GLOBAL MAD SPECTRA

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Abstract. We address the issue of controlling the spectrum of maximal almost disjoint families globally, i.e. for more than one regular cardinal κ simultaneously. Assuming GCH we show that there is a cardinal-preserving generic extension satisfying

$$\forall\kappa \in C(\mathfrak{sp}(\mathfrak{a}_\kappa) = B(\kappa))$$

where C denotes the class of successors of regular cardinals together with $\aleph_0$, $B(\kappa)$ is a prescribed set of cardinals to which we refer as a κ-Blass spectrum and $\mathfrak{sp}(\mathfrak{a}_\kappa)$ is the spectrum of κ-mad families.

1. Introduction

In the following we show that one can simultaneously control the cardinalities of κ-maximal almost disjoint families for many cardinals κ. We start by recalling some well-known definitions and introducing notation which will be used throughout the paper.

Definition 1.1. Let κ be a regular infinite cardinal. Let a and b be subsets of κ of size κ, i.e. $a, b \in [\kappa]^\kappa$.

(1) The sets a and b are *almost disjoint* if $|a \cap b| < \kappa$.
(2) A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is *almost disjoint* if any two distinct elements in $\mathcal{A}$ are almost disjoint. An almost disjoint family is *maximal* (*mad*) if it is maximal with respect to inclusion, i.e. it is not properly contained in another almost disjoint family.
(3) The *almost disjointness number* $\mathfrak{a}_\kappa$ is the minimal size of at least κ-sized mad families:

$$\mathfrak{a}_\kappa = \min\{|\mathcal{A}| \colon |\mathcal{A}| \geq \kappa \text{ and } \mathcal{A} \subseteq [\kappa]^\kappa \text{ is mad}\}.$$

By a diagonal argument it is easily shown that $\kappa < \mathfrak{a}_\kappa \leq \mathfrak{c}_\kappa$, where $\mathfrak{c}_\kappa$ is used to denote 2^κ. It is also well-known that there exists always a κ-mad family of size $\mathfrak{c}_\kappa$. The next definition captures the cardinalities of κ-mad families in a model of set theory.

Definition 1.2. For a regular infinite cardinal κ, the *spectrum of κ-mad families*, denoted $\mathfrak{sp}(\mathfrak{a}_\kappa)$, is defined as follows:

$$\mathfrak{sp}(\mathfrak{a}_\kappa) = \{\delta \leq 2^\kappa \colon \exists\mathcal{A} \in \mathcal{P}([\kappa]^\kappa)\ [|\mathcal{A}| = \delta \wedge \mathcal{A} \text{ is } \kappa\text{-mad}]\}.$$

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It is known that $\mathfrak{sp}(\mathfrak{a}_\kappa)$ is closed under singular limits (see e.g. [12, p. 901]). In [5], S. Hechler showed that consistently $\mathfrak{c}$ is large and there is, for each cardinal $\mu \in [\aleph_1, \mathfrak{c}]$, a ω-mad family of size μ. In [1], A. Blass showed that assuming GCH there is a cardinal-preserving generic extension in which the spectrum of ω-mad families equals any prescribed set B of cardinals with $\min(B) = \aleph_1$, $\forall \mu \in B\ [\mathrm{cof}(\mu) = \omega \to \mu^+ \in B]$ and $|B| \geq \aleph_1 \to [\aleph_1, |B|] \subseteq B$ (such a set is referred to as a ω-Blass spectrum in this article). Making different assumptions on the possible spectrum C of ω-mad families, S. Shelah and O. Spinas showed in [12], that consistently $\mathfrak{sp}(\mathfrak{a}_\omega) = C$ and e.g. $\aleph_1 \notin C$. In [4], V. Fischer generalized the proof of [1] to a regular uncountable cardinal κ, showing that assuming GCH, there is a cardinal-preserving forcing extension in which $\mathfrak{sp}(\mathfrak{a}_\kappa) = B$ for a given κ-Blass spectrum B. In section 3, we will also consider the following invariants:

Definition 1.3. Let κ be regular and infinite. Let f and g be functions from κ to κ, i.e. $f, g \in {}^\kappa\kappa$.

(1) We say that g *eventually dominates* f, written $f <^* g$, if $\exists \alpha < \kappa\ \forall \beta > \alpha\ [f(\beta) < g(\beta)]$.
(2) A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is *dominating* if $\forall g \in {}^\kappa\kappa\ \exists f \in \mathcal{F}\ [g <^* f]$.
(3) A set $\mathcal{F} \subseteq {}^\kappa\kappa$ is *unbounded* if $\forall g \in {}^\kappa\kappa\ \exists f \in \mathcal{F}\ [f \not<^* g]$.
(4) Finally, $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ denote the generalized *bounding* and *dominating numbers* respectively:
$$\mathfrak{b}_\kappa = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq {}^\kappa\kappa \text{ is unbounded}\} \text{ and } \mathfrak{d}_\kappa = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq {}^\kappa\kappa \text{ is dominating}\}.$$

In the above definition, we drop the lower index κ, if $\kappa = \aleph_0$, i.e. $\mathfrak{a} = \mathfrak{a}_{\aleph_0}, \mathfrak{b} = \mathfrak{b}_{\aleph_0}, \mathfrak{d} = \mathfrak{d}_{\aleph_0}, \mathfrak{c} = \mathfrak{c}_{\aleph_0}$. The inequality $\mathfrak{b}_\kappa \leq \mathfrak{a}_\kappa$ holds in ZFC for every regular cardinal κ. The characteristics $\mathfrak{d}$ and $\mathfrak{a}$ are known to be independent: $\mathfrak{a} < \mathfrak{d}$ holds in Cohen's model and the consistency of $\mathfrak{d} < \mathfrak{a}$ was shown in [10]. Without assuming large cardinals, the consistency of even $\mathfrak{b}_\kappa < \mathfrak{a}_\kappa$ is still open for regular uncountable cardinals. However, relative to the existence of supercompact cardinals, an even stronger consistency is established in [9]: If $\aleph_0 < \kappa^{<\kappa} = \kappa < \theta$ and θ is supercompact, then $\theta < \mathfrak{b}_\kappa < \mathfrak{d}_\kappa < \mathfrak{a}_\kappa$ holds in a generic extension.

In Section 3, we show (Theorem 3.10 and 3.13):

Theorem. (GCH) If C is a class of regular infinite cardinals and E is an Easton function on C, then there is a cardinal preserving generic extension, where $\forall \kappa \in C\ [\mathfrak{a}_\kappa = \kappa^+ = \mathfrak{b}_\kappa < \mathfrak{d}_\kappa = \mathfrak{c}_\kappa = E(\kappa)]$ holds. If E additionally satisfies $\forall \kappa \in C\ [\sup\{E(\beta) \colon \beta \in C \cap \kappa\} \leq \kappa^+]$, then $\forall \kappa \in C\ [\mathfrak{sp}(\mathfrak{a}_\kappa) = \{\kappa^+, E(\kappa)\}]$ holds as well.

Finally, in Section 4 we show (Theorem 4.9) that one can control the spectrum on the successors of regular cardinals together with $\aleph_0$:

Theorem. (GCH) Suppose that C is the class of successors of regular cardinals together with $\aleph_0$ and $\{B(\kappa) \colon \kappa \in C\}$ is a family of κ-Blass spectra. Then there is a cardinal preserving generic extension where $\forall \kappa \in C\ [\mathfrak{sp}(\mathfrak{a}_\kappa) = B(\kappa)]$ holds.

The following notation is used throughout the article.

Definition 1.4.

(1) For any class C of ordinals and any ordinal λ, let $C^+_\lambda = \{\kappa \in C \colon \kappa > \lambda\}$ and $C^-_\lambda = \{\kappa \in C \colon \kappa \leq \lambda\}$.

(2) For any function E on a class of ordinals and any ordinal λ, let $E^+_\lambda = E \restriction \{\kappa \in \mathrm{dom}(E) \colon \kappa > \lambda\}$ and $E^-_\lambda = E \restriction \{\kappa \in \mathrm{dom}(E) \colon \kappa \leq \lambda\}$.

Recall the definition of the product of two forcing posets and the Product Lemma. If $(P, \leq_P, 1_Q)$ and $(Q, \leq_Q, 1_Q)$ are forcing posets, then their product $(P \times Q, \leq, 1)$ is defined by $(p, q) \leq (p', q') \Leftrightarrow p \leq_P p' \wedge q \leq_Q q'$ and $1 = (1_P, 1_Q)$. The functions $i \colon P \to P \times Q$ and $j \colon Q \to P \times Q$ are defined as $i(p) = (p, 1_Q)$ and $i(q) = (1_P, q)$. It is known that the mappings i and j in the above definition are complete embeddings. More generally, if $(P_i, \leq_i, 1_i)$, for $i \in I$, are forcings, then their product $\prod_{i \in I}(P_i, \leq_i, 1_i)$ is given by the poset $(\prod_{i \in I} P_i, \leq, 1)$ where the relation is given as follows: For $p, q \in \prod_{i \in I} P_i$, $p \leq q$ iff $\forall i \in I$ $[p(i) \leq_i q(i)]$ and $1 = \langle 1_i \colon i \in I \rangle$. If $(P, \leq_P, 1_Q)$ and $(Q, \leq_Q, 1_Q)$ are forcing posets, then forcing with $P \times Q$ adjoins both a P-generic filter and a Q-generic filter over the ground model (see e.g. [7, Lemma V.1.1.]). By the Product Lemma ([7, Theorem V.1.2.]) we refer to the fact that if P, Q, i and j are as above and $G \subseteq P$ and $H \subseteq Q$ holds, then the following are equivalent:

(1) $G \times H$ is $P \times Q$-generic over V.
(2) G is P-generic over V and H is Q-generic over $V[G]$.
(3) H is Q-generic over V and G is P-generic over $V[H]$.

