma} \subseteq n$,
(2) $p \Vdash \dot{b} \cap a_{\rho^\frown i}$ is finite, for each $i < \gamma$, and
(3) $p \Vdash \dot{b} \cap a_{\rho\restriction\zeta}$ is infinite, for each successor $\zeta \leq |\rho|$.

From now on, whenever we say "almost hereditarily below γ", we shall mean "almost hereditarily below γ except for $\rho^\frown\gamma$". Note that (the canonical name for) $a_{\rho^\frown\gamma}$ is almost hereditarily below γ; also $\dot{b}$ is almost hereditarily below γ (because $\dot{b}$ is hereditarily below γ), and similarly $a_{\rho^\frown i}$ is almost hereditarily below γ for each $i < \gamma$, and $a_{\rho\restriction\zeta}$ is almost hereditarily below γ for each successor $\zeta \leq |\rho|$.

By Lemma 5.12 and Lemma 5.2, we can fix p' which is almost hereditarily below γ such that items (1), (2), and (3) above hold true for p' in place of p. Without loss of generality, we can assume that $\rho^\frown\gamma \in \mathrm{dom}(p'(\eta))$, as well as that p' is a full[38] condition.

Define $R := \mathrm{dom}(p'(\eta)) \cap \{\rho^\frown i \mid i < \gamma\}$, and $R' := \mathrm{dom}(f^{p'(\eta)}_{\rho^\frown\gamma})$. Let $\dot{x}$ be a $\mathbb{P}_\lambda/G_\eta$-name such that

$$\Vdash \dot{x} = \bigcap_{\tau \in R'} (\dot{b} \cap a_\tau) \setminus \bigcup_{\tau \in R} a_\tau;$$

since the conditions which are hereditarily below γ form a complete subforcing of $\mathbb{P}_\lambda / G_\eta$ by Lemma 5.12, and all names which are used to define $\dot{x}$ are hereditarily below γ, we can assume that $\dot{x}$ has been chosen to be hereditarily below γ as well. Note that since R and R' are finite, p' forces $\dot{x}$ to be infinite.

By Lemma 5.7 there is $r \in \mathbb{P}_\lambda / G_\eta$ hereditarily below γ and $m > n, |s^{p'(\eta)}_{\rho^\frown\gamma}|$ such that r is compatible with p', and $r \Vdash m \in \dot{x}$. Apply Lemma 5.18 to obtain $p'' \leq p', r$ such that p'' is almost hereditarily below γ, and moreover

$$p''(\eta)(\rho^\frown\gamma) = p'(\eta)(\rho^\frown\gamma).$$

It follows that $p'' \Vdash m \in \dot{x}$, as well as $p'' \Vdash m \in a_\tau$ for $\tau \in R'$ and $p'' \Vdash m \notin a_\tau$ for $\tau \in R$.

Now extend p'' to a condition q as follows. Let $q(\alpha) = p''(\alpha)$ for $\alpha > \eta$. For $\tau \in (R \cup R') \cap \mathrm{dom}(p''(\eta))$ extend $s^{q(\eta)}_\tau$ such that $|s^{q(\eta)}_\tau| > m$. It follows (for $\tau \in R'$) that $s^{q(\eta)}_\tau(m) = 1$ for $\tau \in R' \cap \mathrm{dom}(p''(\eta))$, and $a_\tau(m) = 1$ for $\tau \in R' \setminus \mathrm{dom}(p''(\eta))$ because $p'' \Vdash m \in a_\tau$ for $\tau \in R'$; moreover, $s^{q(\eta)}_\tau(m) = 0$ for $\tau \in R$ because $p'' \Vdash m \notin a_\tau$ for $\tau \in R$. Additionally fill $s^{q(\eta)}_{\rho^\frown\gamma}$ with 0 for entries smaller than m and with 1 at m. That is possible, because the $s^{q(\eta)}_\tau(m)$ are accordingly for $\tau \in R$ and $\tau \in R' \cap \mathrm{dom}(p''(\eta))$ respectively and $a_\tau(m) = 1$ for $\tau \in R' \setminus \mathrm{dom}(p''(\eta))$.

It follows that $q \Vdash m \in \dot{x} \cap a_{\rho^\frown\gamma}$, which is a contradiction to the fact that p' forces $\dot{x} \cap a_{\rho^\frown\gamma} \subseteq n$. $\square$

[37] Here, again, we use our modifications discussed in Remark 5.10.

[38] Here, again, we use the modification of Definition 4.18, where 0 is replaced by η, i.e., $p'(\eta)$ is full.

This finishes the proof that the generic matrix is a distributivity matrix of height λ for $\mathcal{P}(\omega)/\mathrm{fin}$. To finish the proof of Main Theorem 4.1, it remains to prove that $\mathfrak{b}$ (and hence $\mathfrak{h}$) is small in our final model; this is the subject of Sections 6 and 7.

6. $\mathcal{B}$-Canjar filters

In this section, we will give the neccessary preliminaries about $\mathcal{B}$-Canjar filters and the preservation of unboundedness, which are needed in Section 7.

Definition 6.1. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter containing the Frechét filter. *Mathias forcing with respect to* $\mathcal{F}$ (denoted by $\mathbb{M}(\mathcal{F})$) is the set of pairs (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$, where the order is defined as follows: $(t, B) \leq (s, A)$ if

(1) $t \trianglerighteq s$, i.e., t extends s
(2) $B \subseteq A$
(3) for each $n \geq |s|$, if $t(n) = 1$, then $n \in A$.

Note that $\mathbb{M}(\mathcal{F})$ is σ-centered: for $s \in 2^{<\omega}$, the set $\{(s, A) \mid A \in \mathcal{F}\}$ is clearly centered (i.e., finitely many conditions have a common lower bound). Also note that Mathias forcing with respect to the Frechét filter is forcing equivalent to Cohen forcing $\mathbb{C}$. Recall also that $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$, and that $\mathcal{B} \subseteq \omega^\omega$ is unbounded, if there exists no $g \in \omega^\omega$ with $f \leq^* g$ for all $f \in \mathcal{B}$.

A filter $\mathcal{F}$ is Canjar if $\mathbb{M}(\mathcal{F})$ does not add a dominating real over the ground model (i.e., the ground model reals remain unbounded). We need the following generalization of Canjarness:

Definition 6.2. Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. A filter $\mathcal{F}$ on ω is *$\mathcal{B}$-Canjar* if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$ (i.e., $\mathcal{B}$ is still unbounded in the extension by $\mathbb{M}(\mathcal{F})$).

6.1. A combinatorial characterization of $\mathcal{B}$-Canjarness.

Later, we will prove that certain filters are $\mathcal{B}$-Canjar; for that, we use the following combinatorial characterization of $\mathcal{B}$-Canjarness by Guzmán-Hrušák-Martínez [24]. This characterization generalizes a characterization of Canjarness by Hrušák-Minami [26].

Let $\mathcal{F}$ be a filter on ω; recall that a set $X \subseteq [\omega]^{<\omega}$ is in $(\mathcal{F}^{<\omega})^+$ if and only if for each $A \in \mathcal{F}$ there is an $s \in X$ with $s \subseteq A$. Note that if $\mathcal{G} \subseteq \mathcal{F}$ are filters and $X \in (\mathcal{F}^{<\omega})^+$, then $X \in (\mathcal{G}^{<\omega})^+$.

Given $\bar{X} = \langle X_n \mid n \in \omega \rangle$ (with $X_n \subseteq [\omega]^{<\omega}$ for each $n \in \omega$), and $f \in \omega^\omega$, let

$$\bar{X}_f = \bigcup_{n \in \omega} (X_n \cap \mathcal{P}(f(n))).$$

Theorem 6.3. *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. A filter $\mathcal{F}$ on ω is $\mathcal{B}$-Canjar if and only if the following holds: for each sequence $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$, there exists an $f \in \mathcal{B}$ such that $\bar{X}_f \in (\mathcal{F}^{<\omega})^+$.*

Proof. See [24, Proposition 1]. □

It is well-known that Cohen forcing $\mathbb{C}$ preserves[39] the unboundedness of every unbounded family. As mentioned above, Mathias forcing with respect to the Frechét filter is forcing equivalent to $\mathbb{C}$, and hence the Frechét filter is $\mathcal{B}$-Canjar for every unbounded family $\mathcal{B}$. To illustrate the characterization of $\mathcal{B}$-Canjarness from Theorem 6.3, we also want to provide the following easy combinatorial proof of this fact:

[39] In fact, $\mathbb{C}$ is almost bounding.

Lemma 6.4. *Let $\mathcal{B}$ be an unbounded family. Then the Frechét filter is $\mathcal{B}$-Canjar.*

Proof. Let $\mathcal{F}$ be the Frechét filter. To show that $\mathcal{F}$ is $\mathcal{B}$-Canjar, we use Theorem 6.3. So let $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$. Note that a set $X \subseteq [\omega]^{<\omega}$ is in $(\mathcal{F}^{<\omega})^+$ if and only if for each $n \in \omega$ there is an $s \in X$ with $\min(s) \geq n$. For each $n \in \omega$, pick $s_n \in X_n$ such that $\min(s_n) \geq n$, and let $g \in \omega^\omega$ such that $g(n) > \max(s_n)$ for each $n \in \omega$. Since $\mathcal{B}$ is unbounded, we can pick $f \in \mathcal{B}$ such that $f(n) > g(n)$ for infinitely many n. It is easy to check that $s_n \in \bar{X}_f$ for infinitely many n, and this implies that $\bar{X}_f \in (\mathcal{F}^{<\omega})^+$, as desired. □

The following observation will be crucial later on:

Lemma 6.5. *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family, $\mathcal{F}$ a $\mathcal{B}$-Canjar filter extending the Frechét filter and $\{a_n \mid n < \omega\}$ such that $\mathcal{F} \cup \{a_n \mid n < \omega\}$ is a filter base. Then the filter generated by $\mathcal{F} \cup \{a_n \mid n < \omega\}$ is $\mathcal{B}$-Canjar.*

Proof. Let $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\langle \mathcal{F} \cup \{a_n \mid n < \omega\}\rangle^{<\omega})^+$. Let

$$Y_n := \{s \in X_n \mid s \subseteq \cap_{k<n} a_k\}$$

and $\bar{Y} := \langle Y_n \mid n \in \omega \rangle$. It is easy to see that $Y_n \in (\mathcal{F}^{<\omega})^+$ for each n. By the assumption and Theorem 6.3 there exists $f \in \mathcal{B}$ such that $\bar{Y}_f \in (\mathcal{F}^{<\omega})^+$.

To show that $\bar{Y}_f \in (\langle \mathcal{F} \cup \{a_n \mid n < \omega\}\rangle^{<\omega})^+$ let $B \in \langle \mathcal{F} \cup \{a_n \mid n < \omega\}\rangle$, i.e., there exists $A \in \mathcal{F}$ and $n \in \omega$ with $B \supseteq A \cap \bigcap_{k<n} a_k$. Since $\mathcal{F}$ contains the Frechét filter and $\bar{Y}_f \in (\mathcal{F}^{<\omega})^+$, there exist infinitely many $s \in \bar{Y}_f$ with $s \subseteq A$. So there exists $m \geq n$ and $s \in Y_m \cap \bar{Y}_f$ with $s \subseteq A$; note that $s \in Y_m$ implies $s \subseteq \bigcap_{k<n} a_k$, so $s \subseteq B$, as desired.

Clearly $\bar{Y}_f \subseteq \bar{X}_f$, so $\bar{X}_f \in (\langle \mathcal{F} \cup \{a_n \mid n < \omega\}\rangle^{<\omega})^+$. □

We also get the following:

Lemma 6.6. *Let $\mathcal{B}$ be an unbounded family. Then every countably generated filter is $\mathcal{B}$-Canjar.*

Proof. This follows immediately from Lemma 6.4 and Lemma 6.5. □

6.2. **Preservation of unboundedness at limits.** We will also use the following theorem by[40]Judah-Shelah [28] about preservation of unboundedness in finite support iterations:

Theorem 6.7. *Suppose $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\}$ is a finite support iteration of c.c.c. partial orders of limit length δ, and $\mathcal{B} \subseteq \omega^\omega$ is unbounded and satisfies*

$$\forall \mathcal{A} \subseteq \mathcal{B} \ (|\mathcal{A}| = \aleph_0 \rightarrow \exists f \in \mathcal{B} \ \forall g \in \mathcal{A} \ g \leq^* f); \tag{6}$$

moreover, suppose that

$$\forall \alpha < \delta \ \Vdash_{\mathbb{P}_\alpha} \text{“}\mathcal{B} \textit{ is an unbounded family”}.$$

Then $\Vdash_{\mathbb{P}_\delta}$ “$\mathcal{B}$ is an unbounded family”.

