THE GENERALIZED INDEPENDENCE NUMBER

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ABSTRACT. We present a model in which the statement $\mathfrak{i}(\kappa) < 2^\kappa$ holds, for a supercompact cardinal κ. Here, $\mathfrak{i}(\kappa)$ stands for the *generalized independence number*, which corresponds to the least size of a κ-maximal independent family of subsets of κ. To obtain the model we develop an appropriate generalization in the context of supercompact cardinals to the notion of a *diagonalization filter* for an independent family introduced by Fischer and Shelah in [8]. This permits along a $< \kappa$-support iteration to control the minimal size of a κ-maximal independent family.

1. INTRODUCTION

Independent families on the set of real numbers (i.e. for $\kappa = \omega$) were first defined by Fichtenholz and Kantorovic (see [5]) and since then special interest has been given among others to the study of their possible sizes (for more recent work see [8, 6], [15]). The *independence number*, defined as the minimum size of a maximal independent family of subsets of ω has been well studied (references [10],[16] and [11] are specially relevant for this paper). There are still though, some long standing open problems, for instance whether the inequality $\mathfrak{i} < \mathfrak{a}$ is consistent (here $\mathfrak{a}$ is the almost disjointness number) and whether $\mathfrak{i}$ can be singular.

In the past decade generalizations of many cardinal invariants to the uncountable have been explored and there is already a broad literature in the subject. This paper focuses on the study of the generalized independence number $\mathfrak{i}(\kappa)$ for κ an uncountable regular cardinal.

1.1. Generalized independent families.

Throughout this paper, κ is an uncountable regular cardinal. The following is a straightforward generalization of the classical definition of independent families.

Definition 1. Let $\mathcal{A}$ be a family of unbounded subsets of κ of size $\geq \kappa$:

- We denote by $\mathrm{BF}_\kappa(\mathcal{A})$ the family $\{h : \mathcal{A} \to 2 : |\mathrm{dom}(h)| < \kappa\}$ and call it *the family of bounded functions on $\mathcal{A}$.*
- Given $h \in \mathrm{BF}_\kappa(\mathcal{A})$, we define $\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \mathcal{A} \cap \mathrm{dom}(h)\}$, where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \kappa \setminus A$ otherwise.

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- We refer to $\{\mathcal{A}^h : h \in \mathrm{BF}_\kappa(\mathcal{A})\}$ as the family of *generalized boolean combinations* of $\mathcal{A}$.

Definition 2. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ such that $|\mathcal{A}| \geq \kappa$ is called *κ-independent* if for for every $h \in \mathrm{BF}_\kappa(\mathcal{A})$, the set $\mathcal{A}^h$ is unbounded on κ. A κ-independent family $\mathcal{A}$ is said to be *κ-maximal independent* if it is *not* properly contained in another κ-independent family.

The first question one should address is the existence of these objects in ZFC. In the countable case for instance, one constructs first an independent family of size continuum and then uses the axiom of choice to find a maximal one. The next proposition shows the existence of a κ-independent family (and so a maximal one) of size 2^κ under some large cardinal assumptions.

Proposition 3 (Generalization of Hausdorff's example for the classical case)**.** Let κ be a strongly inaccessible cardinal. There is a κ-independent family of size 2^κ.

Proof. Put $\mathcal{C} = \{(\gamma, A) : \gamma < \kappa \wedge A \subseteq \gamma\}$ and given $X \subseteq \kappa$ define the set $\mathcal{Y}_X = \{(\gamma, A) \in \mathcal{C} : X \cap \gamma \in A\}$. We claim that $\{\mathcal{Y}_X : X \subseteq \kappa\}$ is κ-independent.

Let $\{X_\alpha : \alpha < \delta_1\}$ and $\{Z_\beta : \beta < \delta_2\}$ be two disjoint families of different subsets of κ and $\delta_1, \delta_2 < \kappa$. Note that $(\gamma, A) \in \mathcal{X} = \bigcap_{\alpha<\delta_1} \mathcal{Y}_{X_\alpha} \cap \bigcap_{\beta<\delta_2}(\mathcal{C} \setminus \mathcal{Y}_{Z_\beta})$ if for all $\alpha < \delta_1$, $X_\alpha \cap \gamma \in A$ and for all $\beta < \delta_2$, $Z_\beta \cap \gamma \notin A$. Then it is enough to notice that there are unboundedly many ordinals $\gamma < \kappa$ for which $X_\alpha \cap \gamma \neq X_{\alpha'} \cap \gamma$ when $\alpha \neq \alpha' < \delta_1$, $Z_\beta \cap \gamma \neq Z_{\beta'} \cap \gamma$ when $\beta \neq \beta' < \delta_2$ and $X_\alpha \cap \gamma \neq Z_{\beta'} \cap \gamma$ for all $\alpha < \delta_1$ and $\beta < \delta_2$. Then, for such indexes γ putting $A_\gamma = \{X_\alpha \cap \gamma : \alpha < \delta_1\}$ we get that $(\gamma, A_\gamma) \in \mathcal{X}$ and so that $\mathcal{X}$ has size κ. $\square$

Proposition 4 (Same idea as in Geschke [10])**.** Let κ be uncountable such that $\kappa^{<\kappa} = \kappa$. Then, there is a κ-independent family of size 2^κ.

Proof. Let $\mathcal{B}$ be a κ-almost disjoint family of size 2^κ (take for example, for every $x \in 2^\kappa$ the set $B_x = \{s \in 2^{<\kappa} : s \subseteq x\}$. Then $\{B_x : x \in 2^\kappa\}$ is κ-ad). For each $B \in \mathcal{B}$, define the set $B' = \{a \in [\kappa]^{<\kappa} : a \cap B \neq \emptyset\}$. We claim that the family $\mathcal{A} = \{B' : B \in \mathcal{B}\}$ is κ-independent.

Given $\{B'_\delta : \delta < \gamma_1\}$ and $\{C'_\beta : \beta < \gamma_2\}$ two sequences of elements of $\mathcal{B}$ where $\gamma_1, \gamma_2 < \kappa$, we want to prove that the set $\mathcal{X} = \bigcap_{\alpha<\gamma_1} B'_\alpha \cap \bigcap_{\beta<\delta_2}([\kappa]^{<\kappa} \setminus C'_\beta)$ is unbounded.