Furthermore, if (1), (2) or (3) holds, then $V[G \times H] = V[G][H] = V[H][G]$.

If $p \in \prod_{i \in I} P_i$, then $\mathrm{supp}(p)$ denotes the set $\{i \in I \colon p(i) \neq 1_i\}$, referred to as the support of p.

2. Excluding Values

In this section we show that the spectrum of κ-mad families (where κ is a regular cardinal) can be forced over a model of GCH to be any specified κ-Blass spectrum. Throughout this section let κ be a regular infinite cardinal.

Definition 2.1. A closed set B of cardinals is called a *κ-Blass spectrum* if it satisfies:

(1) $\min B = \kappa^+$,
(2) $\forall \mu \in B$ $[\mathrm{cof}(\mu) \leq \kappa \to \mu^+ \in B]$ and
(3) if $|B| \geq \kappa^+$ then $[\kappa^+, |B|] \subseteq B$.

Let D be a closed set of cardinals such that $\min D \geq \kappa^+$. For each $\xi \in D$ let $\mathcal{I}_\xi = \{(\xi, \eta) : \eta < \xi\}$ be an index set of cardinality ξ ensuring that $\mathcal{I}_{\xi_1} \cap \mathcal{I}_{\xi_2} = \emptyset$ whenever $\xi_1 \neq \xi_2$ and $\xi_1, \xi_2 \in D$. Let $Q_{\mathcal{I}_\xi}$ be the poset for adding a κ-mad family of size $|\mathcal{I}_\xi| = \xi$. That is $Q_{\mathcal{I}_\xi}$ is the poset defined as:

Definition 2.2. The poset $Q_{\mathcal{I}_\xi}$ consists of all functions $p : \Delta^p \to [\kappa]^{<\kappa}$ such that Δ^p is in $[\mathcal{I}_\xi]^{<\kappa}$ and $q \leq p$ iff:

(1) $\Delta^p \subseteq \Delta^q$ and $\forall x \in \Delta^p$ $q(x) \supseteq p(x)$,
(2) whenever (ξ, η_1) and (ξ, η_2) are distinct elements of Δ^p then

$$q(\xi, \eta_1) \cap q(\xi, \eta_2) \subseteq p(\xi, \eta_1) \cap p(\xi, \eta_2).$$

Remark 2.3. Note that in item (2) above, because of item (1), we have in fact, equality, i.e.

$$q(\xi, \eta_1) \cap q(\xi, \eta_2) = p(\xi, \eta_1) \cap p(\xi, \eta_2).$$

Lemma 2.4. Let D be a closed set of cardinals such that $\min D \geq \kappa^+$. Let $\mathbb{P} = \prod^{<\kappa}_{\xi \in D} \mathbb{Q}_{\mathcal{I}_\xi}$ be the product with supports of size less than κ. Then $\mathbb{P}$ has the κ^+-c.c. and is κ-closed, hence $\mathbb{P}$ preserves cardinals.

Proof. The κ-closedness is easily seen due to the regularity of κ and the fact that $\mathbb{Q}_{\mathcal{I}_\xi}$ is κ-closed for each $\xi \in D$. Let $W = \{p_\alpha \colon \alpha \in \kappa^+\} \subseteq \mathbb{P}$ be a set of conditions of size κ^+. As $\kappa^{<\kappa} = \kappa < \kappa^+$, we can apply the Δ-system-lemma to $\{\mathrm{supp}(p_\alpha) \colon \alpha \in \kappa^+\}$ and get an element $U \in [\kappa^+]^{\kappa^+}$ such that $\{\mathrm{supp}(p_\alpha) \colon \alpha \in U\}$ forms a Δ-system with root R, where $|R| < \kappa$. The collection $A = \{\bigcup_{\xi \in R} \Delta^{p_\alpha(\xi)} \colon \alpha \in U\}$ is of size κ^+ and each element in there is of size $< \kappa$. Again by the Δ-system-lemma (applied to A), we get an $U' \in [U]^{\kappa^+}$ such that $A' = \{\bigcup_{\xi \in R} \Delta^{p_\alpha(\xi)} \colon \alpha \in U'\}$ forms a Δ-system with some root $\bar{\Delta}$, where $|\bar{\Delta}| < \kappa$. However, there are at most κ-many functions from $\bar{\Delta}$ to $[\kappa]^{<\kappa}$, since $\kappa^{<\kappa} = \kappa$. So there are at least two distinct $\alpha, \beta \in U'$ such that p_α and p_β coincide on $\bar{\Delta}$. These two conditions are compatible showing, by $\{p_\alpha \colon \alpha \in U'\} \subseteq W$, that W is not an antichain. The following condition $r \in \mathbb{P}$ extends both p_α and p_β: Let $\mathrm{supp}(r) = \mathrm{supp}(p_\alpha) \cup \mathrm{supp}(p_\beta)$, $\forall \xi \in \mathrm{supp}(r)$ $[\Delta^{r(\xi)} = \Delta^{p_\alpha(\xi)} \cup \Delta^{p_\beta(\xi)}]$ and

$$r(\xi)(\xi,\gamma) = \begin{cases} p_\alpha(\xi)(\xi,\gamma) & \text{for } \ \xi \in \mathrm{supp}(p_\alpha) \setminus \mathrm{supp}(p_\beta) \vee (\xi,\gamma) \in \Delta^{p_\alpha(\xi)} \setminus \Delta^{p_\beta(\xi)} \\ p_\beta(\xi)(\xi,\gamma) & \text{for } \ \xi \in \mathrm{supp}(p_\beta) \setminus \mathrm{supp}(p_\alpha) \vee (\xi,\gamma) \in \Delta^{p_\beta(\xi)} \setminus \Delta^{p_\alpha(\xi)} \\ p_\beta(\xi)(\xi,\gamma) = p_\alpha(\xi)(\xi,\gamma) & \text{for } \ \xi \in R \wedge \gamma \in \bar{\Delta} \end{cases}.$$

□

Lemma 2.5. Let D be a closed set of cardinals such that $\min D \geq \kappa^+$. Let $\mathbb{P} = \prod^{<\kappa}_{\xi \in D} \mathbb{Q}_{\mathcal{I}_\xi}$ be the product with supports of size less than κ. In $V^{\mathbb{P}}$ there is a κ-mad family of cardinality ξ for each $\xi \in D$.

Proof. Let $G \subseteq \mathbb{P}$ be generic over V. We show that for each $\xi \in D$, the set $\mathcal{A}^\xi = \{A^\xi_\alpha : \alpha \in \xi\}$ is κ-mad, where $A^\xi_\alpha = \bigcup_{p \in G} p(\xi)(\xi,\alpha)$.

So fix an element ξ in D. First, $\mathcal{A}^\xi$ is almost disjoint: Let $\alpha, \beta \in \xi$ and $\alpha \neq \beta$. The conditions $p \in \mathbb{P}$ such that $(\xi,\alpha), (\xi,\beta) \in \Delta^{p(\xi)}$ are dense in $\mathbb{P}$. So there is $q \in G$ such that $(\xi,\alpha), (\xi,\beta) \in \Delta^{q(\xi)}$. Then $A^\xi_\alpha \cap A^\xi_\beta = p(\xi)(\xi,\alpha) \cap p(\xi)(\xi,\beta)$, which is of size $< \kappa$.

Furthermore $\mathcal{A}^\xi$ is maximal: Let $\dot{B}$ be a nice $\mathbb{P}$-name for an element in $[\kappa]^\kappa$. By the κ^+-c.c. $\dot{B}$ involves only $\leq \kappa$-many conditions. So there is a (ξ,α) such that $(\xi,\alpha) \notin \Delta^{p'(\xi)}$ for any condition p' involved in $\dot{B}$. We show that $V[G] \vDash |\dot{B} \cap \dot{A}^\xi_\alpha| = \kappa$, which will finish the proof. Suppose that there is a $\gamma < \kappa$ and a condition $p \in G$ such that $p \Vdash \dot{B} \cap \dot{A}^\xi_\alpha \subseteq \gamma$. Recall that $|\Delta^{p(\xi)}| < \kappa$ and $p(\xi) \colon \Delta^{p(\xi)} \to [\kappa]^{<\kappa}$. Let $q \in G$ be a condition involved in $\dot{B}$ such that for some $\delta > \gamma$,

$$\delta > \bigcup \{p(\xi)(\xi,\beta) \colon (\xi,\beta) \in \Delta^{p(\xi)}\} \qquad (*)$$

and $q \Vdash \check{\delta} \in \dot{B}$. As $p, q \in G$, p and q are compatible. Now consider the condition $r \in \mathbb{P}$ defined as follows:

- $\mathrm{supp}(r) = \mathrm{supp}(q) \cup \mathrm{supp}(p) \cup \{\xi\}$
- $\Delta^{r(\eta)} = \begin{cases} \Delta^{p(\eta)} \cup \Delta^{q(\eta)} \cup \{(\xi,\alpha)\} & \text{for } \ \eta = \xi \\ \Delta^{p(\eta)} \cup \Delta^{q(\eta)} & \text{for } \ \eta \in \mathrm{supp}(r) \setminus \{\xi\} \end{cases}$

Furthermore, $r(\xi)(\xi,\alpha) = p(\xi)(\xi,\alpha) \cup \{\delta\}$ (note that $(\xi,\alpha) \notin \Delta^{q(\xi)}$ by its choice) and $\forall \eta \in \operatorname{supp}(r)\ \forall(\eta,\mu) \in \Delta^{r(\eta)}\ [(\eta,\mu) \neq (\xi,\alpha) \to r(\eta)(\eta,\mu) = p(\eta)(\eta,\mu) \cup q(\eta)(\eta,\mu)]$. Now r extends both p (by $(*)$) and q and $r \Vdash \delta \in \dot{B}$ (as $r \leq q$) and $r \Vdash \delta \in \dot{A}^{\xi}_{\alpha}$ contradicting that $r \Vdash \dot{B} \cap \dot{A}^{\xi}_{\alpha} \subseteq \gamma$ (as $r \leq p$ and $\delta > \gamma$). □

Until the end of the section we will be occupied with the proof of the following statement.