Proof. See [19, Theorem 3.5.2]. □

[40]In fact, [28, Theorem 2.2] is a much more general version than the one presented here.

6.3. **Preservation of $\mathcal{B}$-Canjarness and finite sums of filters.** The notion of $\mathcal{B}$-Canjarness of a filter seems to be not[41]absolute in general. We will now provide a method how to guarantee that the $\mathcal{B}$-Canjarness of a filter is not destroyed by Mathias forcings with respect to certain other filters. As a tool, we introduce finite sums of filters and consider Mathias forcings with respect to these sums.

Lemma 6.8. *Let[42]$\mathcal{F}$ be a filter, $\mathcal{B} \subseteq \omega^\omega$, and $\mathbb{P}$ be a forcing notion. Then the following are equivalent:*

(1) $\mathbb{P}$ forces that[43]$\mathcal{F}$ is $\mathcal{B}$-Canjar.

(2) $\mathbb{M}(\mathcal{F}) \times \mathbb{P}$ forces that $\mathcal{B}$ is unbounded.

Proof. Let $\mathbb{Q} := \mathbb{M}(\mathcal{F})$. Note that (1) holds if and only if $\mathbb{P}$ forces

$$\mathbb{M}(\langle\check{\mathcal{F}}\rangle) \Vdash \text{“}\mathcal{B} \text{ unbounded”}.$$

Further note that $\mathbb{P}$ forces that $\check{\mathbb{Q}}$ is (dense in, and hence) forcing equivalent to $\mathbb{M}(\langle\check{\mathcal{F}}\rangle)$. So, (1) holds if and only if $\mathbb{P} * \check{\mathbb{Q}}$ forces that $\mathcal{B}$ is unbounded, which is the same as (2) (since $\mathbb{P} * \check{\mathbb{Q}}$ is equivalent to $\mathbb{P} \times \mathbb{Q} = \mathbb{Q} \times \mathbb{P}$). □

Definition 6.9. For two sets $A, B \subseteq \omega$, let $A \oplus B := \{2n \mid n \in A\} \cup \{2m+1 \mid m \in B\}$. For two filters $\mathcal{F}_0$ and $\mathcal{F}_1$, let $\mathcal{F}_0 \oplus \mathcal{F}_1 := \{A \oplus B \mid A \in \mathcal{F}_0, B \in \mathcal{F}_1\}$. More generally, inductively define $\bigoplus_{k<m+1} \mathcal{F}_k := \left(\bigoplus_{k<m} \mathcal{F}_k\right) \oplus \mathcal{F}_m$.

Note that $\mathcal{F}_0 \oplus \mathcal{F}_1$ is a filter if $\mathcal{F}_0$ and $\mathcal{F}_1$ are filters, and hence also the finite sum of filters is a filter. The order of the sum is not important: more presicely, the filter $\bigoplus_{k<m} \mathcal{F}_k$ is isomorphic (based on a bijection on ω) to all reorderings of this sum. For example $(\mathcal{F}_0 \oplus \mathcal{F}_1) \oplus \mathcal{F}_2$ is isomorphic to $(\mathcal{F}_2 \oplus \mathcal{F}_0) \oplus \mathcal{F}_1$. This implies that the $\mathcal{B}$-Canjarness of a finite sum of filters does not depend on the order of the sum.

Lemma 6.10. *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two filters. Then $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$.*

Proof. Let $D_\times \subseteq \mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ be the set of all $((s_0, A_0), (s_1, A_1)) \in \mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ with $|s_0| = |s_1|$, and let $D_\oplus \subseteq \mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ be the set of all $(s, A) \in \mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ with $|s|$ being an even number. Note that $D_\times$ is a dense subforcing of $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$, and $D_\oplus$ is a dense subforcing of $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$.

For $s_0, s_1 \in 2^{<\omega}$ with $L := |s_0| = |s_1|$, let $s_0 \oplus s_1 \in 2^{<\omega}$ be such that $|s_0 \oplus s_1| = 2L$ and satisfies $(s_0 \oplus s_1)(2n) = s_0(n)$ and $(s_0 \oplus s_1)(2n+1) = s_1(n)$.

Define $\iota \colon D_\times \to D_\oplus$ as follows:

$$((s_0, A_0), (s_1, A_1)) \mapsto (s_0 \oplus s_1, A_0 \oplus A_1).$$

It is easy to see that ι is an isomorphism between the forcings $D_\times$ and $D_\oplus$. Consequently, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ and $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ are forcing equivalent. □

[41]As an example, let $\mathcal{B}$ be the ground model reals and $\mathcal{U}$ be a $\mathcal{B}$-Canjar ultrafilter. Let $\mathbb{P}$ be Grigorieff forcing with respect to $\mathcal{U}$, which forces that $\mathcal{U}$ cannot be extended to a P-point. It is well-known that $\mathbb{P}$ preserves the unboundedness of $\mathcal{B}$, and it can be shown that $\mathcal{U}$ is not a P^+-filter in $V[\mathbb{P}]$; since any Canjar filter is a P^+-filter, it follows that $\mathcal{U}$ is no longer $\mathcal{B}$-Canjar. Note that Grigorieff forcing is proper, but not c.c.c.; however, Grigorieff forcing can be decomposed into a σ-closed and a c.c.c. forcing (see [32]). Since a σ-closed forcing does not destroy the $\mathcal{B}$-Canjarness of a filter, the above example also yields an example of a c.c.c. forcing destroying the $\mathcal{B}$-Canjarness of a filter.

[42]We do not have to assume that $\mathcal{B}$ is unbounded or that $\mathcal{F}$ is $\mathcal{B}$-Canjar in the ground model. If these assumptions fail, the equivalence holds trivially (because both (1) and (2) are false).

[43]To be more precise, one should write $\langle\check{\mathcal{F}}\rangle$ instead of $\mathcal{F}$.

The following lemma will be the main ingredient of the "successor step" of the induction (for old filters) in Lemma 7.3:

Lemma 6.11. *If $\mathcal{F}_0 \oplus \mathcal{F}_1$ is $\mathcal{B}$-Canjar, then $\mathbb{M}(\mathcal{F}_1)$ forces that $\mathcal{F}_0$ is $\mathcal{B}$-Canjar.*

Proof. By assumption and Lemma 6.10, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ forces that $\mathcal{B}$ is unbounded; apply Lemma 6.8 to finish the proof. □

The following lemma will be the main ingredient of the "limit step" of the induction (for old filters) in Lemma 7.3:

Lemma 6.12. *Let $\mathcal{B}$ be a family satisfying property* (6) *of Theorem 6.7, let α be a limit, and let $\langle \mathbb{P}_\beta, \mathbb{Q}_\beta \mid \beta < \alpha \rangle$ be a finite support iteration. Suppose that $\mathbb{P}_\beta$ forces that $\mathcal{F}$ is $\mathcal{B}$-Canjar for every $\beta < \alpha$. Then $\mathbb{P}_\alpha$ forces that $\mathcal{F}$ is $\mathcal{B}$-Canjar.*

Proof. By assumption and Lemma 6.8, $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta$ forces that $\mathcal{B}$ is unbounded for every $\beta < \alpha$. Observe that $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$ is the direct limit of the sequence $\langle \mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta \mid \beta < \alpha \rangle$ (and $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta$ is complete in $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$), so it can be written as the limit of a finite support iteration, therefore, by Theorem 6.7, also $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$ forces that $\mathcal{B}$ is unbounded. We obtain the conclusion by again applying Lemma 6.8. □

Lemma 6.13. *Let $\mathcal{F}_0$ be $\mathcal{B}$-Canjar and $\mathcal{F}_1$ be countably generated. Then $\mathcal{F}_0 \oplus \mathcal{F}_1$ is $\mathcal{B}$-Canjar.*

Using the fact that sums can be reordered (see the remark after Definition 6.9), we obtain the following stronger statement: Let $\mathcal{F}_0, \dots, \mathcal{F}_{m-1}$ be filters such that (some of them are countably generated and) the sum of the filters which are not countably generated is $\mathcal{B}$-Canjar; then $\bigoplus_{k<m} \mathcal{F}_k$ is $\mathcal{B}$-Canjar.

Proof of Lemma 6.13. We have to show that $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ forces that $\mathcal{B}$ is unbounded. By Lemma 6.10, $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$.

Since $\mathcal{F}_0$ is $\mathcal{B}$-Canjar by assumption, $\mathcal{B}$ is unbounded in the extension by $\mathbb{M}(\mathcal{F}_0)$. Since $\mathcal{F}_1$ is countably generated, the same holds in the extension by $\mathbb{M}(\mathcal{F}_0)$: more precisely, the filter generated by $\mathcal{F}_1$ is countably generated. Therefore, in the extension by $\mathbb{M}(\mathcal{F}_0)$, (the filter generated by) $\mathcal{F}_1$ is $\mathcal{B}$-Canjar by Lemma 6.6. So, by Lemma 6.8, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ forces that $\mathcal{B}$ is unbounded, as desired. □

7. Preserving unboundedness: $\mathfrak{h} = \mathfrak{b} = \omega_1$

In this section, we want to show that $\mathfrak{b}$ is small (i.e., $\mathfrak{b} = \omega_1$) in our final model W of Main Theorem 4.1. Recall the following well-known ZFC inequalities (see (2) in Section 3.1):

$$\omega_1 \leq \mathfrak{h} \leq \mathfrak{b}$$

So, if we have $\mathfrak{b} = \omega_1$, it follows[44] that $\mathfrak{h} = \omega_1$, and so there exists a distributivity matrix of height ω_1. Since there is also a distributivity matrix of height λ in W, this gives us a model of

$$\{\omega_1, \lambda\} \subseteq \mathsf{COM}.$$

In Section 7.1, we will show that our iteration $\mathbb{P}_\lambda$ can be represented as a finer iteration whose iterands are Mathias forcings with respect to filters. In Section 7.2, we show that the filters which are used are $\mathcal{B}$-Canjar (i.e., the corresponding Mathias forcings preserve the unboundedness of $\mathcal{B}$), where $\mathcal{B}$ is the set of reals of V_0. A similar (but less involved) argument shows that Hechler's original forcings [25] to add a tower or to add a mad family can be represented as an iteration of Mathias forcings with respect to $\mathcal{B}$-Canjar filters as well (see [17]).

7.1. **Finer iteration via filtered Mathias forcings.** As described in Section 4.1, $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\}$ is our main finite support iteration which we force with over V. Its limit $\mathbb{P}_\lambda$ adds a distributivity matrix of height λ. We will now represent our iteration as a "finer" iteration: we write each iterand $\mathbb{Q}_\alpha$ as a finite support iteration of Mathias forcings with respect to certain filters. Fix $\alpha < \lambda$.

As a preparation, we introduce a "nice" enumeration of T_α (recall that $\sigma \in T_\alpha$ if and only if a_σ is added by $\mathbb{Q}_\alpha$). We go through the nodes in T_α level by level, and "blockwise". A *block* is a set of nodes $\{\rho^\frown i \mid i < \lambda\}$ for some $\rho \in \lambda^{<\lambda}$. More precisely, let $\{\sigma_\alpha^\nu \mid \nu < \Lambda_\alpha\}$ be an enumeration of T_α (note that $|T_\alpha| = \mathfrak{c}$ and hence Λ_α is an ordinal with $\mathfrak{c} < \Lambda_\alpha < \mathfrak{c}^+$) such that

(1) ("level by level") $|\sigma_\alpha^{\bar\nu}| < |\sigma_\alpha^\nu| \to \bar\nu < \nu$,
(2) ("blockwise") for each $\rho \in \lambda^{<\lambda}$ with $\{\rho^\frown i \mid i < \lambda\} \subseteq T_\alpha$, there is $\nu < \Lambda_\alpha$ such that

$$\rho^\frown i = \sigma_\alpha^{\nu+i} \text{ for each } i < \lambda.$$

Recall that $\mathbb{Q}_\alpha^C$ denotes $\{p \in \mathbb{Q}_\alpha \mid \mathrm{dom}(p) \subseteq C\}$ (for $C \subseteq \lambda^{<\lambda}$). For any $\beta \leq \Lambda_\alpha$, let

$$\mathbb{Q}_\alpha^{<\beta} := \mathbb{Q}_\alpha^{\{\sigma_\alpha^\nu \mid \nu < \beta\}},$$

and for $\beta < \Lambda_\alpha$,

$$\mathbb{Q}_\alpha^{\leq\beta} := \mathbb{Q}_\alpha^{\{\sigma_\alpha^\nu \mid \nu \leq \beta\}}.$$

Note that $\mathbb{Q}_\alpha^{<\Lambda_\alpha} = \mathbb{Q}_\alpha$, and that $\{\sigma_\alpha^\nu \mid \nu < \beta\}$ is left-up-closed for each $\beta \leq \Lambda_\alpha$ (due to (1) and (2) above). Therefore, $\mathbb{Q}_\alpha^{<\beta}$ is a complete subforcing of $\mathbb{Q}_\alpha$: this is an easy instance of Lemma 4.22, letting $C = E = \{\sigma_\alpha^\nu \mid \nu < \beta\}$. By Lemma 5.1, $\mathbb{Q}_\alpha^{<\beta}$ is a complete subforcing of $\mathbb{Q}_\alpha^{\leq\beta}$, so we can form the quotient $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$. Moreover, because conditions in $\mathbb{Q}_\alpha$ have finite domain,

$$\mathbb{Q}_\alpha^{<\beta} = \bigcup_{\delta<\beta} \mathbb{Q}_\alpha^{<\delta}$$

for each limit ordinal $\beta \leq \Lambda_\alpha$; in other words, $\mathbb{Q}_\alpha^{<\beta}$ is the direct limit of the forcings $\mathbb{Q}_\alpha^{<\delta}$ for $\delta < \beta$. So $\mathbb{Q}_\alpha$ is forcing equivalent to the finite support iteration of the quotients $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$ for $\beta < \Lambda_\alpha$.