Clearly $\bigcap_{\alpha<\gamma_1} B'_\alpha$ is unbounded, and so it is enough to notice that, for every $\delta < \gamma_1$, $|B_\delta \cup \bigcup_{\beta<\gamma_2} C_\beta| < \kappa$ because $\mathcal{B}$ is κ-ad and κ is regular. Thus, there are unboundedly many bounded subsets of $a \subseteq \kappa$ that intersect simultaneously all B_δ's and do not intersect the C_β's. $\square$

Question 5. Are there κ-independent families when κ is uncountable such that $\kappa^{<\kappa} > \kappa$? Note that, in the examples above one uses a set of cardinality κ, such as $[\kappa]^{<\kappa}$ or $\kappa \times 2^{<\kappa}$. Could these assumptions be fully dropped?

Next we study lower bounds of the generalized independence number.

Definition 6. If f, g are functions in κ^κ, we say that $f <^* g$, if there exists an $\alpha < \kappa$ such that for all $\beta > \alpha$, $f(\beta) < g(\beta)$. In this case, we say that g *eventually dominates* f. Also, if $\mathfrak{F}$ is a family of functions from κ to κ we say that:

- $\mathfrak{F}$ is *dominating*, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $g <^* f$.

- $\mathfrak{F}$ is *unbounded*, if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $f \not<^* g$.

Definition 7 (The unbounding and dominating numbers, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$)**.**

- $\mathfrak{b}(\kappa) = \min\{|\mathfrak{F}| \colon \mathfrak{F}$ is an unbounded family of functions in $\kappa^\kappa\}$.
- $\mathfrak{d}(\kappa) = \min\{|\mathfrak{F}| \colon \mathfrak{F}$ is a dominating family of functions in $\kappa^\kappa\}$.

Definition 8. For A and $B \in \mathcal{P}(\kappa)$, say $A \subseteq^* B$ (A is almost contained in B) if $A \setminus B$ has size $< \kappa$. We also say that A *splits* B if both $A \cap B$ and $B \setminus A$ have size κ. A family $\mathcal{A}$ is called a *splitting family* if every unbounded subset of κ is split by a member of $\mathcal{A}$. Finally $\mathcal{A}$ is *unsplit* if no single set splits all members of $\mathcal{A}$.

Definition 9.

- $\mathfrak{s}(\kappa) = \min\{|\mathcal{A}| \colon \mathcal{A}$ is a splitting family of subsets of $\kappa\}$.
- $\mathfrak{r}(\kappa) = \min\{|\mathcal{A}| \colon \mathcal{A}$ is an unsplit family of subsets of $\kappa\}$.

Proposition 10 (Some lower bounds for $\mathfrak{i}(\kappa)$)**.**

(1) $\mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$.
(2) If $\mathfrak{d}(\kappa)$ is such that for every $\gamma < \mathfrak{d}(\kappa)$ we have $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$, then $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$.

Proof. (1) is a consequence from the fact that, the set of boolean combinations of a given κ-maximal independent family $\mathcal{A}$, $\{\mathcal{A}^h : h \in \mathrm{BF}_\kappa(\mathcal{A})\}$ is unsplit. Finally, the proof of (2) can be found in Proposition 27 of [4]. □

Question 11. Can the cardinal arithmetic assumption on κ in the last Proposition be dropped? More precisely, given κ regular uncountable, is it always the case that $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$?

2. A model where $\kappa^+ < \mathfrak{i}(\kappa) < 2^\kappa$ for κ supercompact

2.1. **How to add a κ-independent set.** In this section we present a method to generically add a κ-independent set that diagonalizes the ground model V in a strong way (see Lemma 20). The model is based on a few major ideas. We generalize Shelah's ideals associated to independent families introduced in [16] (see Lemma 12) and develop into the uncountable the notion of a diagonalization filter introduced in [7] (see Definition 14).

Lemma 12. Let κ be a regular uncountable cardinal such that $\kappa^{<\kappa} = \kappa$, $\mathcal{A}$ be a κ-independent family and let $\mathcal{D}_\kappa(X)$ be the set of all functions $h \in \mathrm{BF}_\kappa(\mathcal{A})$ for which $X \cap \mathcal{A}^h$ is bounded in κ. Then:

(1)
$$\begin{aligned}\mathrm{id}_\kappa(\mathcal{A}) &= \{X \subseteq \kappa : \forall h \in \mathrm{BF}_\kappa(\mathcal{A}) \exists h' \supseteq h (|\mathcal{A}^{h'} \cap X| < \kappa)\} \\ &= \{X \subseteq \kappa : \mathcal{D}_\kappa(X) \text{ is dense in } \mathrm{BF}_\kappa(\mathcal{A})\}\end{aligned}$$

is a κ-complete ideal on κ, to which we refer as the *generalized independence density ideal associated to $\mathcal{A}$*.[1]

[1]When we refer to "dense" in $\mathrm{BF}_\kappa(\mathcal{A})$, we mean dense with respect to the inclusion relation.

(2) If $\mathcal{A}_0$, $\mathcal{A}_1$ are κ-independent families such that $\mathcal{A}_0 \subseteq \mathcal{A}_1$ then $\mathrm{id}_\kappa(\mathcal{A}_0) \subseteq \mathrm{id}_\kappa(\mathcal{A}_1)$.

Proof.

(1) We prove that $\mathrm{id}_\kappa(\mathcal{A})$ is κ-complete: Given $\gamma < \kappa$ and $(X_\alpha : \alpha < \gamma) \subseteq \mathrm{id}_\kappa(\mathcal{A})$ we will show that $X = \bigcup_{\alpha<\gamma} X_\alpha \in \mathrm{id}_\kappa(\mathcal{A})$. Take $h \in \mathrm{BF}_\kappa(\mathcal{A})$ arbitrary. There is $h_0 \supseteq h$ for which $|X_0 \cap \mathcal{A}^{h_0}| < \kappa$ and we can find $h_1 \supseteq h_0$ such that $|X_1 \cap \mathcal{A}^{h_1}| < \kappa$. Iterating this process γ-many times gives us a function $h_\gamma = \bigcup_{\alpha<\gamma} h_\alpha$ that is an element of $\mathrm{BF}_\kappa(\mathcal{A})$ and $|X \cap \mathcal{A}^{h_\gamma}| = |\bigcup_{\alpha<\gamma}(X_\alpha \cap \mathcal{A}^{h_\gamma})| < \kappa$ because κ is regular.