Lemma 2.6. Let C be a κ-Blass spectrum. Let $\lambda \notin C$ and let $\mathbb{P} = \prod^{<\kappa}_{\xi\in C} \mathbb{Q}_{\mathcal{I}_\xi}$ be the product with supports of size less than κ. Then in $V^{\mathbb{P}}$ there are no κ-mad families of cardinality λ.

Note that the cofinality of the maximum of a κ-Blass spectrum is greater than κ (by item (2) in Definition 2.1). By counting nice names, it is argued that $V^{\mathbb{P}} \vDash \mathfrak{c}_\kappa = \max(C)$: $V^{\mathbb{P}} \vDash \mathfrak{c}_\kappa \geq \max(C)$ is clear. As $|C| \leq \max(C)$, $\mathbb{P}$ has size $\max(C)$. Then, by the κ^+-c.c. of $\mathbb{P}$, there are no more than $\max(C)^\kappa = \max(C)$-many nice names for subsets of κ.

Proof of Lemma 2.6. Let C be a κ-Blass spectrum and let $\lambda \notin C$. Take $\mu = \max\{\gamma\colon \gamma \in C \text{ and } \gamma < \lambda\}$. Then clearly $\mu \geq \kappa^+$ (by Definition 2.1(1)) and moreover $\kappa^+ \leq \operatorname{cof}(\mu) \leq \mu$ (by Definition 2.1(2)). By GCH in V, we obtain

$$\mu^\kappa = \mu. \qquad (\star)$$

Suppose by way of contradiction that $\dot{\mathcal{A}} = \{\dot{a}_\alpha\colon \alpha < \lambda\}$ is forced by the maximal element in $\mathbb{P}$ to be a κ-mad family of size λ in $V^{\mathbb{P}}$. We may assume that each $\dot{a}_\alpha$ is a nice name.

Definition 2.7.

(1) Whenever $\dot{x}$ is a $\mathbb{P}$-name for an unbounded subset of κ, we can assume that $\dot{x}$ is a nice $\mathbb{P}$-name. That is, we identify $\dot{x}$ with κ-many maximal antichains $\{A_\alpha(\dot{x})\}_{\alpha<\kappa}$ each of cardinality at most κ, such that the conditions in $A_\alpha(\dot{x})$ decide if "$\check{\alpha} \in \dot{x}$". We refer to $\Delta(\dot{x}) = \bigcup_{\alpha\in\kappa} A_\alpha(\dot{x})$ as the *set of conditions involved in* $\dot{x}$.

(2) Let $\dot{x}$ be a $\mathbb{P}$-name for a subset of κ and let $\Delta(\dot{x})$ be the set of conditions involved in $\dot{x}$. The set

$$J(\dot{x}) = \bigcup_{p\in\Delta(\dot{x})} \bigcup_{\xi\in\operatorname{supp}(p)} \Delta^{p(\xi)}$$

is called the *support of* $\dot{x}$.

For each $\alpha \in \lambda$ let J_α denote the support of $\dot{a}_\alpha$.

Let θ be large enough that $\mathbb{P} \in H(\theta)$ and $V \vDash \operatorname{cof}(\theta) > |\mathbb{P}|$. Let $\mathcal{M} \preceq H(\theta)$ be an elementary submodel such that $|M| = \mu$, $\mu \subseteq M$, $M^\kappa \subseteq M$, $C \subseteq M$, $\mathbb{P} \in M$ and M contains all other relevant parameters. The equation $(\star)$ is used here in order to ensure the property $M^\kappa \subseteq M$. The property $C \subseteq M$ requires that $|C| \leq \mu$, which is ensured by Definition 2.1(3).

Let $\bar{\alpha} \in \lambda \setminus M$. Fix a permutation of the index set $\mathcal{I} = \bigcup_{\xi\in C} \mathcal{I}_\xi$ which

- fixes $\mathcal{I}_\xi$ for $\xi \leq \mu$, and
- and for each $\xi > \mu$ maps the $\leq \kappa$-sized set $J_{\bar{\alpha}} \cap \mathcal{I}_\xi \setminus M$ into $(\mathcal{I}_\xi \setminus \bigcup_{i<\lambda} J_i) \cap M$ (otherwise fixing elements of $\mathcal{I}_\xi$).

Such a permutation of the index set exists, because if $\xi > \mu$, then $\xi > \lambda$ as well. Consequently $|\bigcup_{i<\lambda} J_i| = \lambda * \kappa = \lambda$, and $|\mathcal{I}_\xi \setminus \bigcup_{i<\lambda} J_i| = \xi > \kappa$ holds in $H(\theta)$ and by elementarity also in $\mathcal{M}$. This permutation of the index set $\mathcal{I}$ induces an automorphism $\pi : \mathbb{P} \to \mathbb{P}$ of the poset. As names are defined recursively, $\pi \in \mathrm{Aut}(\mathbb{P})$ (where $\mathrm{Aut}(\mathbb{P})$ denotes the automorphism group of $\mathbb{P}$) induces a map $\pi^*\colon V^{(\mathbb{P})} \to V^{(\mathbb{P})}$ (where $V^{(\mathbb{P})}$ denotes the class of all $\mathbb{P}$-names) by $\pi^*(\tau) = \{\langle \pi^*(\sigma), \pi(p)\rangle \colon \langle \sigma, p\rangle \in \tau\}$. The automorphism π preserves antichains and the forcing relation. And as $\dot{a}_{\bar{\alpha}}$ is supposed to be a nice name, and any antichain of $\mathbb{P}$ is of size $\leq \kappa$ (by the κ^+-c.c. of $\mathbb{P}$) and M is closed w.r.t. κ-sequences, we have $\pi^*(\dot{a}_{\bar{\alpha}}) \in M$.

Let G be a generic filter. Then $\pi''(G)$ is a generic filter. It is well-known that $\mathcal{M}[\pi''(G)] \preceq ((H(\theta))^{V[\pi''(G)]}, \in)$ (see [11, Theorem III.2.11.]). As $\dot{\mathcal{A}}$ is forced to be κ-mad, we have

$$\Vdash_{\pi(\mathbb{P})} \forall x \in {}^\kappa\kappa \; \exists \beta < \lambda \; [|x \cap \dot{a}_\beta| = \kappa].$$

We can relativize the statement to $H(\theta)$, so

$$\Vdash_{\pi(\mathbb{P})} \forall x \in {}^\kappa\kappa \cap H(\theta) \; \exists \beta < \lambda \cap H(\theta) \; [|x \cap \dot{a}_\beta| = \kappa].$$

But $\mathcal{M}[\pi''(G)] \preceq ((H(\theta))^{V[\pi''(G)]}, \in)$ and $M \cap Ord = M[\pi''(G)] \cap Ord$, so

$$\Vdash_{\pi(\mathbb{P})} \forall x \in {}^\kappa\kappa \cap M \; \exists \beta < \lambda \cap M \; [|x \cap \dot{a}_\beta| = \kappa].$$

As $\pi^*(\dot{a}_{\bar{\alpha}})$ was in M, we have

$$\Vdash_{\pi(\mathbb{P})} \exists \beta < \lambda \cap M \; [|\pi^*(\dot{a}_{\bar{\alpha}}) \cap \dot{a}_\beta| = \kappa].$$

However $\pi^*(\dot{a}_\beta) = \dot{a}_\beta$ for ordinals $\beta \in M$ as the permutation π fixes the ordinals mentioned in $\dot{a}_\beta$ for $\beta \in M$. Therefore we have

$$\Vdash_{\pi(\mathbb{P})} \exists \beta < \lambda \cap M \; [|\pi^*(\dot{a}_{\bar{\alpha}}) \cap \pi^*(\dot{a}_\beta)| = \kappa]$$

and by applying π^{-1} we have

$$\Vdash_{\mathbb{P}} \exists \beta < \lambda \cap M \; [|\dot{a}_{\bar{\alpha}} \cap \dot{a}_\beta| = \kappa],$$

contradicting the κ-madness of $\dot{\mathcal{A}}$ in the generic extension. □

3. Small Spectra

In this section we give several easy results concerning $\mathfrak{a}_\kappa$ and $\mathfrak{sp}(\mathfrak{a}_\kappa)$. First we show that in the extension by the poset of Definition 2.2, $\mathfrak{a}_\kappa$ is small.