Recall that $\mathbb{M}(\mathcal{F})$ denotes Mathias forcing with respect to the filter $\mathcal{F}$ (see Definition 6.1). We are now going to show that $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha^\beta)$ for a filter $\mathcal{F}_\alpha^\beta$. Work in an extension by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}$, and note that, for each $\tau \in T_\eta$ with $\eta < \alpha$, a set a_τ has been added by $\mathbb{P}_\alpha$, and for each $\nu < \beta$, a set $a_{\sigma_\alpha^\nu}$ has been added by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}$. These sets[45] are used to define $\mathcal{F}_\alpha^\beta$ as follows. Let $\rho \in \lambda^{<\lambda}$ and $i < \lambda$ be such that $\sigma_\alpha^\beta = \rho^\frown i$, and let

$$\mathfrak{F}_\alpha^\beta := \{a_{\rho\restriction(\xi+1)} \mid \xi + 1 \leq |\rho|\} \cup \{\omega \setminus a_{\rho^\frown j} \mid j < i\},$$

i.e., $\mathfrak{F}_\alpha^\beta$ is the collection of all sets assigned to the nodes above σ_α^β and the complements of the sets assigned to the nodes to the left of σ_α^β within the same block. Note that $\mathfrak{F}_\alpha^\beta$ is a filter base, i.e., any

[44] In this paper, we show that $\mathfrak{h}$ is small by showing that its upper bound $\mathfrak{b}$ is small. Sometimes, it is shown directly that $\mathfrak{h}$ is small (by constructing a distributivity matrix of small height); see, e.g., [38].

intersection of finitely many elements is infinite: indeed, for finite $I \subseteq i$ and $\xi + 1 \leq |\rho|$, let $j^* \in \lambda \setminus I$; then $a_{\rho^\frown j^*} \subseteq^* a_{\rho \restriction (\xi+1)} \cap \bigcap_{j \in I}(\omega \setminus a_{\rho^\frown j})$. Then let

$$\mathcal{F}^\beta_\alpha := \langle \mathfrak{F}^\beta_\alpha \rangle_{\text{Frechét}},$$

i.e., $\mathcal{F}^\beta_\alpha$ is the filter generated by taking finite intersections of sets from $\mathfrak{F}^\beta_\alpha$ and the Frechét filter and taking the upwards closure.

The quotient $\mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha$ adds the set a_σ where $\sigma = \sigma^\beta_\alpha$. The following lemma will provide a dense embedding from $\mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha$ to $\mathbb{M}(\mathcal{F}^\beta_\alpha)$ which preserves (the finite appoximations of) the generic real a_σ. Therefore, a_σ is also the generic real for $\mathbb{M}(\mathcal{F}^\beta_\alpha)$. Recall that the generic real for $\mathbb{M}(\mathcal{F})$ is a pseudo-intersection of $\mathcal{F}$, and the definition of $\mathcal{F}^\beta_\alpha$ ensures that a pseudo-intersection of it is almost contained in $a_{\rho \restriction (\xi+1)}$ whenever $\xi + 1 \leq |\rho|$ and almost disjoint from $a_{\rho^\frown j}$ for each $j < i$, as it is the case for the real a_σ.

Lemma 7.1. $\mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha$ *is densely embeddable into* $\mathbb{M}(\mathcal{F}^\beta_\alpha)$.

Proof. For simplicity of notation, let $\sigma := \sigma^\beta_\alpha$ for the rest of this proof. Let G be a generic filter for $\mathbb{Q}^{<\beta}_\alpha$. We work in the extension by G, so

$$\mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha = \{p \in \mathbb{Q}^{\leq\beta}_\alpha \mid \forall q \in G (p \text{ is compatible with } q)\}.$$

Let us define an embedding $\iota\colon \mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha \to \mathbb{M}(\mathcal{F}^\beta_\alpha)$ as follows: for $p \in \mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha$, let $p(\sigma) = (s_\sigma, f_\sigma, h_\sigma)$, and let $\iota(p) = (s_\sigma, A)$, where

$$A = \bigcap_{\tau \in \operatorname{dom}(f_\sigma)} (a_\tau \cup f_\sigma(\tau)) \cap \bigcap_{\rho \in \operatorname{dom}(h_\sigma)} ((\omega \setminus a_\rho) \cup h_\sigma(\rho)) \setminus |s_\sigma|.$$

To see that it is a dense embedding, we have to check the following conditions:

(1) (Density) For every condition $(s, A) \in \mathbb{M}(\mathcal{F}^\beta_\alpha)$, there exists a condition p such that $\iota(p) \leq (s, A)$.
(2) (Incompatibility preserving) If p and p' are incompatible, then so are $\iota(p)$ and $\iota(p')$.
(3) (Order preserving) If $p' \leq p$, then $\iota(p') \leq \iota(p)$.

To show (1), let $(s, A) \in \mathbb{M}(\mathcal{F}^\beta_\alpha)$. Since $A \in \mathcal{F}^\beta_\alpha$, there exist finite sets $\{\rho_i \mid i < m\}$, $\{\tau_j \mid j < l\}$ and $N \in \omega$ such that $\bigcap_{j<l} a_{\tau_j} \cap \bigcap_{i<m}(\omega \setminus a_{\rho_i}) \setminus N \subseteq A$. Extend s with 0's to s_σ such that $|s_\sigma| = \max(|s|, N)$, and let $\operatorname{dom}(h_\sigma) := \{\rho_i \mid i < m\}$ and $h_\sigma(\rho_i) := |s_\sigma|$ for every i, and $\operatorname{dom}(f_\sigma) := \{\tau_j \mid j < l\}$ and $f_\sigma(\tau_j) := |s_\sigma|$ for every j. Let $p := \{(\sigma, (s_\sigma, f_\sigma, h_\sigma))\} \cup \{(\tau, (\langle\rangle, \emptyset, \emptyset)) \mid \tau \in (\operatorname{dom}(f_\sigma) \cap T_\alpha) \cup \operatorname{dom}(h_\sigma)\}$.

To see that p is in the quotient, let $q \in G$ be arbitrary; it is easy to check that $q \cup \{(\tau, (s_\tau, f_\tau, h_\tau)) \mid \tau \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)\} \leq p, q$.

By definition, $\iota(p) = (s_\sigma, A')$, where

$$A' = \bigcap_{\tau \in \operatorname{dom}(f_\sigma)} (a_\tau \cup f_\sigma(\tau)) \cap \bigcap_{\rho \in \operatorname{dom}(h_\sigma)} ((\omega \setminus a_\rho) \cup h_\sigma(\rho)) \setminus |s_\sigma|.$$

It follows that

$$A' \overset{(*)}{=} \bigcap_{\tau \in \operatorname{dom}(f_\sigma)} a_\tau \cap \bigcap_{\rho \in \operatorname{dom}(h_\sigma)} (\omega \setminus a_\rho) \setminus |s_\sigma| \subseteq \bigcap_{j<l} a_{\tau_j} \cap \bigcap_{i<m} (\omega \setminus a_{\rho_i}) \setminus N \subseteq A$$

[45]It is possible (see the base step $\beta^* = 0$ of the proof of Lemma 7.3(3)) that only sets a_τ with $\tau \in T_\eta$ for some $\eta < \alpha$ are used. This is the case if ρ is pre-T_α-minimal and $i = 0$.

(where (*) holds because $|s_\sigma| \geq f_\sigma(\tau), h_\sigma(\rho)$ for every τ, ρ in the respective domains). Therefore $s_\sigma \trianglerighteq s$, $A' \subseteq A$, and $s_\sigma(n) = 0$ for all $n \geq |s|$. So $\iota(p) = (s_\sigma, A') \leq (s, A)$.

We prove (2) by showing the contrapositive. Assume $\iota(p)$ and $\iota(p')$ are compatible. Define q as follows. Let $\mathrm{dom}(q) := \mathrm{dom}(p) \cup \mathrm{dom}(p')$. For every $\tau \in \mathrm{dom}(q)$, let $s^q_\tau := s^p_\tau \cup s^{p'}_\tau$, $\mathrm{dom}(f^q_\tau) := \mathrm{dom}(f^p_\tau) \cup \mathrm{dom}(f^{p'}_\tau)$ and for $\rho \in \mathrm{dom}(f^q_\tau)$ let $f^q_\tau(\rho) = \min(f^p_\tau(\rho), f^{p'}_\tau(\rho))$, and the same for h: $\mathrm{dom}(h^q_\tau) := \mathrm{dom}(h^p_\tau) \cup \mathrm{dom}(h^{p'}_\tau)$ and for $\rho \in \mathrm{dom}(h^q_\tau)$ let $h^q_\tau(\rho) = \min(h^p_\tau(\rho), h^{p'}_\tau(\rho))$. It is easy to check that q is a condition in the quotient and $q \leq p, p'$.

To show (3), let $p' \leq p$. So, by definition, $s^{p'}_\sigma \trianglerighteq s^p_\sigma$, and $\mathrm{dom}(h^{p'}_\sigma) \supseteq \mathrm{dom}(h^p_\sigma)$ and $\mathrm{dom}(f^{p'}_\sigma) \supseteq \mathrm{dom}(f^p_\sigma)$, and $f^{p'}_\sigma(\tau) \leq f^p_\sigma(\tau)$ for $\tau \in \mathrm{dom}(f^p_\sigma)$ and $h^{p'}_\sigma(\rho) \leq h^p_\sigma(\rho)$ for $\rho \in \mathrm{dom}(h^p_\sigma)$; so

$$
\begin{aligned}
A' &:= \bigcap_{\tau \in \mathrm{dom}(f^{p'}_\sigma)} (a_\tau \cup f^{p'}_\sigma(\tau)) \cap \bigcap_{\rho \in \mathrm{dom}(h^{p'}_\sigma)} ((\omega \setminus a_\rho) \cup h^{p'}_\sigma(\rho)) \setminus |s^{p'}_\sigma| \\
&\subseteq \bigcap_{\tau \in \mathrm{dom}(f^{p}_\sigma)} (a_\tau \cup f^{p}_\sigma(\tau)) \cap \bigcap_{\rho \in \mathrm{dom}(h^{p}_\sigma)} ((\omega \setminus a_\rho) \cup h^{p}_\sigma(\rho)) \setminus |s^{p}_\sigma| =: A.
\end{aligned}
$$

By definition, $\iota(p) = (s^p_\sigma, A)$ and $\iota(p') = (s^{p'}_\sigma, A')$. To show that $(s^{p'}_\sigma, A') \leq (s^p_\sigma, A)$, it remains to show that for $n \geq |s^p_\sigma|$ with $s^{p'}_\sigma(n) = 1$, we have $n \in A$. First fix $\rho \in \mathrm{dom}(h^p_\sigma)$ and show that $n \in (\omega \setminus a_\rho) \cup h^p_\sigma(\rho)$. If $n < h^p_\sigma(\rho)$, this is clear. If $n \geq h^p_\sigma(\rho)$, we know that $s^{p'}_\sigma$ respects h^p_σ, and so $n \in \omega \setminus a_\rho$. So in both cases, $n \in (\omega \setminus a_\rho) \cup h^p_\sigma(\rho)$. Now fix $\tau \in \mathrm{dom}(f^p_\sigma)$ and show that $n \in a_\tau \cup f^p_\sigma(\tau)$. This is the same argument as for h. If $n < f^p_\sigma(\tau)$, this is clear. If $n \geq f^p_\sigma(\tau)$, we know that $s^{p'}_\sigma$ respects f^p_σ, and so $n \in a_\tau$. So in both cases, $n \in a_\tau \cup f^p_\sigma(\tau)$, finishing the proof. □

The following fact will be needed in the proof of Claim 7.4:

Corollary 7.2. $\mathbb{P}_\alpha$ *is σ-centered for each $\alpha \leq \lambda$.*

More generally, the same holds for $\mathbb{P}_\alpha \,/\, \mathbb{P}_\eta$ for $\eta < \alpha$.