(2) Clear from the definition of the ideal.

$\square$

Remark 13.
- The degree of completeness that the ideal $\mathrm{id}(\mathcal{A})$ entails might be stronger. Indeed, if $|\mathcal{A}| = \lambda \geq \kappa$ and $\lambda^{<\kappa} = \lambda$, $\mathrm{id}(\mathcal{A})$ is indeed λ-complete. However, it is not λ^+-complete.
- Given $\mathcal{A}$, κ-independent, $X \subseteq \kappa$ and $h \in \mathrm{BF}_\kappa(\mathcal{A})$ such that $|\mathcal{A}^h \cap X| < \kappa$, there is $h_1 \in \mathrm{BF}_\kappa(\mathcal{A})$ such that $h_1 \supseteq h$ and $\mathcal{A}^{h_1} \cap X = \emptyset$.

Definition 14. Let $\mathcal{A}$ be a κ-independent family. A κ-complete filter $\mathcal{F}$ is called a *diagonalization filter* for the family $\mathcal{A}$ if the following hold:

(1) For every $F \in \mathcal{F}$ and $h \in \mathrm{BF}_\kappa(\mathcal{A})$, there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq^* F$.
(2) $\mathcal{F} \cap \{\mathcal{A}^h : h \in \mathrm{BF}_\kappa(\mathcal{A})\} = \emptyset$.

In addition, a *maximal diagonalization filter for $\mathcal{A}$* is a κ-complete filter that is maximal with respect to properties (1) and (2), i.e. there is no κ-complete filter $\mathcal{F}' \supset \mathcal{F}$ satisfying these properties.

Note: The dual filter of $\mathrm{id}_\kappa(\mathcal{A})$, which will be denoted $\mathrm{fil}_\kappa(\mathcal{A})$ satisfies the conditions listed above. Note that $X \in \mathrm{fil}_\kappa(\mathcal{A})$ if for every $h \in \mathrm{BF}_\kappa(\mathcal{A})$, there is $h' \supseteq h$ for which $\mathcal{A}^{h'} \subseteq^* X$ (i.e. $|\mathcal{A}^{h'} \setminus X| < \kappa$).

Remark 15. A maximal diagonalization filter cannot be an ultrafilter.

Proof. Suppose $\mathcal{U}$ is a κ-complete maximal diagonalization ultrafilter, then the set of boolean combinations $\{\mathcal{A}^h : h \in \mathrm{BF}_\kappa(\mathcal{A})\}$ has to be decided by $\mathcal{U}$. Thus, take any $h \in \mathrm{BF}_\kappa(\mathcal{A})$. Then $\kappa \backslash \mathcal{A}^h \in \mathcal{U}$ and by part (1) of Definition 14 there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \kappa \backslash \mathcal{A}^h$, which is a contradiction. $\square$

We are now interested in the existence of maximal diagonalization filters. To this end, we will use the concept of a strongly compact cardinal.

Definition 16.
- Given two cardinals λ, μ the language $\mathcal{L}_{\lambda,\mu}$ corresponds to the usual first order logic with non-logical symbols (relations, functions and constant symbols), however one admits $\max\{\lambda, \mu\}$-many variables. From these terms one constructs atomic formulas as usual. The formulas are generated by conjunctions and disjunctions of less than λ-many terms, as well as by existential and universal quantification of less than μ-many variables.

- A collection Σ of $\mathcal{L}_{\lambda,\mu}$ sentences is satisfiable if it has a model.
- A collection Σ of $\mathcal{L}_{\lambda,\mu}$ sentences is ν-satisfiable if every sub-collection Δ of cardinality $< \nu$ is satisfiable.
- A cardinal κ is *strongly compact* if every collection Σ of $\mathcal{L}_{\kappa,\kappa}$ sentences the following holds: If Σ is κ-satisfiable then Σ is satisfiable.

Proposition 17. Suppose κ is strongly compact and $\mathcal{A}$ is a κ-independent family. Then there is a maximal diagonalization filter for $\mathcal{A}$.

Proof. Let $\mathcal{F}_0 = \mathrm{fil}_\kappa(\mathcal{A})$. Then $\mathcal{F}_0$ is a κ-complete diagonalization filter for $\mathcal{A}$. Consider the $\mathcal{L}_{\kappa,\kappa}$ language of set theory expanded with new constants for each element of $\mathcal{A}$ and each subset X of κ. Denote those $\dot{A}$ and $\dot{X}$ respectively. Let $\mathcal{M}_0$ be a model with universe $\mathcal{P}(\kappa) \cup \kappa$, with the natural interpretation of X and A and such that

$$\mathcal{M}_0 \vDash \mathcal{A} \text{ is independent.}$$

In $\mathcal{M}_0$ both $\mathrm{BF}_\kappa(\mathcal{A})$ and $\mathrm{fil}_\kappa(\mathcal{A})$ are definable and

$$\mathcal{M}_0 \vDash \mathrm{fil}_\kappa(\mathcal{A}) \text{ is a } \kappa\text{-complete, diagonalization filter.}$$

Consider the formulas $\varphi_0(X,h) := \mathcal{A}^h \not\subseteq X$ and $\varphi_1(X,h) := \exists h' \supseteq h(|\mathcal{A}^{h'} \backslash X| < \kappa$. Then the theory of $\mathcal{M}_0$, denoted T_0 includes $\{\varphi_0(X,h), \varphi_1(X,h) : X \in \mathrm{fil}_\kappa(\mathcal{A}), h \in \mathrm{BF}_\kappa(\mathcal{A})\}$.

