

ON HEIGHTS OF DISTRIBUTIVITY MATRICES

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ABSTRACT. We construct a model in which there exists a distributivity matrix of regular height λ larger than \mathfrak{h} ; both $\lambda = \mathfrak{c}$ and $\lambda < \mathfrak{c}$ are possible. A distributivity matrix is a refining system of mad families without common refinement. Of particular interest in our proof is the preservation of \mathcal{B} -Canjarness.

1. INTRODUCTION

The Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ has attracted a lot of attention in the last decades. One of the characteristics of a partial order is its distributivity. The distributivity of $\mathcal{P}(\omega)/\text{fin}$ is the well-known cardinal characteristic \mathfrak{h} (the *distributivity number*) which has been defined in [2], where the famous base matrix theorem is proved, and is tightly connected to many other structural properties of $\mathcal{P}(\omega)/\text{fin}$ involving towers and mad families.

Recall that \mathfrak{h} is the smallest number of mad families such that there is no single mad family refining¹ all of them (or, equivalently, the least cardinal on which $\mathcal{P}(\omega)/\text{fin}$ adds a new function into the ordinals). It is easy to see and well-known that such a system of \mathfrak{h} many mad families can be chosen to be refining (i.e., having property (2) from Definition 1.1).

In this paper, we consider distributivity matrices of arbitrary height:

Definition 1.1. We say that $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ is a *distributivity matrix of height λ* if

- (1) A_ξ is a mad family, for each $\xi < \lambda$,
- (2) A_η refines A_ξ whenever $\eta \geq \xi$, and
- (3) there is no common refinement, i.e., there is no mad family B which refines every A_ξ .

It is straightforward to check that the existence of distributivity matrices is only a matter of cofinality: if δ is a singular cardinal with $\text{cf}(\delta) = \lambda$, then there exists a distributivity matrix of height δ if and only if there exists one of height λ . Therefore, we are only interested in distributivity matrices of regular height.

Note that \mathfrak{h} is the minimal height of a distributivity matrix. On the other hand, it is easy to check that there can never be a distributivity matrix of regular height larger than \mathfrak{c} . Distributivity matrices and similar objects have been studied e.g. in [2], [11], [13], [24], [10], [1], and [32]. However, to the best of our knowledge, all considered distributivity matrices are of height \mathfrak{h} . The natural question arises whether

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¹For basic definitions, see Section 2.

a distributivity matrix is necessarily of height \mathfrak{h} . The main result of this paper shows that it is consistent that there exists a distributivity matrix of regular height λ larger than \mathfrak{h} (so, in particular, the existence of distributivity matrices of two different regular heights is consistent):

Main Theorem 1.2. *Let V_0 be a model of ZFC which satisfies GCH. In V_0 , let $\omega_1 < \lambda \leq \mu$ be cardinals such that λ is regular and $\text{cf}(\mu) > \omega$. Then there is a c.c.c. (and hence cofinality preserving) extension W of V_0 in which there exists a distributivity matrix of height λ , and $\omega_1 = \mathfrak{h} = \mathfrak{b} < \mathfrak{c} = \mu$.*

We construct our model W as follows. We start with V_0 and first go to the Cohen extension in which $\mathfrak{c} = \mu$. In this model V , we define a forcing iteration (see Section 3.1) which adds a distributivity matrix of height λ . Building on ideas from [22], we use c.c.c. iterands which approximate the distributivity matrix by finite conditions; we have to use an iteration, because after a single step of the forcing, new reals are added, which prevents the generically added almost disjoint families from being maximal. We show that the generic object is actually a distributivity matrix: in particular, the branches are towers (see Section 4.3) and the levels are mad families (see Section 4.4); for that, we use complete subforcings (which are based on the notion of eligible set; see Section 3.4) to capture new subsets of ω (see Section 4.2).

To show that $\omega_1 = \mathfrak{h} = \mathfrak{b}$, we show that $\mathfrak{b} = \omega_1$, and use the fact that $\mathfrak{h} \leq \mathfrak{b}$ holds in ZFC. In fact, we show that the ground model reals $\mathcal{B} = \omega^\omega \cap V_0$ remain unbounded. For that, we represent our iteration as a finer iteration of Mathias forcings with respect to filters (see Section 6.1). We use a characterization from [21] to show that these filters are \mathcal{B} -Canjar (see Section 5 and Section 6.2), i.e., that the corresponding Mathias forcings preserve the unboundedness of \mathcal{B} . In [14], the same is done for Hechler's original forcings [22] to add a tower or to add a mad family.

More precisely, we can use a genericity argument to show that the filters are \mathcal{B} -Canjar at the stage where they appear, but we need the \mathcal{B} -Canjarness in later stages of the iteration. Since the notion of \mathcal{B} -Canjarness of a filter is not absolute (see² Example 5.8), we have to develop a method how to guarantee that the \mathcal{B} -Canjarness of a filter is not destroyed by Mathias forcings with respect to certain other filters. One basic ingredient is defining a "sum" $\mathcal{F}_0 \oplus \mathcal{F}_1$ of two (or finitely many) filters \mathcal{F}_0 and \mathcal{F}_1 for which the following holds true (see Lemma 5.12):

Proposition 1.3. *If $\mathcal{B} \subseteq \omega^\omega$ is unbounded and $\mathcal{F}_0 \oplus \mathcal{F}_1$ is \mathcal{B} -Canjar, then Mathias forcing with respect to \mathcal{F}_1 forces " \mathcal{F}_0 is \mathcal{B} -Canjar".*

We conclude the paper with further discussion and some open questions. In Section 7.1, we consider the nature of maximal branches through distributivity matrices. There are two possibilities for a maximal branch: either it is cofinal or not. Consistently, there are distributivity matrices of height \mathfrak{h} without cofinal branches (this was shown in [11] and [13]). In the model of Main Theorem 1.2, all maximal branches of the (generic) distributivity matrix of height $\lambda > \mathfrak{h}$ are cofinal. In the Cohen model, however, there are no distributivity matrices of this type of height larger than \mathfrak{h} . In Section 7.2, we discuss the notion of a distributivity spectrum.

²We thank Osvaldo Guzmán [20] for providing an example of non-absoluteness.

2. PRELIMINARIES

In this section, we recall some very basic and well-known definitions and facts. The reader should feel free to skip this section and only come back if necessary.

Let $[\omega]^\omega$ denote the collection of infinite subsets of ω , and let \subseteq^* denote the pre-order of almost-inclusion: $b \subseteq^* a$ if $b \setminus a$ is finite. We write $a =^* b$ if $a \subseteq^* b$ and $b \subseteq^* a$. We say that a and b are *almost disjoint* if $a \cap b$ is finite. Moreover, we say that $A \subseteq [\omega]^\omega$ is an *almost disjoint family* (or *ad family*) if a and a' are almost disjoint whenever $a, a' \in A$ with $a \neq a'$. An almost disjoint family A is *maximal* (called *mad family*) if for each $b \in [\omega]^\omega$ there exists $a \in A$ such that $|b \cap a| = \aleph_0$ (i.e., if A is a maximal antichain in $([\omega]^\omega, \subseteq^*)$). For two almost disjoint families A and B , we say that B *refines* A if for each $b \in B$ there exists an $a \in A$ with $b \subseteq^* a$. Let

$$\text{spec}(\alpha) := \{\mu \mid \mu \text{ is an infinite cardinal and there is a mad family of size } \mu\}$$

be the *mad spectrum on* ω , and let $\alpha := \min(\text{spec}(\alpha))$ be the *almost disjointness number*. It is well-known and easy to see that there are always mad families of size α , i.e., $\alpha \in \text{spec}(\alpha)$. Indeed, by identifying $2^{<\omega}$ with ω and taking the set of branches through the tree $2^{<\omega}$, we get an almost disjoint family of size α , which can be extended to a mad family (using the axiom of choice).

Recall from Definition 1.1 that a distributivity matrix $\{A_\xi \mid \xi < \lambda\}$ is a refining system of mad families without common refinement. Such a system can be viewed as a tree (which we think of growing downwards): for each $\xi < \lambda$, the elements of the mad family A_ξ form the level ξ of the tree, and for $b \in A_\eta$ and $a \in A_\xi$ with $\eta > \xi$, the element b is below the element a in the tree if and only if $b \subseteq^* a$. Due to the refining structure of the distributivity matrix, each element of A_η is below exactly one element of A_ξ . Note that this tree is necessarily splitting³ at some limit levels: this is because there always appear \subseteq^* -decreasing sequences of limit length which have no weakest lower bound, and so no single element below such a sequence can be enough to get maximality of the next level.

We say that $\langle a_\xi \mid \xi < \delta \rangle$ is a *branch through the distributivity matrix* $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ if $a_\xi \in A_\xi$ for each $\xi < \delta$, and $a_\eta \subseteq^* a_\xi$ for each $\xi \leq \eta < \delta$. We say that the branch is *maximal* if there is no branch through \mathcal{A} strictly extending it. As a matter of fact, a maximal branch through a distributivity matrix can be cofinal or not; for a discussion of different types of distributivity matrices (in particular such without cofinal branches), see Section 7.1.

We say that $b \in [\omega]^\omega$ *intersects* a distributivity matrix $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ if for each $\xi < \lambda$ there is an $a \in A_\xi$ with $b \subseteq^* a$. Note that Definition 1.1(3) is equivalent to

$$(3') \quad \{b \in [\omega]^\omega \mid b \text{ intersects } \mathcal{A}\} \text{ is not dense in } ([\omega]^\omega, \subseteq^*), \text{ i.e., there is an } \bar{a} \in [\omega]^\omega \text{ such that no } b \subseteq^* \bar{a} \text{ intersects } \mathcal{A};$$

in particular, (3') holds if there is *no* b intersecting \mathcal{A} . If this is the case, we call \mathcal{A} *normal*. In fact, a distributivity matrix can always be turned into a normal distributivity matrix of the same height (basically by “restricting” the matrix to a witness \bar{a} for (3')).

We say that a distributivity matrix $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ is a *base matrix* if $\bigcup_{\xi < \lambda} A_\xi$ is dense in $([\omega]^\omega, \subseteq^*)$. It is straightforward to check that a base matrix is always normal.

³See also the discussion in Section 3.1 about the generic distributivity matrix of Main Theorem 1.2, whose underlying tree is splitting everywhere.

For a sequence $\langle a_\xi \mid \xi < \delta \rangle \subseteq [\omega]^\omega$, we say that $b \in [\omega]^\omega$ is a *pseudo-intersection* of $\langle a_\xi \mid \xi < \delta \rangle$ if $b \subseteq^* a_\xi$ for each $\xi < \delta$. We say that $\langle a_\xi \mid \xi < \delta \rangle$ is a *tower of length δ* if $a_\eta \subseteq^* a_\xi$ for any $\eta > \xi$, and it does not have an infinite pseudo-intersection. Let

$$\text{spec}(t) := \{\delta \mid \delta \text{ is regular and there is a tower of length } \delta\}$$

be the *tower spectrum*, and let $t := \min(\text{spec}(t))$ be the *tower number*. Note that whenever $\langle a_\xi \mid \xi < \delta \rangle$ is a tower, then there is a (sub)tower of length $\text{cf}(\delta)$. On the other hand, each tower of length $\text{cf}(\delta)$ can be expanded to one of length δ (by repeating elements). Therefore the restriction to regular cardinals in the definition of the tower spectrum makes sense.

For $f, g \in \omega^\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We say that $\mathcal{B} \subseteq \omega^\omega$ is an *unbounded family*, if there exists no $g \in \omega^\omega$ with $f \leq^* g$ for all $f \in \mathcal{B}$. The *(un)bounding number* \mathfrak{b} is the smallest size of an unbounded family in ω^ω . The following inequalities between the cardinal characteristics are well-known and not too hard to prove (see, e.g., [4] for more details):

$$(1) \quad \omega_1 \leq t \leq \mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}.$$

3. FORCING A DISTRIBUTIVITY MATRIX

In this section, we start with the proof of our main result, i.e., Main Theorem 1.2: we define a forcing (see Section 3.1), show basic properties of the forcing and of the generic object (see Sections 3.2 and 3.3), and prove a crucial lemma about complete subforcings (see Section 3.4). In Section 4, we will finish the proof that the generic object is indeed a distributivity matrix. Section 5 and Section 6 are devoted to the remaining part of the proof of Main Theorem 1.2, i.e., to showing that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ in the final model.

3.1. Definition of the forcing iteration. We will define a forcing for adding a distributivity matrix. The definition has been motivated by the forcing for adding towers and mad families from Hechler's paper [22]. The presentation of our forcing will be somewhat different from the presentation of Hechler's forcings in [22]. In [14], we represent these forcings in a form which is analogous to our definition of \mathbb{Q}_α below.

We proceed as follows. In V_0 , let \mathbb{C}_μ be the usual forcing for adding μ many Cohen reals, and let V be the extension by \mathbb{C}_μ . In V , we perform our main forcing iteration of length λ which is going to add a distributivity matrix of height λ . The iteration is going to be a finite support iteration whose iterands have the countable chain condition (see Lemma 3.5) and are of size continuum; in particular, the size of the continuum stays the same during the whole iteration, and hence $\mathfrak{c} = \mu$ holds true in the final model (see Lemma 3.6).

As discussed in Section 2, a distributivity matrix can be viewed as a tree, where each node is equipped with an element of $[\omega]^\omega$. In fact, our generic distributivity matrix $\{A_{\xi+1} \mid \xi < \lambda\}$ will be based on the tree $\lambda^{<\lambda}$: each node $\sigma \in \lambda^{<\lambda}$ of successor length will carry an infinite set $a_\sigma \subseteq \omega$ such that for each $\xi < \lambda$,

$$A_{\xi+1} = \{a_\sigma \mid \sigma \in \lambda^{\xi+1}\}$$

is a mad family, and $a_\sigma \subseteq^* a_\tau$ if σ extends τ . In particular, all maximal branches of our distributivity matrix will be cofinal.

We write $\tau \leq \sigma$ if $\tau \subseteq \sigma$ (i.e., if σ extends τ); we write $\tau \triangleleft \sigma$ if $\tau \leq \sigma$ and $\tau \neq \sigma$. The length of σ is denoted by $|\sigma|$. We think of the tree $\lambda^{<\lambda}$ as “growing downwards”, i.e., we say that σ is below τ if $\tau \leq \sigma$; moreover, we say that $\sigma \hat{\ } j$ is to the left of $\sigma \hat{\ } i$ whenever $j < i$.

Note that our mad families $A_{\xi+1}$ are indexed by successor ordinals only, for the following reason. Since there are \subseteq^* -decreasing sequences of limit length which do not have weakest lower bounds, and the mad family on the level directly below such a sequence has to be “maximal below the sequence” (i.e., each pseudo-intersection of the sequence is compatible with some element of the mad family), it is necessary that the underlying tree “splits” at such limit levels. However, $\lambda^{<\lambda}$ does not split at limit levels, so it is convenient to equip only nodes σ of successor length with infinite sets a_σ , and use nodes ρ of limit length to talk about the branch $\langle a_\sigma \mid \sigma \triangleleft \rho \rangle$.

Before giving the precise definition of our forcing iteration, let us describe the idea informally. We start with the tree $\lambda^{<\lambda}$ in V , and generically add a set $a_\sigma \subseteq \omega$ for every $\sigma \in \lambda^{<\lambda}$ (of successor length) in such a way that $a_\tau \supseteq^* a_\sigma$ if $\tau \leq \sigma$, and $a_\sigma \cap a_\tau =^* \emptyset$ if $|\sigma| = |\tau|$. This results in a refining system of almost disjoint families. But these antichains are not maximal, which can be seen as follows. The forcing adds new reals (any a_τ is a new infinite subsets of ω), so there are new branches through $\lambda^{<\omega}$. Let ρ be such a new branch of length ω ; then $\langle a_{\rho \upharpoonright n} \mid n < \omega \rangle$ is a \subseteq^* -decreasing sequence of length ω (in the extension), so (since $\omega < \mathfrak{t}$) it has an infinite pseudo-intersection b . It is easy to see that b is incompatible with all a_σ with $\sigma \in \lambda^{\omega+1} \cap V$, so the antichain $\{a_\sigma \mid |\sigma| = \omega + 1\}$ is not maximal.

To solve this problem, we use a finite support iteration

$$\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\}$$

of length λ . At each step \mathbb{Q}_α of the iteration, a set a_σ is added for every new node σ (of successor length) of the tree $\lambda^{<\lambda}$. In the definition below, we will use T_α to denote these new nodes (the nodes σ for which no sets a_σ have been added yet), and we will use T'_α to denote the old nodes (the nodes σ for which there has already been added a set a_σ at an earlier stage $\beta < \alpha$ of the iteration). The sets a_σ with $\sigma \in T'_\alpha$ will be used in the definition of the iterand \mathbb{Q}_α . In the definition of the very first forcing \mathbb{Q}_0 of the iteration, the set T_0 will be the collection of all nodes in $\lambda^{<\lambda}$ (of successor length), and T'_0 will be empty (since no sets a_σ have been defined yet). After λ many steps, we are finished, because no new nodes appear at stage λ (see Lemma 3.7).

As usual, we abuse notation and identify $a_\sigma \subseteq \omega$ with its characteristic function in 2^ω .

Definition 3.1. Let $\alpha < \lambda$, and assume that \mathbb{P}_α has already been defined. For every $\beta \leq \alpha$, let G_β be generic for \mathbb{P}_β . We work in $V[G_\alpha]$ to define our iterand \mathbb{Q}_α . First (letting succ denote the sequences of ordinals of successor length), let

$$T'_\alpha := \bigcup_{\beta < \alpha} (\lambda^{<\lambda} \cap \text{succ})^{V[G_\beta]},$$

and let

$$T_\alpha := (\lambda^{<\lambda} \cap \text{succ})^{V[G_\alpha]} \setminus T'_\alpha.$$

Note that for each $\sigma \in T'_\alpha$, there exists a minimal $\beta < \alpha$ such that $\sigma \in V[G_\beta]$, and hence, by induction, a_σ has been added by \mathbb{Q}_β . For each $\sigma \in T_\alpha$, the set a_σ is not defined yet, and will be added by \mathbb{Q}_α (see below, at the end of the definition).

Now \mathbb{Q}_α is defined as follows: $p \in \mathbb{Q}_\alpha$ if p is a function with finite domain, $\text{dom}(p) \subseteq T_\alpha$, and for each $\sigma \in \text{dom}(p)$, we have

$$p(\sigma) = (s_\sigma^p, f_\sigma^p, h_\sigma^p) = (s_\sigma, f_\sigma, h_\sigma),$$

where⁴

- (1) $s_\sigma \in 2^{<\omega}$,
- (2) for each $\tau \in \text{dom}(p)$ with $\tau \triangleleft \sigma$, $|s_\tau| \geq |s_\sigma|$,
- (3) $\text{dom}(f_\sigma) \subseteq (\text{dom}(p) \cup T'_\alpha) \cap \{\tau \in T_\alpha \cup T'_\alpha \mid \tau \triangleleft \sigma\}$, finite,⁵
- (4) $f_\sigma: \text{dom}(f_\sigma) \rightarrow \omega$,
- (5) whenever $\tau \in \text{dom}(f_\sigma) \cap T_\alpha$, and⁶ $n \in \text{dom}(s_\tau) \cap \text{dom}(s_\sigma)$ with $n \geq f_\sigma(\tau)$, we have

$$s_\tau(n) = 0 \rightarrow s_\sigma(n) = 0,$$

and whenever $\tau \in \text{dom}(f_\sigma) \cap T'_\alpha$ and $n \in \text{dom}(s_\sigma)$ with $n \geq f_\sigma(\tau)$, we have

$$a_\tau(n) = 0 \rightarrow s_\sigma(n) = 0,$$

- (6) $\text{dom}(h_\sigma) \subseteq \text{dom}(p) \cap \{\rho \hat{\ } j \mid j < i\}$ (where $\rho \in \lambda^{<\lambda}$ and $i \in \lambda$ such that $\sigma = \rho \hat{\ } i$),
- (7) $h_\sigma: \text{dom}(h_\sigma) \rightarrow \omega$,
- (8) whenever $\tau \in \text{dom}(h_\sigma)$, and $n \in \text{dom}(s_\tau) \cap \text{dom}(s_\sigma)$ with $n \geq h_\sigma(\tau)$, we have

$$s_\tau(n) = 0 \vee s_\sigma(n) = 0.$$

The order on \mathbb{Q}_α is defined as follows: $q \leq p$ (“ q is stronger than p ”) if

- (i) $\text{dom}(p) \subseteq \text{dom}(q)$,
- (ii) and for each $\sigma \in \text{dom}(p)$, we have
 - (a) $s_\sigma^p \leq s_\sigma^q$,
 - (b) $\text{dom}(f_\sigma^p) \subseteq \text{dom}(f_\sigma^q)$ and $f_\sigma^p(\tau) \geq f_\sigma^q(\tau)$ for each $\tau \in \text{dom}(f_\sigma^p)$,
 - (c) $\text{dom}(h_\sigma^p) \subseteq \text{dom}(h_\sigma^q)$ and $h_\sigma^p(\tau) \geq h_\sigma^q(\tau)$ for each $\tau \in \text{dom}(h_\sigma^p)$.

Given a generic filter G for \mathbb{Q}_α , we define, for each $\sigma \in T_\alpha$,

$$a_\sigma := \bigcup \{s_\sigma^p \mid p \in G \wedge \sigma \in \text{dom}(p)\}.$$

This completes the definition of the forcing.

Let us describe the role of the parts of a condition: s_σ is a finite approximation of the set a_σ assigned to σ , whereas the functions f_σ and h_σ are promises for guaranteeing that the branches through the generic matrix are \subseteq^* -decreasing and the levels are almost disjoint families, respectively. More precisely, f_σ promises that $a_\sigma \setminus f_\sigma(\tau) \subseteq a_\tau$ for each $\tau \in \text{dom}(f_\sigma)$ and h_σ promises that $a_\tau \cap a_\sigma \subseteq h_\sigma(\tau)$ for each $\tau \in \text{dom}(h_\sigma)$ (see Lemma 3.10).

⁴The paragraph after the definition gives a short intuitive explanation of the roles of s_σ , f_σ , and h_σ .

⁵Note that in (6), it automatically follows that $\text{dom}(h_\sigma)$ is finite because $\text{dom}(h_\sigma) \subseteq \text{dom}(p)$, but not here because $\text{dom}(f_\sigma) \subseteq \text{dom}(p) \cup T'_\alpha$.

⁶By (2), here it is actually sufficient to require $n \in \text{dom}(s_\sigma)$.

Remark 3.2. Note that \mathbb{Q}_α is not separative. As an example, we can take p and q as follows: $\text{dom}(p) = \text{dom}(q) = \{\sigma, \tau\}$ (where σ is to the left of τ within the same block), $p(\tau) = q(\tau) = (\langle 1 \rangle, \emptyset, h)$ where $h(\sigma) = 0$ and $p(\sigma) = (\langle \rangle, \emptyset, \emptyset)$ and $q(\sigma) = (\langle 0 \rangle, \emptyset, \emptyset)$. It is easy to see that $p \not\leq q$, but $p \leq^* q$, i.e., any condition stronger than p is compatible with q . Therefore, we later need to provide certain iteration lemmas for the general case of non-separative forcings (see Lemma 4.3).

Remark 3.3. Let us remark that it is possible to derive a bit more from the proof of Main Theorem 1.2 than what is stated in the theorem. Our forcing construction is based on the tree $\lambda^{<\lambda}$ (see Definition 3.1) and therefore results in a specific kind of distributivity matrix of height λ : first, all its maximal branches are cofinal, and second, the underlying tree has λ -splitting, i.e., each node has exactly λ many immediate successors. From the latter property, it immediately follows that $\lambda \in \text{spec}(\alpha)$ (in particular, $\alpha \leq \lambda$).