Definition 3.1. Let $\mathbb{Q}$ be a forcing notion and κ be a regular cardinal. A κ-mad family $\mathcal{A}$ is called $\mathbb{Q}$-indestructible if $\mathcal{A}$ is still maximal in any $\mathbb{Q}$-generic extension of the ground model.

Lemma 3.2. ($2^\kappa = \kappa^+$) Let $\mathbb{P}$ be a poset of cardinality κ for a regular infinite cardinal κ. Then there is a $\mathbb{P}$-indestructible κ-mad family of cardinality κ^+.

Proof. By the assumption $2^\kappa = \kappa^+$ we can fix an enumeration $\langle (p_\xi, \tau_\xi)\colon \kappa \leq \xi < \kappa^+\rangle$ of all pairs (p, τ) such that $p \in \mathbb{P}$ and τ is a nice $\mathbb{P}$-name for a subset of κ (there are κ^+-many nice $\mathbb{P}$-names since $[\mathbb{P}]^{\leq\kappa} = \kappa^+$). Recursively define subsets $\{A_\xi \colon \xi < \kappa^+\}$ of κ as follows: First let $\{A_\xi \colon \xi < \kappa\}$

be any partition of κ into sets of size κ. Let ξ be such that $\kappa \leq \xi < \kappa^+$ and suppose that we already defined A_η for every $\eta < \xi$. Now choose A_ξ such that the following conditions hold:

(1) $\forall \eta < \xi\ [|A_\xi \cap A_\eta| < \kappa]$
(2) If

$$p_\xi \Vdash |\tau_\xi| = \kappa \text{ and } \forall \eta < \xi\ [p_\xi \Vdash |\tau_\xi \cap A_\eta| < \kappa], \tag{$\star$}$$

then

$$\forall \alpha < \kappa\ \forall q \leq p_\xi\ \exists r \leq q\ \exists \beta \geq \alpha\ [\beta \in A_\xi \wedge r \Vdash \beta \in \tau_\xi]$$

To verify that A_ξ can indeed be chosen like above, note that (1) is easily satisfied as there are no κ-mad families of size κ. To satisfy (2), assume ($\star$) and let $\{B_i : i \in \kappa\}$ be an enumeration of $\{A_\eta : \eta < \xi\}$ and let $\langle(\alpha_i, q_i) : i \in \kappa\rangle$ enumerate $\kappa \times \{q : q \leq p_\xi\}$. By ($\star$), for each $i \in \kappa$ we have $q_i \Vdash |\tau_\xi \setminus (\bigcup_{j \leq i} B_j)| = \kappa$, so choose any $r \leq q_i$ and $\beta_i \geq \alpha_i$ such that $\beta_i \notin \bigcup_{j \leq i} B_j$ and $r_i \Vdash \beta_i \in \tau_\xi$. Define A_ξ to be $\{\beta_i : i \in \kappa\}$.

Now consider the family $\mathcal{A} = \{A_\xi : \xi \in \kappa^+\}$ and show that this is κ-mad in $V[G]$, where G is $\mathbb{P}$-generic over V. Suppose not and let (p_ξ, τ_ξ) be such that $p_\xi \in G$ and $p_\xi \Vdash \forall x \in \mathcal{A}\ [|\tau_\xi \cap x| < \kappa]$. Thus ($\star$) holds at ξ; however also $p_\xi \Vdash |\tau_\xi \cap A_\xi| < \kappa$ holds, so there is an extension $q \leq p_\xi$ and an $\alpha < \kappa$ with $q \Vdash \tau_\xi \cap A_\xi \subseteq \alpha$, contradicting $\exists r \leq q\ \exists \beta \geq \alpha\ [\beta \in A_\xi \wedge r \Vdash \beta \in \tau_\xi]$. □

Lemma 3.3. Let $V \vDash \mathrm{GCH}$, let κ be a regular cardinal and $\lambda \geq \kappa^+$. Let $\mathbb{Q}^\kappa_{\mathcal{I}_\lambda}$ denote the poset as in Definition 2.2. Let $\dot{f}$ be a $\mathbb{Q}^\kappa_{\mathcal{I}_\lambda}$-name for a κ-real. Then there is a subset $J \subseteq \lambda$ such that $|J| \leq \kappa$ and $\dot{f}$ is equivalent to a $\mathbb{Q}^\kappa_{\mathcal{I}_J}$-name.

Proof. For each $\alpha < \kappa$, let A_α be a maximal antichain in $\mathbb{Q}^\kappa_{\mathcal{I}_\lambda}$ deciding $f(\alpha)$. By the κ^+-c.c. of $\mathbb{Q}^\kappa_{\mathcal{I}_\lambda}$ any antichain has size $\leq \kappa$. Hence $|\bigcup\{\mathrm{dom}(p) : p \in \bigcup_{\alpha<\kappa} A_\alpha\}| \leq \kappa$. Define $J = \bigcup\{\mathrm{dom}(p) : p \in \bigcup_{\alpha<\kappa} A_\alpha\}$, then $\dot{f}$ is equivalent to a $\mathbb{Q}^\kappa_{\mathcal{I}_J}$-name. □

Theorem 3.4. *Let $V \vDash \mathrm{GCH}$, let κ be a regular cardinal and $\lambda \geq \kappa^+$. Let $\mathbb{Q}^\kappa_{\mathcal{I}_\lambda}$ denote the poset as in Definition 2.2. Then $V^{\mathbb{Q}^\kappa_{\mathcal{I}_\lambda}} \vDash \mathfrak{a}_\kappa = \kappa^+$.*

Proof. Let $K \in [\lambda]^\kappa \cap V$. Since $|\mathbb{Q}^\kappa_{\mathcal{I}_K}| = \kappa$, by Lemma 3.2 (and GCH in V) in the ground model, there is a κ-mad family $\mathcal{A}$ which remains maximal in the generic extension by $\mathbb{Q}^\kappa_{\mathcal{I}_K}$. But then $\mathcal{A}$ remains maximal after forcing with $\mathbb{Q}^\kappa_{\mathcal{I}_J}$ for any $J \in [\lambda]^\kappa$, since any such $\mathbb{Q}^\kappa_{\mathcal{I}_J}$ is forcing equivalent (indeed isomorphic) to $\mathbb{Q}^\kappa_{\mathcal{I}_K}$. However by the previous lemma, any κ-real which might destroy the maximality of $\mathcal{A}$ in $V^{\mathbb{Q}^\kappa_{\mathcal{I}_\lambda}}$ is in fact equivalent to a $\mathbb{Q}^\kappa_{\mathcal{I}_J}$-name for some $J \subseteq \lambda$ such that $|J| \leq \kappa$. □

We further remark that it is implicitly shown that the spectrum of madness can globally exclude the possible minimal values:

Remark 3.5. In [3, Theorem 4] it is shown that for a class of regular cardinals λ the triple $(\mathfrak{b}_\lambda, \mathfrak{d}_\lambda, \mathfrak{c}_\lambda)$ can be controlled by forcing. As $\mathfrak{b}_\lambda \leq \mathfrak{a}_\lambda$ for every regular λ, it is consistently true that for every regular cardinal κ, the spectrum of κ-mad families consists only of $2^\kappa = \mathfrak{b}_\kappa = \mathfrak{d}_\kappa$, which is chosen (forced) to be greater than κ^+.

Recall the following definition.

Definition 3.6.

(1) A function E is called an *index function* if $\mathrm{dom}(E)$ is a class of regular cardinals.

(2) An index function E is called an *Easton function*, if for each $\kappa \in \mathrm{dom}(E)$, $E(\kappa)$ is a cardinal with $\mathrm{cof}(E(\kappa)) > \kappa$ such that $\forall \kappa, \kappa' \in \mathrm{dom}(E)$ $[\kappa < \kappa' \to E(\kappa) \leq E(\kappa')]$.

In the following we consider Easton products. That is:

Definition 3.7. If E is an index function, I is $\mathrm{dom}(E)$ and $\mathbb{R} = \prod_{\kappa \in I} \mathrm{Fn}_\kappa(E(\kappa) \times \kappa, 2)$, then the *Easton poset* $\mathbb{P}(E) \subseteq \mathbb{R}$ consists of those $p \in \mathbb{R}$ such that for all regular cardinals λ,

$$|\{\kappa \in \lambda \cap I : p(\kappa) \neq \mathbb{1}\}| < \lambda.$$

It is well-known that $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$, where $\mathbb{P}(E^+_\lambda)$ is λ^+-closed and the second $\mathbb{P}(E^-_\lambda)$ has the λ^+-c.c. if λ is regular and $2^{<\lambda} = \lambda$. In order to prove Theorem 3.10, which evaluates $\mathfrak{a}_\kappa$, $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ in the Easton extension, we need two easy lemmas.

Lemma 3.8. Suppose E_1, E_2 are index functions such that $\mathrm{dom}(E_1) = \mathrm{dom}(E_2) = I \subseteq \lambda^+$ for some ordinal λ and $\forall \kappa \in I[E_1(\kappa) \cap E_2(\kappa) = \emptyset]$. Further assume that E is an Easton function with $\mathrm{dom}(E) = I$ and $\forall \kappa \in I$ $[E(\kappa) = E_1(\kappa) \cup E_2(\kappa)]$. Let G be $\mathbb{P}(E)$-generic over V and let $G_1 = G \cap \mathbb{P}(E_1)$ and $G_2 = G \cap \mathbb{P}(E_2)$. Then G_1 is $\mathbb{P}(E_1)$-generic over V and G_2 is $\mathbb{P}(E_2)$-generic over $V[G_1]$ and $V[G] = V[G_1][G_2]$.