Now, take a new constant c and expand the language $\mathcal{L}_{\kappa,\kappa}$ to a language $\mathcal{L}' = \mathcal{L}_{\kappa,\kappa} \cup \{c\}$. Let

$$T := T_0 \cup \{\forall h(\varphi_0(\dot{X},h) \wedge \varphi_1(\dot{X},h)) \leftrightarrow c \in \dot{X}\}_{\dot{X} \subseteq \kappa, h \in \mathrm{BF}_\kappa \mathcal{A}}.$$

We claim that T is $< \kappa$-satisfiable. Indeed, consider a family of $< \kappa$-many formulas in T, say

$$\Gamma = \{\forall h(\varphi_0(\dot{X}_i,h) \wedge \varphi_1(\dot{X}_i,h)) \leftrightarrow c \in \dot{X}_i\}_{i<\gamma}$$

where $\{\dot{X}_i\}_{i<\gamma}$ are pairwise distinct constants, $\gamma < \kappa$. In $\mathcal{M}_0$ find γ distinct elements $X_i \in \mathrm{fil}_\kappa(\mathcal{A})$. Since $\mathcal{M}_0 \vDash (\mathrm{fil}_\kappa(\mathcal{A})$ is κ-complete), we have that $\mathcal{M} \vDash \bigcap_{i<\gamma} X_i \neq \emptyset$. Thus for some $a \in \mathcal{M}_0$ we have $a \in \bigcap_{i<\gamma} X_i$. Then, expand $\mathcal{M}_0$ to a $\mathcal{L}'$-structure $\mathcal{M}_0'$ by defining $c^{\mathcal{M}_0'} := a$. Then $\mathcal{M}_0' \vDash \Gamma$.

By strong compactness of κ, the theory T is satisfiable with model $\mathcal{M}$. Now, define $\mathcal{G}$ as follows:

$$X \in \mathcal{G} \text{ if and only if } c \in X.$$

We claim that $\mathcal{G}$ is a κ-complete maximal diagonalization filter. The fact that $\mathcal{G}$ is κ-complete is straightforward. To check that $\mathcal{G}$ is a diagonalization filter, consider an arbitrary $X \in \mathcal{G}$. Then since $c \in X$ and

$$\mathcal{M} \vDash \forall h(\varphi_0(\dot{X},h) \wedge \varphi_1(\dot{X},h)) \leftrightarrow c \in \dot{X},$$

we obtain $\mathcal{M} \vDash \forall h(\varphi_0(\dot{X},h) \wedge \varphi_1(\dot{X},h))$. Thus $\mathcal{G}$ is a diagonalization filter for $\mathcal{A}$. To check maximality, consider any Z such that $Z \notin \mathcal{G}$. Then by definition of $\mathcal{G}$, $\mathcal{M} \vDash c \notin Z$ and so

$$\mathcal{M} \vDash \neg\big(\forall h(\varphi_0(\dot{X},h) \wedge \varphi_1(\dot{X},h))\big).$$

Thus $\mathcal{G} \cup \{Z\}$ is not a diagonalization filter. $\square$

Diagonalization filters will be used to prove that, given an independent family $\mathcal{A}$ in a ground model V, it is possible to add through forcing, a generic set x so that $\mathcal{A} \cup \{x\}$ is still independent. Our goal, is to iterate such a forcing notion, in order to add a maximal independent family of desired size. First, we define the forcing notion we will be using:

Definition 18 (Generalized Mathias forcing)**.** Let κ be a regular cardinal and let $\mathcal{F}$ be a κ-complete filter on κ. The generalized Mathias forcing $\mathbb{M}^{\kappa}_{\mathcal{F}}$ has as its set of conditions pairs $\{(s, A) : s \in [\kappa]^{<\kappa}$ and $A \in \mathcal{F}\}$ and is ordered by $(t, B) \leq (s, A)$ if and only if $t \supseteq s, B \subseteq A$ and $t \setminus s \subseteq A$. We denote by $\mathbb{1}_{\mathcal{F}}$ the maximum element of $\mathbb{M}^{\kappa}_{\mathcal{F}}$, that is $\mathbb{1}_{\mathcal{F}} = (\emptyset, \kappa)$.

Observation 19. The generic real added by the generalized Mathias forcing $\mathbb{M}^{\kappa}_{\mathcal{F}}$ over a model V given by $x^{\kappa}_{\mathcal{F}} = \bigcup\{s : \exists A \in \mathcal{U}(s, A) \in G\}$, where G is $\mathbb{M}^{\kappa}_{\mathcal{F}}$-generic has the following property: If $\mathcal{F}$ is a κ-complete filter, then $\mathbb{M}^{\kappa}_{\mathcal{F}}$ adds generically an unbounded set $x^{\kappa}_{\mathcal{F}} \subseteq \kappa$ such that $x^{\kappa}_{\mathcal{F}} \subseteq^* F$ for all $F \in \mathcal{F}$. We say that the generic set $x^{\kappa}_{\mathcal{F}}$ *diagonalizes* the filter $\mathcal{F}$.

On the other hand, this forcing is κ-centered, so κ^+-cc and κ-closed which ensures that cardinals up to κ^+ are preserved.

Now, we prove the following diagonalization property for the *generalized Mathias forcing with respect to a diagonalization filter* (this is the uncountable version of Lemma 2 in [7]).

Lemma 20. Let κ be a strongly compact cardinal, $\mathcal{A}$ a κ-independent family, $\mathcal{F}$ a diagonalization filter and G an $\mathbb{M}^{\kappa}_{\mathcal{F}}$-generic. Then:

(1) $\mathcal{A} \cup \{x^{\kappa}_{\mathcal{F}}\}$ is κ-independent.
(2) Given $Y \in ([\kappa]^{\kappa} \cap V) \setminus \mathcal{A}$ so that $\mathcal{A} \cup \{Y\}$ is κ-independent, the family $\mathcal{A} \cup \{x^{\kappa}_{\mathcal{F}}, Y\}$ is not κ-independent.

Proof. (1) Let $h \in \mathrm{BF}_{\kappa}(\mathcal{A})$, $\alpha \in \kappa$. Consider the set

$$D_{h,\alpha} = \{(s, F) \in \mathbb{M}^{\kappa}_{\mathcal{F}} : |s \cap \mathcal{A}^h| > |\alpha|\}.$$

Let $(s, F) \in \mathbb{P}^{\kappa}_{\mathcal{F}}$. Then $F \cap \mathcal{A}^h$ is unbounded in κ and so, since κ is strongly inaccessible, we can find $t \subseteq F \cap \mathcal{A}^h$ such that $\sup s < \min t$ and $\kappa > |t| > |\alpha|$ (here $|t| = \sup(t) + 1$). Then $(s \cup t, F \backslash (\sup t + 1))$ extends (s, F) and belongs to $D_{h,\alpha}$. Thus $D_{h,\alpha}$ is dense. Since h, α were arbitrary, the intersection $\mathcal{A}^h \cap x^{\kappa}_{\mathcal{F}}$ is unbounded in κ for each h.