We can modify the construction (by changing the underlying tree) to obtain different kinds of distributivity matrices of height λ . In fact, the following generalization of Main Theorem 1.2 holds true: if $\omega_1 \leq \lambda \leq \text{cf}(\theta)$ and $\theta \leq \mu$ with λ regular and $\text{cf}(\mu) > \omega$, then (using $\theta^{<\lambda}$ as the underlying tree) there is an extension such that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ and $\mu = \mathfrak{c}$, and there exists a distributivity matrix of height λ with θ -splitting (hence, in particular, $\theta \in \text{spec}(\alpha)$). The reason why we have to require $\lambda \leq \text{cf}(\theta)$ is Lemma 4.14 (see Remark 4.16). It would even be possible to have different splitting at different nodes, provided that all the splitting sizes have cofinality at least λ . This way, we can get more values into $\text{spec}(\alpha)$ (similar as in Hechler's paper [22], where he constructs a model in which all uncountable regular cardinals up to \mathfrak{c} are in $\text{spec}(\alpha)$).

Note that even $\lambda = \omega_1$ is possible in our forcing construction. It is true that it does not yield a distributivity matrix of regular height larger than \mathfrak{h} (because $\mathfrak{h} = \omega_1$ holds true in our model), but we can obtain distributivity matrices of height ω_1 with additional features (e.g., by choosing $\theta = \lambda = \omega_1$, resulting in a matrix with ω_1 -splitting). Observe that it is always possible to turn a distributivity matrix with θ -splitting into a distributivity matrix with \mathfrak{c} -splitting (of the same height), by just taking every ω th level (and deleting all other levels). It is not clear whether it is possible to do it the other way round, i.e., to get a distributivity matrix with θ -splitting (for $\theta < \mathfrak{c}$) from a distributivity matrix with \mathfrak{c} -splitting (even if θ happens to be in $\text{spec}(\alpha)$). Therefore, we decided to state and prove Main Theorem 1.2 for $\theta = \lambda$ (i.e., small splitting), and not for $\theta = \mathfrak{c}$.

As a matter of fact, the Cohen model satisfies $\text{spec}(\alpha) = \{\omega_1, \mathfrak{c}\}$ (see, e.g., [6, Proposition 3.1]). Thus, if $\omega_1 < \theta < \mathfrak{c}$, there are no mad families of size θ , and hence no distributivity matrix with θ -splitting in the Cohen model. If we choose, e.g., $\lambda = \omega_1$, $\theta = \omega_2$ and $\mu = \omega_3$ in the generalization of our main theorem described in the above remark, the generic matrix cannot exist in the Cohen model with $\mathfrak{c} = \omega_3$. On the other hand, our forcing construction with $\theta = \lambda = \omega_1$ and $\omega_1 < \mu$ actually results in the Cohen model with $\mathfrak{c} = \mu$: this can be seen by representing the iteration as an iteration of Mathias forcings with respect to filters, as described in Section 6.1; since $\theta = \lambda = \omega_1$, all the filters are countably generated, therefore the respective Mathias forcings are forcing equivalent to Cohen forcing. Therefore, we can in particular derive the following from our proof of Main Theorem 1.2:

Observation 3.4. *Let $\mu > \omega_1$. Then, in the Cohen model with $\mathfrak{c} = \mu$ (i.e., in the extension of a GCH model by \mathbb{C}_μ), there exists a distributivity matrix of height ω_1 which is ω_1 -splitting everywhere.*

In particular, the distributivity matrix is ω_1 -splitting at limit levels; note that being ω_1 -splitting at successors is not so much of interest because it is known that $\omega_1 \in \text{spec}(\alpha)$, and so such splitting at successors can be accomplished by hand.

3.2. Countable chain condition and some implications. We are now going to show that our iterands \mathbb{Q}_α have the c.c.c.; it immediately follows that their finite support iteration \mathbb{P}_λ has the c.c.c. as well, and therefore it does not change cofinalities or cardinalities.

Lemma 3.5. \mathbb{Q}_α is precaliber ω_1 (hence in particular c.c.c.) for every $\alpha < \lambda$.

In fact, \mathbb{Q}_α is even σ -centered: in Section 6, we are going to show that each \mathbb{Q}_α can be represented as a finite support iteration of length strictly less than c^+ of Mathias forcings with respect to certain filters; since filtered Mathias forcings are always σ -centered (see Definition 5.1 and the subsequent remark), and σ -centeredness is preserved under finite support iterations of length strictly less than c^+ , it follows that \mathbb{Q}_α is σ -centered (see also Corollary 6.2).

Proof of Lemma 3.5. Let $\{p_i \mid i < \omega_1\} \subseteq \mathbb{Q}_\alpha$. We want to show that the set cannot be an antichain. First note that it is possible to extend⁷ all s_σ^p (with $\sigma \in \text{dom}(p)$) of a condition $p \in \mathbb{Q}_\alpha$ to the same length $N_p \in \omega$, by just adding 0's at the end. Therefore we can assume without loss of generality that there exists N such that $|s_\sigma^{p_i}| = N$ for each $i \in \omega_1$ and each $\sigma \in \text{dom}(p_i)$ (the reason why we want the $s_\sigma^{p_i}$ to have the same length, is to avoid trouble with Definition 3.1(2)). Since $\text{dom}(p_i) \subseteq T_\alpha \subseteq \lambda^{<\lambda}$ is finite for every i , we can apply the Δ -system lemma to find a subset $X \subseteq \omega_1$ of size ω_1 such that $\{\text{dom}(p_i) \mid i \in X\}$ is a Δ -system with root $R \subseteq T_\alpha$. Also, $\text{dom}(f_\sigma^{p_i}) \cap T'_\alpha$ is finite for each $i \in X$ and each $\sigma \in \text{dom}(p_i)$, so we can repeatedly apply⁸ the Δ -system lemma, for each $\sigma \in R$ (hence finitely many times), to find a subset $Y \subseteq X$ of size ω_1 such that $\{\text{dom}(f_\sigma^{p_i}) \cap T'_\alpha \mid i \in Y\}$ is a Δ -system with root A_σ for each $\sigma \in R$. Moreover, we can assume without loss of generality that for each $\sigma \in R$, there are s_σ^* , f_σ^* , and h_σ^* such that for all $i \in Y$, we have $s_\sigma^{p_i} = s_\sigma^*$, $f_\sigma^{p_i} \upharpoonright (R \cup A_\sigma) = f_\sigma^*$, and $h_\sigma^{p_i} \upharpoonright R = h_\sigma^*$. Now it is straightforward to check that any two conditions from $\{p_i \mid i \in Y\}$ are compatible; in fact, any finitely many of them have a common lower bound. \square

We now show that the size of the continuum in the final model is as desired; in fact, the following holds:

Lemma 3.6. Let $\alpha \leq \lambda$. Then, in $V[\mathbb{P}_\alpha]$, we have $c = \mu$.

Proof. First note that $V \models c = \mu \wedge \mu^{<\mu} = \mu$, because it is the extension after adding μ many Cohen reals over a model which satisfies GCH. We show simultaneously by induction on $\alpha \leq \lambda$ that

- (1) $|\mathbb{P}_\alpha| \leq \mu$ and
- (2) $V[\mathbb{P}_\alpha] \models c = \mu \wedge |T_\alpha| \leq \lambda^{<\lambda} \leq \mu$.

Clearly (1) and (2) hold for \mathbb{P}_0 since \mathbb{P}_0 is the trivial forcing. Now assume that we have shown (1) and (2) for each $\alpha' < \alpha$.

⁷In Lemma 3.14, we will show a stronger fact.

⁸In case $\alpha = 0$, this is not necessary, because $T'_0 = \emptyset$ (in the definition of \mathbb{Q}_0).

To show (1), argue as follows. If α is a limit, then $|\mathbb{P}_\alpha| \leq \mu$, because we use finite support, each $\mathbb{P}_{\alpha'} \leq \mu$, and $\alpha \leq \mu$. If $\alpha = \alpha' + 1$ is a successor, $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'} * \mathbb{Q}_{\alpha'}$. By induction, $\mathbb{P}_{\alpha'} \Vdash |T_{\alpha'}| \leq \lambda^{<\lambda} \leq \mu$, and so it is easy to check that $|\mathbb{Q}_{\alpha'}| \leq \mu$, hence $|\mathbb{P}_{\alpha'} * \mathbb{Q}_{\alpha'}| \leq \mu$.

To show (2), we count nice names. For every real x in $V[\mathbb{P}_\alpha]$, there exists a nice name. Such a nice name consists of antichains X_n in \mathbb{P}_α for each entry $x(n)$. By the c.c.c., each X_n is countable, so the number of nice names for reals is $|\mathbb{P}_\alpha|^{\leq \omega} \leq \mu$, so there are only μ many reals in $V[\mathbb{P}_\alpha]$. Similarly, a nice name for an element of $\lambda^{<\lambda}$ consists of less than μ many countable antichains, and since $|\mathbb{P}_\alpha|^{<\mu} \leq \mu^{<\mu} = \mu$, there are at most μ many elements of $\lambda^{<\lambda}$ in $V[\mathbb{P}_\alpha]$. \square

The following lemma guarantees that, by the end of the iteration of length λ , a set a_σ has been added for every $\sigma \in \lambda^{<\lambda}$ of successor length (so T_λ would be empty, hence \mathbb{Q}_λ would be the trivial forcing – if we would continue the iteration after λ many stages):

Lemma 3.7. *Every node $\sigma \in \lambda^{<\lambda}$ from the final model $V[\mathbb{P}_\lambda]$ already appears in some intermediate model $V[\mathbb{P}_\alpha]$ with $\alpha < \lambda$.*

Proof. Let $\dot{\sigma}$ be a nice \mathbb{P}_λ -name for σ ; more precisely, $\dot{\sigma}$ has the following form. First, $\dot{\sigma}$ contains an antichain which decides the length of $\dot{\sigma}$. Since \mathbb{P}_λ has the c.c.c., this antichain is countable, so there are only countably many values possible for the length; let $\xi < \lambda$ be larger than all the possible values. Now, for all $\xi' < \xi$, there is an antichain deciding the entry of $\dot{\sigma}(\xi')$ (if ξ' is less than the length of $\dot{\sigma}$). Again, by c.c.c. all these antichains are countable. So there are ξ many countable antichains which are in $\dot{\sigma}$; the union of these antichains contains less than λ many elements. Since we use finite support, there exists an $\alpha < \lambda$ such that $\dot{\sigma}$ is a \mathbb{P}_α -name, hence $\sigma \in V[\mathbb{P}_\alpha]$. \square

3.3. The generic distributivity matrix. Let G be a generic filter for the iteration \mathbb{P}_λ . In the final model $V[G]$, we derive our “intended generic object” (which is going to be a distributivity matrix of height λ) from the generic filter G as follows. For each $\sigma \in \lambda^{<\lambda} \cap \text{succ}$, we can fix the minimal $\alpha < \lambda$ such that $\sigma \in V[G_\alpha]$ (see Lemma 3.7). Then in $V[G_\alpha]$, the node σ belongs to T_α , and, letting $G(\alpha)$ be the corresponding filter for \mathbb{Q}_α , the set

$$(2) \quad a_\sigma = \bigcup \{s_\sigma^p \mid p \in G(\alpha) \wedge \sigma \in \text{dom}(p)\}$$

is added by \mathbb{Q}_α . Back in the final model $V[G]$, we let, for each $\xi < \lambda$,

$$A_{\xi+1} := \{a_\sigma \mid |\sigma| = \xi + 1\}$$

(which is going to be a mad family). Here, we are going to show that the generic object $\{A_{\xi+1} \mid \xi < \lambda\}$ is a refining system of almost disjoint families.

Our first lemma guarantees that each a_σ is going to have infinitely many 1's. Note that, whenever we write “ $s(m) = 1$ ”, we actually mean “ $m \in \text{dom}(s)$ and $s(m) = 1$ ”.

Lemma 3.8. *Let $\alpha < \lambda$. For each $\sigma \in T_\alpha$ and each $n \in \omega$, the set*

$$D_{\sigma,n} = \{q \in \mathbb{Q}_\alpha \mid \sigma \in \text{dom}(q) \text{ and } |s_\sigma^q| \geq n\}$$

is dense in \mathbb{Q}_α . In particular, $\text{dom}(a_\sigma) = \omega$ (i.e., a_σ can be viewed as a subset of ω).

Proof. Let $\sigma \in T_\alpha$, $n \in \omega$ and $p \in \mathbb{Q}_\alpha$. If $\sigma \notin \text{dom}(p)$, extend p to $p \cup \{(\sigma, \langle \cdot \rangle, \emptyset, \emptyset)\}$. From now on, we assume that $\sigma \in \text{dom}(p)$.

First note that (2), (5), and (8) in Definition 3.1 give restrictions on how s_τ^p 's can be extended; however, it is always possible to extend an s_τ^p by 0's (provided that all $s_{\tau'}^p$ with $\tau' \triangleleft \tau$ are at least as long, as demanded by (2)): no matter what the $s_{\tau'}^p$ with $\tau' \triangleleft \tau$ are, extending s_τ^p by 0 never makes (5) false (due to (2), s_τ^p for $\tau \triangleleft \bar{\tau}$ are shorter and therefore do not matter at all); similarly, no matter what the s_π^p are with π having the same predecessor as τ , extending s_τ^p by 0 never makes (8) false.

Now, for every $\tau \in \text{dom}(p)$ with $\tau \trianglelefteq \sigma$, if $|s_\tau^p| < n$, extend s_τ^p with 0's to length n . In particular, the resulting condition q satisfies $|s_\sigma^q| \geq n$, as desired. \square

The next lemma shows that we can always assume that the domain of f_σ^p and the domain of h_σ^p is as large as possible. Parts of the lemma will be essential also later, for the notion of “full condition” (see Definition 3.13).

Lemma 3.9. *Let $p \in \mathbb{Q}_\alpha$ and $\sigma \in \text{dom}(p)$. Then there exists a $q \leq p$ such that $\text{dom}(q) = \text{dom}(p)$, and the following holds:*

- (a) $\tau \in \text{dom}(f_\sigma^q)$ for each $\tau \in \text{dom}(q)$ with $\tau \triangleleft \sigma$, and
- (b) (letting $\sigma = \rho \hat{\ } i$) $\rho \hat{\ } j \in \text{dom}(h_\sigma^q)$ for each $j < i$ with $\rho \hat{\ } j \in \text{dom}(q)$.

In particular, the set

$$D := \{q \in \mathbb{Q}_\alpha \mid (a) \text{ and } (b) \text{ holds for each } \sigma \in \text{dom}(q)\}$$

is dense in \mathbb{Q}_α .

Moreover, if $\alpha > 0$, then the following holds: whenever $\tau' \triangleleft \sigma$ with $\tau' \in T'_\alpha$ (i.e., $a_{\tau'}$ has already been added before), there exists $q \leq p$ such that $\text{dom}(q) = \text{dom}(p)$, $q \in D$, and $\tau' \in \text{dom}(f_\sigma^q)$.

Proof. For every $\tau \in \text{dom}(p) \setminus \text{dom}(f_\sigma^p)$ with $\tau \triangleleft \sigma$, let $f_\sigma^q(\tau) := |s_\tau^p|$. For every $\rho \hat{\ } j \in \text{dom}(p) \setminus \text{dom}(h_\sigma^p)$ with $j < i$, let $h_\sigma^q(\rho \hat{\ } j) := |s_\sigma^p|$. For the moreover part, let $f_\sigma^q(\tau') := |s_\sigma^p|$. If we (repeatedly) extend p in this way to q , it is clear that q is a condition with the properties we wanted. \square

The next lemma will be used to show that, for $\tau \triangleleft \sigma$, the set of conditions which force $a_\sigma \subseteq^* a_\tau$ is dense, as well as to show that, for $\rho \hat{\ } i$ and $\rho \hat{\ } j$ with $i \neq j$, the set of conditions which force $a_{\rho \hat{\ } i} \cap a_{\rho \hat{\ } j} =^* \emptyset$ is dense.

Lemma 3.10. *Let $p \in \mathbb{Q}_\alpha$ and $\sigma \in \text{dom}(p)$.*

- (1) *If $\tau \in \text{dom}(f_\sigma^p)$, then $p \Vdash a_\sigma \setminus f_\sigma^p(\tau) \subseteq a_\tau$ (in particular, $p \Vdash a_\sigma \subseteq^* a_\tau$).*
- (2) *If $\tau \in \text{dom}(h_\sigma^p)$, then $p \Vdash a_\tau \cap a_\sigma \subseteq h_\sigma^p(\tau)$ (in particular, $p \Vdash a_\tau \cap a_\sigma =^* \emptyset$).*

Proof. For (1), we use Definition 3.1(5). There are two cases: if $\tau \in T_\alpha$ (i.e., τ is a new node), it follows that $\tau \in \text{dom}(p)$, and that $s_\tau^p(n) = 1$ whenever $n \geq f_\sigma^p(\tau)$ and $s_\sigma^p(n) = 1$; if $\tau \in T'_\alpha$ (i.e., a_τ has already been added by \mathbb{Q}_β for some $\beta < \alpha$), it follows that $a_\tau(n) = 1$ whenever $n \geq f_\sigma^p(\tau)$ and $s_\sigma^p(n) = 1$. In both cases, using (2), we get $p \Vdash a_\sigma \setminus f_\sigma^p(\tau) \subseteq a_\tau$.

The proof of (2) is similar, using Definition 3.1(8). \square

Finally, we can show that in the final model $V[\mathbb{P}_\lambda]$, the sets along branches of $\lambda^{<\lambda}$ are \subseteq^* -decreasing, and the sets on any level of $\lambda^{<\lambda} \cap \text{succ}$ are pairwise almost disjoint.

Corollary 3.11. *In $V[\mathbb{P}_\lambda]$, the following hold:*

- (1) *If $\tau, \sigma \in \lambda^{<\lambda} \cap \text{succ}$ such that $\tau \triangleleft \sigma$, then $a_\sigma \subseteq^* a_\tau$.*
- (2) *If $\rho \in \lambda^{<\lambda}$, and $j < i < \lambda$, then $a_{\rho \smallfrown j} \cap a_{\rho \smallfrown i} =^* \emptyset$. Indeed, the following holds. For each $\sigma, \sigma' \in \lambda^{<\lambda} \cap \text{succ}$ satisfying $|\sigma| = |\sigma'|$ and $\sigma \neq \sigma'$, we have $a_\sigma \cap a_{\sigma'} =^* \emptyset$; in other words, for each $\xi < \lambda$,*

$$A_{\xi+1} = \{a_\sigma \mid \sigma \in \lambda^{\xi+1}\}$$

is an almost disjoint family.

Proof. To show (1), let $\eta < \lambda$ be minimal such that $\sigma \in (\lambda^{<\lambda})^{V[\mathbb{P}_\eta]}$. Lemma 3.8, Lemma 3.9, and Lemma 3.10(1) in particular imply that the set

$$\{q \in \mathbb{Q}_\eta \mid q \Vdash a_\sigma \subseteq^* a_\tau\}$$

is dense. Hence $V[\mathbb{P}_{\eta+1}] \Vdash a_\sigma \subseteq^* a_\tau$, and this remains true in the final model.

To show (2), let $\eta < \lambda$ be minimal such that $\rho \in (\lambda^{<\lambda})^{V[\mathbb{P}_\eta]}$. Lemma 3.8, Lemma 3.9, and Lemma 3.10(2) in particular imply that the set

$$\{q \in \mathbb{Q}_\eta \mid q \Vdash a_{\rho \smallfrown j} \cap a_{\rho \smallfrown i} =^* \emptyset\}$$

is dense; this proves the first assertion of (2). To prove the second assertion of (2), find $\rho \in \lambda^{<\lambda}$ with $\rho \triangleleft \sigma, \sigma'$ and $i, j < \lambda$ with $j \neq i$ such that $\rho \smallfrown j \trianglelefteq \sigma$ and $\rho \smallfrown i \trianglelefteq \sigma'$, and apply the first assertion of (2) as well as (1). \square

Finally, we show that each a_σ is an infinite subset of ω :

Lemma 3.12. *Let $\alpha < \lambda$. For each $\sigma \in T_\alpha$ and each $n \in \omega$, the set*

$$D_{\sigma,n} := \{q \in \mathbb{Q}_\alpha \mid \sigma \in \text{dom}(q) \text{ and } \exists m \geq n (s_\sigma^q(m) = 1)\}$$

is dense in \mathbb{Q}_α . In particular, $a_\sigma \in [\omega]^\omega$ (when viewed as a subset of ω).

Proof. The proof proceeds by induction on $\alpha < \lambda$.

Let $\sigma \in T_\alpha$, $n \in \omega$ and $p \in \mathbb{Q}_\alpha$. By Lemma 3.8, we can assume that $\sigma \in \text{dom}(p)$. Let $N_0 \in \omega$ be bigger than the maximal length of all the s_τ^p with $\tau \in \text{dom}(p)$ and $\tau \trianglelefteq \sigma$, and bigger than n . Let

$$A := \bigcup \{\text{dom}(f_\tau^p) \cap T'_\alpha \mid \tau \in \text{dom}(p) \wedge \tau \trianglelefteq \sigma\}.$$

If A is empty (which in particular holds in case $\alpha = 0$, due to $T'_0 = \emptyset$), let $m \in \omega$ be arbitrary with $m \geq N_0$. Otherwise, let $N_1 \geq N_0$ be large enough such that $a_{\psi'} \setminus N_1 \subseteq a_\psi$ for all $\psi, \psi' \in A$ with $\psi \trianglelefteq \psi'$ (see Corollary 3.11). Moreover, let ψ^* be the longest element of the finite set A , and let $m \geq N_1$ such that $a_{\psi^*}(m) = 1$ (this is possible, since a_{ψ^*} is infinite by induction, due to the fact that $\psi^* \in T'_\alpha$, and hence $\psi^* \in T_\beta$ for some $\beta < \alpha$). Therefore $a_\psi(m) = 1$ for each $\psi \in A$.

Now, for every $\tau \in \text{dom}(p)$ with $\tau \trianglelefteq \sigma$, extend s_τ^p with 0's to length m . Finally, we extend s_τ^p to $s_\tau^p \hat{\ } 1$ for every $\tau \in \text{dom}(p)$ with $\tau \trianglelefteq \sigma$ (in particular, for $\tau = \sigma$). It is easy to check that the resulting $q \leq p$ is indeed a condition, and $s_\sigma^q(m) = 1$, as desired. \square

Altogether, we have proved that $\{A_{\xi+1} \mid \xi < \lambda\}$ is a refining system of ad families, i.e., for each $\xi < \lambda$, $A_{\xi+1}$ is an almost disjoint family, and for all $\xi < \xi' < \lambda$, $A_{\xi'+1}$ refines $A_{\xi+1}$.

To show that $\{A_{\xi+1} \mid \xi < \lambda\}$ is actually a distributivity matrix requires much more work. The proof will be completed in Section 4. After a lot of preparatory work, it will be shown in Section 4.4 that the sets $A_{\xi+1}$ are indeed maximal, and in Section 4.3 that the sets along branches are indeed towers, which implies that there is no set intersecting the whole family $\{A_{\xi+1} \mid \xi < \lambda\}$ (and hence there is no common refinement).

3.4. Eligible sets and complete subforcings. The goal of this section is to show that our forcing \mathbb{Q}_α has complete subforcings which use only part of T_α (see Lemma 3.17). In Section 4.2, this will be extended to the whole iteration (see Lemma 4.12), which will be an important ingredient of the proof that the generic object is a distributivity matrix (see Section 4.3 and Section 4.4). Moreover, we will show in Section 6 that each \mathbb{Q}_α (and hence our whole iteration) can be seen as an iteration of Mathias forcings with respect to certain filters; to show that these filters are \mathcal{B} -Canjar, we will again use Lemma 3.17. Let us start with a concept which is going to be very useful:

Definition 3.13. A condition $p \in \mathbb{Q}_\alpha$ is called *full* if there exists an $N \in \omega$ such that for all $\sigma \in \text{dom}(p)$

- (1) $|s_\sigma^p| = N$,
- (2) $N > \max(\text{rng}(f_\sigma^p))$ and $N > \max(\text{rng}(h_\sigma^p))$,
- (3) $\tau \in \text{dom}(f_\sigma^p)$ for each $\tau \in \text{dom}(p)$ with $\tau \triangleleft \sigma$, and
- (4) (letting $\sigma = \rho \hat{\ } i$) $\rho \hat{\ } j \in \text{dom}(h_\sigma^p)$ for each $j < i$ with $\rho \hat{\ } j \in \text{dom}(p)$.