Proof of Corollary 7.2. Since Mathias forcing with respect to a filter is always σ-centered (see the remark after Definition 6.1) and $\mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha$ is densely embeddable into such a forcing by the above lemma, also $\mathbb{Q}^{\leq\beta}_\alpha / \mathbb{Q}^{<\beta}_\alpha$ is σ-centered.

Recall that $\Lambda_\eta < \mathfrak{c}^+$ for every $\eta < \alpha$, and $\alpha \leq \lambda \leq \mathfrak{c}$, so $\mathbb{P}_\alpha$ is a finite support iteration of σ-centered forcings of length strictly less than $\mathfrak{c}^+$. As a matter of fact, the finite support iteration of σ-centered forcings of length strictly less than $\mathfrak{c}^+$ is σ-centered (the result was mentioned without proof in [39, proof of Lemma 2]; for a proof, see [7] or [23, Lemma 5.3.8]). □

7.2. **The filters are $\mathcal{B}$-Canjar.** To finish the proof of Main Theorem 4.1, we have to show that $\mathfrak{b} = \omega_1$ holds true in the final extension.

Recall that the setup is the following. Our very ground model V_0 is a model of CH; therefore, its set of reals

$$\mathcal{B} = \omega^\omega \cap V_0$$

has size ω_1. Clearly, $\mathcal{B}$ is an unbounded family in V_0. We will show that $\mathcal{B}$ remains unbounded in the course of the iteration, thereby witnessing $\mathfrak{b} = \omega_1$ in the final model.

First, observe that $\mathcal{B}$ is still unbounded in V, the extension of V_0 by μ many ω-Cohen reals (due to the fact that $\mathbb{C}_\mu$ does not add dominating reals).

In Section 7.1, we have defined filters $\mathcal{F}_\alpha^\beta$ for $\alpha < \lambda$ and $\beta < \Lambda_\alpha$ (and their canonical filter bases $\mathfrak{F}_\alpha^\beta$) and have shown that $\mathbb{Q}_\alpha$ is equivalent to the finite support iteration of the Mathias forcings $\mathbb{M}(\mathcal{F}_\alpha^\beta)$. In particular, $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta} * \mathbb{M}(\mathcal{F}_\alpha^\beta) = \mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq\beta}$, and $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\Lambda_\alpha} = \mathbb{P}_{\alpha+1}$. Note that $\mathcal{B}$ satisfies the closure property (6) from Theorem 6.7; therefore it suffices to show that $\mathcal{B}$ remains unbounded at successor steps of our "fine" iteration: for each $\alpha < \lambda$ and each $\beta < \Lambda_\alpha$, the unboundedness of $\mathcal{B}$ is preserved by $\mathbb{M}(\mathcal{F}_\alpha^\beta)$. To achieve this, we will show that the filters $\mathcal{F}_\alpha^\beta$ are $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}]$ (see Lemma 7.3(2)). Actually, we show for every β^* that $\mathcal{F}_\alpha^\beta$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ whenever $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$, i.e., we show the $\mathcal{B}$-Canjarness of a filter $\mathcal{F}_\alpha^\beta$ as soon as it exists.

In many cases, we use a genericity argument to show that the filters are $\mathcal{B}$-Canjar at the stage where they appear, but we need the $\mathcal{B}$-Canjarness in later stages of the iteration, for two reasons: first, we want to force with this filter in a later stage, and second, we want to use the $\mathcal{B}$-Canjarness of an older filter to show the $\mathcal{B}$-Canjarness of a filter which appears later. As mentioned earlier, the notion of $\mathcal{B}$-Canjarness of a filter is not absolute, therefore we will use our method from Section 6.3 to guarantee that the $\mathcal{B}$-Canjarness of the filter is not destroyed by the other Mathias forcings along the iteration. This method is based on finite sums of filters, therefore we show that all finite sums of filters which exist in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ are $\mathcal{B}$-Canjar (see Lemma 7.3(3)).

Lemma 7.3. *For every $\alpha < \lambda$, for every $\beta^* < \Lambda_\alpha$,*

(1) $\mathcal{B}$ is unbounded in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$,
(2) $\mathcal{F}_\alpha^{\beta^*}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$,
(3) if $m \in \omega$ and $\beta_0, \dots, \beta_{m-1} < \Lambda_\alpha$ with $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ for every $k < m$, then $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$.

Proof. First note that, for each $\alpha < \lambda$ and $\beta^* < \Lambda_\alpha$, (2) is a special instance of (3): in fact, $\mathfrak{F}_\alpha^{\beta^*} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$, so (3) for $m = 1$ and $\beta_0 = \beta^*$ is (2). However, we need (3) in order to carry out the induction (to preserve $\mathcal{B}$-Canjarness of our filters).

We prove (1) and (3) (and hence (2)) by (simultaneous) induction on the pairs (α, β^*) (with the lexicographical ordering). So suppose that (1) and (3) hold for each $(\alpha', \beta') <_{lex} (\alpha, \beta^*)$, i.e., for each pair with $\alpha' < \alpha$ (and β' arbitrary) or $\alpha' = \alpha$ and $\beta' < \beta^*$.

Proof of (1):

For $\alpha = \beta^* = 0$, just note that $\mathcal{B}$ is unbounded in $V[\mathbb{P}_0 * \mathbb{Q}_0^{<0}] = V$, since V is the extension by Cohen forcing $\mathbb{C}_\mu$ (which does not add dominating reals) of our GCH ground model V_0.

In case $\beta^* = \beta' + 1$ is a successor ordinal, we use the fact that (1) holds for $\alpha = \alpha'$ and β' by induction, so $\mathcal{B}$ is unbounded in the extension by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta'}$; by Lemma 7.1, $\mathbb{Q}_\alpha^{\leq\beta'} / \mathbb{Q}_\alpha^{<\beta'}$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha^{\beta'})$, i.e., $\mathbb{Q}_\alpha^{<\beta^*} = \mathbb{Q}_\alpha^{<\beta'} * \mathbb{M}(\mathcal{F}_\alpha^{\beta'})$; since (2) holds for β' by induction, $\mathbb{M}(\mathcal{F}_\alpha^{\beta'})$ preserves the unboundedness of $\mathcal{B}$, hence the same is true for $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}$, as desired.

In case (α, β^*) is a limit point of the lexicographical ordering (i.e., $\beta^* = 0$ or β^* is a limit ordinal), we use the fact that $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}$ is the limit of a finite support iteration of c.c.c. forcings, and that (1) holds for each $(\alpha', \beta') <_{lex} (\alpha, \beta^*)$; so we can apply Theorem 6.7 to conclude (1) for (α, β^*).

Proof of (3):

Fix α. By (1), $\mathcal{B}$ is unbounded in $V[\mathbb{P}_\alpha]$. We say that $\rho \in \lambda^{<\lambda}$ is *pre-T_α-minimal* if it is the predecessor of a minimal node of T_α; it is straightforward to check that this is the case if and only if

- $\rho \in V[\mathbb{P}_\alpha]$,
- $\rho \notin V[\mathbb{P}_\eta]$ for any $\eta < \alpha$, and
- for every $\gamma < |\rho|$, there exists $\eta < \alpha$ with $\rho \restriction \gamma \in V[\mathbb{P}_\eta]$.

Note that for $\alpha = 0$, the only pre-T_α-minimal node is the root $\langle\rangle$, and for $\alpha > 0$, all pre-T_α-minimal nodes have limit length.

We proceed by induction on β^*.

Base step $\beta^* = 0$:

Let $\beta_0, \ldots, \beta_{m-1}$ be such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<0}]$ for each $k < m$. Since $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<0}] = V[\mathbb{P}_\alpha]$, it follows that $\sigma_\alpha^{\beta_k} = \rho_k^\frown 0$ for some pre-T_α-minimal node ρ_k: indeed, observe that $\mathfrak{F}_\alpha^\beta$ contains elements which are only added by $\mathbb{Q}_\alpha$ (and hence $\mathfrak{F}_\alpha^\beta \notin V[\mathbb{P}_\alpha]$) whenever $\sigma_\alpha^\beta = \rho^\frown i$ with ρ not pre-T_α-minimal or $i > 0$ (because $a_\tau \in \mathfrak{F}_\alpha^\beta$ or $\omega \setminus a_\tau \in \mathfrak{F}_\alpha^\beta$ for some $\tau \in T_\alpha$).

If $\mathrm{cf}(|\rho_k|)$ is countable for all $k < m$, the filter $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is countably generated, hence it follows by Lemma 6.6 that it is $\mathcal{B}$-Canjar.

In particular, for $\alpha = 0$, the only pre-T_α-minimal node is $\rho = \langle\rangle$, hence this finishes the proof for $\alpha = \beta^* = 0$. So assume $\alpha > 0$ for the rest of the proof of the base step.

If $\mathrm{cf}(\alpha) \leq \omega$ (and $\alpha > 0$), all pre-T_α-minimal nodes ρ have $\mathrm{cf}(|\rho|) = \omega$:

Claim 7.4. *Let ρ be a pre-T_α-minimal node and* $\mathrm{cf}(|\rho|) > \omega$. *Then*

(1) $\mathrm{cf}(\alpha) > \omega$, *and*
(2) there exists no $\eta < \alpha$ such that $\rho \restriction \gamma \in V[\mathbb{P}_\eta]$ for all $\gamma < |\rho|$.

Proof. Let us first show (2). Assume it does not hold, i.e., we can fix $\eta < \alpha$ such that $\rho \restriction \gamma \in V[\mathbb{P}_\eta]$ for all $\gamma < |\rho|$: so ρ is fresh over $V[\mathbb{P}_\eta]$. This is not possible, because $\mathbb{P}_\alpha \,/\, \mathbb{P}_\eta$ is σ-centered (see Corollary 7.2), hence in particular $(\mathbb{P}_\alpha \,/\, \mathbb{P}_\eta) \times (\mathbb{P}_\alpha \,/\, \mathbb{P}_\eta)$ has the c.c.c., so by Theorem 2.8 (see also the discussion after Definition 2.6), $\mathbb{P}_\alpha \,/\, \mathbb{P}_\eta$ does not add a fresh function on any ordinal of uncountable cofinality.

Now let us show (1). Assume towards contradiction that $\mathrm{cf}(\alpha) \leq \omega$, and let $\langle \alpha_n \mid n \in \omega \rangle$ be increasing cofinal in α (in case α is a successor, let α_n be its predecessor for every n). For every $\gamma < |\rho|$, let $n \in \omega$ be such that $\rho \restriction \gamma \in V[\mathbb{P}_{\alpha_n}]$, which is possible since ρ is pre-T_α-minimal. Since $\mathrm{cf}(|\rho|) > \omega$, there exists $n^* \in \omega$ such that $\rho \restriction \gamma \in V[\mathbb{P}_{\alpha_{n^*}}]$ for cofinally many $\gamma < |\rho|$ (and hence for all $\gamma < |\rho|$), contradicting (2). □

So we can assume that $\mathrm{cf}(\alpha) > \omega$. We first argue that $\mathrm{cf}(|\rho|) > \omega$ for all pre-T_α-minimal nodes ρ. Assume towards contradiction that $\mathrm{cf}(|\rho|) = \omega$ and ρ is pre-T_α-minimal. Let $\langle \gamma_n \mid n \in \omega \rangle$ be increasing cofinal in $|\rho|$. For every $n < \omega$, let $\alpha_n < \alpha$ be such that $\rho \restriction \gamma_n \in V[\mathbb{P}_{\alpha_n}]$. Since $\mathrm{cf}(\alpha) > \omega$, there exists $\alpha' < \alpha$ with $\alpha_n < \alpha'$ for every n. There are no new countable sequences of elements of $V[\mathbb{P}_{\alpha'}]$ in $V[\mathbb{P}_\alpha]$, because $V[\mathbb{P}_\alpha]$ is a limit of uncountable cofinality of a c.c.c. iteration, hence there exists $\alpha'' < \alpha$ such that $\langle \rho \restriction \gamma_n \mid n \in \omega \rangle \in V[\mathbb{P}_{\alpha''}]$. Hence also $\rho \in V[\mathbb{P}_{\alpha''}]$, so it is not pre-$T_\alpha$-minimal, a contradiction.