Again, fix h, α as above and consider the set

$$E_{h,\alpha} := \{(s, F) : |(\min F \backslash \sup s) \cap \mathcal{A}^h| > |\alpha|\}.$$

Now, consider an arbitrary $(s, F) \in \mathbb{M}^{\kappa}_{\mathcal{F}}$ and find an initial segment of $\mathcal{A}^h \backslash (\sup s + 1)$ such that $|t| > |\alpha|$. Then $(s, F \backslash (\sup t + 1)) \leq (s, F)$ and belongs to $E_{h,\alpha}$. Thus $E_{h,\alpha}$ is dense. Again, since h, α are arbitrary, we obtain that $\mathcal{A}^h \backslash x^{\kappa}_{\mathcal{F}}$ is unbounded for each h.

(2) Let $y \in ([\kappa]^{\kappa} \backslash \mathcal{A}) \cap V$ such that $\mathcal{A} \cup \{y\}$ is κ-independent. If $y \in \mathcal{F}$, then $|x^{\kappa}_{\mathcal{F}} \backslash y| < \kappa$. If $y \notin \mathcal{F}$ then either there is $F \in \mathcal{F}$ such that $|F \cap y| < \kappa$ and so $x^{\kappa}_{\mathcal{F}} \cap y$ is bounded in κ, or there is $F \in \mathcal{F}$ and $h \in \mathrm{BF}_{\kappa}(\mathcal{A})$ such that $F \cap y \subseteq \mathcal{A}^h$. Let $C \in \mathrm{dom}(h)$ and wlog assume $h(C) = 1$. Then $(x^{\kappa}_{\mathcal{F}} \cap y) \backslash C$ is bounded. Thus in each of the above cases $\mathcal{A} \cup \{x^{\kappa}_{\mathcal{A}}, y\}$ is not κ-independent. $\square$

In joint work with Brooke-Taylor and Friedman [4], we constructed a generic extension in which the value of the generalized ultrafilter $\mathfrak{u}(\kappa)$ can be separated from the value of 2^κ. Moreover we computed the value of other cardinal invariants in this model:

Theorem 21 (Main result of [4])**.** *Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$ and Γ satisfies $\Gamma^\kappa = \Gamma$. Then there is forcing extension in which cardinals have not been changed satisfying:*

$$\begin{aligned}\kappa^* &= \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \operatorname{cov}(\mathcal{M}_\kappa) \\ &= \operatorname{add}(\mathcal{M}_\kappa) = \operatorname{non}(\mathcal{M}_\kappa) = \operatorname{cof}(\mathcal{M}_\kappa) \text{ and } 2^\kappa = \Gamma.\end{aligned}$$

If in addition $(\Gamma)^{<\kappa^*} \leq \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_{\mathcal{W}}(\kappa) = \kappa^*$ where $\mathcal{W}$ is a κ-complete ultrafilter on κ.

One of the questions left open is to evaluate the generalized independence number in the above model. Lemma 20 allows us to modify the construction of [4] and obtain a generic extension in which in addition $\mathfrak{i}(\kappa) = \kappa^*$.

To present the explicit argument we make a short review of the construction of [4].

2.2. **The main model.**

Let Γ be such that $\Gamma^\kappa = \Gamma$, we define an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \Gamma^+, \beta < \Gamma^+ \rangle$ of length Γ^+ recursively as follows:

(1) *If α is an even ordinal* (abbreviated $\alpha \in$ EVEN), let NUF denote the set of normal ultrafilters on κ in $V^{\mathbb{P}_\alpha}$. Then let $\mathbb{Q}_\alpha$ be the poset with underlying set of conditions $\{\mathbb{1}_{\mathbb{Q}_\alpha}\} \cup \{\{\mathcal{U}\} \times \mathbb{M}^\kappa_{\mathcal{U}} : \mathcal{U} \in \text{NUF}\}$ and extension relation stating that $q \leq p$ if and only if either $p = \mathbb{1}_{\mathbb{Q}_\alpha}$, or there is $\mathcal{U} \in$ NUF such that $p = (\mathcal{U}, p_1)$, $q = (\mathcal{U}, q_1)$ and $q_1 \leq_{\mathbb{M}^\kappa_{\mathcal{U}}} p_1$.

(2) *If α is an odd ordinal* (abbreviated $\alpha \in$ ODD), let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for a κ-centered, κ-directed closed forcing notion of size at most Γ.

Definition 22 (The support)**.** We define three different kinds of support for conditions $p \in \mathbb{P}_\alpha$, $\alpha < \Gamma^+$:

- The *Ultrafilter Support* USupt(p), that corresponds to the set of ordinals $\beta \in \operatorname{dom}(p) \cap$ EVEN such that $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \neq \mathbb{1}_{\mathbb{Q}_\beta}$.
- The *Essential Support* SSupt(p), which consists of all $\beta \in \operatorname{dom}(p) \cap$EVEN such that $\neg(p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \{\check{\mathbb{1}}_{\mathbb{Q}_\beta}\} \cup \{(\mathcal{U}, \mathbb{1}_{\mathcal{U}}) : \mathcal{U} \in \text{NUF}\})$ (for the definition of $\mathbb{1}_{\mathcal{U}}$ see Definition 18).
- The *Directed Support* RSupt(p), consists of all $\beta \in \operatorname{dom}(p) \cap$ ODD such that $\neg(p \restriction \beta \Vdash p(\beta) = \mathbb{1}_{\dot{\mathbb{Q}}_\beta})$.

We require that the conditions in $\mathbb{P}_{\Gamma^+}$ have support bounded below Γ^+ and also that given $p \in \mathbb{P}_{\Gamma^+}$ if $\beta \in$ USupt(p) then for all $\alpha \in \beta \cap$EVEN, $\alpha \in$ USupt(p). Finally we demand that both SSupt(p) and RSupt(p) have size $< \kappa$ and are contained in sup(USupt(p)), i.e. Supt(p) (the entire support of p) and USupt(p) have the same supremum.