Proof. The mapping $j : \mathbb{P}(E_1) \times \mathbb{P}(E_2) \to \mathbb{P}(E)$ with $j((s_0, s_1, ...), (t_0, t_1, ...)) = (s_0 \cup t_0, s_1 \cup t_1, ...)$ is an isomorphism. So by [6, VII Corollary 7.6], $j^{-1}(G) = H$ is $\mathbb{P}(E_1) \times \mathbb{P}(E_2)$-generic over V and $V[G] = V[H]$. By [6, VII Lemma 1.3], $H = H_1 \times H_2$, where $H_j = i_j^{-1}(H)$ for $j \in \{1, 2\}$ and $i_1 \colon \mathbb{P}(E_1) \to \mathbb{P}(E_1) \times \mathbb{P}(E_2)$ and $i_2 \colon \mathbb{P}(E_2) \to \mathbb{P}(E_1) \times \mathbb{P}(E_2)$ are the complete embeddings defined as $i_1(p_1) = (p_1, 1_{\mathbb{P}(E_2)})$ and $i_2(p_2) = (1_{\mathbb{P}(E_1)}, p_2)$. By the Product Lemma, H_1 is $\mathbb{P}(E_1)$-generic over V, H_2 is $\mathbb{P}(E_2)$-generic over $V[G_1]$ and $V[H] = V[H_1][H_2]$. However

$$H_1 = \{p_1 \in \mathbb{P}(E_1) \colon ((s_0, s_1, ...), 1_{\mathbb{P}(E_2)}) \in H\} = \{p_1 \in \mathbb{P}(E_1) \colon (s_0 \cup \emptyset, s_1 \cup \emptyset, ...) \in G\} = G_1$$

and the same for H_2 and G_2. $\square$

Lemma 3.9. Assume that E is an Easton function with $\mathrm{dom}(E) = I \subseteq \lambda^+$ for a regular λ with $2^{<\lambda} = \lambda$. Let $\dot{f}$ be a $\mathbb{P}(E)$-name for a λ-real. Then there is an index function E' with $\mathrm{dom}(E') = I$ and $\forall \kappa \in I$ $[E'(\kappa) \subseteq E(\kappa)]$ such that $\forall \kappa \in I$ $|E'(\kappa)| \leq \lambda$ and $\dot{f}$ is equivalent to a $\mathbb{P}(E')$-name.

Proof. For each $\alpha < \lambda$ let A_α be a maximal antichain in $\mathbb{P}(E)$ deciding the value of $\dot{f}(\alpha)$. As $\mathbb{P}(E)$ has the λ^+-c.c. each maximal antichain A_α is of size at most λ. So $|\bigcup\{\{\kappa\} \times \mathrm{dom}(p(\kappa)) \colon \kappa \in I,\ p \in \bigcup_{\alpha<\lambda} A_\alpha\}| \leq \lambda$. Then $\dot{f}$ is equivalent to a $\mathbb{P}(E')$-name where $\forall \kappa \in I$ $[E'(\kappa) = \bigcup\{\mathrm{dom}(p(\kappa)) \colon p \in \bigcup_{\alpha<\lambda} A_\alpha\}]$. $\square$

In the next theorem consider the special case in which E is strictly increasing, $E(\kappa) \geq \kappa^{++}$, aiming to establish the consistency of $\mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \kappa^+ < \mathfrak{d}_\kappa = \mathfrak{c}_\kappa$ globally.

Theorem 3.10. (GCH) *Let E be an Easton function such that $\forall\kappa \in \mathrm{dom}(E)$ $[E(\kappa) > \kappa^+]$ and let $\mathbb{P}(E)$ be the Easton product. Then $V^{\mathbb{P}(E)} \vDash \forall\kappa \in \mathrm{dom}(E)$ $[\mathfrak{a}_\kappa = \kappa^+ = \mathfrak{b}_\kappa < \mathfrak{d}_\kappa = \mathfrak{c}_\kappa]$.*

Proof. Let $\kappa \in \mathrm{dom}(E)$ be arbitrary. Consider $\mathbb{P}(E)$ as $\mathbb{P}(E^-_\kappa) \times \mathbb{P}(E^+_\kappa)$. Let K be a $\mathbb{P}(E)$-generic over V. By the Product Lemma, $V[K] = V[H][G]$, where H is $\mathbb{P}(E^+_\kappa)$-generic over V and G is $\mathbb{P}(E^-_\kappa)$-generic over $V[H]$. $\mathbb{P}(E^+_\kappa)$ is κ^+-closed in V, so it preserves GCH at and below κ. Now consider $V[H] =: V_1$ as the ground model. In $V[H]$ there is a $\mathbb{P}(E^-_\kappa)$-indestructible κ-mad family of size κ^+, denoted $\mathcal{A}_\kappa$: By the above lemma it suffices to show maximality in an extension by $\mathbb{P}(E')$ for some index set E' such that $\forall\gamma \in \mathrm{dom}(E')$ $|E'(\gamma)| \leq \kappa$. This poset $\mathbb{P}(E')$ can be completely embedded into $\mathbb{P}(\bar{E})$, where $\bar{E}$ is an index function with domain $\mathrm{dom}(E)$ and $\forall\gamma \in \mathrm{dom}(E)$ $[\bar{E}(\gamma) = \kappa]$. So it suffices to show maximality in the extension by $\mathbb{P}(\bar{E})$. On the other hand $\mathbb{P}(\bar{E})$ is of size κ. However we saw that there is a κ-mad family of size κ^+ whose maximality is preserved in an extension by a poset of that size. Therefore in $V^{\mathbb{P}(E)}$ we have that for every $\kappa \in \mathrm{dom}(E)$,

$$\mathfrak{a}_\kappa = \kappa^+ = \mathfrak{b}_\kappa < \mathfrak{c}_\kappa = E(\kappa).$$

because $\mathfrak{b}_\kappa \leq \mathfrak{a}_\kappa$ is provable in ZFC and it is well-known that $\mathfrak{c}_\kappa = E(\kappa)$ holds in the Easton extension.

To show that $\mathfrak{d}_\kappa \geq E(\kappa)$ let D be a family of κ-reals of size less than $E(\kappa)$. By the previous lemma, there is an index set E' such that $\mathbb{P}(E')$ is of size less than $E(\kappa)$ and $D \in V_1^{\mathbb{P}(E')}$. If $\alpha \in E(\kappa) \setminus E'(\kappa)$ than, by the Product Lemma, the real c_α added by $\mathrm{Fn}_\kappa(E(\kappa) \times \kappa, 2)$ is Cohen over $V_1^{\mathbb{P}(E')}$, in particular unbounded and hence D is not dominating. □

Remark 3.11. By the result in [8], it was sufficient to show that for each $\kappa \in \mathrm{dom}(E)$ we have $\mathfrak{b}_\kappa = \kappa^+$ in the generic Easton extension, as this implies $\mathfrak{a}_\kappa = \kappa^+$.

Theorem 3.12. (GCH at and below κ) *Assume that λ is a cardinal such that* $\mathrm{cof}(\lambda) > \kappa$. *Then in the generic extension by $\mathbb{C}(\kappa)_\lambda = (\mathrm{Fn}_{<\kappa}(\kappa \times \lambda, 2), \subseteq)$, every κ-mad family is either of size κ^+ or of size λ.*

Proof. Let δ be such that $\kappa^+ < \delta < \lambda$, and for each $\alpha < \delta$ let $\dot{X}^\alpha$ be a $\mathbb{C}(\kappa)_\lambda$-name for an element in $[\kappa]^\kappa$. We can identify any $\mathbb{C}(\kappa)_\lambda$-name $\dot{X}$ for a κ-real with κ-many maximal antichains $\{A^{\dot{X}}_\beta : \beta \in \kappa\}$ such that $A^{\dot{X}}_\beta$ decides "$\check{\beta} \in \dot{X}$" in the generic extension. For such a name $\dot{X}$, let $S^{\dot{X}} = \bigcup\{\mathrm{dom}(p) \colon \exists\beta < \kappa\ [p \in A^{\dot{X}}_\beta]\}$, called the support of $\dot{X}$. By the κ^+-c.c. of $\mathbb{C}(\kappa)_\lambda$, each maximal antichain has size at most κ, so $|S^{\dot{X}}| \leq \kappa$ for each name $\dot{X}$ for a κ-real. For each $\alpha < \delta$, let S^α be the support for $\dot{X}^\alpha$. Consequently $|\bigcup\{S^\alpha : \alpha < \delta\}| \leq \delta$ and $|(\kappa \times \lambda) \setminus \bigcup\{S^\alpha : \alpha < \delta\}| = \lambda$. Now consider the set $\{S^\alpha : \alpha < \kappa^{++}\}$. As GCH holds at and below κ and $|S^\alpha| < \kappa^+$, there is, by the Δ-System Lemma, an index set $B \in [\kappa^{++}]^{\kappa^{++}}$ such that $\{S^\alpha : \alpha \in B\}$ forms a Δ-System with root R. Further, for any two $\alpha, \beta \in B$, let $\varphi_{\alpha,\beta} \colon S^\alpha \to S^\beta$ be a bijection fixing the root R. Each such bijection $\varphi_{\alpha,\beta}$ induces an isomorphism $\psi_{\alpha,\beta} \colon (\mathrm{Fn}_{<\kappa}(S^\alpha, 2), \subseteq) \to (\mathrm{Fn}_{<\kappa}(S^\beta, 2), \subseteq)$ between the corresponding restrictions of the Cohen forcing by:

(1) $\forall p \in \mathrm{Fn}_{<\kappa}(S^\alpha, 2)$ $[\mathrm{dom}(\psi_{\alpha,\beta}(p)) = \varphi_{\alpha,\beta}(\mathrm{dom}(p))]$ and
(2) $\forall x \in \mathrm{dom}(p)$ $[(\psi_{\alpha,\beta}(p)((\varphi_{\alpha,\beta}(x)) = p(x)]$.