Moreover, $p \in \mathbb{P}_\lambda$ is full if $p(0)$ is full.

Later, we will consider quotients $\mathbb{P}_\lambda / \mathbb{P}_\eta$ and therefore use a modification, where 0 is replaced by η , i.e., $p(\eta)$ is full; see Remark 4.10.

The set of full conditions is dense:

Lemma 3.14. *For every condition $p \in \mathbb{Q}_\alpha$ there exists a full condition q with $q \leq p$ and $\text{dom}(q) = \text{dom}(p)$. Hence the set of full conditions in \mathbb{P}_λ is dense in \mathbb{P}_λ .*

Proof. We can assume that p belongs to the dense set D from Lemma 3.9, i.e., p fulfills (3) and (4) for each $\sigma \in \text{dom}(p)$. Now let

$$N > \max(\text{rng}(f_\sigma^p), \text{rng}(h_\sigma^p), |s_\sigma^p|)$$

for every $\sigma \in \text{dom}(p)$. Finally, for every $\sigma \in \text{dom}(p)$, extend s_σ^p with 0's to length N . It is easy to see that this results in a condition q which is full. \square

We now introduce a notation for the collection of conditions in \mathbb{Q}_α whose domain is contained in a prescribed set of nodes:

Definition 3.15. Let $C \subseteq \lambda^{<\lambda}$. Define

$$\mathbb{Q}_\alpha^C := \{p \in \mathbb{Q}_\alpha \mid \text{dom}(p) \subseteq C\}.$$

In our completeness lemma below we are going to show that \mathbb{Q}_α^C is a complete subforcing of \mathbb{Q}_α provided that C has a certain form.

Definition 3.16. Let $E \subseteq \lambda^{<\lambda}$. We call E α -left-up-closed if

- for each $\sigma \in E$ and each $\tau \triangleleft \sigma$ with $\tau \in T_\alpha$, we have $\tau \in E$,
- for each ρ and i with $\rho \hat{\ } i \in E$ and each $j < i$, we have $\rho \hat{\ } j \in E$.

We say that C is α -eligible if $C = E \cup \bar{C}$, where $E \subseteq \lambda^{<\lambda}$ is α -left-up-closed, and either \bar{C} is empty, or the following holds: $E \subseteq \lambda^{<\gamma}$ for some $\gamma < \lambda$ and $\bar{C} \subseteq \lambda^{\gamma'}$ for some $\gamma' \geq \gamma$ (with γ' successor), and, for $\sigma, \sigma' \in \bar{C}$,

- (1) either there exist ρ, i and j such that $\rho \hat{\ } i = \sigma$ and $\rho \hat{\ } j = \sigma'$ (i.e., σ and σ' are in the same “block”),
- (2) or there exist two incomparable nodes $\tau, \tau' \in E$ with $\tau \triangleleft \sigma$ and $\tau' \triangleleft \sigma'$ (i.e., σ and σ' split within E).

So an α -eligible set consists of an α -left-up-closed part together with nodes from one single later level. Clearly, each α -left-up-closed set is α -eligible.

The purpose of items (1) and (2) in the definition above is to ensure that after forcing with \mathbb{Q}_α^C , the two sets a_σ and $a_{\sigma'}$ are almost disjoint (which is necessary for \mathbb{Q}_α^C being a complete subforcing); this is guaranteed by either item: the two sets are forced to be almost disjoint either because it happens in the same block, or because they are almost contained in almost disjoint sets which are already added by \mathbb{Q}_α^E .

For $p \in \mathbb{Q}_\alpha$, let $p \upharpoonright C$ be the condition p' with $\text{dom}(p') = \text{dom}(p) \cap C$, and $s_\sigma^{p'} = s_\sigma^p$, $f_\sigma^{p'} = f_\sigma^p \upharpoonright (C \cup T'_\alpha)$ and $h_\sigma^{p'} = h_\sigma^p \upharpoonright C$ for each $\sigma \in \text{dom}(p')$. Clearly, p' is a condition in \mathbb{Q}_α^C . Note that if C is α -left-up-closed, then $p \upharpoonright C = p \upharpoonright C$, because for every $\sigma \in \text{dom}(p) \cap C$ clearly $f_\sigma^p \upharpoonright (C \cup T'_\alpha) = f_\sigma^p$ and $h_\sigma^p \upharpoonright C = h_\sigma^p$.

The following crucial completeness lemma is given in a quite general form. This way, it can be used in Section 4.2 as well as in Section 6.2. For Section 4.2, a somewhat easier version would be enough (see the proof of Lemma 4.12).

Lemma 3.17. *Let C be α -eligible. Then \mathbb{Q}_α^C is a complete subforcing of \mathbb{Q}_α . Moreover, if $p \in \mathbb{Q}_\alpha$ is a full condition, then $p \upharpoonright C$ is a reduction of p to \mathbb{Q}_α^C .*

Note that the sets $\lambda^1 = \{\sigma \in \lambda^{<\lambda} \mid |\sigma| = 1\}$ and $1^{<\lambda} = \{\sigma \in \lambda^{<\lambda} \mid \sigma(\xi) = 0 \text{ for every } \xi\}$ are 0-left-up-closed, hence the forcings $\mathbb{Q}_0^{(\lambda^1)}$ and $\mathbb{Q}_0^{(1^{<\lambda})}$ are complete subforcings of \mathbb{Q}_0 by the lemma. These forcings are isomorphic to the forcings introduced by Hechler [22] to add a mad family and a tower, respectively (compare with the respective definitions in [14]).

Proof of Lemma 3.17. We give the proof only for the case $\alpha = 0$, and leave the (only slightly different) general case to the reader.

We first show that $\mathbb{Q}_0^C \subseteq_{ic} \mathbb{Q}_0$. Let $p_0, p_1 \in \mathbb{Q}_0^C$ and $q \in \mathbb{Q}_0$ with $q \leq p_0, p_1$. We have to show that there exists a condition $q' \in \mathbb{Q}_0^C$ with $q' \leq p_0, p_1$. Let $q' := q \upharpoonright C$. It is very easy to check that q' is as we wanted.

Let $p \in \mathbb{Q}_0$. To find a reduction, let $p' \leq p$ be a full condition with $\text{dom}(p') = \text{dom}(p)$ (see Lemma 3.14). We will show that $p' \upharpoonright C$ is a reduction of p to \mathbb{Q}_0^C . Let $q \leq p' \upharpoonright C$ with $q \in \mathbb{Q}_0^C$. We have to show that q is compatible with p . To show this, we define a witness r as follows. Let $\text{dom}(r) := \text{dom}(p') \cup \text{dom}(q)$. For $\sigma \in \text{dom}(q)$, let $s_\sigma^r := s_\sigma^q$, and for $\sigma \in \text{dom}(p') \setminus \text{dom}(q)$, let $f_\sigma^r := f_\sigma^{p'}$ and $h_\sigma^r := h_\sigma^{p'}$.

For $\sigma \in \text{dom}(q) \cap \text{dom}(p')$, let $\text{dom}(f_\sigma^r) := \text{dom}(f_\sigma^q) \cup \text{dom}(f_\sigma^{p'})$ and let $f_\sigma^r(\sigma') := \min(f_\sigma^q(\sigma'), f_\sigma^{p'}(\sigma'))$ for every $\sigma' \in \text{dom}(f_\sigma^q) \cap \text{dom}(f_\sigma^{p'})$ and $f_\sigma^r(\sigma') := f_\sigma^q(\sigma')$ for $\sigma' \in \text{dom}(f_\sigma^q) \setminus \text{dom}(f_\sigma^{p'})$. Similarly, let

$\text{dom}(h_\sigma^r) := \text{dom}(h_\sigma^q) \cup \text{dom}(h_\sigma^{p'})$ and let $h_\sigma^r(\sigma') := \min(h_\sigma^q(\sigma'), h_\sigma^{p'}(\sigma'))$ for every $\sigma' \in \text{dom}(h_\sigma^q) \cap \text{dom}(h_\sigma^{p'})$ and $h_\sigma^r(\sigma') := h_\sigma^{p'}(\sigma')$ for $\sigma' \in \text{dom}(h_\sigma^{p'}) \setminus \text{dom}(h_\sigma^q)$.

For $\sigma \in \text{dom}(p') \setminus \text{dom}(q)$, make the following definition. Let $f_\sigma^r := f_\sigma^{p'}$ and $h_\sigma^r := h_\sigma^{p'}$. If there is no $\tau \in \text{dom}(q)$ with $\sigma \preceq \tau$, let $s_\sigma^r := s_\sigma^{p'}$. If there exists $\tau \in \text{dom}(q)$ with $\sigma \preceq \tau$, extend $s_\sigma^{p'}$ to the maximal length of the s_τ^q for $\tau \in \text{dom}(q)$ with $\sigma \triangleleft \tau$ in the following way: if $n \geq |s_\sigma^{p'}|$ and there exists $\tau \in \text{dom}(p')$ which extends σ with $s_\tau^q(n) = 1$, let $s_\sigma^r(n) = 1$, and let $s_\sigma^r(n) = 0$ otherwise. This makes sure that $s_\sigma^r(n) = 1$ whenever $s_\tau^r(n) = 1$ for $\sigma \triangleleft \tau$ and $\sigma \in \text{dom}(f_\tau^{p'})$.

Claim. r is a condition.

Proof. It is very easy to check that s_σ^r , f_σ^r and h_σ^r are well-defined with the right domains and ranges for all $\sigma \in \text{dom}(r)$.

If $\sigma \preceq \tau$, then $|s_\sigma^r| \geq |s_\tau^r|$: if σ and τ are both in $\text{dom}(q)$, so $s_\sigma^r = s_\sigma^q$ and $s_\tau^r = s_\tau^q$, so the length is ok, because they are both from q ; if $\sigma \notin \text{dom}(q)$, we lengthened s_σ^r to make it as long as all the s 's of τ 's which extend it.

Let $\sigma, \tau \in \text{dom}(r)$ with $\tau \in \text{dom}(f_\sigma^r)$ and $m \geq f_\sigma^r(\tau)$ and $s_\sigma^r(m) = 1$; we have to show that $s_\tau^r(m) = 1$ (note that, in the general case, i.e., for \mathbb{Q}_α with $\alpha > 0$, one also has to deal with the case where $\tau \in \text{dom}(f_\sigma^r)$, but $\tau \notin \text{dom}(r)$, which is analogous, but concerned with $a_\tau^r(m)$ in place of $s_\tau^r(m)$). *Case 1:* σ and τ are both in $\text{dom}(q)$. It follows that $\tau \in C \cap \text{dom}(f_\sigma^r) = \text{dom}(f_\sigma^q)$, $f_\sigma^r(\tau) = f_\sigma^q(\tau)$ and $s_\sigma^r = s_\sigma^q$ and $s_\tau^r = s_\tau^q$, so they fit together, because q is a condition. *Case 2:* $\sigma \in \text{dom}(q)$, $\tau \notin \text{dom}(q)$. Since $\tau \notin \text{dom}(q)$ and $\tau \in \text{dom}(f_\sigma^r)$, it follows that $\tau \in \text{dom}(f_\sigma^{p'})$. In particular $f_\sigma^{p'}$ is defined, so $\sigma \in \text{dom}(p')$. If $m < |s_\tau^{p'}|$, it follows that $s_\tau^r(m) = s_\tau^{p'}(m)$ and $s_\sigma^r(m) = s_\sigma^{p'}(m)$, and $f_\sigma^r(\tau) = f_\sigma^{p'}(\tau)$. So $s_\tau^r(m) = 1$, because p' is a condition. If $m \geq |s_\tau^{p'}| = |s_\sigma^{p'}|$, then $s_\sigma^r(m) = 1$ implies that $s_\rho^q(m) = 1$ for some $\rho \succeq \sigma$, but then $\rho \succeq \tau$, and therefore also $s_\tau^r(m) = 1$. *Case 3:* $\sigma \notin \text{dom}(q)$. So $f_\sigma^r = f_\sigma^{p'}$, and it follows that $\tau \in \text{dom}(f_\sigma^{p'}) \subseteq \text{dom}(p')$. If $m < |s_\tau^{p'}|$, it follows that $s_\tau^{p'}(m) = s_\tau^r(m) = 1$, because p' is a condition and $\tau \in \text{dom}(f_\sigma^{p'})$. If $m \geq |s_\tau^{p'}|$, then our definition implies that there exists a ρ with $\rho \triangleright \sigma$, $\rho \in \text{dom}(p')$ and $s_\rho^q(m) = 1$. So both ρ and τ are in $\text{dom}(p')$, $\tau \in \text{dom}(f_\rho^{p'})$ and $m \geq f_\rho^{p'}(\tau)$, hence $s_\rho^r(m) = 1$ implies that $s_\tau^r(m) = 1$ by definition of s_τ^r . This finishes the proof that s_σ^r and s_τ^r fit together (with respect to f_σ^r).

Assume $\rho, \rho' \in \text{dom}(r)$ and $\rho' \in \text{dom}(h_\rho^r)$, $m \geq h_\rho^r(\rho')$ and $s_\rho^r(m) = 1$; we have to show that $s_{\rho'}^r(m) = 0$, if it is defined. *Case 1:* $\rho, \rho' \in \text{dom}(q)$. The requirement follows, because q is a condition. *Case 2:* $\rho, \rho' \in \text{dom}(p') \setminus \text{dom}(q)$. In this case the requirement holds, because p' is a condition. *Case 3:* One of them is in $\text{dom}(q)$, the other one not. Since $\rho' \in \text{dom}(h_\rho^r)$, both are in $\text{dom}(p')$ and $h_\rho^r(\rho') = h_\rho^{p'}(\rho')$ (because q cannot provide an h -value for a pair of two nodes if not both of them are in $\text{dom}(q)$). So for $m < |s_\rho^{p'}|$, the requirement holds, because it depends only on p' . The form of C implies that for at most one of the two nodes ρ and ρ' there exists a node in C extending it. Therefore, for $m \geq |s_\rho^{p'}|$, only one of $s_\rho^r(m)$ and $s_{\rho'}^r(m)$ is defined, and we have nothing to show. This finishes the proof that s_ρ^r and $s_{\rho'}^r$ fit together (with respect to h_ρ^r). \square

It is straightforward to check that r extends both q and p' (and therefore p). \square

4. NO REFINEMENT, AND MADNESS OF LEVELS

This section is dedicated to the central part of the proof that the generic object added by our forcing iteration is a distributivity matrix of height λ : we will show that the levels are mad families and that there is no further refinement. This will be done in Section 4.4 and Section 4.3, respectively. Before that, we provide several preliminary lemmas and concepts.

4.1. On forcing iterations and correct systems. In this section, we give some lemmas about forcing iterations (and completeness) in general, i.e., they are not specific for our forcing from Definition 3.1. We will need them for our proofs. For a good source about forcing iteration, see [18]. Here \mathbb{P} , \mathbb{Q} , etc. are arbitrary forcing notions.

For two forcing notions \mathbb{P} and \mathbb{P}' , let $\mathbb{P}' \triangleleft \mathbb{P}$ denote that \mathbb{P}' is a complete subforcing of \mathbb{P} . Recall that $\mathbb{P}' \triangleleft \mathbb{P}$ if and only if

- (1) $\mathbb{P}' \subseteq_{ic} \mathbb{P}$, i.e., for each $q, q' \in \mathbb{P}'$, we have $q \perp_{\mathbb{P}'} q' \implies q \perp_{\mathbb{P}} q'$, and
- (2) for each condition $p \in \mathbb{P}$, there is $q \in \mathbb{P}'$ such that q is a *reduction of p to \mathbb{P}'* , i.e., for each $r \in \mathbb{P}'$ with $r \leq q$, we have $r \not\perp_{\mathbb{P}} p$.

Let us first recall two easy facts:

Lemma 4.1. *Suppose that $\mathbb{P}_0 \triangleleft \mathbb{P}$ and $\mathbb{P}_1 \triangleleft \mathbb{P}$ satisfying $\mathbb{P}_0 \subseteq \mathbb{P}_1$. Then $\mathbb{P}_0 \triangleleft \mathbb{P}_1$. Moreover, if $q \in \mathbb{P}_0$ is a reduction of $p \in \mathbb{P}_1$ from \mathbb{P} to \mathbb{P}_0 , then q is also a reduction of p from \mathbb{P}_1 to \mathbb{P}_0 .*

Lemma 4.2. *Suppose that $\mathbb{P}' \triangleleft \mathbb{P}$. Let φ be some formula, let \dot{x}, \dot{y} , etc. be \mathbb{P}' -names, and let $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \varphi(\dot{x}, \dot{y}, \dots)$. Then for each $p' \in \mathbb{P}'$ which is a reduction of p , we have $p' \Vdash_{\mathbb{P}'} \varphi(\dot{x}, \dot{y}, \dots)$.*

Let us now recall the following well-known fact (see, e.g., [9]): If $\{\mathbb{P}_\alpha, \dot{Q}_\alpha \mid \alpha < \delta\}$ and $\{\mathbb{P}'_\alpha, \dot{Q}'_\alpha \mid \alpha < \delta\}$ are finite support iterations such that $\mathbb{P}'_\alpha \triangleleft_{\mathbb{P}_\alpha} \dot{Q}'_\alpha \triangleleft \dot{Q}_\alpha$ for each $\alpha < \delta$, then \mathbb{P}'_δ is complete in \mathbb{P}_δ . We will need the following technical strengthening of this fact.

Lemma 4.3. *Let $\{\mathbb{P}_\alpha, \dot{Q}_\alpha \mid \alpha < \delta\}$ and $\{\mathbb{P}'_\alpha, \dot{Q}'_\alpha \mid \alpha < \delta\}$ be¹⁰ finite support iterations such that for each $\alpha < \delta$,*

$$\mathbb{P}'_\alpha \triangleleft_{\mathbb{P}_\alpha} \dot{Q}'_\alpha \triangleleft \dot{Q}_\alpha.$$

Then \mathbb{P}'_δ is a complete subforcing of \mathbb{P}_δ .

Moreover, if $\text{RED}: \mathbb{Q}_0 \rightarrow \mathbb{Q}'_0$ is a map such that $\text{RED}(q)$ is a reduction of q for each $q \in \mathbb{Q}_0$, then for each $p \in \mathbb{P}_\delta$, there is a $p' \in \mathbb{P}'_\delta$ such that p' is a reduction of p , and $p'(0) = \text{RED}(p(0))$, and, if $\alpha \geq 1$ and $p(\alpha)$ is a \mathbb{P}'_α -name with $p \upharpoonright \alpha \Vdash p(\alpha) \in \dot{Q}'_\alpha$, then $p'(\alpha) = p(\alpha)$.

In fact, the iterands in the above lemma need not be separative, which is essential, because we are going to apply it to our forcings \mathbb{Q}_α from Definition 3.1, which are not separative (see Remark 3.2).

The following concept has been introduced by Brendle (see, e.g., [7] and [8]):

⁹In fact, $\mathbb{P}_1 \subseteq_{ic} \mathbb{P}$ is sufficient for the proof to go through.

¹⁰We do not seem to need here that our finite support iterations are c.c.c.; however, finite support iterations of non-c.c.c. iterands collapse cardinals. We will use the lemma for our forcings from Definition 3.1, so, in our application, everything will have the c.c.c. anyway.

Definition 4.4. A system of forcings $\mathbb{R}_0, \mathbb{R}_1 < \mathbb{R}$ with $\mathbb{R}_0 \cap \mathbb{R}_1 < \mathbb{R}_0, \mathbb{R}_1$ is *correct* if any two conditions $p_0 \in \mathbb{R}_0$ and $p_1 \in \mathbb{R}_1$ which have a common reduction in $\mathbb{R}_0 \cap \mathbb{R}_1$ are compatible in \mathbb{R} .

In the following lemma, we are considering a system where $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$, $\mathbb{R}_0 = \mathbb{P}$, and $\mathbb{R}_1 = \mathbb{P}' * \dot{\mathbb{Q}}'$. It is easy to check that, under the assumptions of the lemma, this is a correct system. We do not know, however, whether the conclusion of the lemma holds for every correct system.

Lemma 4.5. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ and $\mathbb{P}' * \dot{\mathbb{Q}}'$ be two-step iterations satisfying $\mathbb{P}' < \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}' < \dot{\mathbb{Q}}$. Then*

$$V[\mathbb{P}' * \dot{\mathbb{Q}}'] \cap V[\mathbb{P}] = V[\mathbb{P}'].$$

Proof. We will only show the special case which we will need later (it is straightforward to extend the proof to the general case): for any $\delta, \varepsilon \in \text{Ord}$,

$$\delta^\varepsilon \cap V[\mathbb{P}' * \dot{\mathbb{Q}}'] \cap V[\mathbb{P}] \subseteq V[\mathbb{P}'].$$

Let G be a generic filter for \mathbb{P}' , and let \dot{f}_0 be a \mathbb{P} -name, and let \dot{f}_1 be a $\mathbb{P}' * \dot{\mathbb{Q}}'$ -name. Work in $V[G]$. Assume towards contradiction that there is a condition $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ with $p \in \mathbb{P}/G$ such that

$$(3) \quad (p, \dot{q}) \Vdash \dot{f}_0 = \dot{f}_1 \in \text{Ord}^{< \text{Ord}} \wedge \dot{f}_0 \notin V[G].$$

Let $p' \in G$ be a reduction of p to \mathbb{P}' . By standard arguments, we can fix a \mathbb{P}' -name \dot{q}' such that $p' \Vdash \dot{q}'$ is a reduction of \dot{q} and $(p', \dot{q}') \in \mathbb{P}' * \dot{\mathbb{Q}}'$.

Since p is reduction of (p, \dot{q}) to \mathbb{P} , it follows from (3) and Lemma 4.2 that $p \Vdash \dot{f}_0 \notin V[G]$. Therefore, we can fix $\gamma \in \varepsilon$ such that p does not decide $\dot{f}_0(\gamma)$ in \mathbb{P}/G . Let $(p_1, \dot{q}_1) \leq (p', \dot{q}')$ and $\xi_1 \in \delta$ such that $p_1 \in G$ and $(p_1, \dot{q}_1) \Vdash \dot{f}_1(\gamma) = \xi_1$. Since p does not decide \dot{f}_0 at γ , we can fix $p_0 \in \mathbb{P}/G$ with $p_0 \leq p$ and $\xi_0 \in \delta$ with $\xi_0 \neq \xi_1$ such that $p_0 \Vdash \dot{f}_0(\gamma) = \xi_0$. Now we want to find a condition (p^*, \dot{q}^*) which is stronger than (p, \dot{q}) , (p_1, \dot{q}_1) and $(p_0, \mathbb{1})$.

First note that p_0 and p_1 are compatible, because $p_0 \in \mathbb{P}/G$ and $p_1 \in G$, and fix $p^* \leq p_0, p_1$. Since $p^* \leq p, p_1$ it follows that $p^* \Vdash \dot{q}'$ is a reduction of \dot{q} and $\dot{q}_1 \leq \dot{q}'$ hence $p^* \Vdash \dot{q}_1 \not\leq \dot{q}$. Let \dot{q}^* be a \mathbb{P} -name such that $p^* \Vdash \dot{q}^* \leq \dot{q}_1, \dot{q}$. It is easy to check that $(p^*, \dot{q}^*) \leq (p, \dot{q}), (p_1, \dot{q}_1), (p_0, \mathbb{1})$. Now $(p^*, \dot{q}^*) \Vdash \dot{f}_0 = \dot{f}_1 \wedge \dot{f}_0(\gamma) = \xi_0 \wedge \dot{f}_1(\gamma) = \xi_1$, but $\xi_0 \neq \xi_1$, a contradiction. \square

We conclude with an easy observation we will need later on:

Lemma 4.6. *Suppose that $\mathbb{P}' < \mathbb{P}$, and \dot{b} is a \mathbb{P}' -name, and $p \in \mathbb{P}$ is such that $p \Vdash \dot{b} \in [\omega]^\omega$. Then for each $N \in \omega$ there exists $r \in \mathbb{P}'$ and $m > N$ such that $r \Vdash m \in \dot{b}$ and r is compatible with p .*

4.2. Complete subforcings: hereditarily below γ . In this section, we give some technical definitions and lemmas as a preparation for the main proofs in Section 4.3 and Section 4.4. More precisely, we define, for each $\gamma < \lambda$, the subforcings of “hereditarily below γ ” conditions of our iteration and show that they form complete subforcings (see Lemma 4.12). Furthermore, we show that each condition is hereditarily below γ for some $\gamma < \lambda$ (see Lemma 4.14).