Lemma 23 (Main Lemma, [4])**.** The poset $\mathbb{P} = \mathbb{P}_{\Gamma^+}$ preserves cardinals and has the following properties:

- Let κ be a supercompact cardinal and κ^* be a cardinal satisfying $\kappa < \kappa^* \leq \Gamma$, κ^* regular. Suppose that $p \in \mathbb{P}$ is such that $p \Vdash \dot{\mathcal{U}}$ is a normal ultrafilter on κ. Then for some $\alpha < \Gamma^+$ there is an extension $q \leq p$ such that $q \Vdash \dot{\mathcal{U}}_\alpha = \dot{\mathcal{U}} \cap V[G_\alpha]$. Moreover this can be done for a set of ordinals $S \subseteq \Gamma^+$ of order type κ^* in such a way that $\forall \alpha \in S(\dot{\mathcal{U}} \cap V_\alpha \in V[G_\alpha])$ and $\dot{\mathcal{U}} \cap V[G_{\sup S}] \in V[G_{\sup S}]$. Here $\dot{\mathcal{U}}_\alpha$ is the canonical name for the ultrafilter generically chosen at stage α.
- If $\alpha = \sup(S)$, then in $V^{\mathbb{P}_\alpha}$, $\mathfrak{u}(\kappa) = \kappa^*$ while $2^\kappa = \Gamma$.

Until the end of the section let $\underline{\mathbb{P}^* = \mathbb{P}_\alpha}$. Using the notion of a generalized diagonalization filters, we will modify the above construction to evaluate $\mathfrak{i}(\kappa)$:

Theorem 24. *Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$, for every $\gamma < \mathfrak{d}(\kappa)$ we have $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$ and Γ satisfies $\Gamma^\kappa = \Gamma$. Then there is forcing extension in which cardinals have not been changed satisfying:*

$$\underline{\underline{\kappa^* = \mathfrak{i}(\kappa)}} = \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa)$$
$$= \mathrm{cof}(\mathcal{M}_\kappa) \text{ and } \underline{\underline{2^\kappa = \Gamma}}.$$

If in addition $(\Gamma)^{<\kappa^*} \leq \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_{\mathcal{W}}(\kappa) = \kappa^*$ where $\mathcal{W}$ is a κ-complete ultrafilter on κ.

Important note: The assumption of supercompactness was originally used (for the purposes of Theorem 21) to preserve the measurability of κ along the iteration. When evaluating $\mathfrak{i}(\kappa)$, we use more: the preservation of strong compactness along the iteration.

Proof. We modify to iteration $\mathbb{P}^*$ to an iteration $\bar{\mathbb{P}}^*$ by specifying the iterands $\dot{\mathbb{Q}}_j$ for every odd ordinal $j < \alpha$. Let $\bar{\gamma} = \langle \gamma_i \rangle_{i<\kappa^*}$ and $\tilde{\gamma} = \langle \tilde{\gamma}_i \rangle_{i<\kappa^*}$ be two disjoint strictly increasing cofinal in α sequence of odd ordinals. The stages outside of $\tilde{\gamma}$ will be used to evaluate all cardinal characteristics except $\mathfrak{i}(\kappa)$ in the same way as in Theorem 36 of [4]. The stages in $\tilde{\gamma}$ will be used to adjoin a κ-maximal independent family of size κ^*. Let $\mathcal{A}_0$ be a κ-independent family of size κ in the ground model, let $\mathcal{F}_0$ be a diagonalization filter for $\mathcal{A}_0$ and let $\dot{Q}_{\tilde{\gamma}_0} = \mathbb{P}^\kappa_{\mathcal{F}_0}$.

For each $i < \kappa^*$, in $V^{\bar{\mathbb{P}}^*}_{\gamma_i}$ the poset $\dot{\mathbb{Q}}_{\tilde{\gamma}_i}$ will be defined as follows: $\dot{\mathbb{Q}}_{\gamma_0} = \mathbb{P}^\kappa_{\mathcal{F}_0}$ where $\mathcal{F}_0$ is a diagonalization filter associated to $\mathcal{A}_0$ in $V^{\mathbb{P}_{\gamma_0}}$. Put also $\mathcal{A}_1 = \mathcal{A}_0 \cup \{x^\kappa_{\mathcal{F}_0}\}$. If $\dot{\mathbb{Q}}_{\tilde{\gamma}_j}$ and $\mathcal{A}_j$ have been defined for some $j < \kappa^*$ and $i = j+1$, define $\mathbb{Q}_{\tilde{\gamma}_i} = \mathbb{P}^\kappa_{\mathcal{F}_i}$ where $\mathcal{F}_i$ is a diagonalization filter associated to $\mathcal{A}_j$ in $V^{\bar{\mathbb{P}}^*_{\tilde{\gamma}_j}}$ and let $\mathcal{A}_i = \mathcal{A}_j \cup \{x^\kappa_{\mathcal{F}_i}\}$. Finally, in the limit steps $i < \kappa^*$ it is enough to put $\mathcal{A}_i = \bigcup_{j<i} \mathcal{A}_j$.

Thus, in the generic extension $V^{\bar{\mathbb{P}}^*}$ the family $\mathcal{A} = \bigcup\{\mathcal{A}_i : i < \kappa^*\}$ is maximal κ-independent: If $\dot{X}$ is a $\bar{\mathbb{P}}^*$-name for a subset of κ, using the properties of the forcing, there is $i < \kappa^*$ such that $\dot{X}$ is a $\bar{\mathbb{P}}_{\gamma_i}$-name. By the diagonalization property of Lemma 20 we get that $\mathcal{A}_i \cup \{x^\kappa_{\mathcal{F}_i}, X\}$ is not κ-independent. Hence $\mathcal{A}$ is maximal of size κ^* and so, $\mathfrak{i}(\kappa) \leq \kappa^*$.

To see that $\mathfrak{i}(\kappa) \geq \kappa^*$ proceed as follows: By results of [4], if $\mathfrak{d}(\kappa)$ is such that for every $\gamma < \mathfrak{d}(\kappa)$ we have $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$, then $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$. Since in the extension $\mathfrak{d}(\kappa) = \kappa^*$ we obtain the lower bound we were looking for. □

3. Final comments and open questions

We conclude with some natural questions which remain of interest. Throughout this final section, we assume κ to be strongly inaccessible.

Question 25. Can the assumption of supercompactness of κ be weakened? In other words, what is the consistency strength of the statement $\mathrm{Con}(\mathfrak{i}(\kappa) < 2^\kappa)$?