Furthermore, if for $J \subseteq \kappa \times \lambda$, $V^{(\mathrm{Fn}_{<\kappa}(J,2))}$ denotes the class of all $\mathrm{Fn}_{<\kappa}(J,2)$-names, then, as names are defined recursively, $\psi_{\alpha,\beta}$ induces a mapping $\psi^*_{\alpha,\beta} \colon V^{(\mathrm{Fn}_{<\kappa}(S^\alpha,2))} \to V^{(\mathrm{Fn}_{<\kappa}(S^\beta,2))}$ by $\psi^*_{\alpha,\beta}(\tau) = \{\langle \psi^*_{\alpha,\beta}(\sigma), \psi_{\alpha,\beta}(p)\rangle \colon \langle \sigma, p\rangle \in \tau\}$. The isomorphism $\psi_{\alpha,\beta}$ preserves maximal antichains, as well as the forcing relation. Note that for a fixed set $T \subseteq \kappa \times \lambda$ of cardinality κ, there are, by $[\kappa]^\kappa = \kappa^+$, at most κ^+-many names for κ-reals with the same support T. By this reason and the fact that $|[B]^2| = \kappa^{++} > \kappa^+$, we can assume w.l.o.g. that for any two $\alpha, \beta \in B$, $\psi^*_{\alpha,\beta}$ maps $\dot{X}^\alpha$ to $\dot{X}^\beta$ (if this was not true for B, thin B out so that a subset $B' \in [B]^{\kappa^{++}}$ satisfies this property).

Now define a new $\mathbb{C}(\kappa)_\lambda$-name $\dot{X}^\delta$ for a κ-real such that its support S^δ satisfies $S^\delta \cap \bigcup_{\alpha<\delta} S^\alpha = R$ and for any $\alpha \in B$, S^α is mapped to S^δ by a bijection $\varphi_{\alpha,\delta}$ fixing the root R and again assume that the induced functions $\psi^*_{\alpha,\delta}$ map $\dot{X}^\alpha$ to $\dot{X}^\delta$.

Suppose that $\Vdash_{\mathbb{C}(\kappa)_\lambda} \forall \alpha, \beta \in \delta\ [|\dot{X}^\alpha \cap \dot{X}^\beta| < \kappa]$. We will reach a contradiction by showing that the family $\{X^\alpha \colon \alpha < \delta\}$ is not maximal in the generic extension, witnessed by X^δ. So fix an arbitrary $\beta < \delta$. As $|S^\beta| = \kappa$, $[\kappa]^\kappa = \kappa^+$ and $|B| = \kappa^{++}$, there are at least two distinct elements $\alpha, \alpha' \in B$ such that the supports S^α and $S^{\alpha'}$ have the same intersection with S^β, i.e. $S^\alpha \cap S^\beta = S^{\alpha'} \cap S^\beta$. Fix an $\alpha \in B$ with this property. Then $S^\alpha \cap S^\beta \subseteq R$, because if $I = S^\alpha \cap S^\beta = S^{\alpha'} \cap S^\beta$, then $I \subseteq S^\alpha$ and $I \subseteq S^{\alpha'}$ and consequently $I \subseteq S^\alpha \cap S^{\alpha'} = R$. On the other hand we have $S^\delta \cap S^\beta = R \cap S^\beta = S^\alpha \cap S^\beta$, where the first equality holds because $S^\delta \cap \bigcup_{\alpha<\delta} S^\alpha = R$ and the second holds because $S^\alpha \cap S^\beta \subseteq R$. Now, as $S^\delta \cap S^\beta = S^\alpha \cap S^\beta \subseteq R$, the canonical bijection $\varphi_{\alpha,\delta} \colon S^\alpha \to S^\delta$ extends to a bijection Φ between $S^\alpha \cup S^\beta$ and $S^\delta \cup S^\beta$, where Φ further induces an isomorphism $\Psi \colon (\mathrm{Fn}_{<\kappa}(S^\alpha \cup S^\beta, 2), \subseteq) \to (\mathrm{Fn}_{<\kappa}(S^\delta \cup S^\beta, 2), \subseteq)$ and Ψ itself induces a map $\Psi^* \colon V^{(\mathrm{Fn}_{<\kappa}(S^\alpha \cup S^\beta,2))} \to V^{(\mathrm{Fn}_{<\kappa}(S^\delta \cup S^\beta,2))}$. By the assumption $\Vdash_{\mathbb{C}(\kappa)_\lambda} \forall \alpha, \beta \in \delta\ [|\dot{X}^\alpha \cap \dot{X}^\beta| < \kappa]$ and as S^α (resp. S^β) is the support for $\dot{X}^\alpha$ (resp. $\dot{X}^\beta$), $\Vdash_{\mathrm{Fn}_{<\kappa}(S^\alpha \cup S^\beta,2)} |\dot{X}^\alpha \cap \dot{X}^\beta| < \kappa$ must hold. Then, as Ψ is an isomorphism such that Ψ^* identifies $\dot{X}^\alpha$ with $\dot{X}^\delta$, $\Vdash_{\mathrm{Fn}_{<\kappa}(S^\delta \cup S^\beta,2)} |\dot{X}^\delta \cap \dot{X}^\beta| < \kappa$ is true. So $\Vdash_{\mathbb{C}(\kappa)_\lambda} |\dot{X}^\delta \cap \dot{X}^\beta| < \kappa$, showing that $\{X^\alpha \colon \alpha < \delta\}$ is not maximal in the generic extension. □

Theorem 3.13. (GCH) *Let E be an Easton function such that* $\forall \kappa \in \mathrm{dom}(E)$ $[\sup\{E(\beta) \colon \beta \in \mathrm{dom}(E) \cap \kappa\} \leq \kappa^+]$ *and let $\mathbb{P}(E)$ be the Easton product. Then*

$$V^{\mathbb{P}(E)} \models \forall \kappa \in \mathrm{dom}(E)[\mathfrak{sp}(\mathfrak{a}_\kappa) = \{\kappa^+, E(\kappa)\}].$$

Proof. Let $\kappa \in \mathrm{dom}(E)$ be arbitrary. Consider $\mathbb{P}(E)$ as $\mathbb{P}(E^-_\kappa) \times \mathbb{P}(E^+_\kappa)$. Let K be a $\mathbb{P}(E)$-generic over V. By the Product Lemma, $V[K] = V[H][G]$, where H is $\mathbb{P}(E^+_\kappa)$-generic over V and G is $\mathbb{P}(E^-_\kappa)$-generic over $V[H]$. The poset $(\mathbb{P}(E^-_\kappa))^V$ has the κ^+-c.c. and $(\mathbb{P}(E^+_\kappa))^V$ is κ^+-closed. Furthermore, the closure property of $(\mathbb{P}(E^+_\kappa))^V$ ensures that $(\mathbb{P}(E^-_\kappa))^V = (\mathbb{P}(E^-_\kappa))^{V[H]}$. Consider $V_0 := V[H]$ as the ground model and let δ be a cardinal in V_0 such that $\kappa^+ < \delta < E(\kappa)$.

Define $\mathcal{I}$ to be the index set $\bigcup_{\alpha \leq \kappa} E(\alpha) \times \alpha \times \{\alpha\}$, which is a disjoint union.

Suppose by way of contradiction that $\dot{\mathcal{X}} = \{\dot{X}^\alpha \colon \alpha < \delta\}$ is forced by the maximal element in $\mathbb{P}(E^-_\kappa)$ to be a κ-mad family of size δ in $V_0^{\mathbb{P}(E^-_\kappa)}$. We can identify any $\mathbb{P}(E^-_\kappa)$-name $\dot{X}$ for a κ-real with κ-many maximal antichains $\{A^{\dot{X}}_\beta \colon \beta \in \kappa\}$ such that $A^{\dot{X}}_\beta$ decides "$\check{\beta} \in \dot{X}$" in the generic extension. For such a name $\dot{X}$, let $S^{\dot{X}} = \bigcup_{\alpha \leq \kappa}\{\mathrm{dom}(p(\alpha)) \colon \exists \beta < \kappa\ [p \in A^{\dot{X}}_\beta]\} \subseteq \mathcal{I}$, called the support of $\dot{X}$. By the κ^+-c.c. of $\mathbb{P}(E^-_\kappa)$, each maximal antichain has size at most κ, so $|S^{\dot{X}}| \leq \kappa$ for each name $\dot{X}$ for a κ-real. Now for each $\alpha < \delta$, let S^α be the support of $\dot{X}^\alpha$.