Let us now provide the following recursive definition (we give the definition for the entire iteration but we will actually need it for tails of the iteration; see Remark 4.10):

Definition 4.7. Let $\gamma < \lambda$. By recursion on $\alpha \leq \lambda$ we define when a condition $p \in \mathbb{P}_\alpha$ is *hereditarily below γ* (and introduce the notation ${}^{< \gamma} \mathbb{P}_\alpha$):

- (1) $p \in \mathbb{Q}_0$ is hereditarily below γ , if $\text{dom}(p) \subseteq \gamma^{<\gamma}$.
- (2) Let ${}^{<\gamma}\mathbb{P}_\alpha := \{p \in \mathbb{P}_\alpha \mid p \text{ hereditarily below } \gamma\}$.
- (3) $p \in \mathbb{P}_{\alpha+1}$ is hereditarily below γ , if $p \upharpoonright \alpha$ is hereditarily below γ and $p(\alpha)$ is a ${}^{<\gamma}\mathbb{P}_\alpha$ -name, and $p \upharpoonright \alpha \Vdash \text{dom}(p(\alpha)) \subseteq \gamma^{<\gamma}$.
- (4) For α limit, $p \in \mathbb{P}_\alpha$ is hereditarily below γ , if $p \upharpoonright \beta$ is hereditarily below γ , for every $\beta < \alpha$.

For $\alpha \leq \lambda$ a \mathbb{P}_α -name \dot{b} is *hereditarily below* γ , if for all $(\dot{x}, p) \in \dot{b}$, $p \in {}^{<\gamma}\mathbb{P}_\alpha$ and \dot{x} is hereditarily below γ (this is by recursion).

Clearly, if $p \in \mathbb{P}_\alpha$ is hereditarily below γ and $\gamma' > \gamma$, then p is also hereditarily below γ' . The same holds for a \mathbb{P}_α -name \dot{b} .

Remark 4.8. In the more general situation described in Remark 3.3, i.e., if we work with the tree $\theta^{<\lambda}$ in place of $\lambda^{<\lambda}$, we would rather need a pair of ordinals (ε, δ) in place of γ , where $\varepsilon < \theta$ and $\delta < \lambda$, and $\gamma^{<\gamma}$ in the definition above (as well as in Definition 4.9 below) would be replaced by $\varepsilon^{<\delta}$ (see also Remark 4.16).

Definition 4.9. Let $\gamma < \lambda$ and $\tau \in \lambda^{<\lambda}$. By recursion on $\alpha \leq \lambda$ we define when a condition $p \in \mathbb{P}_\alpha$ is *almost hereditarily below* γ *except for* τ (and introduce the notation ${}^{<\gamma+\tau}\mathbb{P}_\alpha$):

- (1) $p \in \mathbb{Q}_0$ is almost hereditarily below γ except for τ , if $\text{dom}(p) \subseteq \gamma^{<\gamma} \cup \{\tau\}$.
- (2) Let ${}^{<\gamma+\tau}\mathbb{P}_\alpha := \{p \in \mathbb{P}_\alpha \mid p \text{ almost hereditarily below } \gamma \text{ except for } \tau\}$.
- (3) $p \in \mathbb{P}_{\alpha+1}$ is almost hereditarily below γ except for τ , if $p \upharpoonright \alpha$ is almost hereditarily below γ except for τ and $p(\alpha)$ is a¹¹ ${}^{<\gamma}\mathbb{P}_\alpha$ -name, and $p \upharpoonright \alpha \Vdash \text{dom}(p(\alpha)) \subseteq \gamma^{<\gamma}$.
- (4) For α limit, $p \in \mathbb{P}_\alpha$ is almost hereditarily below γ except for τ , if $p \upharpoonright \beta$ is almost hereditarily below γ except for τ , for every $\beta < \alpha$.

For $\alpha \leq \lambda$ a \mathbb{P}_α -name \dot{b} is *almost hereditarily below* γ *except for* τ , if for all $(\dot{x}, p) \in \dot{b}$, both p and \dot{x} are almost hereditarily below γ except for τ . We will write *almost hereditarily below* γ and omit the τ if it is clear from the context which τ is meant.

Clearly, if $p \in \mathbb{P}_\alpha$ is almost hereditarily below γ and $\gamma' > \gamma$, then p is also almost hereditarily below γ' and if $p \in \mathbb{P}_\alpha$ is hereditarily below γ , then it is almost hereditarily below γ except for τ for every τ . The same holds for a \mathbb{P}_α -name \dot{b} .

Remark 4.10. As mentioned above, we will need several of our concepts for tails of the iteration instead of the whole iteration. We will later have the following situation: $\eta < \lambda$ will be fixed, and we will work in $V[G_\eta]$ for a fixed generic filter $G_\eta \subseteq \mathbb{P}_\eta$. We will use variants of the above definitions and the subsequent lemmas for the tail iteration $\{\mathbb{P}_\alpha / G_\eta, \mathbb{Q}_\alpha \mid \eta \leq \alpha < \lambda\}$. In the definitions and lemmas, \mathbb{Q}_η plays the role of \mathbb{Q}_0 (see for example Lemma 4.18(3)). So, e.g., in the definition of almost hereditarily below γ except for τ (with $\tau \in (\lambda^{<\lambda})^{V[G_\eta]}$), we want $\text{dom}(p(\eta)) \subseteq (\gamma^{<\gamma})^{V[G_\eta]} \cup \{\tau\}$.

Before proving completeness, let us recall that $\gamma^{<\gamma}$ is α -left-up-closed; actually, we will need a bit more:

Lemma 4.11. *Assume \mathbb{P}'_α is a complete subforcing of \mathbb{P}_α , and G is generic for \mathbb{P}_α . Then in $V[G]$, the set $(\gamma^{<\gamma})^{V[G \cap \mathbb{P}'_\alpha]}$ is α -left-up-closed.*

¹¹This is not a typo: we really require $p(\alpha)$ to be a ${}^{<\gamma}\mathbb{P}_\alpha$ -name, not just a ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ -name.

Proof. Suppose σ and $\rho \hat{\ } i$ belong to $(\gamma^{<\gamma})^{V[G \cap \mathbb{P}'_\alpha]}$. Note that the following holds in $V[G]$: $\sigma \upharpoonright (\xi + 1) \in V[G \cap \mathbb{P}'_\alpha]$ for each $\xi < |\sigma|$, and $\rho \hat{\ } j \in V[G \cap \mathbb{P}'_\alpha]$ for each $j < i$. Therefore $(\gamma^{<\gamma})^{V[G \cap \mathbb{P}'_\alpha]}$ is α -left-up-closed. \square

We can now show that the subforcing of conditions which are (almost) hereditarily below γ is a complete subforcing:

Lemma 4.12. *Let $\gamma < \lambda$. Then ${}^{<\gamma}\mathbb{P}_\lambda$ is a complete subforcing of \mathbb{P}_λ .*

Also, if $\tau \in \lambda^{<\lambda}$ is such that either

- (1) $|\tau| \geq \gamma$, or
- (2) τ is such that $\gamma^{<\gamma} \cup \{\tau\}$ is 0-left-up-closed,

then ${}^{<\gamma+\tau}\mathbb{P}_\lambda$ is a complete subforcing of \mathbb{P}_λ .

Moreover, if p is full and almost hereditarily below γ except for τ , then

$$(p(0) \upharpoonright \gamma^{<\gamma}, p(1), p(2), \dots)$$

is a reduction of p to ${}^{<\gamma}\mathbb{P}_\lambda$.

Proof. We show by induction on α that ${}^{<\gamma}\mathbb{P}_\alpha$ (as well as ${}^{<\gamma+\tau}\mathbb{P}_\alpha$) is a complete subforcing of \mathbb{P}_α , for $1 \leq \alpha \leq \lambda$. In fact, we will define ${}^{<\gamma}\mathbb{P}_\alpha$ -names \dot{Q}'_α such that ${}^{<\gamma}\mathbb{P}_\delta$ (or ${}^{<\gamma+\tau}\mathbb{P}_\delta$, respectively) is the finite support iteration of the \dot{Q}'_α 's; the only difference of the two iterations will be the first iterand \dot{Q}'_0 .

(Initial step $\alpha = 1$) Note that ${}^{<\gamma}\mathbb{P}_1 = \mathbb{Q}_0^{\gamma^{<\gamma}}$ is a complete subforcing of $\mathbb{P}_1 = \mathbb{Q}_0$: this is an easy instance of Lemma 3.17, because $\gamma^{<\gamma}$ is 0-left-up-closed. Similarly, ${}^{<\gamma+\tau}\mathbb{P}_1 = \mathbb{Q}_0^{\gamma^{<\gamma} \cup \{\tau\}}$ is a complete subforcing of \mathbb{Q}_0 : in case (2) holds, $\gamma^{<\gamma} \cup \{\tau\}$ is 0-left-up-closed by assumption; in case (1) holds, $\gamma^{<\gamma} \cup \{\tau\}$ is easily seen to be 0-eligible. Take $\dot{Q}'_0 = \mathbb{Q}_0^{\gamma^{<\gamma}}$ in the iteration representing ${}^{<\gamma}\mathbb{P}_\lambda$, and take $\dot{Q}'_0 = \mathbb{Q}_0^{\gamma^{<\gamma} \cup \{\tau\}}$ in the iteration representing ${}^{<\gamma+\tau}\mathbb{P}_\lambda$.

(Successor step $\alpha + 1$) Assume that ${}^{<\gamma}\mathbb{P}_\alpha$ and ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ are complete subforcings of \mathbb{P}_α . We show that ${}^{<\gamma}\mathbb{P}_{\alpha+1}$ and ${}^{<\gamma+\tau}\mathbb{P}_{\alpha+1}$ are complete subforcings of $\mathbb{P}_{\alpha+1}$. In $V[G]$, for G generic for \mathbb{P}_α , let¹² $E := (\gamma^{<\gamma})^{V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]}$; by Lemma 4.11, E is α -left-up-closed, so Lemma 3.17 implies that in $V[G]$, \mathbb{Q}_α^E is a complete subforcing of \mathbb{Q}_α . We use the following, which we will prove after finishing the proof of the lemma:

Claim 4.13. \mathbb{Q}_α^E is an element of $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$.

Using the claim, we can fix a ${}^{<\gamma}\mathbb{P}_\alpha$ -name \dot{Q}'_α for \mathbb{Q}_α^E . Since ${}^{<\gamma}\mathbb{P}_\alpha \subseteq {}^{<\gamma+\tau}\mathbb{P}_\alpha$ and both are complete subforcings of \mathbb{P}_α , Lemma 4.1 implies that ${}^{<\gamma}\mathbb{P}_\alpha$ is a complete subforcing of ${}^{<\gamma+\tau}\mathbb{P}_\alpha$, so the ${}^{<\gamma}\mathbb{P}_\alpha$ -name \dot{Q}'_α is also a ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ -name. So we can apply Lemma 4.3 to obtain that ${}^{<\gamma}\mathbb{P}_\alpha * \dot{Q}'_\alpha$ and ${}^{<\gamma+\tau}\mathbb{P}_\alpha * \dot{Q}'_\alpha$ are complete subforcings of $\mathbb{P}_{\alpha+1}$. By definition, ${}^{<\gamma}\mathbb{P}_\alpha * \dot{Q}'_\alpha$ is equivalent to ${}^{<\gamma}\mathbb{P}_{\alpha+1}$, and ${}^{<\gamma+\tau}\mathbb{P}_\alpha * \dot{Q}'_\alpha$ is equivalent to ${}^{<\gamma+\tau}\mathbb{P}_{\alpha+1}$, so the successor step is finished.

(Limit step α) It follows by Lemma 4.3 that the limit of the finite support iteration of the $\dot{Q}'_{\alpha'}$, with $\alpha' < \alpha$ is a complete subforcing of \mathbb{P}_α , and by definition, ${}^{<\gamma}\mathbb{P}_\alpha$ (or ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ in the other case) is equivalent to the limit of this finite support iteration.

¹²Note that E is really defined this way for both cases (see also footnote 11).

Now let us show the moreover part. By the moreover part of Lemma 3.17, $p(0) \upharpoonright \gamma^{<\gamma}$ is a reduction of $p(0)$ to ${}^{<\gamma}\mathbb{P}_1$ (which is \mathbb{Q}'_0 in the iteration representing ${}^{<\gamma}\mathbb{P}_\lambda$). Since $p \in {}^{<\gamma+\tau}\mathbb{P}_\lambda$, which is the iteration of the \mathbb{Q}'_α (for $\alpha \geq 1$ the iterands of the two iterations coincide), so $p \upharpoonright \alpha \Vdash p(\alpha) \in \mathbb{Q}'_\alpha$ for $\alpha \geq 1$, therefore Lemma 4.3 completes the proof. \square

Proof of Claim 4.13. We work in $V[G]$. Let $G_\beta := G \cap \mathbb{P}_\beta$. Let $T'_\alpha = \bigcup_{\beta < \alpha} (\lambda^{<\lambda} \cap \text{succ})^{V[G_\beta]}$ and $T_\alpha = (\lambda^{<\lambda} \cap \text{succ})^{V[G] \setminus T'_\alpha}$, as in the definition of \mathbb{Q}_α .

It is straightforward to check that \mathbb{Q}_α^E can be defined in $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$ provided that $E \cap T'_\alpha$ (and hence also $E \cap T_\alpha$) belongs to $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$. First note that

$$E = (\gamma^{<\gamma})^{V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]} = \gamma^{<\gamma} \cap V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$$

and

$$T'_\alpha = \bigcup_{\beta < \alpha} (\lambda^{<\lambda} \cap \text{succ})^{V[G_\beta]} = \bigcup_{\beta < \alpha} (\lambda^{<\lambda} \cap \text{succ} \cap V[G_\beta]).$$

So

$$E \cap T'_\alpha = \bigcup_{\beta < \alpha} (\gamma^{<\gamma} \cap \text{succ} \cap V[G \cap {}^{<\gamma}\mathbb{P}_\alpha] \cap V[G_\beta]).$$

Apply Lemma 4.5 to $\mathbb{P}_\beta * \mathbb{Q}$, where \mathbb{Q} is the quotient $\mathbb{P}_\alpha / \mathbb{P}_\beta$, and ${}^{<\gamma}\mathbb{P}_\beta * \mathbb{Q}'$, where \mathbb{Q}' is the quotient ${}^{<\gamma}\mathbb{P}_\alpha / {}^{<\gamma}\mathbb{P}_\beta$ (which is possible since ${}^{<\gamma}\mathbb{P}_\beta \triangleleft \mathbb{P}_\beta$ by induction hypothesis, and $\Vdash_{\mathbb{P}_\beta} \mathbb{Q}' \triangleleft \mathbb{Q}$ by Lemma 4.3 for the tail iterations) to obtain

$$\gamma^{<\gamma} \cap V[G \cap {}^{<\gamma}\mathbb{P}_\alpha] \cap V[G_\beta] = \gamma^{<\gamma} \cap V[G_\beta \cap {}^{<\gamma}\mathbb{P}_\alpha].$$

Therefore,

$$E \cap T'_\alpha = \bigcup_{\beta < \alpha} (\gamma^{<\gamma} \cap \text{succ})^{V[G_\beta \cap {}^{<\gamma}\mathbb{P}_\alpha]},$$

which clearly belongs to $V[G \cap {}^{<\gamma}\mathbb{P}_\alpha]$, as desired. \square

The next lemma shows that every condition in \mathbb{P}_λ is (essentially) hereditarily below γ for some $\gamma < \lambda$.

Lemma 4.14. *For every $p \in \mathbb{P}_\lambda$, there exists a $\gamma < \lambda$ and a condition $p' \in {}^{<\gamma}\mathbb{P}_\lambda$ which is forcing equivalent to p .*

Proof. We will actually show by induction on α that for every $p \in \mathbb{P}_\alpha$, there exists a $\gamma < \lambda$ and a condition p' (forcing) equivalent to p such that $p' \in {}^{<\gamma}\mathbb{P}_\alpha$.

(Initial step $\alpha = 1$) Given $p \in \mathbb{P}_1 = \mathbb{Q}_0$, note that $\text{dom}(p) \subseteq \lambda^{<\lambda}$ is finite. So the maximum length of the nodes σ in the domain as well as the maximal entry of the nodes are bounded, i.e., there is $\gamma < \lambda$ such that $\text{dom}(p) \subseteq \gamma^{<\gamma}$. So $p \in {}^{<\gamma}\mathbb{P}_1$.

(Limit step α) Let $p \in \mathbb{P}_\alpha$. By induction hypothesis, for each $\beta < \alpha$ there exists γ_β such that $p \upharpoonright \beta \in {}^{<\gamma_\beta}\mathbb{P}_\beta$. Since we are using finite support, there exists $\beta^* < \alpha$ which is an upper bound of the support of p . Then $p \in {}^{<\gamma_{\beta^*}}\mathbb{P}_\alpha$.

(Successor step $\alpha + 1$) Let $(p, \dot{q}) \in \mathbb{P}_\alpha * \mathbb{Q}_\alpha$. First, by the induction hypothesis, we can assume without loss of generality that there exists $\gamma_p < \lambda$ such that $p \in {}^{<\gamma_p}\mathbb{P}_\alpha$. We will describe a name \dot{q}' which is equivalent to \dot{q} (more precisely, $p \Vdash \dot{q} = \dot{q}'$) and analyze it, to find a $\gamma < \lambda$ such that \dot{q}' is a ${}^{<\gamma}\mathbb{P}_\alpha$ -name and $p \Vdash \text{dom}(\dot{q}') \subseteq \gamma^{<\gamma}$.

Claim 4.15. *Let $\dot{\sigma}$ be a \mathbb{P}_α -name of a sequence of ordinals of length less than λ ; then there exists a $\gamma < \lambda$ such that there exists a ${}^{<\gamma}\mathbb{P}_\alpha$ -name which is equivalent to $\dot{\sigma}$. The same holds true if $\dot{\sigma}$ is a name for a finite sequence of such sequences.*

Proof. Clearly, by c.c.c., there exists a $\delta < \lambda$ which is an upper bound for the length of $\dot{\sigma}$. Since \mathbb{P}_α has the c.c.c., for each $\xi < \delta$, $\dot{\sigma}(\xi)$ is represented by a countable antichain. So only $|\delta| \cdot \aleph_0$ many (hence less than λ many) conditions appear in $\dot{\sigma}$. By inductive hypothesis we can assume that each of these conditions belongs to ${}^{<\gamma}\mathbb{P}_\alpha$ for some $\gamma < \lambda$, so we can fix a $\gamma_\sigma < \lambda$ which is an upper bound of all the appearing γ . So $\dot{\sigma}$ is actually a ${}^{<\gamma_\sigma}\mathbb{P}_\alpha$ -name. The statement about names for finite sequences of sequences follows easily. \square

Now, let \dot{N} be a \mathbb{P}_α -name such that $p \Vdash |\text{dom}(\dot{q})| = \dot{N}$; by (a simple instance of) Claim 4.15, we can fix $\gamma_{\dot{N}} < \lambda$ and assume that \dot{N} is a ${}^{<\gamma_{\dot{N}}}\mathbb{P}_\alpha$ -name. To represent \dot{q} , we provide ω -sequences $\langle \dot{\sigma}_k \mid k \in \omega \rangle$ (of names for potential members of $\text{dom}(\dot{q})$) and $\langle (\dot{s}_k, \dot{f}_k, \dot{h}_k) \mid k \in \omega \rangle$ such that

$$p \Vdash \text{dom}(\dot{q}) = \{\dot{\sigma}_k \mid k \in \dot{N}\} \wedge \forall k \in \dot{N} (\dot{q}(\dot{\sigma}_k) = (\dot{s}_k, \dot{f}_k, \dot{h}_k)),$$

where $\dot{\sigma}_k$ is forced to be a sequence of ordinals of length less than λ , and \dot{f}_k and \dot{h}_k can be represented as finite sequences of such sequences, together with finite sequences of natural numbers, and \dot{s}_k is forced to be an element of $2^{<\omega}$. Using Claim 4.15, we can find $\gamma' < \lambda$, larger than $\gamma_{\dot{N}}$, such that there exist ${}^{<\gamma'}\mathbb{P}_\alpha$ -names $\dot{\sigma}'_k, \dot{s}'_k, \dot{f}'_k$, and \dot{h}'_k which are equivalent to $\dot{\sigma}_k, \dot{s}_k, \dot{f}_k$, and \dot{h}_k , respectively. By replacing all $\dot{\sigma}_k, \dot{s}_k, \dot{f}_k$, and \dot{h}_k in \dot{q} by their respective equivalent names, we get a ${}^{<\gamma'}\mathbb{P}_\alpha$ -name \dot{q}' such that $p \Vdash \dot{q} = \dot{q}'$.

Again by the c.c.c., there exist $\varepsilon, \delta < \lambda$ such that $p \Vdash \dot{\sigma}_k \in \varepsilon^{<\delta}$ for every $k < \omega$. Let $\gamma := \max(\gamma_p, \gamma', \varepsilon, \delta) < \lambda$. Then $(p, \dot{q}') \in {}^{<\gamma}\mathbb{P}_{\alpha+1}$, and it is equivalent to (p, \dot{q}) , which finishes the proof. \square

Remark 4.16. In the more general situation described in Remark 3.3, i.e., if we work with the tree $\theta^{<\lambda}$ in place of $\lambda^{<\lambda}$ (see also Remark 4.8), we have to require that $\text{cf}(\theta) \geq \lambda$. The reason is that $|\sigma_k|$ can be arbitrarily large below λ : if $\text{cf}(\theta) < \lambda$, it could happen that there does not exist an $\varepsilon < \theta$ which is needed in the end of the generalization of the above proof.

Lemma 4.17. *Let G be \mathbb{P}_λ -generic and $V[G] \models b \subseteq \omega$. Then there exists a $\gamma < \lambda$ and a \mathbb{P}_λ -name \dot{b} for b which is hereditarily below γ .*

Proof. For every condition $p \in \mathbb{P}_\lambda$, let $\gamma_p < \lambda$ be such that there exists a condition in ${}^{<\gamma_p}\mathbb{P}_\lambda$ which is forcing equivalent to p (which is possible by Lemma 4.14). Let \dot{b}' be a nice name for b , and let \dot{b} be a name where every condition p appearing in \dot{b}' is replaced by an equivalent condition in ${}^{<\gamma_p}\mathbb{P}_\lambda$. Since b is a countable set and \mathbb{P}_λ has the c.c.c., the set B of conditions which appeared in \dot{b}' is countable. Let $\gamma := \sup\{\gamma_p \mid p \in B\} < \lambda$; then \dot{b} is a ${}^{<\gamma}\mathbb{P}_\lambda$ -name. \square

We conclude with a technical lemma which will be crucial later on:

Lemma 4.18. *Suppose $\tau \in \lambda^{<\lambda} \setminus \gamma^{<\gamma}$. Let $p, r \in \mathbb{P}_\lambda$ such that p is a full condition which is almost hereditarily below γ except for τ , and r is hereditarily below γ , and p and r are compatible (in \mathbb{P}_λ). Then there exists a $p^* \in \mathbb{P}_\lambda$ such that*

- (1) p^* is almost hereditarily below γ except for τ ,
- (2) $p^* \leq p, r$, and

$$(3) \quad p^*(0)(\tau) = p(0)(\tau).$$

Proof. Without loss of generality we can assume that $\text{dom}(p(0)) \supseteq \{\tau\}$. Since p is full and almost hereditarily below γ , by (the “moreover part” of) Lemma 4.12

$$p^{\text{RED}} := (p(0) \upharpoonright \gamma^{<\gamma}, p(1), p(2), \dots)$$

is a reduction of p to ${}^{<\gamma}\mathbb{P}_\lambda$. We show that $p^{\text{RED}} \not\leq_{{}^{<\gamma}\mathbb{P}_\lambda} r$. Assume not. Since ${}^{<\gamma}\mathbb{P}_\lambda$ is a complete subforcing of \mathbb{P}_λ , it follows that $p^{\text{RED}} \perp_{\mathbb{P}_\lambda} r$. But $p \leq p^{\text{RED}}$, so $p \perp_{\mathbb{P}_\lambda} r$, which is a contradiction to the assumption of the lemma.