In the theory of combinatorial characteristics of the classical Baire space, Sacks model (the model obtained as a countable support iteration of length $\aleph_2$ of Sacks forcing) is a typical generic extension in which many cardinal characteristics take the value $\aleph_1$ while the continuum $\mathfrak{c}$ has value $\aleph_2$. For constructions of Sacks indestructible maximal independent families see [16, 6]).

An approach towards answering Question 25 is to generalize the construction of such indestructible maximal independent families in the context of uncountable cardinals. In [6], building upon work of Shelah [16], we define a class of maximal independent families, to which we refer as densely maximal independent families, and show that such families are Sacks indestructible, both under products and iterations of Sacks forcing.

Dense maximality has its higher analogue:

Definition 26 (κ-dense maximality)**.** Let $\mathcal{A}$ be a maximal κ-independent family of subsets of κ. We say that $\mathcal{A}$ is *densely maximal* if for all $h \in \mathrm{BF}_\kappa(\mathcal{A})$ and $X \in [\kappa]^\kappa \setminus \mathrm{id}(\mathcal{A})$, there is $h' \supseteq h$, $h' \in \mathrm{BF}_\kappa(\mathcal{A})$ for which either $\mathcal{A}^{h'} \cap X$ or $\mathcal{A}^{h'} \setminus X$ is bounded on κ.

In the countable case, dense maximal families exists under the hypothesis of CH [16], or we can force them without adding reals [6]. The following question arises:

Question 27. Assume $2^\kappa = \kappa^+$. Is there a dense independent family of subsets of κ?

Moreover, assuming that such family exists, it is not clear whether they remain maximal even after a single κ-Sacks forcing, where κ is an inaccessible cardinal (see Kanamori [14]). More specifically:

Question 28. Let V be a model of GCH, let $\mathcal{A}$ be a κ-densely maximal independent family and let $\mathbb{S}_\kappa$ be the generalized κ-Sacks forcing for an inaccessible cardinal κ. Is $\mathcal{A}$ still maximal in $V^{\mathbb{S}_\kappa}$?

As a final remark, we want to mention, that it is possible to construct a κ-maximal almost disjoint family of subsets of κ, which is indestructible after forcing with $\mathbb{S}_\kappa$. For this, we introduce first the basic definitions of the generalization of Sacks forcing for uncountable cardinals.

3.1. κ-Sacks forcing.

The generalization of Sacks forcing for uncountable cardinals was first studied by Kanamori [14] and since then, it has been used to prove many consistency results (see [3]). Let κ be an uncountable regular cardinal.

Recall that $T \subseteq 2^{<\kappa}$ is a *tree* if it is closed under initial segments. That is, $u \in T$ and $v \subseteq u$ imply $v \in T$. A node $u \in T$ *splits* in T if both $u^\frown 0$ and $u^\frown 1$ belong to T.

Definition 29. For strongly inaccessible κ, let $\mathbb{S}_\kappa$ be the following forcing notion: *κ-Sacks forcing* whose conditions are sub-trees T of $2^{<\kappa}$ satisfying the following conditions:

(1) Each $u \in T$ has a splitting extension in $t \in T$, that is $u \subseteq t$ and t splits in T.
(2) For any $\alpha < \kappa$, if $(u_\beta : \beta < \alpha)$ is a sequence of elements in T such that $\beta < \gamma < \alpha \to u_\beta \subseteq u_\gamma$, then $\bigcup\{u_\beta : \beta < \alpha\} \in T$.
(3) If $\delta < \kappa$ is a limit ordinal, $u \in 2^\delta$ and for arbitrarily large $\beta < \delta$ if $u \restriction \beta$ splits in T, then u splits in T.

The extension relation is defined by $T \leq S$ if and only if $T \subseteq S$.

Note: It is clear that in the definition of the forcing one does not need to assume κ to be strongly inaccessible (it is enough κ regular with $2^{<\kappa} = \kappa$). Nonetheless, in order to have some useful properties of the forcing (like fusion, for instance) it is necessary to make this assumption.

As in the countable case, we define the stem(T) where T is a condition in $\mathbb{S}_\kappa$ as the unique splitting node that is comparable with all elements in T. In addition, by recursion on κ we define:

Definition 30 (The α-th *splitting level* of T)**.** Given $T \in \mathbb{S}_\kappa$ define:

- $\mathrm{split}_0(T) = \mathrm{stem}(T)$.
- $\mathrm{split}_{\alpha+1}(T) = \{\mathrm{stem}(T_{u^\frown i}) : u \in \mathrm{split}_\alpha(T) \text{ and } i \in 2\}$.
- If δ is a limit ordinal $< \kappa$, we define $\mathrm{split}_\delta(T) = \{s \in T : s \text{ is a limit of nodes in } \bigcup_{\alpha<\delta} \mathrm{split}_\alpha(T)\}$.

Since there is a canonical bijection b between $2^{<\kappa}$ and $\bigcup_{\alpha<\kappa} \mathrm{split}_\alpha(T)$ sending elements of 2^α to $\mathrm{split}_\alpha(T)$ and recursively defined by:

- $b(\emptyset) = \mathrm{stem}(T)$,
- $b(u^\frown i) = \mathrm{stem}(T_{b(u)^\frown i})$ for $u \in 2^\alpha$ and $i \in 2$,
- $b(u) = \bigcup_{\beta<\alpha} b(u \restriction \beta)$ if α is a limit ordinal and $u \in 2^\alpha$,

One of the main properties one expects when working with tree-like forcing notions is that they have fusion, which in particular implies that the cardinal κ^+ is preserved after forcing with $\mathbb{S}_\kappa$. To get this, we use the splitting levels defined above and we define the *fusion orderings* as follows: given S and $T \in \mathbb{S}_\kappa$, $S \leq_\alpha T$ if and only if $S \leq T$ and $\mathrm{split}_\alpha(T) = \mathrm{split}_\alpha(S)$.

Definition 31. A *fusion sequence* of conditions $(T_\alpha : \alpha < \kappa) \subseteq \mathbb{S}_\kappa$ is sequence of conditions in $\mathbb{S}_\kappa$ such that $T_{\alpha+1} \leq_\alpha T_\alpha$ for all $\alpha < \kappa$ and for a given limit ordinal $\delta < \kappa$, $T_\delta \leq_\alpha T_\alpha$ for all $\alpha < \delta$.