Let θ be large enough that $\mathbb{P}(E^-_\kappa) \in H(\theta)$ and $V_0 \vDash \mathrm{cof}(\theta) > |\mathbb{P}(E^-_\kappa)|$. Let $\mathcal{M} \preceq H(\theta)$ be an elementary submodel such that $|M| = \kappa^+$, $\kappa^+ \subseteq M$, $M^\kappa \subseteq M$, $\{E(\alpha)\colon \alpha \leq \kappa\} \subseteq M$, $\mathbb{P}(E^-_\kappa) \in M$, $\forall \alpha < \kappa \cap \mathrm{dom}(E)$ $[E(\alpha) \times \alpha \times \{\alpha\} \subseteq M]$ and M contains all other relevant parameters. The hypothesis of the theorem is used here in order to ensure the property $\forall \alpha < \kappa \cap \mathrm{dom}(E)$ $[E(\alpha) \times \alpha \times \{\alpha\} \subseteq M]$, which makes the choice of the permutation of the index set possible (in the next paragraph) and makes it easy to find the desired automorphism of the forcing.

Let $\bar{\alpha} \in \delta \setminus M$. Now fix a permutation φ of the index set $\mathcal{I}$ with $\varphi \restriction (E(\alpha) \times \alpha \times \{\alpha\}) = E(\alpha) \times \alpha \times \{\alpha\}$ (for each $\alpha \leq \kappa$) which maps the $\leq \kappa$-sized set $[S^{\bar{\alpha}} \cap (E(\alpha) \times \alpha \times \{\alpha\})] \setminus M$ into $[(E(\alpha) \times \alpha \times \{\alpha\}) \setminus \bigcup_{i<\delta} S^i] \cap M$ (otherwise fixing elements of $E(\alpha) \times \alpha \times \{\alpha\}$). This permutation φ of the index set induces an automorphism $\pi : \mathbb{P}(E^-_\kappa) \to \mathbb{P}(E^-_\kappa)$ of the poset. As names are defined recursively, $\pi \in \mathrm{Aut}(\mathbb{P}(E^-_\kappa))$ induces a map $\pi^*\colon V_0^{(\mathbb{P}(E^-_\kappa))} \to V_0^{(\mathbb{P}(E^-_\kappa))}$ (where $V_0^{(\mathbb{P}(E^-_\kappa))}$ denotes the class of all $\mathbb{P}(E^-_\kappa)$-names) by $\pi^*(\tau) = \{\langle \pi^*(\sigma), \pi(p)\rangle \colon \langle \sigma, p\rangle \in \tau\}$. The automorphism π preserves antichains and the forcing relation. And as $\dot{X}^{\bar{\alpha}}$ is supposed to be a nice name, and any antichain of $\mathbb{P}(E^-_\kappa)$ is of size $\leq \kappa$ and M is closed w.r.t. κ-sequences, we have $\pi^*(\dot{X}^{\bar{\alpha}}) \in M$.

Let G be a generic filter. Then $\pi''(G)$ is a generic filter. It is well-known that $\mathcal{M}[\pi''(G)] \preceq ((H(\theta))^{V_0[\pi''(G)]}, \in)$ (see [11, Theorem III.2.11.]). As $\dot{\mathcal{X}}$ is forced to be κ-mad, we have

$$\Vdash_{\pi(\mathbb{P}(E^-_\kappa))} \forall x \in {}^\kappa\kappa \; \exists \beta < \delta \; [|x \cap \dot{X}^\beta| = \kappa].$$

We can relativize the statement to $H(\theta)$, so

$$\Vdash_{\pi(\mathbb{P}(E^-_\kappa))} \forall x \in {}^\kappa\kappa \cap H(\theta) \; \exists \beta < \delta \cap H(\theta) \; [|x \cap \dot{X}^\beta| = \kappa].$$

But $\mathcal{M}[\pi''(G)] \preceq ((H(\theta))^{V_0[\pi''(G)]}, \in)$ and $M \cap \mathit{Ord} = M[\pi''(G)] \cap \mathit{Ord}$, so

$$\Vdash_{\pi(\mathbb{P}(E^-_\kappa))} \forall x \in {}^\kappa\kappa \cap M \; \exists \beta < \delta \cap M \; [|x \cap \dot{X}^\beta| = \kappa].$$

As $\pi^*(\dot{X}^{\bar{\alpha}})$ was in $M \subseteq \mathcal{M}[\pi''(G)]$, we have

$$\Vdash_{\pi(\mathbb{P}(E^-_\kappa))} \exists \beta < \delta \cap M \; [|\pi^*(\dot{X}^{\bar{\alpha}}) \cap \dot{X}^\beta| = \kappa].$$

However $\pi^*(\dot{X}^\beta) = \dot{X}^\beta$ for ordinals $\beta \in M$ as the permutation π fixes the ordinals mentioned in $\dot{X}^\beta$ for $\beta \in M$. Therefore we have

$$\Vdash_{\pi(\mathbb{P}(E^-_\kappa))} \exists \beta < \delta \cap M \; [|\pi^*(\dot{X}^{\bar{\alpha}}) \cap \pi^*(\dot{X}^\beta)| = \kappa]$$

and by applying π^{-1} we have

$$\Vdash_{\mathbb{P}(E^-_\kappa)} \exists \beta < \delta \cap M \; [|\dot{X}^{\bar{\alpha}} \cap \dot{X}^\beta| = \kappa],$$

contradicting the κ-madness of $\dot{\mathcal{X}}$ in the generic extension. □

4. Global Spectra

In this section we show that the spectrum of κ-mad families at successors of regular cardinals together with $\aleph_0$ can be forced to be any prescribed family of κ-Blass spectra. We first give a lemma which we use later.

Lemma 4.1. Let κ be a regular cardinal. Any κ-c.c. forcing poset $\mathbb{Q}$ preserves κ-mad families.

Proof. Suppose $\mathcal{X} = \{X_i \in [\kappa]^\kappa \colon i < \lambda\}$ is a kappa-mad family in the ground model. Suppose by way of contradiction that a condition $p \in \mathbb{Q}$ forces that $\dot{A}$ is unbounded in κ and almost disjoint from X_i for each $i \in \lambda$, i.e. $p \Vdash \dot{A} \in [\kappa]^\kappa \wedge \forall i < \lambda\ [|\dot{A} \cap \check{X}_i| < \kappa]$. Let $X = \{\alpha \in \kappa \colon \exists q \leq p\ [q \Vdash \alpha \in \dot{A}]\}$. Then X is in the ground model and is unbounded in κ. But as $\mathbb{Q}$ is κ-cc, for each $i < \lambda$ there is $\alpha_i < \kappa$ such that $p \Vdash \dot{A} \cap X_i \subseteq \alpha_i$. It follows that $X \cap X_i$ is also bounded by α_i for each i, contradicting the maximality of $\mathcal{X}$ in the ground model. □

Next, we simultaneously add, for each regular cardinal κ of a class C, κ-mad families of sizes determined by closed sets $C(\kappa)$. Since (for now) we are aiming only to add (and not to exclude) sizes of κ-mad families, we do not have to require the sets $C(\kappa)$ to be κ-Blass spectra.

Definition 4.2. Let C be a class of regular cardinals, and for each $\kappa \in C$ let $C(\kappa)$ be a closed set of cardinals such that $\min(C(\kappa)) \geq \kappa^+$, $\mathrm{cof}(\max(C(\kappa))) > \kappa$ and $\forall \kappa, \kappa' \in C\ [\kappa < \kappa' \rightarrow \max(C(\kappa)) \leq \max(C(\kappa'))]$.

(1) For each $\kappa \in C$ and any well-ordered set ξ let $\mathcal{I}_{\kappa,\xi} = \{(\kappa, \xi, \eta) : \eta < \xi\}$ be an index set of cardinality $|\xi|$ ensuring that $\mathcal{I}_{\kappa_1,\xi_1} \cap \mathcal{I}_{\kappa_2,\xi_2} = \emptyset$ whenever $\kappa_1 \neq \kappa_2$ or $\xi_1 \neq \xi_2$ where $\kappa_1, \kappa_2 \in C$, $\xi_1 \in C(\kappa_1), \xi_2 \in C(\kappa_2)$.

(2) For a cardinal α and a well-ordered set β, let the poset $\mathbb{Q}_{\mathcal{I}_{\alpha,\beta}}$ consists of all functions $p \colon \Delta^p \rightarrow [\alpha]^{<\alpha}$ such that Δ^p is in $[\mathcal{I}_{\alpha,\beta}]^{<\alpha}$ and $q \leq p$ iff:
   - $\Delta^p \subseteq \Delta^q$ and $\forall x \in \Delta^p\ [q(x) \supseteq p(x)]$,
   - whenever (α, β, η_1) and (α, β, η_2) are distinct elements of Δ^p then
   $$q(\alpha, \beta, \eta_1) \cap q(\alpha, \beta, \eta_2) \subseteq p(\alpha, \beta, \eta_1) \cap p(\alpha, \beta, \eta_2).$$

(3) For each $\kappa \in C$, let $\mathbb{P}(C(\kappa)) = \prod^{<\kappa}_{\xi \in C(\kappa)} \mathbb{Q}_{\mathcal{I}_{\kappa,\xi}}$ be the product with supports of size less than κ.

(4) The forcing poset $\mathbb{P}(C)$ consists of elements $p \in \prod_{\kappa \in C} \mathbb{P}(C(\kappa))$ with Easton support, i.e. such that for every regular cardinal λ we have $|\{\alpha \in \lambda \cap C \colon p(\kappa) \neq \mathbb{1}\}| < \lambda$.