Let $q^* \in {}^{<\gamma}\mathbb{P}_\lambda$ be such that $q^* \leq p^{\text{RED}}, r$; without loss of generality, we can assume that $q^*(0)$ is full. Since $q^*(0) \leq p^{\text{RED}}(0) = p(0) \upharpoonright \gamma^{<\gamma}$ and $p(0) \upharpoonright \gamma^{<\gamma}$ is a reduction of $p(0)$ by Lemma 3.17 (recall that $p(0) \upharpoonright \gamma^{<\gamma} = p(0) \upharpoonright \gamma^{<\gamma}$ because $\gamma^{<\gamma}$ is 0-left-up-closed), it follows that $q^*(0)$ is compatible with $p(0)$. Let $\bar{q}(0)$ be a full witness for that. So $\bar{q}(0) \leq p(0), r(0), q^*(0)$.

Let $p^*(0) := \bar{q}(0) \upharpoonright \gamma^{<\gamma} \cup \{(\tau, p(0)(\tau))\}$, and for $\alpha > 0$, let $p^*(\alpha) := q^*(\alpha)$.

Claim. $p^*(0)$ is a condition.

Proof. For $\sigma, \sigma' \in \text{dom}(\bar{q}(0) \upharpoonright \gamma^{<\gamma})$, it is clear that the requirements for being a condition are fulfilled, because $\bar{q}(0)$ is a condition.

Let $\sigma \leq \tau$ and $\sigma \in \text{dom}(p^*(0))$. Let $\sigma' \in \text{dom}(p(0)) \setminus \{\tau\}$. Clearly, $s_\sigma^{p^*(0)} = s_\sigma^{\bar{q}(0)}$ and $s_\tau^{p^*(0)} = s_\tau^{p(0)}$. Since $\bar{q}(0)$ and p are full, it follows that $|s_\sigma^{\bar{q}(0)}| = |s_{\sigma'}^{\bar{q}(0)}|$ and hence $|s_\sigma^{p^*(0)}| = |s_{\sigma'}^{p^*(0)}| \geq |s_{\sigma'}^{p(0)}| = |s_\tau^{p(0)}| = |s_\tau^{p^*(0)}|$.

Let $\sigma \in \text{dom}(f_\tau^{p^*(0)})$ and assume that $s_\tau^{p^*(0)}(m) = 1$ for some $m \geq f_\tau^{p^*(0)}(\sigma)$. We have to show that $s_\sigma^{p^*(0)}(m) = 1$. Since $\bar{q}(0)$ extends $p(0)$, we have $s_\tau^{\bar{q}(0)}(m) = 1$ and $\text{dom}(f_\tau^{p^*(0)}) \subseteq \text{dom}(f_\tau^{\bar{q}(0)})$, and for $\sigma \in \text{dom}(f_\tau^{p^*(0)})$ it holds that $f_\tau^{p^*(0)}(\sigma) \geq f_\tau^{\bar{q}(0)}(\sigma)$, so $\sigma \in \text{dom}(f_\tau^{\bar{q}(0)})$ and $m \geq f_\tau^{\bar{q}(0)}(\sigma)$. Since $\bar{q}(0)$ is a condition, it follows that $s_\sigma^{p^*(0)}(m) = s_\sigma^{\bar{q}(0)}(m) = 1$.

Let $\sigma \in \text{dom}(h_\tau^{p^*(0)})$ and assume that $s_\tau^{p^*(0)}(m) = 1$ for some $m \geq h_\tau^{p^*(0)}(\sigma)$. We have to show that $s_\sigma^{p^*(0)}(m) = 0$. Since $\bar{q}(0)$ extends $p(0)$, we have $s_\tau^{\bar{q}(0)}(m) = 1$ and $\text{dom}(h_\tau^{p^*(0)}) \subseteq \text{dom}(h_\tau^{\bar{q}(0)})$, and for $\sigma \in \text{dom}(h_\tau^{p^*(0)})$ it holds that $h_\tau^{p^*(0)}(\sigma) \geq h_\tau^{\bar{q}(0)}(\sigma)$, so $\sigma \in \text{dom}(h_\tau^{\bar{q}(0)})$ and $m \geq h_\tau^{\bar{q}(0)}(\sigma)$. Since $\bar{q}(0)$ is a condition, it follows that $s_\sigma^{p^*(0)}(m) = s_\sigma^{\bar{q}(0)}(m) = 0$. \square

Moreover, $p^*(0) \leq q^*(0)$, because $\bar{q}(0) \leq q^*(0)$ and q^* hereditarily below γ except for τ . So p^* is a condition. Clearly p^* is almost hereditarily below γ and $p^*(0)(\tau) = p(0)(\tau)$.

Since $r(0)$ is hereditarily below γ and $p(0)$ is almost hereditarily below γ , and $\bar{q}(0) \leq r(0), p(0)$, it is clear that $p^*(0)$ extends $r(0)$ and $p(0)$. So clearly $p^* \leq r, p$. \square

4.3. No refinement: branches are towers. Now we are ready to prove that the generic matrix has no refinement. More precisely, we show that the sets along any branch in our tree have no pseudo-intersection, i.e., they form a tower.

Lemma 4.19. *In $V[\mathbb{P}_\lambda]$, the sequence $\langle a_{\sigma \upharpoonright \xi} \mid \xi < \lambda \rangle$ is a tower for each $\sigma \in \lambda^\lambda$.*

Proof. Let G_λ be generic for \mathbb{P}_λ and work in $V[G_\lambda]$. Fix $\sigma \in \lambda^\lambda$. By Corollary 3.11(1), $\langle a_{\sigma \upharpoonright \xi} \mid \xi < \lambda \rangle$ is \subseteq^* -decreasing. Let us show that $\langle a_{\sigma \upharpoonright \xi} \mid \xi < \lambda \rangle$ is actually a tower. Let $b \subseteq \omega$ be infinite, and assume towards a contradiction that $b \subseteq^* a_{\sigma \upharpoonright \xi}$ for every $\xi < \lambda$.

Apply Lemma 4.17 to get $\gamma < \lambda$ and a \mathbb{P}_λ -name \dot{b} for b which is hereditarily below γ . Without loss of generality we can assume that γ is a successor ordinal. Fix $\eta < \lambda$ minimal such that $\sigma \upharpoonright \gamma \in V[G_\eta]$ (such an η exists by Lemma 3.7). From now on, we work in $V[G_\eta]$, and we consider¹³ the tail forcing $\mathbb{P}_\lambda/G_\eta$. The \mathbb{P}_λ -name \dot{b} can be understood as a $\mathbb{P}_\lambda/G_\eta$ -name for b which is hereditarily below γ .

Since $b \subseteq^* a_{\sigma \upharpoonright \gamma}$ holds in $V[G_\lambda]$, we can pick $n \in \omega$ and $p \in \mathbb{P}_\lambda/G_\eta$ such that

$$p \Vdash \dot{b} \setminus n \subseteq a_{\sigma \upharpoonright \gamma}.$$

From now on, whenever we say “almost hereditarily below γ ”, we shall mean “almost hereditarily below γ except for $\sigma \upharpoonright \gamma$ ”. Note that (the canonical name for) $a_{\sigma \upharpoonright \gamma}$ is almost hereditarily below γ ; also \dot{b} is almost hereditarily below γ (because \dot{b} is hereditarily below γ).

By Lemma 4.12 and Lemma 4.2, we can fix p' which is almost hereditarily below γ such that

$$p' \Vdash \dot{b} \setminus n \subseteq a_{\sigma \upharpoonright \gamma}.$$

Recall that η is minimal with $\sigma \upharpoonright \gamma \in V[G_\eta]$, so \mathbb{Q}_η will assign a set $a_{\sigma \upharpoonright \gamma}$ to $\sigma \upharpoonright \gamma$. Therefore we can assume without loss of generality that $\sigma \upharpoonright \gamma \in \text{dom}(p'(\eta))$, and we can assume that p' is a full¹⁴ condition.

By Lemma 4.6 there is $r \in \mathbb{P}_\lambda/G_\eta$ hereditarily below γ and $m > n, |s_{\sigma \upharpoonright \gamma}^{p'(\eta)}|$ such that r is compatible with p' , and $r \Vdash m \in \dot{b}$. Apply Lemma 4.18 to obtain $p'' \leq p', r$ such that p'' is almost hereditarily below γ , and moreover

$$p''(\eta)(\sigma \upharpoonright \gamma) = p'(\eta)(\sigma \upharpoonright \gamma).$$

It follows that $p'' \Vdash m \in \dot{b}$. In particular $m > |s_{\sigma \upharpoonright \gamma}^{p''(\eta)}|$, thus we can strengthen p'' to a condition q (only strengthening $p''(\eta)$) by extending $s_{\sigma \upharpoonright \gamma}^{p''(\eta)}$ to length $> m$ with $s_{\sigma \upharpoonright \gamma}^{q(\eta)}(m) = 0$. Then $q \Vdash m \in \dot{b} \wedge m \notin a_{\sigma \upharpoonright \gamma}$, which is a contradiction to the fact that p' forces $\dot{b} \setminus n \subseteq a_{\sigma \upharpoonright \gamma}$. \square

4.4. Levels are mad families. Finally, we want to show that the levels of the generic matrix form mad families.

Lemma 4.20. *In $V[\mathbb{P}_\lambda]$, the family $A_{\xi+1} = \{a_\sigma \mid |\sigma| = \xi + 1\}$ is mad for each $\xi < \lambda$.*

Proof. Let G_λ be generic for \mathbb{P}_λ and work in $V[G_\lambda]$. The main work lies in the following claim, which guarantees “local madness” below branches. We will prove it after finishing the proof of the lemma.

Claim 4.21. *Let $\rho \in \lambda^{<\lambda}$, and let $b \subseteq \omega$ be infinite such that $b \cap a_{\rho \upharpoonright \zeta}$ is infinite for every successor $\zeta \leq |\rho|$. Then there exists an $i < \lambda$ such that $b \cap a_{\rho \upharpoonright i}$ is infinite.*

Fix $\xi < \lambda$. By Corollary 3.11(2), $A_{\xi+1}$ is an almost disjoint family. Using the claim, we will show that $A_{\xi+1}$ is actually mad. Let $b \subseteq \omega$ be infinite. To find $\sigma \in \lambda^{\xi+1}$ such that $b \cap a_\sigma$ is infinite, we construct, by induction on ζ , a branch $\langle \rho_\zeta \mid \zeta \leq \xi + 1 \rangle$ with $|\rho_\zeta| = \zeta$ for each ζ , and $\rho_{\zeta'} \trianglelefteq \rho_\zeta$ for $\zeta' \leq \zeta$, such that $b \cap a_{\rho_\zeta}$ is infinite for every successor $\zeta \leq \xi + 1$.

Let $\rho_0 := \langle \rangle$. Now assume we have constructed $\langle \rho_{\zeta'} \mid \zeta' < \zeta \rangle$. If ζ is a limit, just let $\rho_\zeta := \bigcup \{\rho_{\zeta'} \mid \zeta' < \zeta\}$. If $\zeta = \zeta' + 1$ is a successor, $\rho_{\zeta'}$ fulfills the assumptions of the claim by induction. Let $i < \lambda$ be given by the claim, and let $\rho_\zeta := \rho_{\zeta'} \hat{\ } i$. Then $b \cap a_{\rho_\zeta}$ is infinite, as required. Finally, $\sigma := \rho_{\xi+1}$ is as desired. \square

¹³Here we use our modifications discussed in Remark 4.10.

¹⁴Here we use the modification of Definition 3.13, where 0 is replaced by η , i.e., $p'(\eta)$ is full.

Proof of Claim 4.21. Assume towards contradiction that $b \cap a_{\rho \upharpoonright \zeta}$ is infinite for every successor $\zeta \leq |\rho|$, but $b \cap a_{\rho^{-i}}$ is finite for every $i < \lambda$.

Let η be minimal with $\rho \in V[G_\eta]$ (such an η exists by Lemma 3.7). Thus $a_{\rho^{-i}}$ (for any i) is not defined in $V[G_\eta]$ but it will get defined in the next step of the forcing iteration. From now on, we work in $V[G_\eta]$, and we consider¹⁵ the tail forcing $\mathbb{P}_\lambda / G_\eta$, and apply Lemma 4.17 to get a $\mathbb{P}_\lambda / G_\eta$ -name \dot{b} for b and $\gamma' < \lambda$ such that \dot{b} is hereditarily below γ' . Let $\gamma < \lambda$ be any ordinal strictly above $|\rho| + 1$, $\text{sup}(\text{rng}(\rho))$, and γ' .

Note that we can pick $n \in \omega$ and $p \in \mathbb{P}_\lambda / G_\eta$ such that

- (1) $p \Vdash \dot{b} \cap a_{\rho^{-\gamma}} \subseteq n$,
- (2) $p \Vdash \dot{b} \cap a_{\rho^{-i}}$ is finite, for each $i < \gamma$, and
- (3) $p \Vdash \dot{b} \cap a_{\rho \upharpoonright \zeta}$ is infinite, for each successor $\zeta \leq |\rho|$.

From now on, whenever we say “almost hereditarily below γ ”, we shall mean “almost hereditarily below γ except for $\rho^{-\gamma}$ ”. Note that (the canonical name for) $a_{\rho^{-\gamma}}$ is almost hereditarily below γ ; also \dot{b} is almost hereditarily below γ (because \dot{b} is hereditarily below γ), and similarly $a_{\rho^{-i}}$ is almost hereditarily below γ for each $i < \gamma$, and $a_{\rho \upharpoonright \zeta}$ is almost hereditarily below γ for each successor $\zeta \leq |\rho|$.

By Lemma 4.12 and Lemma 4.2, we can fix p' which is almost hereditarily below γ such that items (1), (2), and (3) above hold true for p' in place of p . Without loss of generality, we can assume that $\rho^{-\gamma} \in \text{dom}(p'(\eta))$, as well as that p' is a full¹⁶ condition.

Define $R := \text{dom}(p'(\eta)) \cap \{\rho^{-i} \mid i < \gamma\}$, and $R' := \text{dom}(f_{\rho^{-\gamma}}^{p'(\eta)})$. Let \dot{x} be a $\mathbb{P}_\lambda / G_\eta$ -name such that

$$\Vdash \dot{x} = \bigcap_{\tau \in R'} (\dot{b} \cap a_\tau) \setminus \bigcup_{\tau \in R} a_\tau;$$

since the conditions which are hereditarily below γ form a complete subforcing of $\mathbb{P}_\lambda / G_\eta$ by Lemma 4.12, and all names which are used to define \dot{x} are hereditarily below γ , we can assume that \dot{x} has been chosen to be hereditarily below γ as well. Note that since R and R' are finite, p' forces \dot{x} to be infinite.

By Lemma 4.6 there is $r \in \mathbb{P}_\lambda / G_\eta$ hereditarily below γ and $m > n, |s_{\rho^{-\gamma}}^{p'(\eta)}|$ such that r is compatible with p' , and $r \Vdash m \in \dot{x}$. Apply Lemma 4.18 to obtain $p'' \leq p', r$ such that p'' is almost hereditarily below γ , and moreover

$$p''(\eta)(\rho^{-\gamma}) = p'(\eta)(\rho^{-\gamma}).$$

It follows that $p'' \Vdash m \in \dot{x}$, as well as $p'' \Vdash m \in a_\tau$ for $\tau \in R'$ and $p'' \Vdash m \notin a_\tau$ for $\tau \in R$.

Now extend p'' to a condition q as follows. Let $q(\alpha) = p''(\alpha)$ for $\alpha > \eta$. For $\tau \in (R \cup R') \cap \text{dom}(p''(\eta))$ extend $s_\tau^{q(\eta)}$ such that $|s_\tau^{q(\eta)}| > m$. It follows (for $\tau \in R'$) that $s_\tau^{q(\eta)}(m) = 1$ for $\tau \in R' \cap \text{dom}(p''(\eta))$, and $a_\tau(m) = 1$ for $\tau \in R' \setminus \text{dom}(p''(\eta))$ because $p'' \Vdash m \in a_\tau$ for $\tau \in R'$; moreover, $s_\tau^{q(\eta)}(m) = 0$ for $\tau \in R$ because $p'' \Vdash m \notin a_\tau$ for $\tau \in R$. Additionally fill $s_{\rho^{-\gamma}}^{q(\eta)}$ with 0 for entries smaller than m and with 1 at m . That is possible, because the $s_\tau^{q(\eta)}(m)$ are accordingly for $\tau \in R$ and $\tau \in R' \cap \text{dom}(p''(\eta))$ respectively and $a_\tau(m) = 1$ for $\tau \in R' \setminus \text{dom}(p''(\eta))$.

It follows that $q \Vdash m \in \dot{x} \cap a_{\rho^{-\gamma}}$, which is a contradiction to the fact that p' forces $\dot{x} \cap a_{\rho^{-\gamma}} \subseteq n$. \square

¹⁵Here, again, we use our modifications discussed in Remark 4.10.

¹⁶Here, again, we use the modification of Definition 3.13, where 0 is replaced by η , i.e., $p'(\eta)$ is full.

This finishes the proof that the generic matrix is a distributivity matrix of height λ . To finish the proof of Main Theorem 1.2, it remains to prove that \mathfrak{b} (and hence \mathfrak{h}) is small in our final model; this is the subject of Sections 5 and 6.

5. \mathcal{B} -CANJAR FILTERS

In this section, we will give the necessary preliminaries about \mathcal{B} -Canjar filters and the preservation of unboundedness, which are needed in Section 6.

For $\mathfrak{F} \subseteq \mathcal{P}(\omega)$, let $\langle \mathfrak{F} \rangle$ denote the filter generated by \mathfrak{F} together with the Frechét filter.

Definition 5.1. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter containing the Frechét filter. *Mathias forcing with respect to \mathcal{F}* (denoted by $\mathbb{M}(\mathcal{F})$) is the set of pairs (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$, where the order is defined as follows: $(t, B) \leq (s, A)$ if

- (1) $t \supseteq s$, i.e., t extends s
- (2) $B \subseteq A$
- (3) for each $n \geq |s|$, if $t(n) = 1$, then $n \in A$.

Note that $\mathbb{M}(\mathcal{F})$ is σ -centered: for $s \in 2^{<\omega}$, the set $\{(s, A) \mid A \in \mathcal{F}\}$ is clearly centered (i.e., finitely many conditions have a common lower bound). Also note that Mathias forcing with respect to the Frechét filter is forcing equivalent to Cohen forcing \mathbb{C} .

A filter \mathcal{F} is Canjar if $\mathbb{M}(\mathcal{F})$ does not add a dominating real over the ground model (i.e., the ground model reals remain unbounded). We need the following generalization of Canjarness:

Definition 5.2. Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. A filter \mathcal{F} on ω is *\mathcal{B} -Canjar* if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of \mathcal{B} (i.e., \mathcal{B} is still unbounded in the extension by $\mathbb{M}(\mathcal{F})$).

5.1. A combinatorial characterization of \mathcal{B} -Canjarness. Later, we will prove that certain filters are \mathcal{B} -Canjar; for that, we use the following combinatorial characterization of \mathcal{B} -Canjarness by Guzmán-Hrušák-Martínez [21]. This characterization generalizes a characterization of Canjarness by Hrušák-Minami [23].

Let \mathcal{F} be a filter on ω ; recall that a set $X \subseteq [\omega]^{<\omega}$ is in $(\mathcal{F}^{<\omega})^+$ if and only if for each $A \in \mathcal{F}$ there is an $s \in X$ with $s \subseteq A$. Note that if $\mathcal{G} \subseteq \mathcal{F}$ are filters and $X \in (\mathcal{F}^{<\omega})^+$, then $X \in (\mathcal{G}^{<\omega})^+$.

Given $\bar{X} = \langle X_n \mid n \in \omega \rangle$ (with $X_n \subseteq [\omega]^{<\omega}$ for each $n \in \omega$), and $f \in \omega^\omega$, let

$$\bar{X}_f = \bigcup_{n \in \omega} (X_n \cap \mathcal{P}(f(n))).$$

Theorem 5.3. *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. A filter \mathcal{F} on ω is \mathcal{B} -Canjar if and only if the following holds: for each sequence $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$, there exists an $f \in \mathcal{B}$ such that $\bar{X}_f \in (\mathcal{F}^{<\omega})^+$.*

Proof. See [21, Proposition 1]. □

It is well-known that Cohen forcing \mathbb{C} preserves the unboundedness of every unbounded family (in fact, \mathbb{C} is even almost bounding). As mentioned above, Mathias forcing with respect to the Frechét filter is forcing equivalent to \mathbb{C} , and hence the Frechét filter is \mathcal{B} -Canjar for every unbounded family \mathcal{B} . To illustrate the characterization of \mathcal{B} -Canjarness from Theorem 5.3, we also want to provide the following easy combinatorial proof of this fact:

Lemma 5.4. *Let \mathcal{B} be an unbounded family. Then the Frechét filter is \mathcal{B} -Canjar.*

Proof. Let \mathcal{F} be the Frechét filter. To show that \mathcal{F} is \mathcal{B} -Canjar, we use Theorem 5.3. So let $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$. Note that a set $X \subseteq [\omega]^{<\omega}$ is in $(\mathcal{F}^{<\omega})^+$ if and only if for each $n \in \omega$ there is an $s \in X$ with $\min(s) \geq n$. For each $n \in \omega$, pick $s_n \in X_n$ such that $\min(s_n) \geq n$, and let $g \in \omega^\omega$ such that $g(n) > \max(s_n)$ for each $n \in \omega$. Since \mathcal{B} is unbounded, we can pick $f \in \mathcal{B}$ such that $f(n) > g(n)$ for infinitely many n . It is easy to check that $s_n \in \bar{X}_f$ for infinitely many n , and this implies that $\bar{X}_f \in (\mathcal{F}^{<\omega})^+$, as desired. \square

The following observation will be crucial later on:

Lemma 5.5. *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family, \mathcal{F} a \mathcal{B} -Canjar filter extending the Frechét filter and $\{a_n \mid n < \omega\}$ such that $\mathcal{F} \cup \{a_n \mid n < \omega\}$ is a filter base. Then $\langle \mathcal{F} \cup \{a_n \mid n < \omega\} \rangle$ is \mathcal{B} -Canjar.*

Proof. Let $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\langle \mathcal{F} \cup \{a_n \mid n < \omega\} \rangle^{<\omega})^+$. Let

$$Y_n := \{s \in X_n \mid s \subseteq \bigcap_{k < n} a_k\}$$

and $\bar{Y} := \langle Y_n \mid n \in \omega \rangle$. It is easy to see that $Y_n \in (\mathcal{F}^{<\omega})^+$ for each n . By the assumption and Theorem 5.3 there exists $f \in \mathcal{B}$ such that $\bar{Y}_f \in (\mathcal{F}^{<\omega})^+$.

To show that $\bar{Y}_f \in (\langle \mathcal{F} \cup \{a_n \mid n < \omega\} \rangle^{<\omega})^+$ let $B \in \langle \mathcal{F} \cup \{a_n \mid n < \omega\} \rangle$, i.e., there exists $A \in \mathcal{F}$ and $n \in \omega$ with $B \supseteq A \cap \bigcap_{k < n} a_k$. Since \mathcal{F} contains the Frechét filter and $\bar{Y}_f \in (\mathcal{F}^{<\omega})^+$, there exist infinitely many $s \in \bar{Y}_f$ with $s \subseteq A$. So there exists $m \geq n$ and $s \in Y_m \cap \bar{Y}_f$ with $s \subseteq A$; note that $s \in Y_m$ implies $s \subseteq \bigcap_{k < m} a_k$, so $s \subseteq B$, as desired.