Now we will show that $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha]$, using the characterization from Theorem 6.3. Let $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\alpha]$ be positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. We want to show that there exists $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. Since $\langle X_n \mid n \in \omega\rangle$ is hereditarily countable and $\mathrm{cf}(\alpha) > \omega$, there exists $\eta < \alpha$ with $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\eta]$. Moreover, let η be large enough such that for all $j < k < m$ with $\rho_j \neq \rho_k$, there exists a successor $\delta < |\rho_j|, |\rho_k|$ such that $\rho_j \restriction \delta \neq \rho_k \restriction \delta$ and $a_{\rho_j \restriction \delta}, a_{\rho_k \restriction \delta} \in V[\mathbb{P}_\eta]$. For every $k < m$, let $\gamma_k < |\rho_k|$ be such that $a_{\rho_k \restriction \gamma_k} \notin V[\mathbb{P}_\eta]$. Such γ_k exist, bexause the ρ_k are pre-T_α-minimal, using (2) from the above claim. Clearly $a_{\rho_k \restriction \gamma_k} \in V[\mathbb{P}_\alpha]$ for every $k < m$. The filter $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle$ is countably generated and hence $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha]$ (see Lemma 6.6). Note that $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle \subseteq \bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$, hence $\langle X_n \mid n \in \omega\rangle$ is positive for $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle$. Therefore we can fix $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle$. Note that $\bar{X}_f \in V[\mathbb{P}_\eta]$.

We will use a genericity argument to show that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. It is enough to show that for all successors $\delta_k < |\rho_k|$, for all $l_k \in \omega$ there exists $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} (a_{\rho_k \restriction \delta_k} \setminus l_k)$, because sets of this form are a basis for the filter. If $\delta_k \leq \gamma_k$ for all k, this holds by the choice of f.

We show by induction on $\eta \leq \eta' < \alpha$ that for all successors $\delta_k < |\rho_k|$ and all $l_k < \omega$, if all $a_{\rho_k \restriction \delta_k} \in V[\mathbb{P}_{\eta'}]$ then $V[\mathbb{P}_{\eta'}] \models$ "$\exists s \in \bar{X}_f \ s \subseteq \bigoplus_{k<m} (a_{\rho_k \restriction \delta_k} \setminus l_k)$". Note that this holds for $\eta' = \eta$ by choice of f, and that at limit steps of the induction no new $a_{\rho_k \restriction \delta_k}$ appear, so we only have to show it for successors. Assume that it holds for η' and show it for $\eta' + 1$.

For every $k < m$, let $\delta_k < |\rho_k|$ with $a_{\rho_k \restriction \delta_k} \in V[\mathbb{P}_{\eta'+1}]$ and $l_k \in \omega$ be given. Let $p \in \mathbb{Q}_{\eta'}$. We will show that there exists $q \leq p$ and $s \in \bar{X}_f$ such that $q \Vdash s \subseteq \bigoplus_{k<m} (a_{\rho_k \restriction \delta_k} \setminus l_k)$. Without loss of generality we can assume that $\rho_k \restriction \delta_k \in \mathrm{dom}(p)$ for all $k < m$ with $\rho_k \restriction \delta_k \in T_{\eta'}$, and that p is a full condition.

For every $k < m$, define Σ_k. If $\rho_k \restriction \delta_k \in T_{\eta'}$, let

$$\Sigma_k := \bigcup \{\mathrm{dom}(f^p_{\rho_k \restriction \gamma}) \cap T'_{\eta'} \mid \gamma \leq \delta_k \wedge \rho_k \restriction \gamma \in \mathrm{dom}(p)\}.$$

If $\rho_k \restriction \delta_k \notin T_{\eta'}$, let $\Sigma_k := \{\rho_k \restriction \delta_k\}$. Let $\Sigma := \bigcup_{k<m} \Sigma_k$. For every $k < m$, let σ_k be the longest initial segment of ρ_k which belongs to Σ (if there exists one; let[46] $\sigma_k := \rho_k \restriction 1$ otherwise). Note that $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$, and that $a_{\sigma_k} \in V[\mathbb{P}_{\eta'}]$ for every $k < m$. Now let $N \in \omega$ be large enough such that

- $N \geq l_k$ for every $k < m$,
- $N \geq |s^p_\sigma|$ for every $\sigma \in \mathrm{dom}(p)$,
- $a_{\sigma_k} \setminus N \subseteq a_\tau$ for all $\tau \in \Sigma_k$, for all $k < m$.

By hypothesis, in $V[\mathbb{P}_{\eta'}]$, we can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} (a_{\sigma_k} \setminus N)$.

To get q, extend p as follows. For every $k < m$, for every $\gamma \leq \delta_k$ with $\rho_k \restriction \gamma \in \mathrm{dom}(p)$, let

$$s^q_{\rho_k \restriction \gamma} := s^p_{\rho_k \restriction \gamma} {}^\frown (\mathbf{0} \restriction [|s^p_{\rho_k \restriction \gamma}|, N))^\frown (a_{\sigma_k} \restriction [N, \max(s)]).$$

Observe that, if $\rho_k \neq \rho_j$, there is no $\tau \in \mathrm{dom}(p)$ with $\tau \trianglelefteq \rho_k$ and $\tau \trianglelefteq \rho_j$ (by choice of η and since $\eta' \geq \eta$); so the above is well-defined, since $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$.

Note that η was chosen large enough such that for all $\gamma_k \leq \delta_k$ and $\gamma_j \leq \delta_j$, if $\rho_j \restriction \gamma_j \neq \rho_k \restriction \gamma_k$, then they are not in the same block; so, in particular, $\rho_j \restriction \gamma_j \notin \mathrm{dom}(h^p_{\rho_k \restriction \gamma_k})$ and $\rho_k \restriction \gamma_k \notin \mathrm{dom}(h^p_{\rho_j \restriction \gamma_j})$.

[46]We just have to choose any initial segment of ρ_k which belongs to $T'_{\eta'}$ and make sure that $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$. Alternatively, in such cases, we could replace a_{σ_k} by ω below.

Therefore, the requirement (8) from Definition 4.5 is fulfilled. It is easy to see that the other requirements of Definition 4.5 are fulfilled as well, hence q is a condition.

It is easy to check that q forces $s \subseteq \bigoplus_{k<m}(a_{\rho_k \restriction \delta_k} \setminus l_k)$, as desired.

Successor step:

Let us say that a filter $\mathcal{F}^{\beta}_{\alpha}$ (and its filter base $\mathfrak{F}^{\beta}_{\alpha}$) is *new in* $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$ if $\mathfrak{F}^{\beta}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$ and $\mathfrak{F}^{\beta}_{\alpha} \notin V[\mathbb{P}_\alpha * \mathbb{Q}^{<\delta^*}_{\alpha}]$ for all $\delta^* < \beta^*$.

Now assume that we have shown (3) for β^*; let us show it for $\beta^* + 1$.

If $\mathfrak{F}^{\beta_k}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$ for every $k < m$, then by induction hypothesis $\left(\bigoplus_{k<m} \mathcal{F}^{\beta_k}_{\alpha}\right) \oplus \mathcal{F}^{\beta^*}_{\alpha}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$, hence, by Lemma 6.11, $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_{\alpha}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha} * \mathbb{M}(\mathcal{F}^{\beta^*}_{\alpha})] = V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*+1}_{\alpha}]$.

It is easy to check that there are exactly two new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*+1}_{\alpha}]$: $\mathcal{F}^{\beta}_{\alpha}$, where β is such that $\sigma^{\beta}_{\alpha} = \sigma^{\beta^*}_{\alpha} {}^\frown 0$, and $\mathcal{F}^{\beta'}_{\alpha}$, where $\beta' = \beta^* + 1$ (i.e., $\sigma^{\beta'}_{\alpha} = \rho^\frown(i+1)$ and $\sigma^{\beta^*}_{\alpha} = \rho^\frown i$). Both $\mathcal{F}^{\beta}_{\alpha}$ and $\mathcal{F}^{\beta'}_{\alpha}$ are extensions of $\mathcal{F}^{\beta^*}_{\alpha}$ by one new set. Therefore, the filter $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_{\alpha}$ is an extension of $\bigoplus_{k<m} \mathcal{F}^{\tilde\beta_k}_{\alpha}$ by finitely many sets, where $\tilde\beta_k = \beta^*$ if $\beta_k = \beta$ or $\beta_k = \beta'$, and $\tilde\beta_k = \beta_k$ otherwise. By the above, $\bigoplus_{k<m} \mathcal{F}^{\tilde\beta_k}_{\alpha}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*+1}_{\alpha}]$, hence, by Lemma 6.5, also $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_{\alpha}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*+1}_{\alpha}]$.

Limit step:

Now assume that β^* is a limit, and that we have shown (3) for all $\delta^* < \beta^*$; let us show it for β^*. If for each $k < m$ there exists $\delta^*_k < \beta^*$ such that $\mathfrak{F}^{\beta_k}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\delta^*_k}_{\alpha}]$, then there exists $\bar\delta^* < \beta^*$ such that $\mathfrak{F}^{\beta_k}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\bar\delta^*}_{\alpha}]$ for all $k < m$. By induction hypothesis, $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_{\alpha}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\delta^*}_{\alpha}]$ for every $\bar\delta^* \leq \delta^* < \beta^*$, hence, by Lemma 6.12, it is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$.

Now we have to consider new filters. There are two cases: either β^* is such that $\sigma^{\beta^*}_{\alpha}$ has 0 as its last entry, or such that it has a limit ordinal i as its last entry.

First case: $\sigma^{\beta^*}_{\alpha} = \rho^\frown 0$ for some ρ

Let us first argue that there are no new filters unless $|\rho|$ is a limit and $\sigma^{\beta^*}_{\alpha}$ is the first node of its level in the enumeration (i.e., $|\sigma^{\delta}_{\alpha}| < |\rho|$ for each $\delta < \beta^*$). If $\sigma^{\beta^*}_{\alpha}$ is not the first node of the level in the enumeration, then there are no new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$: If $\mathfrak{F}^{\beta}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$, then there exists $\delta^* < \beta^*$ such that $\mathfrak{F}^{\beta}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\delta^*}_{\alpha}]$, because $\mathfrak{F}^{\beta}_{\alpha}$ contains – from the sets of this level – only sets within one block and only boundedly many sets within this block. Similarly, if $|\rho|$ is a successor (and $\sigma^{\beta^*}_{\alpha}$ is the first node of its level in the enumeration), there are no new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$: If $\mathfrak{F}^{\beta}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$, then there exists $\delta^* < \beta^*$ such that $\mathfrak{F}^{\beta}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\delta^*}_{\alpha}]$, because $\mathfrak{F}^{\beta}_{\alpha}$ contains – from the sets of level $|\rho|$ – only boundedly many sets.

So we assume from now on that $|\rho|$ is a limit and $\sigma^{\beta^*}_{\alpha}$ is the first node of its level in the enumeration. In this case, there are many new filters $\mathcal{F}^{\beta}_{\alpha}$ in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$; in fact, it is easy to check that $\mathcal{F}^{\beta}_{\alpha}$ is new if and only if the following holds: $\sigma^{\beta}_{\alpha} = \bar\rho^\frown 0$ for some $\bar\rho$ with $|\bar\rho| = |\rho|$ and $\bar\rho$ not pre-T_α-minimal. Observe that $\mathfrak{F}^{\beta}_{\alpha} = \{a_{\bar\rho \restriction \gamma} \mid \gamma < |\rho|\}$. Let $\beta_0, \ldots, \beta_{m-1}$ be such that $\mathfrak{F}^{\beta_k}_{\alpha} \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$ for each $k < m$. We want to show that $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_{\alpha}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_{\alpha}]$.

In case $\mathrm{cf}(|\rho|) = \omega$, we can use Lemma 6.13 and the remark afterwards to finish the proof: $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is a sum of filters, in which the new filters are countably generated, whereas the sum of the filters which are not new is $\mathcal{B}$-Canjar (see the first paragraph of the limit step).

So let us assume from now on that $\mathrm{cf}(|\rho|) > \omega$. Let $\mathsf{new} \subseteq m$ be the set of $k < m$ such that $\mathcal{F}_\alpha^{\beta_k}$ is a new filter, and $\mathsf{old} = m \setminus \mathsf{new}$ be the set of $k < m$ such that $\mathcal{F}_\alpha^{\beta_k}$ is not new. For each $k \in \mathsf{new}$, we can fix ρ_k such that $\sigma_\alpha^{\beta_k} = \rho_k^\frown 0$ (with $|\rho_k| = |\rho|$ and ρ_k not pre-T_α-minimal).