The study of the values of many cardinal invariants in the classical Sacks model has been well studied. We know, for instance that all cardinals in the classical Cichoń diagram remain small (with value $\aleph_1$), and that this indeed is a consequence of the *Sacks property* which states that given an $\mathbb{S}$-name for a real $\dot{f}$, a condition $T \in \mathbb{S}$ and a function $x \in \omega^\omega \cap V$ such that $\sup_{n\in\omega} x(n) = \omega$, then there are a slalom $\varphi \in ([\omega]^\omega)^\omega$ such that $|\varphi(n)| \leq x(n)$ and a condition $S \leq T$ such that $S \Vdash \dot{f}(n) \in \varphi(n)$. In [3] we proved that this property holds in the uncountable case for a suitable choice in the size of the slaloms. More specifically:

Definition 32 (Generalized Sacks property)**.** Let $h \in \kappa^\kappa$ with $\sup_{\alpha<\kappa} h(\alpha) = \kappa$. A forcing notion $\mathbb{P}$ has the h-generalized Sacks Property if for every condition $p \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{f}$ for an element in κ^κ there are a condition $q \leq p$ and a h-slalom $F : \kappa \to [\kappa]^{<\kappa} \in \mathrm{Loc}_h(\kappa)$ [2] such that $q \Vdash \dot{f}(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.

Unlike the countable case, the choice of the bounding function for the slaloms affects whether or not we get the generalized Sacks property. One has, for instance that if h is a function on 2^κ for which $h(\alpha) = |\alpha|$, then $\mathbb{S}_\kappa$ does not have the h-generalized Sacks property. However:

Proposition 33 (See [3])**.** Let $h \in 2^\kappa$ be the function defined by $h(\alpha) = 2^\alpha$. Then $\mathbb{S}_\kappa$ has the h-generalized Sacks property. Note that, in particular this implies that κ^+ is preserved.

Proposition 34. Assume $2^\kappa = \kappa^+$. Then there is a κ-mad family $\mathcal{B}$ of subsets of κ for which $\Vdash_{\mathbb{S}_\kappa}$ ($\mathcal{B}$ is maximal).

Proof. This is a straightforward generalization of the proof for the classical case. Use the fact that $2^\kappa = \kappa^+$ in order to list in a κ^+-sequence $\{(\dot{x}_\alpha, T_\alpha) : \alpha < \kappa^+\}$ all pairs of the form $(\dot{x}, T)$ where $T \in \mathbb{S}_\kappa$ and $\dot{x}$ is an $\mathbb{S}_\kappa$-name such that $T \Vdash \dot{x} \subseteq \kappa$.

We define the desired κ-mad family $\mathcal{B} = \{B_\alpha : \alpha < \kappa^+\}$ inductively, making sure that at a given step $\alpha < \kappa^+$, given the corresponding pair $(\dot{x}_\alpha, T_\alpha)$, there is some extension $S \leq T_\alpha$ which forces either:

(1) $|\dot{x}_\alpha \cap \check{B}_\alpha| = \kappa$, or
(2) $\dot{x}_\alpha \subseteq^* \bigcup_{i \in F_0} B_{\delta_i}$, where $F_0 \subseteq \kappa^+$ and $|F_0| < \kappa$. In other words, $\dot{x}_\alpha$ is forced to be an element of the ideal generated by the family $\mathcal{B}$.

In any case, T_α does not force "$\mathcal{B} \cup \{\dot{x}_\alpha\}$ is κ-ad", provided that $\dot{x}_\alpha \notin \mathcal{B}$. Thus, $\mathcal{B}$ will be preserved to be maximal in the extension $V^{\mathbb{S}_\kappa}$.

Suppose now that we have defined B_β for all $\beta < \alpha$. Without loss of generality the B_β's can be seen as the columns $B_\beta \simeq \{\beta\} \times \kappa$ for $\beta < \kappa$. Also, we can assume that $T_\alpha \Vdash$ ($\dot{x}_\alpha$ meets the columns $\{\delta\} \times \kappa$ for unboundedly many δ). Otherwise the condition (2) would be already fulfilled.

The desired set B_α has the form $B_\alpha = \{(i, j) \in \kappa \times \kappa : j < f(i)\}$ for some suitable function $f : \kappa \to \kappa$. Clearly, this implies $|B_\alpha \cap B_\beta| < \kappa$ for all $\beta < \alpha$. The rest of the construction shows that f can be chosen so that condition (1) is satisfied.

Given the pair $(\dot{x}_\alpha, T_\alpha)$, we construct the following two $\mathbb{S}_\kappa$-names: First, let $\dot{Y}_\alpha$ be an $\mathbb{S}_\kappa$-name for the set $Y_\alpha = \{\alpha < \kappa : (\{\alpha\} \times \kappa) \cap \dot{x}_\alpha \neq \emptyset\}$ and secondly, let $\dot{g}_\alpha$ be an $\mathbb{S}_\kappa$-name for a function with domain Y_α and values in κ such that

$$T_\alpha \Vdash \dot{Y}_\alpha \text{ is unbounded on } \kappa \text{ and } \forall \delta \in \dot{Y}_\alpha \, \exists \gamma < \dot{g}(\delta)((\delta, \gamma) \in \dot{x}_\alpha).$$

Specifically, if $\delta \in Y_\alpha$, $\dot{g}(\delta)$ can be defined as $\min\{\gamma < \kappa : \exists T' \leq T \, (T' \Vdash (\delta, \gamma) \in \dot{x}_\alpha)\} + 1$. Then, the condition is satisfied.

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[2]Given $h \in \kappa^\kappa$ such that $\sup_{\alpha<\kappa} h(\alpha) = \kappa$, $\mathrm{Loc}_h(\kappa) = \{F \in ([\kappa]^{<\kappa})^\kappa : \forall \alpha < \kappa \, |F(\alpha)| < |h(\alpha)|\}$

Using the κ-Sacks property, there is a slalom $F_\alpha : \kappa \to [\kappa]^{<\kappa}$ and a condition $S \leq T_\alpha$ forcing

$$S \Vdash \forall \delta < \kappa \, (\dot{g}_\alpha(\delta) \in F(\delta))$$

and $|F(\delta)| < 2^{|\delta|}$. Then if we define (in the ground model) $f(\delta) = \sup(F(\delta)) + 1$ we get that $S \Vdash |B_\alpha \cap \dot{x}_\alpha| = \kappa$. $\square$

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