Clearly $\bar{Y}_f \subseteq \bar{X}_f$, so $\bar{X}_f \in (\langle \mathcal{F} \cup \{a_n \mid n < \omega\} \rangle^{<\omega})^+$. \square

We also get the following:

Lemma 5.6. *Let \mathcal{B} be an unbounded family. Then every countably generated filter is \mathcal{B} -Canjar.*

Proof. This follows immediately from Lemma 5.4 and Lemma 5.5. \square

5.2. Preservation of unboundedness at limits. We will use the following theorem by Judah-Shelah [25] about preservation of unboundedness in finite support iterations. In fact, [25, Theorem 2.2] is a much more general version than the theorem presented here.

Theorem 5.7. *Suppose $\{\mathbb{P}_\alpha, \dot{Q}_\alpha \mid \alpha < \delta\}$ is a finite support iteration of c.c.c. partial orders of limit length δ , and $\mathcal{B} \subseteq \omega^\omega$ is unbounded; also suppose that \mathcal{B} is countably directed, i.e., it satisfies*

$$(4) \quad \forall \mathcal{A} \subseteq \mathcal{B} \ (|\mathcal{A}| = \aleph_0 \rightarrow \exists f \in \mathcal{B} \ \forall g \in \mathcal{A} \ g \leq^* f);$$

moreover, suppose that

$$\forall \alpha < \delta \ \Vdash_{\mathbb{P}_\alpha} \text{“}\mathcal{B} \text{ is an unbounded family”}.$$

Then $\Vdash_{\mathbb{P}_\delta}$ “ \mathcal{B} is an unbounded family”.

Proof. See [16, Theorem 3.5.2]. \square

5.3. Preservation of \mathcal{B} -Canjarness and finite sums of filters. The notion of \mathcal{B} -Canjarness of a filter is not absolute in general:

Example 5.8 (from [20]). *Let \mathcal{B} be the ground model reals and \mathcal{U} be a \mathcal{B} -Canjar ultrafilter. Let \mathbb{P} be Grigorieff forcing with respect to \mathcal{U} , which forces that \mathcal{U} cannot be extended to a P -point. It is well-known that \mathbb{P} preserves the unboundedness of \mathcal{B} , and it can be shown that \mathcal{U} is not a P^+ -filter in $V[\mathbb{P}]$; since any Canjar filter is a P^+ -filter, it follows that \mathcal{U} is no longer \mathcal{B} -Canjar.*

Note that Grigorieff forcing is proper, but not c.c.c.; however, Grigorieff forcing can be decomposed into a σ -closed and a c.c.c. forcing (see [28]). Since a σ -closed forcing does not destroy the \mathcal{B} -Canjarness of a filter, the above example also yields an example of a c.c.c. forcing destroying the \mathcal{B} -Canjarness of a filter.

We will now provide a method how to guarantee that the \mathcal{B} -Canjarness of a filter is not destroyed by Mathias forcings with respect to certain other filters. As a tool, we introduce finite sums of filters and consider Mathias forcings with respect to these sums.

Lemma 5.9. *Let \mathcal{F} be a filter, $\mathcal{B} \subseteq \omega^\omega$, and \mathbb{P} be a forcing notion. Then the following are equivalent:*

- (1) \mathbb{P} forces that¹⁷ \mathcal{F} is \mathcal{B} -Canjar.
- (2) $\mathbb{M}(\mathcal{F}) \times \mathbb{P}$ forces that \mathcal{B} is unbounded.

Even though we will apply the lemma only in case \mathcal{B} is unbounded and \mathcal{F} is \mathcal{B} -Canjar in the ground model, this is not necessary for the proof. If one of these assumptions fails, both (1) and (2) are false.

Proof of Lemma 5.9. Let $\mathbb{Q} := \mathbb{M}(\mathcal{F})$. Note that (1) holds if and only if \mathbb{P} forces

$$\mathbb{M}(\check{\mathcal{F}}) \Vdash \text{“}\mathcal{B} \text{ unbounded”}.$$

Further note that \mathbb{P} forces that $\check{\mathbb{Q}}$ is (dense in, and hence) forcing equivalent to $\mathbb{M}(\check{\mathcal{F}})$. So, (1) holds if and only if $\mathbb{P} * \check{\mathbb{Q}}$ forces that \mathcal{B} is unbounded, which is the same as (2) (since $\mathbb{P} * \check{\mathbb{Q}}$ is equivalent to $\mathbb{P} \times \mathbb{Q} = \mathbb{Q} \times \mathbb{P}$). \square

Definition 5.10. For two sets $A, B \subseteq \omega$, let $A \oplus B := \{2n \mid n \in A\} \cup \{2m + 1 \mid m \in B\}$. For two filters \mathcal{F}_0 and \mathcal{F}_1 , let $\mathcal{F}_0 \oplus \mathcal{F}_1 := \{A \oplus B \mid A \in \mathcal{F}_0, B \in \mathcal{F}_1\}$. More generally, inductively define $\bigoplus_{k < m+1} \mathcal{F}_k := \left(\bigoplus_{k < m} \mathcal{F}_k\right) \oplus \mathcal{F}_m$.

Note that $\mathcal{F}_0 \oplus \mathcal{F}_1$ is a filter if \mathcal{F}_0 and \mathcal{F}_1 are filters, and hence also the finite sum of filters is a filter. The order of the sum is not important: more precisely, the filter $\bigoplus_{k < m} \mathcal{F}_k$ is isomorphic (based on a bijection on ω) to all reorderings of this sum. For example $(\mathcal{F}_0 \oplus \mathcal{F}_1) \oplus \mathcal{F}_2$ is isomorphic to $(\mathcal{F}_2 \oplus \mathcal{F}_0) \oplus \mathcal{F}_1$. This implies that the \mathcal{B} -Canjarness of a finite sum of filters does not depend on the order of the sum.

Lemma 5.11. *Let \mathcal{F}_0 and \mathcal{F}_1 be two filters. Then $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$.*

Proof. Let $D_\times \subseteq \mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ be the set of all $((s_0, A_0), (s_1, A_1)) \in \mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ with $|s_0| = |s_1|$, and let $D_\oplus \subseteq \mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ be the set of all $(s, A) \in \mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ with $|s|$ being an even number. Note that D_\times is a dense subforcing of $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$, and D_\oplus is a dense subforcing of $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$.

¹⁷To be more precise, one should write $\langle \check{\mathcal{F}} \rangle$ instead of \mathcal{F} .

For $s_0, s_1 \in 2^{<\omega}$ with $L := |s_0| = |s_1|$, let $s_0 \oplus s_1 \in 2^{<\omega}$ be such that $|s_0 \oplus s_1| = 2L$ and satisfies $(s_0 \oplus s_1)(2n) = s_0(n)$ and $(s_0 \oplus s_1)(2n+1) = s_1(n)$.

Define $\iota: D_\times \rightarrow D_\oplus$ as follows:

$$((s_0, A_0), (s_1, A_1)) \mapsto (s_0 \oplus s_1, A_0 \oplus A_1).$$

It is easy to see that ι is an isomorphism between the forcings D_\times and D_\oplus . Consequently, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ and $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ are forcing equivalent. \square

The following lemma will be the main ingredient of the ‘‘successor step’’ of the induction (for old filters) in Lemma 6.4:

Lemma 5.12. *If $\mathcal{F}_0 \oplus \mathcal{F}_1$ is \mathcal{B} -Canjar, then $\mathbb{M}(\mathcal{F}_1)$ forces that \mathcal{F}_0 is \mathcal{B} -Canjar.*

Proof. By assumption and Lemma 5.11, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ forces that \mathcal{B} is unbounded; apply Lemma 5.9 to finish the proof. \square

The following lemma will be the main ingredient of the ‘‘limit step’’ of the induction (for old filters) in Lemma 6.4:

Lemma 5.13. *Let \mathcal{B} be a countably directed family (see (4) of Theorem 5.7), let α be a limit, and let $\{\mathbb{P}_\beta, \dot{Q}_\beta \mid \beta < \alpha\}$ be a finite support iteration. Suppose that \mathbb{P}_β forces that \mathcal{F} is \mathcal{B} -Canjar for every $\beta < \alpha$. Then \mathbb{P}_α forces that \mathcal{F} is \mathcal{B} -Canjar.*

Proof. By assumption and Lemma 5.9, $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta$ forces that \mathcal{B} is unbounded for every $\beta < \alpha$. Observe that $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$ is the direct limit of the sequence $\langle \mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta \mid \beta < \alpha \rangle$ (and $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta$ is complete in $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$), so it can be written as the limit of a finite support iteration, therefore, by Theorem 5.7, also $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$ forces that \mathcal{B} is unbounded. We obtain the conclusion by again applying Lemma 5.9. \square

Lemma 5.14. *Let \mathcal{F}_0 be \mathcal{B} -Canjar and \mathcal{F}_1 be countably generated. Then $\mathcal{F}_0 \oplus \mathcal{F}_1$ is \mathcal{B} -Canjar.*

Using the fact that sums can be reordered (see the remark after Definition 5.10), we obtain the following stronger statement: Let $\mathcal{F}_0, \dots, \mathcal{F}_{m-1}$ be filters such that (some of them are countably generated and) the sum of the filters which are not countably generated is \mathcal{B} -Canjar; then $\bigoplus_{k < m} \mathcal{F}_k$ is \mathcal{B} -Canjar.

Proof of Lemma 5.14. We have to show that $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ forces that \mathcal{B} is unbounded. By Lemma 5.11, $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$.

Since \mathcal{F}_0 is \mathcal{B} -Canjar by assumption, \mathcal{B} is unbounded in the extension by $\mathbb{M}(\mathcal{F}_0)$. Since \mathcal{F}_1 is countably generated, the same holds in the extension by $\mathbb{M}(\mathcal{F}_0)$: more precisely, the filter generated by \mathcal{F}_1 is countably generated. Therefore, in the extension by $\mathbb{M}(\mathcal{F}_0)$, (the filter generated by) \mathcal{F}_1 is \mathcal{B} -Canjar by Lemma 5.6. So, by Lemma 5.9, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ forces that \mathcal{B} is unbounded, as desired. \square

6. PRESERVING UNBOUNDEDNESS: $\mathfrak{h} = \mathfrak{b} = \omega_1$

In this section, we will finish the proof of Main Theorem 1.2 by showing that \mathfrak{b} is small (i.e., $\mathfrak{b} = \omega_1$) in the final model W . Recall the following well-known ZFC inequalities (see (1) in Section 2):

$$\omega_1 \leq \mathfrak{h} \leq \mathfrak{b}$$

So, if we have $\mathfrak{b} = \omega_1$, it follows that $\mathfrak{h} = \omega_1$ (in particular, there exists a distributivity matrix of height ω_1). We have shown that there is a distributivity matrix of height $\lambda > \omega_1$ in W ; so there exists a distributivity matrix of regular height larger than \mathfrak{h} .

In Section 6.1, we will show that our iteration \mathbb{P}_λ can be represented as a finer iteration whose iterands are Mathias forcings with respect to filters. In Section 6.2, we show that the filters which are used are \mathcal{B} -Canjar (i.e., the corresponding Mathias forcings preserve the unboundedness of \mathcal{B}), where \mathcal{B} is the set of reals of V_0 . A similar (but less involved) argument shows that Hechler's original forcings [22] to add a tower or to add a mad family can be represented as an iteration of Mathias forcings with respect to \mathcal{B} -Canjar filters as well (see [14]).

6.1. Finer iteration via filtered Mathias forcings. As described in Section 3.1, $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\}$ is our main finite support iteration which we force with over V . Its limit \mathbb{P}_λ adds a distributivity matrix of height λ . We will now represent our iteration as a “finer” iteration: we write each iterand \mathbb{Q}_α as a finite support iteration of Mathias forcings with respect to certain filters. Fix $\alpha < \lambda$.

As a preparation, we introduce a “nice” enumeration of T_α (recall that $\sigma \in T_\alpha$ if and only if a_σ is added by \mathbb{Q}_α). We go through the nodes in T_α level by level, and “blockwise”. A *block* is a set of nodes $\{\rho \hat{\ } i \mid i < \lambda\}$ for some $\rho \in \lambda^{<\lambda}$. More precisely, let $\{\sigma_\alpha^\nu \mid \nu < \Lambda_\alpha\}$ be an enumeration of T_α (note that $|T_\alpha| = \mathfrak{c}$ and hence Λ_α is an ordinal with $\mathfrak{c} < \Lambda_\alpha < \mathfrak{c}^+$) such that

- (1) (“level by level”) $|\sigma_\alpha^{\bar{\nu}}| < |\sigma_\alpha^\nu| \rightarrow \bar{\nu} < \nu$,
- (2) (“blockwise”) for each $\rho \in \lambda^{<\lambda}$ with $\{\rho \hat{\ } i \mid i < \lambda\} \subseteq T_\alpha$, there is $\nu < \Lambda_\alpha$ such that

$$\rho \hat{\ } i = \sigma_\alpha^{\nu+i} \text{ for each } i < \lambda.$$

Recall that \mathbb{Q}_α^C denotes $\{p \in \mathbb{Q}_\alpha \mid \text{dom}(p) \subseteq C\}$ (for $C \subseteq \lambda^{<\lambda}$). For any $\beta \leq \Lambda_\alpha$, let

$$\mathbb{Q}_\alpha^{<\beta} := \mathbb{Q}_\alpha^{\{\sigma_\alpha^\nu \mid \nu < \beta\}},$$

and for $\beta < \Lambda_\alpha$,

$$\mathbb{Q}_\alpha^{\leq\beta} := \mathbb{Q}_\alpha^{\{\sigma_\alpha^\nu \mid \nu \leq \beta\}}.$$

Note that $\mathbb{Q}_\alpha^{<\Lambda_\alpha} = \mathbb{Q}_\alpha$, and that $\{\sigma_\alpha^\nu \mid \nu < \beta\}$ is α -left-up-closed for each $\beta \leq \Lambda_\alpha$ (due to (1) and (2) above). Therefore, by Lemma 3.17, $\mathbb{Q}_\alpha^{<\beta}$ is a complete subforcing of \mathbb{Q}_α . By Lemma 4.1, $\mathbb{Q}_\alpha^{<\beta}$ is a complete subforcing of $\mathbb{Q}_\alpha^{\leq\beta}$, so we can form the quotient $\mathbb{Q}_\alpha^{\leq\beta} / \mathbb{Q}_\alpha^{<\beta}$. Moreover, because conditions in \mathbb{Q}_α have finite domain,

$$\mathbb{Q}_\alpha^{\leq\beta} = \bigcup_{\delta < \beta} \mathbb{Q}_\alpha^{<\delta}$$

for each limit ordinal $\beta \leq \Lambda_\alpha$; in other words, $\mathbb{Q}_\alpha^{<\beta}$ is the direct limit of the forcings $\mathbb{Q}_\alpha^{<\delta}$ for $\delta < \beta$. So \mathbb{Q}_α is forcing equivalent to the finite support iteration of the quotients $\mathbb{Q}_\alpha^{\leq\beta} / \mathbb{Q}_\alpha^{<\beta}$ for $\beta < \Lambda_\alpha$.

Recall that $\mathbb{M}(\mathcal{F})$ denotes Mathias forcing with respect to the filter \mathcal{F} (see Definition 5.1). We are now going to show that $\mathbb{Q}_\alpha^{\leq\beta} / \mathbb{Q}_\alpha^{<\beta}$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha^\beta)$ for a filter \mathcal{F}_α^β . Work in an extension by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}$, and note that, for each $\tau \in T_\eta$ with $\eta < \alpha$, a set a_τ has been added by \mathbb{P}_α , and for each $\nu < \beta$, a set $a_{\sigma_\alpha^\nu}$ has been added by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}$. These sets¹⁸ are used to define \mathcal{F}_α^β as follows. Let $\rho \in \lambda^{<\lambda}$ and $i < \lambda$ be such that $\sigma_\alpha^\beta = \rho \hat{\ } i$, and let

$$\mathfrak{F}_\alpha^\beta := \{a_{\rho \hat{\ }(\xi+1)} \mid \xi + 1 \leq |\rho|\} \cup \{a_{\rho \hat{\ }j} \mid j < i\},$$

i.e., $\mathfrak{F}_\alpha^\beta$ is the collection of all sets assigned to the nodes above σ_α^β and the complements of the sets assigned to the nodes to the left of σ_α^β within the same block. Note that $\mathfrak{F}_\alpha^\beta$ is a filter base, i.e., any intersection of finitely many elements is infinite: indeed, for finite $I \subseteq i$ and $\xi + 1 \leq |\rho|$, let $j^* \in \lambda \setminus I$; then $a_{\rho^* j^*} \subseteq^* a_{\rho \uparrow (\xi+1)} \cap \bigcap_{j \in I} (\omega \setminus a_{\rho^- j})$. Then let

$$\mathcal{F}_\alpha^\beta := \langle \mathfrak{F}_\alpha^\beta \rangle,$$

i.e., \mathcal{F}_α^β is the filter generated by taking finite intersections of sets from $\mathfrak{F}_\alpha^\beta$ and the Frechét filter and taking the upwards closure.

The quotient $\mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta}$ adds the set a_σ where $\sigma = \sigma_\alpha^\beta$. The following lemma will provide a dense embedding from $\mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta}$ to $\mathbb{M}(\mathcal{F}_\alpha^\beta)$ which preserves (the finite approximations of) the generic real a_σ . Therefore, a_σ is also the generic real for $\mathbb{M}(\mathcal{F}_\alpha^\beta)$. Recall that the generic real for $\mathbb{M}(\mathcal{F})$ is a pseudo-intersection of \mathcal{F} , and the definition of \mathcal{F}_α^β ensures that a pseudo-intersection of it is almost contained in $a_{\rho \uparrow (\xi+1)}$ whenever $\xi + 1 \leq |\rho|$ and almost disjoint from $a_{\rho^- j}$ for each $j < i$, as it is the case for the real a_σ .

Lemma 6.1. $\mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta}$ is densely embeddable into $\mathbb{M}(\mathcal{F}_\alpha^\beta)$.

Proof. For simplicity of notation, let $\sigma := \sigma_\alpha^\beta$ for the rest of this proof. Let G be a generic filter for $\mathbb{Q}_\alpha^{\leq \beta}$. We work in the extension by G , so

$$\mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta} = \{p \in \mathbb{Q}_\alpha^{\leq \beta} \mid \forall q \in G (p \text{ is compatible with } q)\}.$$

Let us define an embedding $\iota: \mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta} \rightarrow \mathbb{M}(\mathcal{F}_\alpha^\beta)$ as follows: for $p \in \mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta}$, let $p(\sigma) = (s_\sigma, f_\sigma, h_\sigma)$, and let $\iota(p) = (s_\sigma, A)$, where

$$A = \bigcap_{\tau \in \text{dom}(f_\sigma)} (a_\tau \cup f_\sigma(\tau)) \cap \bigcap_{\rho \in \text{dom}(h_\sigma)} ((\omega \setminus a_\rho) \cup h_\sigma(\rho)) \setminus |s_\sigma|.$$

To see that it is a dense embedding, we have to check the following conditions:

- (1) (Density) For every condition $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha^\beta)$, there exists a condition p such that $\iota(p) \leq (s, A)$.
- (2) (Incompatibility preserving) If p and p' are incompatible, then so are $\iota(p)$ and $\iota(p')$.
- (3) (Order preserving) If $p' \leq p$, then $\iota(p') \leq \iota(p)$.

To show (1), let $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha^\beta)$. Since $A \in \mathcal{F}_\alpha^\beta$, there exist finite sets $\{\rho_i \mid i < m\}$, $\{\tau_j \mid j < l\}$ and $N \in \omega$ such that $\bigcap_{j < l} a_{\tau_j} \cap \bigcap_{i < m} (\omega \setminus a_{\rho_i}) \setminus N \subseteq A$. Extend s with 0's to s_σ such that $|s_\sigma| = \max(|s|, N)$, and let $\text{dom}(h_\sigma) := \{\rho_i \mid i < m\}$ and $h_\sigma(\rho_i) := |s_\sigma|$ for every i , and $\text{dom}(f_\sigma) := \{\tau_j \mid j < l\}$ and $f_\sigma(\tau_j) := |s_\sigma|$ for every j . Let $p := \{(\sigma, (s_\sigma, f_\sigma, h_\sigma))\} \cup \{(\tau, (\langle \cdot \rangle, \emptyset, \emptyset)) \mid \tau \in (\text{dom}(f_\sigma) \cap T_\alpha) \cup \text{dom}(h_\sigma)\}$.

To see that p is in the quotient, let $q \in G$ be arbitrary; it is easy to check that $q \cup \{(\tau, (s_\tau, f_\tau, h_\tau)) \mid \tau \in \text{dom}(p) \setminus \text{dom}(q)\} \leq p, q$.

By definition, $\iota(p) = (s_\sigma, A')$, where

$$A' = \bigcap_{\tau \in \text{dom}(f_\sigma)} (a_\tau \cup f_\sigma(\tau)) \cap \bigcap_{\rho \in \text{dom}(h_\sigma)} ((\omega \setminus a_\rho) \cup h_\sigma(\rho)) \setminus |s_\sigma|.$$

¹⁸It is possible (see the base step $\beta^* = 0$ of the proof of Lemma 6.4(3)) that only sets a_τ with $\tau \in T_\eta$ for some $\eta < \alpha$ are used. This is the case if ρ is pre- T_α -minimal and $i = 0$.

It follows that

$$A' \stackrel{(*)}{=} \bigcap_{\tau \in \text{dom}(f_\sigma)} a_\tau \cap \bigcap_{\rho \in \text{dom}(h_\sigma)} (\omega \setminus a_\rho) \setminus |s_\sigma| \subseteq \bigcap_{j < l} a_{\tau_j} \cap \bigcap_{i < m} (\omega \setminus a_{\rho_i}) \setminus N \subseteq A$$

(where $(*)$ holds because $|s_\sigma| \geq f_\sigma(\tau), h_\sigma(\rho)$ for every τ, ρ in the respective domains). Therefore $s_\sigma \geq s$, $A' \subseteq A$, and $s_\sigma(n) = 0$ for all $n \geq |s|$. So $\iota(p) = (s_\sigma, A') \leq (s, A)$.

We prove (2) by showing the contrapositive. Assume $\iota(p)$ and $\iota(p')$ are compatible. Define q as follows. Let $\text{dom}(q) := \text{dom}(p) \cup \text{dom}(p')$. For every $\tau \in \text{dom}(q)$, let $s_\tau^q := s_\tau^p \cup s_\tau^{p'}$, $\text{dom}(f_\tau^q) := \text{dom}(f_\tau^p) \cup \text{dom}(f_\tau^{p'})$ and for $\rho \in \text{dom}(f_\tau^q)$ let $f_\tau^q(\rho) = \min(f_\tau^p(\rho), f_\tau^{p'}(\rho))$, and the same for h : $\text{dom}(h_\tau^q) := \text{dom}(h_\tau^p) \cup \text{dom}(h_\tau^{p'})$ and for $\rho \in \text{dom}(h_\tau^q)$ let $h_\tau^q(\rho) = \min(h_\tau^p(\rho), h_\tau^{p'}(\rho))$. It is easy to check that q is a condition in the quotient and $q \leq p, p'$.