Let $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ be positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. Since $\langle X_n \mid n \in \omega\rangle$ is hereditarily countable and $\mathbb{Q}_\alpha^{<\beta^*}$ has the c.c.c., there exists a hereditarily countable name $\dot{X}$ for $\langle X_n \mid n \in \omega\rangle$. Since the conditions in $\mathbb{Q}_\alpha^{<\beta^*}$ have finite domain, the union of all the domains of conditions which occur in the name $\dot{X}$ is countable. Let $\gamma < |\rho|$ be a successor ordinal large enough such that the following hold:

- $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$ (this is possible due to $\mathrm{cf}(|\rho|) > \omega$).
- For all $j, k \in \mathsf{new}$, if $\rho_j \neq \rho_k$, then ρ_j and ρ_k split before γ.
- For all $k \in \mathsf{old}$, either $|\sigma_\alpha^{\beta_k}| < \gamma$ or[47] $|\sigma_\alpha^{\beta_k}| > |\rho|$.
- $a_{\rho_k \restriction \gamma} \notin V[\mathbb{P}_\alpha]$ for all $k \in \mathsf{new}$, i.e., $\rho_k \restriction \gamma \in T_\alpha$ (which is possible because ρ_k is not pre-T_α-minimal).

For every $k < m$ with $k \in \mathsf{old}$, we have $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$ by choice of γ; let $\tilde{\mathfrak{F}}_\alpha^{\beta_k} := \mathfrak{F}_\alpha^{\beta_k}$. For $k \in \mathsf{new}$, let $\tilde{\mathfrak{F}}_\alpha^{\beta_k} := \{a_{\rho_k \restriction \gamma}\}$.

As above, we can use Lemma 6.13 and the remark afterwards to show that $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. Indeed, $\bigoplus_{k \in \mathsf{old}} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ by the first paragraph of the limit step, and for each $k \in \mathsf{new}$, $\langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle$ is countably generated. Moreover, $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle \subseteq \bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$, hence $\langle X_n \mid n \in \omega\rangle$ is positive for $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle$. So we can fix $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle$. Since $\langle X_n \mid n \in \omega\rangle$ and $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle$ are in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$, this holds in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$.

Now we use a genericity argument in $\mathbb{Q}_\alpha^{<\beta^*} / \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}$ to show that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. We have to show that for all $\langle A_k \mid k < m\rangle$ with $A_k \in \mathcal{F}_\alpha^{\beta_k}$, there exists $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} A_k$. For $k \in \mathsf{new}$, we can assume that $A_k = a_{\rho_k \restriction \delta_k} \setminus l_k$ with $\gamma < \delta_k < |\rho|$ and $l_k \in \omega$, because these sets form filter bases (with respect to upwards closure). For $k \in \mathsf{old}$, let $B_k := A_k$, and for $k \in \mathsf{new}$ (in this case $|\sigma_\alpha^{\beta_k}| = |\rho| + 1 > \gamma$), let $B_k := a_{\rho_k \restriction \gamma}$. By the choice of f, there exists, for all $N \in \omega$, an $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} (B_k \setminus N)$.

Let $p \in \mathbb{Q}_\alpha^{<\beta^*} / \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}$. Without loss of generality we can assume that $\rho_k \restriction \delta_k \in \mathrm{dom}(p)$ if $k \in \mathsf{new}$. For every $k \in \mathsf{new}$, define

$$\Sigma_k := \bigcup \{\mathrm{dom}(f^p_{\rho_k \restriction \delta}) \cap \lambda^{\leq\gamma} \mid \delta \leq \delta_k \wedge \rho_k \restriction \delta \in \mathrm{dom}(p)\}.$$

Now let $N \in \omega$ be large enough such that

- $N \geq l_k$ for every $k \in \mathsf{new}$,
- $N \geq |s^p_\sigma|$ for every $\sigma \in \mathrm{dom}(p)$,
- $a_{\rho_k \restriction \gamma} \setminus N \subseteq a_\tau$ for all $\tau \in \Sigma_k$, for all $k \in \mathsf{new}$.

By the above, we can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} (B_k \setminus N)$.

[47] Note that $|\sigma_\alpha^{\beta_k}| > |\rho|$ is only possible if $\sigma_\alpha^{\beta_k} = \tilde{\rho}^\frown 0$ for a pre-T_α-minimal node $\tilde{\rho}$.

To get q, extend p as follows. For every $k \in \mathsf{new}$, for every $\delta \leq \delta_k$ with $\rho_k \restriction \delta \in \mathrm{dom}(p)$, let

$$s^q_{\rho_k \restriction \delta} := s^p_{\rho_k \restriction \delta}{}^\frown(\mathbf{0} \restriction [|s^p_{\rho_k \restriction \delta}|, N))^\frown(a_{\rho_k \restriction \gamma} \restriction [N, \max(s)]).$$

Note that γ was chosen large enough so that for $j, k \in \mathsf{new}$, if $\rho_j \neq \rho_k$, then they split before γ, therefore for $\gamma < \delta < |\rho|$ either $\rho_k \restriction \delta = \rho_j \restriction \delta$ or they are not in the same block. In particular $\rho_j \restriction \delta \notin \mathrm{dom}(h^p_{\rho_k \restriction \delta})$ and $\rho_k \restriction \delta \notin \mathrm{dom}(h^p_{\rho_j \restriction \delta})$. So the requirement (8) from Definition 4.5 is fulfilled. It is easy to see that the other requirements of Definition 4.5 are fulfilled as well, hence q is a condition.

It is easy to check that q forces $s \subseteq \bigoplus_{k<m} A_k$, as desired.

Second case: $\sigma^{\beta^*}_\alpha = \rho^\frown i$ with $i > 0$ limit

In this case, $\mathcal{F}^{\beta^*}_\alpha$ is the only new filter in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_\alpha]$. Indeed, for all other β either $\mathfrak{F}^\beta_\alpha$ appeared in an earlier step already or it is not in this model, because for $\beta \neq \beta^*$ the filter base $\mathfrak{F}^\beta_\alpha$ either uses only boundedly many elements of $\{a_{\rho^\frown j} \mid j < i\}$ or it uses $a_{\rho^\frown i}$ as well. Let $\beta_0, \ldots, \beta_{m-1}$ be such that $\mathfrak{F}^{\beta_k}_\alpha \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_\alpha]$ for each $k < m$. We want to show that $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_\alpha$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_\alpha]$.

Let $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_\alpha]$ be positive for $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_\alpha$. Since $\langle X_n \mid n \in \omega\rangle$ is hereditarily countable and $\mathbb{Q}^{<\beta^*}_\alpha$ has the c.c.c., there exists a hereditarily countable name $\dot{X}$ for $\langle X_n \mid n \in \omega\rangle$. Since the conditions in $\mathbb{Q}^{<\beta^*}_\alpha$ have finite domain, the union D of all the domains of conditions which occur in the name $\dot{X}$ is countable. Let $\beta^{**} < \beta^*$ be such that $\rho^\frown 0 = \sigma^{\beta^{**}}_\alpha$. Let $\bar{D} := \{\tau \in D \mid \exists j < i\ (\tau = \rho^\frown j)\}$. Let $C := \{\sigma^\nu_\alpha \mid \nu < \beta^{**}\} \cup \bar{D}$. Note that

$$C = \{\sigma^\nu_\alpha \mid \nu < \beta^{**} \wedge |\sigma^\nu_\alpha| \leq |\rho|\} \cup \bar{C}$$

with $\bar{C} = \{\sigma^\nu_\alpha \mid \nu < \beta^{**} \wedge |\sigma^\nu_\alpha| = |\rho| + 1\} \cup \bar{D}$. Since $\{\sigma^\nu_\alpha \mid \nu < \beta^{**} \wedge |\sigma^\nu_\alpha| \leq |\rho|\} = \lambda^{\leq|\rho|} \cap T_\alpha$ is left-up-closed, C has the form which is needed in Lemma 4.22, so $\mathbb{Q}^C_\alpha$ is a complete subforcing of $\mathbb{Q}_\alpha$ and a subset of $\mathbb{Q}^{<\beta^*}_\alpha$; recall that $\mathbb{Q}^{<\beta^*}_\alpha$ is also a complete subforcing of $\mathbb{Q}_\alpha$, so, by Lemma 5.1, $\mathbb{Q}^C_\alpha$ is complete in $\mathbb{Q}^{<\beta^*}_\alpha$.

Observe that for all $k < m$ either $\mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^{**}}_\alpha] = \mathfrak{F}^{\beta_k}_\alpha$ or $\mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^{**}}_\alpha] = \mathfrak{F}^{\beta^{**}}_\alpha$. In particular, for every $k < m$ there exists β'_k such that $\mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^{**}}_\alpha] = \mathfrak{F}^{\beta'_k}_\alpha$ and $\mathfrak{F}^{\beta'_k}_\alpha \in V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^{**}}_\alpha]$. Hence $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^{**}}_\alpha]\rangle$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_\alpha]$. For every $k < m$, $\mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]$ is the set $\mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^{**}}_\alpha]$ together with countably many new sets (some of the sets $\omega \setminus a_\tau$ with $\tau \in \bar{D}$), therefore $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]\rangle$ is a filter generated by $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^{**}}_\alpha]\rangle$ together with countably many new sets. Hence, by Lemma 6.5, $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]\rangle$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}^{<\beta^*}_\alpha]$. Since $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]\rangle \subseteq \bigoplus_{k<m} \mathcal{F}^{\beta_k}_\alpha$, the sets $\langle X_n \mid n \in \omega\rangle$ are also positive for $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]\rangle$. So we can fix $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]\rangle$. Since $\langle X_n \mid n \in \omega\rangle$ and $\bigoplus_{k<m}\langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]\rangle$ are in $V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]$, this holds in $V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]$.

Now we use a genericity argument in $\mathbb{Q}^{<\beta^*}_\alpha / \mathbb{Q}^C_\alpha$ to show that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}^{\beta_k}_\alpha$. We have to show that for all $\langle A_k \mid k < m\rangle$ with $A_k \in \mathcal{F}^{\beta_k}_\alpha$ there exists $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} A_k$. For easier notation, assume that there exists $m' \leq m$ such that $A_k \in V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]$ if and only if $k < m'$.

For $m' \leq k < m$ there exists $B_k \in \langle \mathfrak{F}^{\beta_k}_\alpha \cap V[\mathbb{P}_\alpha * \mathbb{Q}^C_\alpha]\rangle$, $\ell_k \in \omega$ and $\langle j^k_r \mid r < \ell_k\rangle \subseteq i$ such that $A_k = B_k \cap \bigcap_{r<\ell_k}(\omega \setminus a_{\rho^\frown j^k_r})$. So $\bigoplus_{k<m} A_k = \bigoplus_{k<m'} A_k \oplus \bigoplus_{m' \leq k<m}(B_k \cap \bigcap_{r<\ell_k}(\omega \setminus a_{\rho^\frown j^k_r}))$. Let $p \in \mathbb{Q}^{<\beta^*}_\alpha / \mathbb{Q}^C_\alpha$. Without loss of generality assume that $\rho^\frown j^k_r \in \mathrm{dom}(p)$ if $\rho^\frown j^k_r \notin C$. Let $N > |s^p_\tau|$ for every $\tau \in \mathrm{dom}(p)$. We can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m'} A_k \oplus \bigoplus_{m' \leq k<m} (B_k \setminus N)$. To get q, extend each $s^p_{\rho^\frown j^k_r}$ with 0's to have length $\max(s) + 1$.

It is easy to check that q is a condition, and that q forces $s \subseteq \bigoplus_{k<m} A_k$, as desired. □

By the above Lemma 7.3(1), $\mathcal{B}$ is unbounded in $V[\mathbb{P}_\alpha]$ for every $\alpha < \lambda$, so by applying Theorem 6.7 once again, it follows that $\mathcal{B}$ is unbounded in our final model $V[\mathbb{P}_\lambda]$. Since $|\mathcal{B}| = \omega_1$, the bounding number $\mathfrak{b}$ is ω_1 in our final model, and so is $\mathfrak{h}$ (see Proposition 3.2), as desired.

This concludes the proof of Main Theorem 4.1.

8. Considerations about $\mathcal{P}(\kappa)/{<}\kappa$

In this section, we study the distributivity spectrum of $\mathcal{P}(\kappa)/{<}\kappa$ for regular uncountable cardinals κ. We compute the fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$ in Section 8.2, and we discuss the combinatorial distributivity spectrum of $\mathcal{P}(\kappa)/{<}\kappa$ in Section 8.3.

These considerations are also an attempt towards defining a κ-analogue of $\mathfrak{h}$. As a matter of fact, the distributivity of $\mathcal{P}(\kappa)/{<}\kappa$ is ω (see Proposition 8.1), i.e., the straightforward generalization of $\mathfrak{h}$ does not work. The same problem occurs in the context of the tower number $\mathfrak{t}$, but its κ-analogue $\mathfrak{t}_\kappa$ can be defined by excluding towers of short length. We do not know whether a similar approach can work for defining $\mathfrak{h}_\kappa$ (see Tentative Definition 8.5). The problem of generalizing $\mathfrak{h}$ to κ has also been considered by Galgon in his PhD thesis (see [20]).

8.1. Distributivity of $\mathcal{P}(\kappa)/{<}\kappa$ and the tower number $\mathfrak{t}_\kappa$.

Let $a, b \in [\kappa]^\kappa$. We write $b \subseteq^* a$ if $b \setminus a$ has size $< \kappa$ (i.e., the equivalence class of b is stronger than the equivalence class of a in $\mathcal{P}(\kappa)/{<}\kappa$). We say that $b \in [\kappa]^\kappa$ is a *pseudo-intersection* of $\langle a_\xi \mid \xi < \delta \rangle \subseteq [\kappa]^\kappa$ if $b \subseteq^* a_\xi$ for each $\xi < \delta$. We say that $\langle a_\xi \mid \xi < \delta \rangle$ is a *tower of length δ in $\mathcal{P}(\kappa)/{<}\kappa$* if $a_\eta \subseteq^* a_\xi$ for any $\eta > \xi$, and it does not have a pseudo-intersection of size κ.

There are always towers of length ω in $\mathcal{P}(\kappa)/{<}\kappa$: indeed, let $\{x_n \mid n \in \omega\}$ be a partition of κ into countably many pieces each of size κ, and define $a_n := \bigcup_{m \geq n} x_m$; then the sequence $\langle a_n \mid n < \omega \rangle$ is $\subseteq^*$-decreasing and has no pseudo-intersection of size κ. The analogous proof works for any other regular cardinal below κ, hence for any regular $\lambda < \kappa$, there is a tower of length λ in $\mathcal{P}(\kappa)/{<}\kappa$.

Note that the above actually says that $\mathcal{P}(\kappa)/{<}\kappa$ is not σ-closed. Let us now argue that the following stronger statement holds (a slightly different presentation can also be found in [20, Theorem 3.5.2]):

Proposition 8.1. *$\mathcal{P}(\kappa)/{<}\kappa$ is not ω-distributive (i.e., $\mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa) = \omega$).*

Proof. We use the game characterization of distributivity given in Section 2.3. We define a strategy σ for Player I in $G_\omega(\mathcal{P}(\kappa)/{<}\kappa)$ and show that it is a winning strategy.

For $b \in [\kappa]^\kappa$, let $b =: \{\alpha_i \mid i < \kappa\}$ be the increasing enumeration of b, and let $b' := \{\alpha_{i+1} \mid i \in \kappa\}$. Let $\sigma(\langle\rangle) := \kappa$ and[48] $\sigma(\langle a_0, b_0, \ldots, b_n \rangle) := b'_n$.

Assume towards a contradiction that σ is not a winning strategy. Fix a run $\langle a_0, b_0, a_1, b_1, \ldots \rangle$ of the game, where Player I played according to σ, yet Player II wins, witnessed by $b \in [\kappa]^\kappa$, i.e., $b \subseteq^* a_n$ for

[48] Here and in the rest of the proof, we work with arbitrary representatives a_n for the equivalence classes in $\mathcal{P}(\kappa)/{<}\kappa$ instead of the equivalence classes (compare with Remark 3.1).

every $n \in \omega$. Note that there exists $\gamma < \kappa$ such that $b_n \setminus \gamma \subseteq a_n$ and $a_{n+1} \setminus \gamma \subseteq b_n$ as well as $b \setminus \gamma \subseteq a_n$ holds true for each $n \in \omega$. Now let $\alpha \in b \setminus \gamma$ be the ωth element of $b \setminus \gamma$. Then $\alpha \in a_n \setminus \gamma$ for every $n \in \omega$, so there exists $i_n \in \kappa$ such that α is the i_nth element of $a_n \setminus \gamma$. It is straightforward to check that $\langle i_n \mid n \in \omega\rangle$ is a strictly decreasing sequence of ordinals; a contradiction. □

The above proposition shows that $\mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa)$ cannot be taken as a suitable generalization of $\mathfrak{h}$ to κ.

Similarly, the tower number $\mathfrak{t}$ cannot be generalized to κ by simply taking the minimal length of a tower in $\mathcal{P}(\kappa)/{<}\kappa$ (which would always yield value ω). However, one can overcome this problem by excluding towers of length less than κ (see, e.g., [36]). We present this approach by looking at the corresponding spectrum. Let

$$spec(\mathfrak{t}_\kappa) := \{\delta \mid \delta \text{ is regular}^{49} \text{and there is a tower of length } \delta \text{ in } \mathcal{P}(\kappa)/{<}\kappa\}$$

be the *tower spectrum of* $\mathcal{P}(\kappa)/{<}\kappa$. As discussed above, each regular cardinal below κ belongs to $spec(\mathfrak{t}_\kappa)$. On the other hand, the usual diagonalization argument shows that there is no tower of length κ in $\mathcal{P}(\kappa)/{<}\kappa$, i.e., κ does not belong to $spec(\mathfrak{t}_\kappa)$. So the part of the spectrum up to κ is not really interesting, and we define

$$\mathfrak{t}_\kappa := \min(\{\delta \in spec(\mathfrak{t}_\kappa) \mid \delta > \kappa\}).$$

Note that $\mathfrak{t}_\kappa$ is well-defined, since the above set is non-empty: in fact, it is quite easy to construct a maximal $\subseteq^*$-decreasing sequence consisting only of club subsets of κ (taking diagonal intersections at limits of cofinality κ), which yields a tower of regular[50] length above κ.

Clearly, $\kappa^+ \leq \mathfrak{t}_\kappa \leq 2^\kappa$. Actually, $\mathfrak{t}_\kappa$ can attain any regular value from κ^+ to 2^κ in suitable models of ZFC; in particular, $\kappa^+ = \mathfrak{t}_\kappa < 2^\kappa$ as well as $\kappa^+ < \mathfrak{t}_\kappa = 2^\kappa$ are consistent (see [36, Theorem 3.7]).

8.2. **The fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$.** As for $\mathcal{P}(\omega)/\mathrm{fin}$ (see Proposition 2.11), the fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$ is an interval. Under the assumption that there are antichains of size θ in $\mathcal{P}(\kappa)/{<}\kappa$, Shelah [33] has shown that the forcing $\mathcal{P}(\kappa)/{<}\kappa$ collapses θ to[51] ω (based on work of Balcar-Simon [3] which shows that it collapses the generalized bounding number $\mathfrak{b}_\kappa$ to ω).

We will use Lemma 2.10 to compute the fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$. As a preparation, we prove the following lemma:

Lemma 8.2. *If $\mathcal{P}(\kappa)/{<}\kappa$ has the χ-c.c., then $(\mathcal{P}(\kappa)/{<}\kappa) \times (\mathcal{P}(\kappa)/{<}\kappa)$ has the χ-c.c..*

Proof. Let A be an antichain in $(\mathcal{P}(\kappa)/{<}\kappa) \times (\mathcal{P}(\kappa)/{<}\kappa)$. We will show that there is an antichain in $\mathcal{P}(\kappa)/{<}\kappa$ of the same size.

Fix a bijection $\iota\colon \kappa \times \kappa \to \kappa$. Define a mapping $\varphi\colon [\kappa]^\kappa \times [\kappa]^\kappa \to [\kappa]^\kappa$ as follows. For $(a, b) \in [\kappa]^\kappa \times [\kappa]^\kappa$, let $a =: \{\alpha_i \mid i < \kappa\}$, and $b =: \{\beta_i \mid i < \kappa\}$ be the increasing enumerations of a and b. Let $\varphi(a, b) := \{\iota(\alpha_i, \beta_i) \mid i < \kappa\}$. It is straightforward to check that φ preserves incompatibility, i.e., if (a_0, b_0) and (a_1, b_1) are incompatible, then $\varphi(a_0, b_0)$ and $\varphi(a_1, b_1)$ are incompatible.

[49] As for the tower spectrum $spec(\mathfrak{t})$ of $\mathcal{P}(\omega)/\mathrm{fin}$ (see Section 3.1), the existence of towers is only a matter of cofinality, and hence the restriction to regular cardinals makes sense.

[50] Actually, the resulting tower might be of singular length, but its cofinality will be a regular cardinal above κ.

[51] Note that this again yields $\mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa) = \omega$, which was directly proved above (by a much easier argument, see Proposition 8.1).

Therefore, $\{\varphi(a,b) \mid (a,b) \in A\}$ is an antichain in $\mathcal{P}(\kappa)/{<}\kappa$ of the same size as A. □

We can now compute the fresh function spectrum of $\mathcal{P}(\kappa)/{<}\kappa$. It turns out that it only depends on the size of the antichains:

Proposition 8.3. *If χ is minimal such that $\mathcal{P}(\kappa)/{<}\kappa$ has the χ-c.c., then*[52]

$$\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \chi)_{Reg}.$$

Proof. By Shelah [33], $\mathcal{P}(\kappa)/{<}\kappa$ collapses θ to ω, if there exists an antichain of size θ. Hence, every cardinal smaller than χ is collapsed to ω. Since $\mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa) = \omega$, it follows by Lemma 2.10 that every regular cardinal smaller than χ belongs to $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa)$.

On the other hand, nothing larger belongs to $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa)$: by Lemma 8.2, $(\mathcal{P}(\kappa)/{<}\kappa) \times (\mathcal{P}(\kappa)/{<}\kappa)$ has the χ-c.c., hence by Theorem 2.8 no regular cardinal $\geq \chi$ belongs to $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa)$. □

Note that there are always antichains of size κ^+ (actually even of size[53] $\mathfrak{b}_\kappa$) in $\mathcal{P}(\kappa)/{<}\kappa$, hence

$$[\omega, \kappa^+]_{Reg} \subseteq \mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa).$$

If $2^{<\kappa} = \kappa$, then there are antichains of size 2^κ (by the same argument as for ω, i.e., by identifying $2^{<\kappa}$ with κ and taking the set of branches through the tree $2^{<\kappa}$), so

$$\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, 2^\kappa]_{Reg} \text{ whenever } 2^{<\kappa} = \kappa.$$

To get a model of $2^{<\kappa} = \kappa$ where 2^κ is large, we can start with a model of GCH and add many κ-Cohen reals.

On the other hand, it is also consistent that 2^κ is large and there are no large antichains in $\mathcal{P}(\kappa)/{<}\kappa$. In fact, the following was shown in [4]:

Proposition 8.4. *Let V be a model of $2^\kappa = \theta$ and $\mu > \theta$ with $\mathrm{cf}(\mu) > \kappa$. Then there exists a cofinality preserving extension of V such that in $\mathcal{P}(\kappa)/{<}\kappa$, there are no antichains of size θ^+, and $2^\kappa = \mu$.*

Note that each antichain in V remains an antichain of the same size in the extension (since cardinalities are preserved).

Also note that, starting with a model of GCH, the proposition easily yields a model satisfying $\kappa^+ < 2^\kappa$ and $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \kappa^+]_{Reg}$.

Proof of Proposition 8.4. We add μ many ω-Cohen reals,[54] i.e., force with $\mathbb{C}_\mu$. Then in the extension, clearly $2^\kappa = \mu$ holds true, and there are no antichains of size θ^+ in $\mathcal{P}(\kappa)/{<}\kappa$, which can be seen as follows.

Assume towards a contradiction that $A = \{a_i \mid i < \theta^+\}$ is an antichain of size θ^+ in $V[\mathbb{C}_\mu]$. Now work in V and fix names $\dot{a}_i$ for the sets a_i. For $i, j < \theta^+$, let $\dot{\zeta}_{i,j}$ be such that it is forced that $\dot{a}_i \cap \dot{a}_j \subseteq \dot{\zeta}_{i,j} < \kappa$. Since $\mathbb{C}_\mu$ has the c.c.c., there exist countable sets $Z_{i,j} \subseteq \kappa$ in the ground model such that it is forced that $\dot{\zeta}_{i,j} \in Z_{i,j}$. Now let $\gamma_{i,j} := \sup(Z_{i,j}) < \kappa$. This defines a mapping from $\theta^+ \times \theta^+$ to κ. Since $\theta^+ = (2^\kappa)^+$ in V,

[52] $[\omega, \chi)_{Reg}$ denotes $\{\lambda \mid \lambda$ is a regular cardinal with $\omega \leq \lambda < \chi\}$.

[53] It is easy to construct a $<^*$-increasing family of functions in κ^κ of size $\mathfrak{b}_\kappa$; the graphs of these functions form an antichain on $\kappa \times \kappa$.