To show (3), let $p' \leq p$. So, by definition, $s_\sigma^{p'} \geq s_\sigma^p$, and $\text{dom}(h_\sigma^{p'}) \supseteq \text{dom}(h_\sigma^p)$ and $\text{dom}(f_\sigma^{p'}) \supseteq \text{dom}(f_\sigma^p)$, and $f_\sigma^{p'}(\tau) \leq f_\sigma^p(\tau)$ for $\tau \in \text{dom}(f_\sigma^p)$ and $h_\sigma^{p'}(\rho) \leq h_\sigma^p(\rho)$ for $\rho \in \text{dom}(h_\sigma^p)$; so

$$\begin{aligned} A' &:= \bigcap_{\tau \in \text{dom}(f_\sigma^{p'})} (a_\tau \cup f_\sigma^{p'}(\tau)) \cap \bigcap_{\rho \in \text{dom}(h_\sigma^{p'})} ((\omega \setminus a_\rho) \cup h_\sigma^{p'}(\rho)) \setminus |s_\sigma^{p'}| \\ &\subseteq \bigcap_{\tau \in \text{dom}(f_\sigma^p)} (a_\tau \cup f_\sigma^p(\tau)) \cap \bigcap_{\rho \in \text{dom}(h_\sigma^p)} ((\omega \setminus a_\rho) \cup h_\sigma^p(\rho)) \setminus |s_\sigma^p| =: A. \end{aligned}$$

By definition, $\iota(p) = (s_\sigma^p, A)$ and $\iota(p') = (s_\sigma^{p'}, A')$. To show that $(s_\sigma^{p'}, A') \leq (s_\sigma^p, A)$, it remains to show that for $n \geq |s_\sigma^p|$ with $s_\sigma^{p'}(n) = 1$, we have $n \in A$. First fix $\rho \in \text{dom}(h_\sigma^p)$ and show that $n \in (\omega \setminus a_\rho) \cup h_\sigma^p(\rho)$. If $n < h_\sigma^p(\rho)$, this is clear. If $n \geq h_\sigma^p(\rho)$, we know that $s_\sigma^{p'}$ respects h_σ^p , and so $n \in \omega \setminus a_\rho$. So in both cases, $n \in (\omega \setminus a_\rho) \cup h_\sigma^p(\rho)$. Now fix $\tau \in \text{dom}(f_\sigma^p)$ and show that $n \in a_\tau \cup f_\sigma^p(\tau)$. This is the same argument as for h . If $n < f_\sigma^p(\tau)$, this is clear. If $n \geq f_\sigma^p(\tau)$, we know that $s_\sigma^{p'}$ respects f_σ^p , and so $n \in a_\tau$. So in both cases, $n \in a_\tau \cup f_\sigma^p(\tau)$, finishing the proof. \square

The following fact will be needed in the proof of Claim 6.5:

Corollary 6.2. \mathbb{P}_α is σ -centered for each $\alpha \leq \lambda$.

More generally, the same holds for $\mathbb{P}_\alpha / \mathbb{P}_\eta$ for $\eta < \alpha$.

Proof of Corollary 6.2. Since Mathias forcing with respect to a filter is always σ -centered (see the remark after Definition 5.1) and $\mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta}$ is densely embeddable into such a forcing by the above lemma, also $\mathbb{Q}_\alpha^{\leq \beta} / \mathbb{Q}_\alpha^{< \beta}$ is σ -centered.

Recall that $\Lambda_\eta < c^+$ for every $\eta < \alpha$, and $\alpha \leq \lambda \leq c$, so \mathbb{P}_α is a finite support iteration of σ -centered forcings of length strictly less than c^+ . As a matter of fact, the finite support iteration of σ -centered forcings of length strictly less than c^+ is σ -centered (the result was mentioned without proof in [33, proof of Lemma 2]; for a proof, see [5] or [19, Lemma 5.3.8]). \square

Also the following lemma will be used in the proof of Claim 6.5. It is similar to a well-known fact about branches of certain trees; see, e.g., [26, Lemma 3.8] or [27]. For the convenience of the reader, we provide an explicit proof here.

Lemma 6.3. *If $\mathbb{P} \times \mathbb{P}$ has the c.c.c. and $\text{cf}(\delta) > \omega$, then, in $V[\mathbb{P}]$, every new function from δ to the ordinals has an initial segment which is new.*

Proof. Assume towards a contradiction that there exists $p \in \mathbb{P}$ and a \mathbb{P} -name \dot{f} such that p forces $\dot{f}: \delta \rightarrow \text{Ord}$ is not in V and $\dot{f} \upharpoonright \gamma \in V$ for each $\gamma < \delta$. Therefore, we can, by induction on $i < \omega_1$, construct $\alpha_i < \delta$, $p_i \leq p$, and $q_i \leq p$ such that p_i and q_i decide \dot{f} up to α_i , and α_i is the first point about which p_i and q_i disagree; more precisely, there is $s_i: \alpha_i + 1 \rightarrow \text{Ord}$ and $t_i: \alpha_i + 1 \rightarrow \text{Ord}$ such that

- (1) $\alpha_j < \alpha_i$ for each $j < i$,
- (2) $p_i \Vdash \dot{f} \upharpoonright (\alpha_i + 1) = s_i$,
- (3) $q_i \Vdash \dot{f} \upharpoonright (\alpha_i + 1) = t_i$,
- (4) $s_i \neq t_i$, and $s_i \upharpoonright \alpha_i = t_i \upharpoonright \alpha_i$.

Consider $\langle (p_i, q_i) \mid i < \omega_1 \rangle$ and use that $\mathbb{P} \times \mathbb{P}$ has the c.c.c. to obtain $i_0 < i_1$ such that (p_{i_0}, q_{i_0}) and (p_{i_1}, q_{i_1}) are compatible, and fix (\bar{p}, \bar{q}) with $(\bar{p}, \bar{q}) \leq (p_{i_0}, q_{i_0})$ and $(\bar{p}, \bar{q}) \leq (p_{i_1}, q_{i_1})$. It follows that both \bar{p} and \bar{q} force that $\dot{f} \upharpoonright \alpha_{i_1} = s_{i_1} \upharpoonright \alpha_{i_1}$. Moreover, $\bar{p} \Vdash \dot{f} \upharpoonright (\alpha_{i_0} + 1) = s_{i_0}$ and $\bar{q} \Vdash \dot{f} \upharpoonright (\alpha_{i_0} + 1) = t_{i_0}$, but $s_{i_0} \neq t_{i_0}$, which easily yields (using $\alpha_{i_0} < \alpha_{i_1}$) a contradiction. \square

6.2. The filters are \mathcal{B} -Canjar. To finish the proof of Main Theorem 1.2, we have to show that $\mathfrak{b} = \omega_1$ holds true in the final extension.

Recall that the setup is the following. Our very ground model V_0 is a model of CH; therefore, its set of reals

$$\mathcal{B} = \omega^\omega \cap V_0$$

has size ω_1 . Clearly, \mathcal{B} is an unbounded family in V_0 . We will show that \mathcal{B} remains unbounded in the course of the iteration, thereby witnessing $\mathfrak{b} = \omega_1$ in the final model.

First, observe that \mathcal{B} is still unbounded in V , the extension of V_0 by μ many Cohen reals (due to the fact that \mathbb{C}_μ does not add dominating reals).

In Section 6.1, we have defined filters \mathcal{F}_α^β for $\alpha < \lambda$ and $\beta < \Lambda_\alpha$ (and their canonical filter bases \mathcal{B}_α^β) and have shown that \mathbb{Q}_α is equivalent to the finite support iteration of the Mathias forcings $\mathbb{M}(\mathcal{F}_\alpha^\beta)$. In particular, $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta} * \mathbb{M}(\mathcal{F}_\alpha^\beta) = \mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}$, and $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\Lambda_\alpha} = \mathbb{P}_{\alpha+1}$. Note that \mathcal{B} is countably directed (see (4) from Theorem 5.7); therefore it suffices to show that \mathcal{B} remains unbounded at successor steps of our “fine” iteration: for each $\alpha < \lambda$ and each $\beta < \Lambda_\alpha$, the unboundedness of \mathcal{B} is preserved by $\mathbb{M}(\mathcal{F}_\alpha^\beta)$. To achieve this, we will show that the filters \mathcal{F}_α^β are \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}]$ (see Lemma 6.4(2)). Actually, we show for every β^* that \mathcal{F}_α^β is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ whenever $\mathcal{B}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$, i.e., we show the \mathcal{B} -Canjarness of a filter \mathcal{F}_α^β as soon as it exists.

In many cases, we use a genericity argument to show that the filters are \mathcal{B} -Canjar at the stage where they appear, but we need the \mathcal{B} -Canjarness in later stages of the iteration, for two reasons: first, we want to force with this filter in a later stage, and second, we want to use the \mathcal{B} -Canjarness of an older filter to show the \mathcal{B} -Canjarness of a filter which appears later. As mentioned earlier, the notion of \mathcal{B} -Canjarness of a filter is not absolute, therefore we will use our method from Section 5.3 to guarantee that the \mathcal{B} -Canjarness of the filter is not destroyed by the other Mathias forcings along the iteration. This method is based on finite sums of filters, therefore we show that all finite sums of filters which exist in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ are \mathcal{B} -Canjar (see Lemma 6.4(3)).

Lemma 6.4. *For every $\alpha < \lambda$, for every $\beta^* < \Lambda_\alpha$,*

- (1) \mathcal{B} is unbounded in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$,
- (2) $\mathcal{F}_\alpha^{\beta^*}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$,
- (3) if $m \in \omega$ and $\beta_0, \dots, \beta_{m-1} < \Lambda_\alpha$ with $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta_k}]$ for every $k < m$, then $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$.

Proof. First note that, for each $\alpha < \lambda$ and $\beta^* < \Lambda_\alpha$, (2) is a special instance of (3): in fact, $\mathfrak{F}_\alpha^{\beta^*} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$, so (3) for $m = 1$ and $\beta_0 = \beta^*$ is (2). However, we need (3) in order to carry out the induction (to preserve \mathcal{B} -Canjarness of our filters).

We prove (1) and (3) (and hence (2)) by (simultaneous) induction on the pairs (α, β^*) (with the lexicographical ordering). So suppose that (1) and (3) hold for each $(\alpha', \beta') <_{lex} (\alpha, \beta^*)$, i.e., for each pair with $\alpha' < \alpha$ (and β' arbitrary) or $\alpha' = \alpha$ and $\beta' < \beta^*$.

Proof of (1):

For $\alpha = \beta^* = 0$, just note that \mathcal{B} is unbounded in $V[\mathbb{P}_0 * \mathbb{Q}_0^{<0}] = V$, since V is the extension by Cohen forcing \mathbb{C}_μ (which does not add dominating reals) of our GCH ground model V_0 .

In case $\beta^* = \beta' + 1$ is a successor ordinal, we use the fact that (1) holds for $\alpha = \alpha'$ and β' by induction, so \mathcal{B} is unbounded in the extension by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta'}$; by Lemma 6.1, $\mathbb{Q}_\alpha^{<\beta'} / \mathbb{Q}_\alpha^{<\beta'}$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha^{\beta'})$, i.e., $\mathbb{Q}_\alpha^{<\beta^*} = \mathbb{Q}_\alpha^{<\beta'} * \mathbb{M}(\mathcal{F}_\alpha^{\beta'})$; since (2) holds for β' by induction, $\mathbb{M}(\mathcal{F}_\alpha^{\beta'})$ preserves the unboundedness of \mathcal{B} , hence the same is true for $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}$, as desired.

In case (α, β^*) is a limit point of the lexicographical ordering (i.e., $\beta^* = 0$ or β^* is a limit ordinal), we use the fact that $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}$ is the limit of a finite support iteration of c.c.c. forcings, and that (1) holds for each $(\alpha', \beta') <_{lex} (\alpha, \beta^*)$; so we can apply Theorem 5.7 to conclude (1) for (α, β^*) .

Proof of (3):

Fix α . By (1), \mathcal{B} is unbounded in $V[\mathbb{P}_\alpha]$. We say that $\rho \in \lambda^{<\lambda}$ is *pre- T_α -minimal* if it is the predecessor of a minimal node of T_α ; it is straightforward to check that this is the case if and only if

- $\rho \in V[\mathbb{P}_\alpha]$,
- $\rho \notin V[\mathbb{P}_\eta]$ for any $\eta < \alpha$, and
- for every $\gamma < |\rho|$, there exists $\eta < \alpha$ with $\rho \upharpoonright \gamma \in V[\mathbb{P}_\eta]$.

Note that for $\alpha = 0$, the only pre- T_α -minimal node is the root $\langle \rangle$, and for $\alpha > 0$, all pre- T_α -minimal nodes have limit length.

We proceed by induction on β^* .

Base step $\beta^* = 0$:

Let $\beta_0, \dots, \beta_{m-1}$ be such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<0}]$ for each $k < m$. Since $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<0}] = V[\mathbb{P}_\alpha]$, it follows that $\sigma_\alpha^{\beta_k} = \rho_k \frown 0$ for some pre- T_α -minimal node ρ_k : indeed, observe that $\mathfrak{F}_\alpha^{\beta_k}$ contains elements which are only added by \mathbb{Q}_α (and hence $\mathfrak{F}_\alpha^{\beta_k} \notin V[\mathbb{P}_\alpha]$) whenever $\sigma_\alpha^\beta = \rho \frown i$ with ρ not pre- T_α -minimal or $i > 0$ (because $a_\tau \in \mathfrak{F}_\alpha^\beta$ or $\omega \setminus a_\tau \in \mathfrak{F}_\alpha^\beta$ for some $\tau \in T_\alpha$).

If $\text{cf}(|\rho_k|)$ is countable for all $k < m$, the filter $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is countably generated, hence it follows by Lemma 5.6 that it is \mathcal{B} -Canjar.

In particular, for $\alpha = 0$, the only pre- T_α -minimal node is $\rho = \langle \rangle$, hence this finishes the proof for $\alpha = \beta^* = 0$. So assume $\alpha > 0$ for the rest of the proof of the base step.

If $\text{cf}(\alpha) \leq \omega$ (and $\alpha > 0$), all pre- T_α -minimal nodes ρ have $\text{cf}(|\rho|) = \omega$:

Claim 6.5. *Let ρ be a pre- T_α -minimal node and $\text{cf}(|\rho|) > \omega$. Then*

- (1) $\text{cf}(\alpha) > \omega$, and
- (2) *there exists no $\eta < \alpha$ such that $\rho \upharpoonright \gamma \in V[\mathbb{P}_\eta]$ for all $\gamma < |\rho|$.*

Proof. Let us first show (2). Fix some $\eta < \alpha$. Since ρ is pre- T_α -minimal, $\rho \in V[\mathbb{P}_\alpha] \setminus V[\mathbb{P}_\eta]$, i.e., ρ is new (with respect to $\mathbb{P}_\alpha / \mathbb{P}_\eta$). By Corollary 6.2, $\mathbb{P}_\alpha / \mathbb{P}_\eta$ is σ -centered, hence in particular $(\mathbb{P}_\alpha / \mathbb{P}_\eta) \times (\mathbb{P}_\alpha / \mathbb{P}_\eta)$ has the c.c.c.. Therefore, by Lemma 6.3, the new function ρ has an initial segment which is not in $V[\mathbb{P}_\eta]$ (because it is new with respect to $\mathbb{P}_\alpha / \mathbb{P}_\eta$).

Now let us show (1). Assume towards contradiction that $\text{cf}(\alpha) \leq \omega$, and let $\langle \alpha_n \mid n \in \omega \rangle$ be increasing cofinal in α (in case α is a successor, let α_n be its predecessor for every n). For every $\gamma < |\rho|$, let $n \in \omega$ be such that $\rho \upharpoonright \gamma \in V[\mathbb{P}_{\alpha_n}]$, which is possible since ρ is pre- T_α -minimal. Since $\text{cf}(|\rho|) > \omega$, there exists $n^* \in \omega$ such that $\rho \upharpoonright \gamma \in V[\mathbb{P}_{\alpha_{n^*}}]$ for cofinally many $\gamma < |\rho|$ (and hence for all $\gamma < |\rho|$), contradicting (2). \square

So we can assume that $\text{cf}(\alpha) > \omega$. We first argue that $\text{cf}(|\rho|) > \omega$ for all pre- T_α -minimal nodes ρ . Assume towards contradiction that $\text{cf}(|\rho|) = \omega$ and ρ is pre- T_α -minimal. Let $\langle \gamma_n \mid n \in \omega \rangle$ be increasing cofinal in $|\rho|$. For every $n < \omega$, let $\alpha_n < \alpha$ be such that $\rho \upharpoonright \gamma_n \in V[\mathbb{P}_{\alpha_n}]$. Since $\text{cf}(\alpha) > \omega$, there exists $\alpha' < \alpha$ with $\alpha_n < \alpha'$ for every n . There are no new countable sequences of elements of $V[\mathbb{P}_{\alpha'}]$ in $V[\mathbb{P}_\alpha]$, because $V[\mathbb{P}_\alpha]$ is a limit of uncountable cofinality of a c.c.c. iteration, hence there exists $\alpha'' < \alpha$ such that $\langle \rho \upharpoonright \gamma_n \mid n \in \omega \rangle \in V[\mathbb{P}_{\alpha''}]$. Hence also $\rho \in V[\mathbb{P}_{\alpha''}]$, so it is not pre- T_α -minimal, a contradiction.

Now we will show that $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha]$, using the characterization from Theorem 5.3. Let $\langle X_n \mid n \in \omega \rangle \in V[\mathbb{P}_\alpha]$ be positive for $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$. We want to show that there exists $f \in \mathcal{B}$ such that \bar{X}_f is positive for $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$. Since $\langle X_n \mid n \in \omega \rangle$ is hereditarily countable and $\text{cf}(\alpha) > \omega$, there exists $\eta < \alpha$ with $\langle X_n \mid n \in \omega \rangle \in V[\mathbb{P}_\eta]$. Moreover, let η be large enough such that for all $j < k < m$ with $\rho_j \neq \rho_k$, there exists a successor $\delta < |\rho_j|, |\rho_k|$ such that $\rho_j \upharpoonright \delta \neq \rho_k \upharpoonright \delta$ and $a_{\rho_j \upharpoonright \delta}, a_{\rho_k \upharpoonright \delta} \in V[\mathbb{P}_\eta]$. For every $k < m$, let $\gamma_k < |\rho_k|$ be such that $a_{\rho_k \upharpoonright \gamma_k} \notin V[\mathbb{P}_\eta]$. Such γ_k exist, because the ρ_k are pre- T_α -minimal, using (2) from the above claim. Clearly $a_{\rho_k \upharpoonright \gamma_k} \in V[\mathbb{P}_\alpha]$ for every $k < m$. The filter $\bigoplus_{k < m} \langle a_{\rho_k \upharpoonright \gamma_k} \rangle$ is countably generated and hence \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha]$ (see Lemma 5.6). Note that $\bigoplus_{k < m} \langle a_{\rho_k \upharpoonright \gamma_k} \rangle \subseteq \bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$, hence $\langle X_n \mid n \in \omega \rangle$ is positive for $\bigoplus_{k < m} \langle a_{\rho_k \upharpoonright \gamma_k} \rangle$. Therefore we can fix $f \in \mathcal{B}$ such that \bar{X}_f is positive for $\bigoplus_{k < m} \langle a_{\rho_k \upharpoonright \gamma_k} \rangle$. Note that $\bar{X}_f \in V[\mathbb{P}_\eta]$.

We will use a genericity argument to show that \bar{X}_f is positive for $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$. It is enough to show that for all successors $\delta_k < |\rho_k|$, for all $l_k \in \omega$ there exists $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k < m} (a_{\rho_k \upharpoonright \delta_k} \setminus l_k)$, because sets of this form are a basis for the filter. If $\delta_k \leq \gamma_k$ for all k , this holds by the choice of f .

We show by induction on $\eta \leq \eta' < \alpha$ that for all successors $\delta_k < |\rho_k|$ and all $l_k < \omega$, if all $a_{\rho_k \upharpoonright \delta_k} \in V[\mathbb{P}_{\eta'}]$ then $V[\mathbb{P}_{\eta'}] \models \text{“}\exists s \in \bar{X}_f \ s \subseteq \bigoplus_{k < m} (a_{\rho_k \upharpoonright \delta_k} \setminus l_k)\text{”}$. Note that this holds for $\eta' = \eta$ by choice of f , and that at limit steps of the induction no new $a_{\rho_k \upharpoonright \delta_k}$ appear, so we only have to show it for successors. Assume that it holds for η' and show it for $\eta' + 1$.

For every $k < m$, let $\delta_k < |\rho_k|$ with $a_{\rho_k \upharpoonright \delta_k} \in V[\mathbb{P}_{\eta'+1}]$ and $l_k \in \omega$ be given. Let $p \in \mathbb{Q}_{\eta'}$. We will show that there exists $q \leq p$ and $s \in \bar{X}_f$ such that $q \Vdash s \subseteq \bigoplus_{k < m} (a_{\rho_k \upharpoonright \delta_k} \setminus l_k)$. Without loss of generality we can assume that $\rho_k \upharpoonright \delta_k \in \text{dom}(p)$ for all $k < m$ with $\rho_k \upharpoonright \delta_k \in T_{\eta'}$, and that p is a full condition.

For every $k < m$, define Σ_k . If $\rho_k \upharpoonright \delta_k \in T_{\eta'}$, let

$$\Sigma_k := \bigcup \{ \text{dom}(f_{\rho_k \upharpoonright \gamma}^p) \cap T_{\eta'} \mid \gamma \leq \delta_k \wedge \rho_k \upharpoonright \gamma \in \text{dom}(p) \}.$$

If $\rho_k \upharpoonright \delta_k \notin T_{\eta'}$, let $\Sigma_k := \{ \rho_k \upharpoonright \delta_k \}$. Let $\Sigma := \bigcup_{k < m} \Sigma_k$. For every $k < m$, let σ_k be the longest initial segment of ρ_k which belongs to Σ (if there exists one; let¹⁹ $\sigma_k := \rho_k \upharpoonright 1$ otherwise). Note that $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$, and that $a_{\sigma_k} \in V[\mathbb{P}_{\eta'}]$ for every $k < m$. Now let $N \in \omega$ be large enough such that

- $N \geq l_k$ for every $k < m$,
- $N \geq |s_\sigma^p|$ for every $\sigma \in \text{dom}(p)$,
- $a_{\sigma_k} \setminus N \subseteq a_\tau$ for all $\tau \in \Sigma_k$, for all $k < m$.

By hypothesis, in $V[\mathbb{P}_{\eta'}]$, we can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k < m} (a_{\sigma_k} \setminus N)$.

To get q , extend p as follows. For every $k < m$, for every $\gamma \leq \delta_k$ with $\rho_k \upharpoonright \gamma \in \text{dom}(p)$, let

$$s_{\rho_k \upharpoonright \gamma}^q := s_{\rho_k \upharpoonright \gamma}^p \hat{\ } (\mathbf{0} \upharpoonright [|s_{\rho_k \upharpoonright \gamma}^p|, N]) \hat{\ } (a_{\sigma_k} \upharpoonright [N, \max(s)]).$$

Observe that, if $\rho_k \neq \rho_j$, there is no $\tau \in \text{dom}(p)$ with $\tau \leq \rho_k$ and $\tau \leq \rho_j$ (by choice of η and since $\eta' \geq \eta$); so the above is well-defined, since $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$.

Note that η was chosen large enough such that for all $\gamma_k \leq \delta_k$ and $\gamma_j \leq \delta_j$, if $\rho_j \upharpoonright \gamma_j \neq \rho_k \upharpoonright \gamma_k$, then they are not in the same block; so, in particular, $\rho_j \upharpoonright \gamma_j \notin \text{dom}(h_{\rho_k \upharpoonright \gamma_k}^p)$ and $\rho_k \upharpoonright \gamma_k \notin \text{dom}(h_{\rho_j \upharpoonright \gamma_j}^p)$. Therefore, the requirement (8) from Definition 3.1 is fulfilled. It is easy to see that the other requirements of Definition 3.1 are fulfilled as well, hence q is a condition.

It is easy to check that q forces $s \subseteq \bigoplus_{k < m} (a_{\rho_k \upharpoonright \delta_k} \setminus l_k)$, as desired.

Successor step:

Let us say that a filter \mathcal{F}_α^β (and its filter base $\mathfrak{F}_\alpha^\beta$) is *new in* $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ if $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ and $\mathfrak{F}_\alpha^\beta \notin V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta^*}]$ for all $\delta^* < \beta^*$.

Now assume that we have shown (3) for β^* ; let us show it for $\beta^* + 1$.

If $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ for every $k < m$, then by induction hypothesis $(\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}) \oplus \mathcal{F}_\alpha^{\beta^*}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*} * \mathbb{M}(\mathcal{F}_\alpha^{\beta^*})] = V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$.