[54] In fact, the proof shows that we only have to demand that the forcing to blow up 2^κ has the κ-c.c. and preserves cofinalities (for example, adding many ν-Cohen reals with $\nu < \kappa$ would work).

we can apply the Erdős-Rado Theorem to get a set $Y \subseteq \theta^+$ of size κ^+ and $\gamma < \kappa$ such that $\gamma_{i,j} = \gamma$ for all $i, j \in Y$. Therefore, $\{a_i \setminus \gamma \mid i \in Y\}$ is a family of κ^+ many disjoint subsets of κ in $V[\mathbb{C}_\mu]$, a contradiction. □

It also follows from the above that the size of the largest antichain in $\mathcal{P}(\kappa)/{<}\kappa$ can be strictly between κ^+ and 2^κ. We can proceed as follows. Start with a ground model V satisfying GCH, and let $\kappa^+ \leq \theta \leq \mu$ with $\mathrm{cf}(\theta) > \kappa$ and $\mathrm{cf}(\mu) > \kappa$. First add θ many κ-Cohen reals and then add μ many ω-Cohen reals, then in the resulting model, $\mathcal{P}(\kappa)/{<}\kappa$ has the θ^+-c.c. and there exists an antichain of size θ, hence

$$\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \theta]_{Reg}$$

holds true in this model.

8.3. **Distributivity matrices for $\mathcal{P}(\kappa)/{<}\kappa$ and $\mathfrak{h}_\kappa$.** We now turn to the the combinatorial distributivity spectrum of $\mathcal{P}(\kappa)/{<}\kappa$.

Recall that the straightforward generalization of $\mathfrak{h}$ to κ, using $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$, does not work: in fact, Proposition 8.1 yields

$$\min(\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)) = \mathfrak{h}(\mathcal{P}(\kappa)/{<}\kappa) = \omega.$$

Together with Proposition 2.9 and Proposition 8.3, we get

$$\{\omega\} \subseteq \mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa) \subseteq \mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \chi)_{Reg}$$

(where χ is minimal such that $\mathcal{P}(\kappa)/{<}\kappa$ has the χ-c.c.).

In his PhD thesis [20], Galgon explored systems of maximal antichains in $\mathcal{P}(\kappa)/{<}\kappa$. He did not explicitly define $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$, but it follows from one of his results (see [20, Proposition 3.7.3]) that if there exists a κ-Aronszajn tree, then κ belongs to $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$. In fact, the witnessing distributivity matrix is directly derived from the κ-Aronszajn tree, by working on the nodes of the tree (as the κ-sized set in place of κ), and assigning to each node the cone of its extensions. Since the levels of the original tree are of size less than κ, the same holds true for all antichains of the resulting distributivity matrix. Moreover, it has no cofinal branches since the original tree is Aronszajn.

Using the definition of the tower number $\mathfrak{t}_\kappa$ as a blueprint, one could attempt to define $\mathfrak{h}_\kappa$ as follows:

Tentative Definition 8.5.

$$\mathfrak{h}_\kappa := \min(\{\delta \in \mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa) \mid \delta > \kappa\}).$$

However, we do not know much about $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$, so this definition might be problematic. Most importantly, it is not even clear whether $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$ is non-empty above κ: indeed, we do not have any example of a distributivity matrix of regular height strictly above κ for $\mathcal{P}(\kappa)/{<}\kappa$. At least, the existence of a distributivity matrix of height κ^+ is never excluded by the fresh function spectrum because κ^+ always belongs to $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa)$.

Recall from Proposition 8.4 and the discussion afterwards that there are models satisfying $\kappa^+ < 2^\kappa$ and $\mathsf{FRESH}(\mathcal{P}(\kappa)/{<}\kappa) = [\omega, \kappa^+]_{Reg}$. Therefore, provided that $\mathfrak{h}_\kappa$ is well-defined in such a model,

$$\kappa^+ = \mathfrak{h}_\kappa < 2^\kappa$$

is consistent. Note that even if $\mathfrak{h}_\kappa$ exists, it is not so clear whether $\mathfrak{t}_\kappa \leq \mathfrak{h}_\kappa$ would hold true in general, because the argument of Proposition 3.3 does not go through: the length of all maximal branches of a distributivity matrix of height $\mathfrak{h}_\kappa$ could perhaps be of cofinality less than κ. By Galgon's above mentioned result, the tree property is necessary for κ being not in $\mathcal{P}(\kappa)/{<}\kappa$; we do not know whether this is consistent (e.g., for large cardinals at least as strong as weakly compact). In case $\kappa \notin \mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$, we could also write $\delta \geq \kappa$ (as in the definition of $\mathfrak{t}_\kappa$) instead of $\delta > \kappa$ in the above attempt to define $\mathfrak{h}_\kappa$. Also, we do not know whether the regular cardinals strictly between ω and κ belong to $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$ (as in case of $spec(\mathfrak{t}_\kappa)$).

9. Questions

Recall that Main Theorem 4.1 yields a model in which $\mathfrak{h} = \omega_1$ and COM also contains a cardinal larger than $\mathfrak{h}$.

Question 9.1. Is it consistent that COM contains more than 2 elements?

In particular, we get a model in which $\mathsf{COM} = [\mathfrak{h}, \mathfrak{c}]_{Reg}$ and $\mathfrak{h} < \mathfrak{c} = \omega_2$ (see Corollary 4.2).

Question 9.2. Is it consistent that $\mathsf{COM} \neq [\mathfrak{h}, \mathfrak{c}]_{Reg}$? Is it even consistent that $\mathfrak{h} < \mathfrak{c}$ and $\mathsf{COM} = \{\mathfrak{h}\}$?

Recall that all maximal branches of the generic matrix of Main Theorem 4.1 are cofinal. In the Cohen model, $\mathfrak{h} = \omega_1$, and the tower spectrum contains only ω_1. Consequently, all branches of a distributivity matrix of regular height larger than ω_1 (if there exists any) are dying out.

Question 9.3. Is there a distributivity matrix of regular height larger than $\mathfrak{h}$ in the Cohen model?

ZFC proves that each base matrix of regular height larger than $\mathfrak{h}$ (if there exists any) has branches which are dying out (see Theorem 3.7).

Question 9.4. Is there (consistently) a base matrix of regular height larger than $\mathfrak{h}$?

Let us now turn to questions about the combinatorial distributivity spectrum of $\mathcal{P}(\kappa)/{<}\kappa$ for regular uncountable cardinals κ (see Section 8.3). We do not know whether the set in Tentative Definition 8.5 is actually non-empty:

Question 9.5. Does $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$ (always) contain an element above κ?

By Galgon's result from [20], the existence of a κ-Aronszajn tree implies that κ belongs to $\mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$.

Question 9.6. Is it consistent that $\kappa \notin \mathsf{COM}(\mathcal{P}(\kappa)/{<}\kappa)$?

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References

[1] Bohuslav Balcar, Michal Doucha, and Michael Hrušák. Base tree property. *Order*, 32(1):69–81, 2015.
[2] Bohuslav Balcar, Jan Pelant, and Petr Simon. The space of ultrafilters on **N** covered by nowhere dense sets. *Fund. Math.*, 110(1):11–24, 1980.
[3] Bohuslav Balcar and Petr Simon. On collections of almost disjoint families. *Comment. Math. Univ. Carolin.*, 29(4):631–646, 1988.

[4] James E. Baumgartner. Almost-disjoint sets, the dense set problem and the partition calculus. *Ann. Math. Logic*, 9(4):401–439, 1976.

[5] Andreas Blass. Simple cardinal characteristics of the continuum. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 63–90. Bar-Ilan Univ., Ramat Gan, 1993.

[6] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[7] Andreas Blass. Finite support iterations of σ-centered forcing notions. MathOverflow, 2011. http://mathoverflow.net/questions/84129.

[8] Jörg Brendle. Mad families and iteration theory. In *Logic and algebra*, volume 302 of *Contemp. Math.*, pages 1–31. Amer. Math. Soc., Providence, RI, 2002.

[9] Jörg Brendle. Templates and iterations: Luminy 2002 lecture notes. *RIMS Kōkyūroku*, 1423:1–12, 2005.

[10] Jörg Brendle. Some forcing techniques: Ultrapowers, templates, and submodels. 2021. http://arxiv.org/abs/2101.11494.

[11] Jörg Brendle and Vera Fischer. Mad families, splitting families and large continuum. *J. Symbolic Logic*, 76(1):198–208, 2011.

[12] A. D. Brooke-Taylor, V. Fischer, S. D. Friedman, and D. C. Montoya. Cardinal characteristics at κ in a small $\mathfrak{u}(\kappa)$ model. *Ann. Pure Appl. Logic*, 168(1):37–49, 2017.

[13] Peter Lars Dordal. A model in which the base-matrix tree cannot have cofinal branches. *J. Symbolic Logic*, 52(3):651–664, 1987.

[14] Peter Lars Dordal. Towers in $[\omega]^\omega$ and ${}^\omega\omega$. *Ann. Pure Appl. Logic*, 45(3):247–276, 1989.

[15] Alan Dow. Tree π-bases for $\beta\mathbf{N} - \mathbf{N}$ in various models. *Topology Appl.*, 33(1):3–19, 1989.

[16] Vera Fischer, Marlene Koelbing, and Wolfgang Wohofsky. Fresh functions. *In preparation*.

[17] Vera Fischer, Marlene Koelbing, and Wolfgang Wohofsky. Towers, mad families, and unboundedness. *In preparation*.

[18] Vera Fischer and Saharon Shelah. The spectrum of independence. *Arch. Math. Logic*, 58(7-8):877–884, 2019.

[19] Vera V. Fischer. *The consistency of arbitrarily large spread between the bounding and the splitting numbers*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–York University (Canada).

[20] Geoff Galgon. *Trees, Refining, and Combinatorial Characteristics*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–University of California, Irvine.

[21] Shimon Garti, Menachem Magidor, and Saharon Shelah. On the spectrum of characters of ultrafilters. *Notre Dame J. Form. Log.*, 59(3):371–379, 2018.

[22] Martin Goldstern. Tools for your forcing construction. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 305–360. Bar-Ilan Univ., Ramat Gan, 1993.

[23] Fiorella Guichardaz. *Forcing over ord-transitive models*. 2019. Thesis (Ph.D.)–Albert-Ludwigs-Universität Freiburg, Germany.

[24] Osvaldo Guzmán, Michael Hrušák, and Arturo Martínez-Celis. Canjar filters II. *Proc. of the 2014 RIMS meeting on Reflection principles and set theory of large cardinals, Kyoto, Japan*, 1895:59–67, 2014.

[25] Stephen H. Hechler. Short complete nested sequences in $\beta N \backslash N$ and small maximal almost-disjoint families. *General Topology and Appl.*, 2:139–149, 1972.

[26] Michael Hrušák and Hiroaki Minami. Mathias-Prikry and Laver-Prikry type forcing. *Ann. Pure Appl. Logic*, 165(3):880–894, 2014.

[27] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[28] Haim Judah and Saharon Shelah. The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing). *J. Symbolic Logic*, 55(3):909–927, 1990.

[29] Kenneth Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs.

[30] William Mitchell. Aronszajn trees and the independence of the transfer property. *Ann. Math. Logic*, 5:21–46, 1972/73.

[31] William John Mitchell. *ARONSZAJN TREES AND THE INDEPENDENCE OF THE TRANSFER PROPERTY*. ProQuest LLC, Ann Arbor, MI, 1970. Thesis (Ph.D.)–University of California, Berkeley.

[32] Miroslav Repický. Collapsing of cardinals in generalized Cohen's forcing. *Acta Univ. Carolin. Math. Phys.*, 29(2):67–74, 1988.

[33] Saharon Shelah. Power set modulo small, the singular of uncountable cofinality. *J. Symbolic Logic*, 72(1):226–242, 2007.

[34] Saharon Shelah. The spectrum of characters of ultrafilters on ω. *Colloq. Math.*, 111(2):213–220, 2008.

[35] Saharon Shelah. The character spectrum of $\beta(\mathbb{N})$. *Topology Appl.*, 158(18):2535–2555, 2011.

[36] Saharon Shelah and Zoran Spasojević. Cardinal invariants $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$. *Publ. Inst. Math. (Beograd) (N.S.)*, 72(86):1–9, 2002.

[37] Saharon Shelah and Otmar Spinas. Mad spectra. *J. Symb. Log.*, 80(3):901–916, 2015.

[38] Otmar Spinas and Wolfgang Wohofsky. A Sacks amoeba preserving distributivity of $\mathcal{P}(\omega)/fin$. *To appear in Fund. Math.*

[39] Franklin D. Tall. σ-centred forcing and reflection of (sub)metrizability. *Proc. Amer. Math. Soc.*, 121(1):299–306, 1994.

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