It is easy to check that there are exactly two new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$: \mathcal{F}_α^β , where β is such that $\sigma_\alpha^\beta = \sigma_\alpha^{\beta^*} \hat{\ } 0$, and $\mathcal{F}_\alpha^{\beta'}$, where $\beta' = \beta^* + 1$ (i.e., $\sigma_\alpha^{\beta'} = \rho \hat{\ } (i+1)$ and $\sigma_\alpha^{\beta^*} = \rho \hat{\ } i$). Both \mathcal{F}_α^β and $\mathcal{F}_\alpha^{\beta'}$ are extensions of $\mathcal{F}_\alpha^{\beta^*}$ by one new set. Therefore, the filter $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is an extension of $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ by finitely many sets, where $\tilde{\beta}_k = \beta^*$ if $\beta_k = \beta$ or $\beta_k = \beta'$, and $\tilde{\beta}_k = \beta_k$ otherwise. By the above, $\bigoplus_{k < m} \mathcal{F}_\alpha^{\tilde{\beta}_k}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$, hence, by Lemma 5.5, also $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$.

Limit step:

¹⁹We just have to choose any initial segment of ρ_k which belongs to $T_{\eta'}$ and make sure that $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$. Alternatively, in such cases, we could replace a_{σ_k} by ω below.

Now assume that β^* is a limit, and that we have shown (3) for all $\delta^* < \beta^*$; let us show it for β^* . If for each $k < m$ there exists $\delta_k^* < \beta^*$ such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta_k^*}]$, then there exists $\bar{\delta}^* < \beta^*$ such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\bar{\delta}^*}]$ for all $k < m$. By induction hypothesis, $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\bar{\delta}^*}]$ for every $\bar{\delta}^* \leq \delta^* < \beta^*$, hence, by Lemma 5.13, it is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$.

Now we have to consider new filters. There are two cases: either β^* is such that $\sigma_\alpha^{\beta^*}$ has 0 as its last entry, or such that it has a limit ordinal i as its last entry.

First case: $\sigma_\alpha^{\beta^*} = \rho \frown 0$ for some ρ

Let us first argue that there are no new filters unless $|\rho|$ is a limit and $\sigma_\alpha^{\beta^*}$ is the first node of its level in the enumeration (i.e., $|\sigma_\alpha^\delta| < |\rho|$ for each $\delta < \beta^*$). If $\sigma_\alpha^{\beta^*}$ is not the first node of the level in the enumeration, then there are no new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$: If $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$, then there exists $\delta^* < \beta^*$ such that $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta^*}]$, because $\mathfrak{F}_\alpha^\beta$ contains – from the sets of this level – only sets within one block and only boundedly many sets within this block. Similarly, if $|\rho|$ is a successor (and $\sigma_\alpha^{\beta^*}$ is the first node of its level in the enumeration), there are no new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$: If $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$, then there exists $\delta^* < \beta^*$ such that $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta^*}]$, because $\mathfrak{F}_\alpha^\beta$ contains – from the sets of level $|\rho|$ – only boundedly many sets.

So we assume from now on that $|\rho|$ is a limit and $\sigma_\alpha^{\beta^*}$ is the first node of its level in the enumeration. In this case, there are many new filters \mathcal{F}_α^β in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$; in fact, it is easy to check that \mathcal{F}_α^β is new if and only if the following holds: $\sigma_\alpha^\beta = \bar{\rho} \frown 0$ for some $\bar{\rho}$ with $|\bar{\rho}| = |\rho|$ and $\bar{\rho}$ not pre- T_α -minimal. Observe that $\mathfrak{F}_\alpha^\beta = \{a_{\bar{\rho} \upharpoonright \gamma} \mid \gamma < |\rho|\}$. Let $\beta_0, \dots, \beta_{m-1}$ be such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ for each $k < m$. We want to show that $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$.

In case $\text{cf}(|\rho|) = \omega$, we can use Lemma 5.14 and the remark afterwards to finish the proof: $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is a sum of filters, in which the new filters are countably generated, whereas the sum of the filters which are not new is \mathcal{B} -Canjar (see the first paragraph of the limit step).

So let us assume from now on that $\text{cf}(|\rho|) > \omega$. Let $\text{new} \subseteq m$ be the set of $k < m$ such that $\mathcal{F}_\alpha^{\beta_k}$ is a new filter, and $\text{old} = m \setminus \text{new}$ be the set of $k < m$ such that $\mathcal{F}_\alpha^{\beta_k}$ is not new. For each $k \in \text{new}$, we can fix ρ_k such that $\sigma_\alpha^{\beta_k} = \rho_k \frown 0$ (with $|\rho_k| = |\rho|$ and ρ_k not pre- T_α -minimal).

Let $\langle X_n \mid n \in \omega \rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ be positive for $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$. Since $\langle X_n \mid n \in \omega \rangle$ is hereditarily countable and $\mathbb{Q}_\alpha^{<\beta^*}$ has the c.c.c., there exists a hereditarily countable name \dot{X} for $\langle X_n \mid n \in \omega \rangle$. Since the conditions in $\mathbb{Q}_\alpha^{<\beta^*}$ have finite domain, the union of all the domains of conditions which occur in the name \dot{X} is countable. Let $\gamma < |\rho|$ be a successor ordinal large enough such that the following hold:

- $\langle X_n \mid n \in \omega \rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\gamma}]$ (this is possible due to $\text{cf}(|\rho|) > \omega$).
- For all $j, k \in \text{new}$, if $\rho_j \neq \rho_k$, then ρ_j and ρ_k split before γ .
- For all $k \in \text{old}$, either $|\sigma_\alpha^{\beta_k}| < \gamma$ or²⁰ $|\sigma_\alpha^{\beta_k}| > |\rho|$.
- $a_{\rho_k \upharpoonright \gamma} \notin V[\mathbb{P}_\alpha]$ for all $k \in \text{new}$, i.e., $\rho_k \upharpoonright \gamma \in T_\alpha$ (which is possible because ρ_k is not pre- T_α -minimal).

For every $k < m$ with $k \in \text{old}$, we have $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\gamma}]$ by choice of γ ; let $\tilde{\mathfrak{F}}_\alpha^{\beta_k} := \mathfrak{F}_\alpha^{\beta_k}$. For $k \in \text{new}$, let $\tilde{\mathfrak{F}}_\alpha^{\beta_k} := \{a_{\rho_k \upharpoonright \gamma}\}$.

²⁰Note that $|\sigma_\alpha^{\beta_k}| > |\rho|$ is only possible if $\sigma_\alpha^{\beta_k} = \bar{\rho} \frown 0$ for a pre- T_α -minimal node $\bar{\rho}$.

As above, we can use Lemma 5.14 and the remark afterwards to show that $\bigoplus_{k < m} \langle \tilde{\mathcal{F}}_\alpha^{\beta_k} \rangle$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \beta^*}]$. Indeed, $\bigoplus_{k \in \text{old}} \langle \tilde{\mathcal{F}}_\alpha^{\beta_k} \rangle$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \beta^*}]$ by the first paragraph of the limit step, and for each $k \in \text{new}$, $\langle \tilde{\mathcal{F}}_\alpha^{\beta_k} \rangle$ is countably generated. Moreover, $\bigoplus_{k < m} \langle \tilde{\mathcal{F}}_\alpha^{\beta_k} \rangle \subseteq \bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$, hence $\langle X_n \mid n \in \omega \rangle$ is positive for $\bigoplus_{k < m} \langle \tilde{\mathcal{F}}_\alpha^{\beta_k} \rangle$. So we can fix $f \in \mathcal{B}$ such that \bar{X}_f is positive for $\bigoplus_{k < m} \langle \tilde{\mathcal{F}}_\alpha^{\beta_k} \rangle$. Since $\langle X_n \mid n \in \omega \rangle$ and $\bigoplus_{k < m} \langle \tilde{\mathcal{F}}_\alpha^{\beta_k} \rangle$ are in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \gamma}]$, this holds in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \gamma}]$.

Now we use a genericity argument in $\mathbb{Q}_\alpha^{\leq \beta^*} / \mathbb{Q}_\alpha^{\leq \gamma}$ to show that \bar{X}_f is positive for $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$. We have to show that for all $\langle A_k \mid k < m \rangle$ with $A_k \in \mathcal{F}_\alpha^{\beta_k}$, there exists $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k < m} A_k$. For $k \in \text{new}$, we can assume that $A_k = a_{\rho_k \upharpoonright \delta_k} \setminus l_k$ with $\gamma < \delta_k < |\rho|$ and $l_k \in \omega$, because these sets form filter bases (with respect to upwards closure). For $k \in \text{old}$, let $B_k := A_k$, and for $k \in \text{new}$ (in this case $|\sigma_\alpha^{\beta_k}| = |\rho| + 1 > \gamma$), let $B_k := a_{\rho_k \upharpoonright \gamma}$. By the choice of f , there exists, for all $N \in \omega$, an $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k < m} (B_k \setminus N)$.

Let $p \in \mathbb{Q}_\alpha^{\leq \beta^*} / \mathbb{Q}_\alpha^{\leq \gamma}$. Without loss of generality we can assume that $\rho_k \upharpoonright \delta_k \in \text{dom}(p)$ if $k \in \text{new}$. For every $k \in \text{new}$, define

$$\Sigma_k := \bigcup \{ \text{dom}(f_{\rho_k \upharpoonright \delta}^p) \cap \lambda^{\leq \gamma} \mid \delta \leq \delta_k \wedge \rho_k \upharpoonright \delta \in \text{dom}(p) \}.$$

Now let $N \in \omega$ be large enough such that

- $N \geq l_k$ for every $k \in \text{new}$,
- $N \geq |s_\sigma^p|$ for every $\sigma \in \text{dom}(p)$,
- $a_{\rho_k \upharpoonright \gamma} \setminus N \subseteq a_\tau$ for all $\tau \in \Sigma_k$, for all $k \in \text{new}$.

By the above, we can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k < m} (B_k \setminus N)$.

To get q , extend p as follows. For every $k \in \text{new}$, for every $\delta \leq \delta_k$ with $\rho_k \upharpoonright \delta \in \text{dom}(p)$, let

$$s_{\rho_k \upharpoonright \delta}^q := s_{\rho_k \upharpoonright \delta}^p \hat{\ } (\mathbf{0} \upharpoonright [s_{\rho_k \upharpoonright \delta}^p, N]) \hat{\ } (a_{\rho_k \upharpoonright \gamma} \upharpoonright [N, \max(s)]).$$

Note that γ was chosen large enough so that for $j, k \in \text{new}$, if $\rho_j \neq \rho_k$, then they split before γ , therefore for $\gamma < \delta < |\rho|$ either $\rho_k \upharpoonright \delta = \rho_j \upharpoonright \delta$ or they are not in the same block. In particular $\rho_j \upharpoonright \delta \notin \text{dom}(h_{\rho_k \upharpoonright \delta}^p)$ and $\rho_k \upharpoonright \delta \notin \text{dom}(h_{\rho_j \upharpoonright \delta}^p)$. So the requirement (8) from Definition 3.1 is fulfilled. It is easy to see that the other requirements of Definition 3.1 are fulfilled as well, hence q is a condition.

It is easy to check that q forces $s \subseteq \bigoplus_{k < m} A_k$, as desired.

Second case: $\sigma_\alpha^{\beta^*} = \rho \hat{\ } i$ with $i > 0$ limit

In this case, $\mathcal{F}_\alpha^{\beta^*}$ is the only new filter in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \beta^*}]$. Indeed, for all other β either \mathcal{F}_α^β appeared in an earlier step already or it is not in this model, because for $\beta \neq \beta^*$ the filter base \mathcal{F}_α^β either uses only boundedly many elements of $\{a_{\rho \hat{\ } j} \mid j < i\}$ or it uses $a_{\rho \hat{\ } i}$ as well. Let $\beta_0, \dots, \beta_{m-1}$ be such that $\mathcal{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \beta^*}]$ for each $k < m$. We want to show that $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \beta^*}]$.

Let $\langle X_n \mid n \in \omega \rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\leq \beta^*}]$ be positive for $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$. Since $\langle X_n \mid n \in \omega \rangle$ is hereditarily countable and $\mathbb{Q}_\alpha^{\leq \beta^*}$ has the c.c.c., there exists a hereditarily countable name \dot{X} for $\langle X_n \mid n \in \omega \rangle$. Since the conditions in $\mathbb{Q}_\alpha^{\leq \beta^*}$ have finite domain, the union D of all the domains of conditions which occur in the name \dot{X} is countable. Let $\beta^{**} < \beta^*$ be such that $\rho \hat{\ } 0 = \sigma_\alpha^{\beta^{**}}$. Let $\bar{D} := \{ \tau \in D \mid \exists j < i (\tau = \rho \hat{\ } j) \}$. Let $C := \{ \sigma_\alpha^\nu \mid \nu < \beta^{**} \} \cup \bar{D}$. Note that

$$C = \{ \sigma_\alpha^\nu \mid \nu < \beta^{**} \wedge |\sigma_\alpha^\nu| \leq |\rho| \} \cup \bar{C}$$

with $\bar{C} = \{\sigma_\alpha^\nu \mid \nu < \beta^{**} \wedge |\sigma_\alpha^\nu| = |\rho| + 1\} \cup \bar{D}$. Since $\{\sigma_\alpha^\nu \mid \nu < \beta^{**} \wedge |\sigma_\alpha^\nu| \leq |\rho|\} = \lambda^{\leq |\rho|} \cap T_\alpha$ is α -left-up-closed, C is α -eligible, so, by Lemma 3.17, \mathbb{Q}_α^C is a complete subforcing of \mathbb{Q}_α , and it is a subset of $\mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}$; recall that $\mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}$ is also a complete subforcing of \mathbb{Q}_α , so, by Lemma 4.1, \mathbb{Q}_α^C is complete in $\mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}$.

Observe that for all $k < m$ either $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}] = \mathfrak{F}_\alpha^{\beta_k}$ or $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}] = \mathfrak{F}_\alpha^{\beta^{**}}$. In particular, for every $k < m$ there exists β'_k such that $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}] = \mathfrak{F}_\alpha^{\beta'_k}$ and $\mathfrak{F}_\alpha^{\beta'_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}]$. Hence $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}] \rangle$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}]$. For every $k < m$, $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$ is the set $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}]$ together with countably many new sets (some of the sets $\omega \setminus a_\tau$ with $\tau \in \bar{D}$), therefore $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle$ is a filter generated by $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}] \rangle$ together with countably many new sets. Hence, by Lemma 5.5, $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle$ is \mathcal{B} -Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle}]$. Since $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle \subseteq \bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$, the sets $\langle X_n \mid n \in \omega \rangle$ are also positive for $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle$. So we can fix $f \in \mathcal{B}$ such that \bar{X}_f is positive for $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle$. Since $\langle X_n \mid n \in \omega \rangle$ and $\bigoplus_{k < m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle$ are in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$, this holds in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$.

Now we use a genericity argument in $\mathbb{Q}_\alpha^{\langle \beta^{**} \rangle} / \mathbb{Q}_\alpha^C$ to show that \bar{X}_f is positive for $\bigoplus_{k < m} \mathcal{F}_\alpha^{\beta_k}$. We have to show that for all $\langle A_k \mid k < m \rangle$ with $A_k \in \mathcal{F}_\alpha^{\beta_k}$ there exists $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k < m} A_k$. For easier notation, assume that there exists $m' \leq m$ such that $A_k \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$ if and only if $k < m'$.

For $m' \leq k < m$ there exists $B_k \in \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle$, $\ell_k \in \omega$ and $\langle j_r^k \mid r < \ell_k \rangle \subseteq i$ such that $A_k = B_k \cap \bigcap_{r < \ell_k} (\omega \setminus a_{\rho^- j_r^k})$. So $\bigoplus_{k < m} A_k = \bigoplus_{k < m'} A_k \oplus \bigoplus_{m' \leq k < m} (B_k \cap \bigcap_{r < \ell_k} (\omega \setminus a_{\rho^- j_r^k}))$. Let $p \in \mathbb{Q}_\alpha^{\langle \beta^{**} \rangle} / \mathbb{Q}_\alpha^C$. Without loss of generality assume that $\rho^- j_r^k \in \text{dom}(p)$ if $\rho^- j_r^k \notin C$. Let $N > |s_\tau^p|$ for every $\tau \in \text{dom}(p)$. We can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k < m'} A_k \oplus \bigoplus_{m' \leq k < m} (B_k \setminus N)$. To get q , extend each $s_{\rho^- j_r^k}^p$ with 0's to have length $\max(s) + 1$.

It is easy to check that q is a condition, and that q forces $s \subseteq \bigoplus_{k < m} A_k$, as desired. \square

By the above Lemma 6.4(1), \mathcal{B} is unbounded in $V[\mathbb{P}_\alpha]$ for every $\alpha < \lambda$, so by applying Theorem 5.7 once again, it follows that \mathcal{B} is unbounded in our final model $V[\mathbb{P}_\lambda]$. Since $|\mathcal{B}| = \omega_1$, the bounding number \mathfrak{b} is ω_1 in our final model, and therefore also \mathfrak{h} is ω_1 , as desired.

This concludes the proof of Main Theorem 1.2.

7. FURTHER DISCUSSION AND QUESTIONS

In this section, we discuss the structure of distributivity matrices as well as a notion of distributivity spectrum. For basic definitions and facts, see Section 2.

7.1. Branches through distributivity matrices. For the nature of a maximal branch through a distributivity matrix, there are two possibilities: either it is cofinal or not. It is straightforward to check that every maximal branch which is not cofinal is a tower. We call a distributivity matrix normal if no single element of $[\omega]^\omega$ intersects it. Recall from Section 2 that whenever there is a distributivity matrix of height λ , then there is also a normal distributivity matrix of height λ . It is easy to see that a distributivity matrix is normal if and only if all its maximal branches are towers.

In case $\mathfrak{t} = \mathfrak{h}$ (so in particular under $\mathfrak{h} = \omega_1$) there are no towers of length strictly less than \mathfrak{h} , hence all maximal branches of a distributivity matrix of height \mathfrak{h} are cofinal.

On the other hand, it is possible to have a distributivity matrix of height \mathfrak{h} which has no cofinal branches. In fact, it was shown by Dow that this is the case in the Mathias model (see [13, Lemma 2.17]):

Theorem 7.1. *Assume CH. In the extension by the countable support iteration of Mathias forcing of length ω_2 , there exists a distributivity matrix of height \mathfrak{h} without cofinal branches²¹ (and $\omega_1 = \mathfrak{t} < \mathfrak{h} = \mathfrak{c} = \omega_2$).*

We do not know whether there exists a normal²² distributivity matrix of height \mathfrak{h} with cofinal branches in the Mathias model; this would imply that $\mathfrak{h} = \omega_2 \in \text{spec}(\mathfrak{t})$. We also do not know whether $\omega_2 \in \text{spec}(\mathfrak{t})$ holds in the Mathias model.

It is actually consistent that *no* normal distributivity matrix of height \mathfrak{h} has cofinal branches. This was proved by Dordal by constructing a model in which \mathfrak{h} does not belong to the tower spectrum (see [11] or²³ [12, Corollary 2.6]):

Theorem 7.2. *It is consistent with ZFC that $\text{spec}(\mathfrak{t}) = \{\omega_1\}$ and $\mathfrak{h} = \omega_2 = \mathfrak{c}$.*

Let us now discuss distributivity matrices of regular height strictly above \mathfrak{h} . Recall that $\omega_1 = \mathfrak{t} = \mathfrak{h}$ holds true in the model of Main Theorem 1.2; in particular, there are distributivity matrices of height ω_1 (all whose maximal branches are cofinal). All maximal branches through the generic distributivity matrix of height $\lambda > \omega_1$ are cofinal, because the forcing construction is based on the tree $\lambda^{<\lambda}$. Moreover, as shown in Section 4.3, all these maximal branches are actually towers (i.e., the matrix is normal). In particular, λ belongs to $\text{spec}(\mathfrak{t})$.

In the Cohen model, the situation is different. Again, $\omega_1 = \mathfrak{t} = \mathfrak{h}$ holds true, so there are distributivity matrices of height ω_1 (all whose maximal branches are cofinal). We do not know the following:

Question 7.3. Is there a distributivity matrix of regular height larger than \mathfrak{h} in the Cohen model?

In any case, there is a crucial difference to the model of our main theorem: in the Cohen model, there is no normal distributivity matrix of regular height $\lambda > \omega_1$ with cofinal branches, due to the following well-known fact.

Proposition 7.4. *Assume CH, and let μ be a cardinal with $\text{cf}(\mu) > \omega$. Then $\text{spec}(\mathfrak{t}) = \{\omega_1\}$ holds true²⁴ in the extension by \mathbb{C}_μ (where \mathbb{C}_μ is the forcing for adding μ many Cohen reals).*

Finally, let us remark that the generic distributivity matrix from Main Theorem 1.2 cannot be a base matrix. This can be seen by a slight generalization of the proof of Lemma 3.12 which yields the following. For each infinite ground model set $b \subseteq \omega$, each a_σ has infinitely many 1's (and also infinitely many 0's) within b . If b is infinite and co-infinite, it follows that b splits a_σ . In particular, $a_\sigma \not\subseteq^* b$, so b witnesses that the generic matrix is not a base matrix.

²¹In fact, there is even a base matrix of this kind.

²²Here and in similar cases, it is necessary to demand that the distributivity matrix is normal, due to the fact that there are always trivial examples of distributivity matrices with constant cofinal branches.

²³In fact, [12, Corollary 2.6]) also works for getting $\mathfrak{h} = \mathfrak{c}$ larger than ω_2 , and for certain tower spectra which are more complicated than $\{\omega_1\}$.

²⁴In fact, the following stronger statement holds true in the extension: Let $\lambda > \omega_1$ be regular, and let $\langle a_\alpha \mid \alpha < \lambda \rangle$ be a \subseteq^* -decreasing sequence; then there exists an $\alpha_0 < \lambda$ such that $a_{\alpha_0} =^* a_\beta$ for every $\beta \geq \alpha_0$.

7.2. The distributivity spectrum. The study of distributivity matrices of various heights naturally gives rise to the following notion. Let

$$\text{spec}(\mathfrak{h}) := \{\lambda \mid \lambda \text{ is regular and there is a distributivity matrix of height } \lambda\}$$

be the *distributivity spectrum*. Recall that the existence of distributivity matrices is only a matter of cofinality, i.e., there exists a distributivity matrix of height δ if and only if there exists one of height $\text{cf}(\delta)$. Therefore, the restriction of the definition of $\text{spec}(\mathfrak{h})$ to regular cardinals makes sense. Clearly, the minimum of $\text{spec}(\mathfrak{h})$ is the distributivity number \mathfrak{h} .

Spectra have been considered for several cardinal characteristics, but not for \mathfrak{h} . For example, spectra for the tower number \mathfrak{t} have been investigated in [22] and [12], spectra for the almost disjointness number \mathfrak{a} in [22], [6], and [31], spectra for the bounding number \mathfrak{b} in [12], spectra for the ultrafilter number \mathfrak{u} in [29], [30], and [17], and spectra for the independence number \mathfrak{i} in [15]. Furthermore, [3] develops a framework for dealing with several spectra.

Let $[\mathfrak{h}, \mathfrak{c}]_{\text{Reg}}$ denote the set of regular cardinals δ with $\mathfrak{h} \leq \delta \leq \mathfrak{c}$. As already mentioned, it is easy to check that there can never be a distributivity matrix of regular height larger than \mathfrak{c} , hence $\text{spec}(\mathfrak{h}) \subseteq [\mathfrak{h}, \mathfrak{c}]_{\text{Reg}}$. Recall that the model of Main Theorem 1.2 satisfies

$$\{\omega_1, \lambda\} \subseteq \text{spec}(\mathfrak{h})$$

(where $\lambda > \omega_1$ is the regular cardinal chosen there). In particular, by choosing $\lambda = \mu = \omega_2$, we obtain a model in which $\{\omega_1, \omega_2\} = \text{spec}(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{c}]_{\text{Reg}}$.

Question 7.5. Is it consistent that $\text{spec}(\mathfrak{h})$ contains more than 2 elements?

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