Lemma 4.3. Suppose λ is a regular cardinal and $\lambda^{<\lambda} = \lambda$. Let $C \subseteq \lambda^+$ and let $C(\kappa)$ (for each $\kappa \in C$) and $\mathbb{P}(C)$ be as in Definition 4.2. Then $\mathbb{P}(C)$ has the λ^+-c.c..

Proof. Let $D = \{p_\alpha \colon \alpha < \lambda^+\} \subseteq \mathbb{P}(C)$ be a set of conditions; we have to show that D is not an antichain. For each $\alpha \in \lambda^+$, let $D_\alpha := \bigcup\{\mathrm{dom}(p_\alpha(\kappa)(\delta)) \colon \kappa \in C, \delta \in C(\kappa)\}$. By definition of the conditions in $\mathbb{P}(C)$, $|D_\alpha| < \lambda$ for each $\alpha < \lambda^+$. By the assumption $\lambda^{<\lambda} = \lambda$, we can apply the Δ-System Lemma and conclude that there is a set $A \in [\lambda^+]^{\lambda^+}$ and a root R such that $\forall \alpha, \beta \in A\ [\alpha \neq \beta \rightarrow D_\alpha \cap D_\beta = R]$. However, as $\lambda^{|R|} \leq \lambda^{<\lambda} = \lambda < \lambda^+$, there must exist

$\alpha', \beta' \in A$, such that $\alpha' \neq \beta'$ and for each $(\kappa, \delta, \gamma) \in R$, $p_{\alpha'}(\kappa)(\delta)(\kappa, \delta, \gamma) = p_{\beta'}(\kappa)(\delta)(\kappa, \delta, \gamma)$. This implies that $p_{\alpha'} \not\perp p_{\beta'}$ showing that D is not an antichain. □

Lemma 4.4. If C, $\{C(\kappa)\colon \kappa \in C\}$ and $\mathbb{P}(C)$ are as in Definition 4.2 and λ is an ordinal, then $\mathbb{P}(C) \cong \mathbb{P}(C^+_\lambda) \times \mathbb{P}(C^-_\lambda)$.

Lemma 4.5. (GCH) If C, $\{C(\kappa)\colon \kappa \in C\}$ and $\mathbb{P}(C)$ are as in Definition 4.2, then $\mathbb{P}(C)$ preserves cardinals.

Proof. It suffices to show that any regular uncountable cardinal δ of the ground model V, remains regular in $V[K]$, where K is $\mathbb{P}(C)$-generic over V. Suppose by way of contradiction that there is a cardinal δ such that $\gamma = (\mathrm{cof}(\delta))^{V[K]} < \delta$. As cofinalities are regular and regularity is downwards absolute, γ is regular in $V[K]$ and V. Let $f \in V[K]$ be such that $f\colon \gamma \to \delta$ and $\sup(\mathrm{ran}(f)) = \delta$. By Lemma 4.4 and the Product Lemma, $V[K] = V[H][G]$ holds, where H is $\mathbb{P}(C^+_\gamma)^V$-generic over V and G is $\mathbb{P}(C^-_\gamma)^V$-generic over $V[H]$. However, as $\mathbb{P}(C^+_\gamma)^V$ is γ^+-closed in V, $V \vDash \mathrm{GCH}$ and γ is regular, $\gamma^{<\gamma} = \gamma$ holds in $V[H]$ and $\mathbb{P}(C^-_\gamma)^V = \mathbb{P}(C^-_\gamma)^{V[H]}$. So by Lemma 4.3, $\mathbb{P}(C^-_\gamma)^V$ has the γ^+-c.c. in $V[H]$. By the Approximation Lemma (see [7, Lemma IV.7.8]) there is a function $F \in V[H]$ such that $F\colon \gamma \to \mathcal{P}(\delta)$ and $\forall \xi \in \gamma\ [f(\xi) \in F(\xi) \wedge (|F(\xi)| \leq \gamma)^{V[H]}]$. However, $\mathbb{P}(C^+_\gamma)^V$ was γ^+-closed in V, so $F \in V$ and $\forall \xi \in \gamma\ [f(\xi) \in F(\xi) \wedge (|F(\xi)| \leq \gamma)^V]$. This is contradicting the regularity of γ in V, because $|\bigcup_{\xi<\gamma} F(\gamma)| \leq \gamma$ and $\sup(\bigcup_{\xi<\gamma} F(\gamma)) = \delta$. □

Theorem 4.6. *Let C, $\{C(\kappa)\colon \kappa \in C)\}$ and $\mathbb{P}(C)$ be as in Definition 4.2. Then:*

$$V^{\mathbb{P}(C)} \vDash \forall \kappa \in C\ [\mathfrak{sp}(\mathfrak{a}_\kappa) \supseteq C(\kappa)].$$

Proof. Let K be $\mathbb{P}(C)$-generic over the ground model. For each $\kappa \in C$, $\delta \in C(\kappa)$ and $\xi \in \delta$, let $A^\kappa_{\delta,\xi} = \bigcup\{p(\kappa)(\delta)(\kappa, \delta, \xi)\colon p \in K\}$. For each $\kappa \in C$ and $\delta \in C(\kappa)$ let $\mathcal{A}^\kappa_\delta = \{A^\kappa_{\delta,\xi}\colon \xi \in \delta\}$. We show that for each $\kappa \in C$ and $\delta \in C(\kappa)$, $V^{\mathbb{P}(C)} \vDash \mathcal{A}^\kappa_\delta$ is κ-mad. Let $\kappa \in C$ and $\delta \in C(\kappa)$ be fixed.

The set $\mathcal{A}^\kappa_\delta$ is almost disjoint: Let $\alpha, \beta \in \delta$ and $\alpha \neq \beta$. The conditions $p \in \mathbb{P}(C)$ such that $(\kappa, \delta, \alpha), (\kappa, \delta, \beta) \in \Delta^{p(\kappa)(\delta)}$ are dense in $\mathbb{P}(C)$. So there is $q \in K$ such that $(\kappa, \delta, \alpha), (\kappa, \delta, \beta) \in \Delta^{q(\kappa)(\delta)}$. Then $A^\kappa_{\delta,\alpha} \cap A^\kappa_{\delta,\beta} = p(\kappa)(\delta)(\kappa, \delta, \alpha) \cap p(\kappa)(\delta)(\kappa, \delta, \beta)$, which is of size $<\kappa$.

Furthermore, $\mathcal{A}^\kappa_\delta$ is maximal: Let $\dot{X}$ be a $\mathbb{P}(C)$-name for an element in $[\kappa]^\kappa$. Again by Lemma 4.4, $\mathbb{P}(C) \cong \mathbb{P}(C^+_\kappa) \times \mathbb{P}(C^-_\kappa)$, $V[K] = V[H][G]$, where H is $\mathbb{P}(C^+_\kappa)^V$-generic over V and G is $\mathbb{P}(C^-_\kappa)^V$-generic over $V[H]$. By the same reason as in the previous proof, $\mathbb{P}(C^-_\kappa)^V = \mathbb{P}(C^-_\kappa)^{V[H]}$ and $\mathbb{P}(C^-_\kappa)^V$ has the κ^+-c.c. in $V[H]$. The first part $\mathbb{P}(C^+_\kappa)$ is κ^+-closed in V, so it does not add new subsets of κ. Hence it suffices to show that $\mathcal{A}^\kappa_\delta$ is κ-mad in the extension by $\mathbb{P}(C^-_\kappa)$ regarding $V[H]$ as the ground model. By the κ^+-c.c. of $\mathbb{P}(C^-_\kappa)$, $\dot{X}$ involves only $\leq\kappa$-many conditions and $\delta \geq \kappa^+$. So there is an $(\kappa, \delta, \alpha) \notin \Delta^{p'(\kappa)(\delta)}$ for any condition p' involved in $\dot{X}$. We show that $V[H][G] \vDash |X \cap A^\kappa_{\delta,\alpha}| = \kappa$, which will finish the proof.

Suppose that there is a $\gamma < \kappa$ and a condition $p \in G$ such that $p \Vdash \dot{X} \cap \dot{A}^\kappa_{\delta,\alpha} \subseteq \gamma$.

Recall that $|\Delta^{p(\kappa)(\delta)}| < \kappa$ and $p(\kappa)(\delta)\colon \Delta^{p(\kappa)(\delta)} \to [\kappa]^{<\kappa}$. Let $q \in G$ be a condition involved in $\dot{X}$ such that for some $\rho > \gamma$ and

$$\rho > \bigcup\{p(\kappa)(\delta)(\kappa, \delta, \mu)\colon (\kappa, \delta, \mu) \in \Delta^{p(\kappa)(\delta)}\}, \qquad (*)$$

$q \Vdash \check{\rho} \in \dot{X}$. As $p, q \in G$, p and q are compatible. Now consider the condition $r \in \mathbb{P}(C_\kappa^-)$ defined as follows:

- $\operatorname{supp}(r) = \operatorname{supp}(q) \cup \operatorname{supp}(p) \cup \{\kappa\}$
- $\operatorname{supp}(r(\eta)) = \begin{cases} \operatorname{supp}(p(\eta)) \cup \operatorname{supp}(q(\eta)) \cup \{\delta\} & \text{for } \eta = \kappa \\ \operatorname{supp}(p(\eta)) \cup \operatorname{supp}(q(\eta)) & \text{for } \eta \in \operatorname{supp}(r) \setminus \{\kappa\} \end{cases}$
- $\Delta^{r(\eta)(\theta)} = \begin{cases} \Delta^{p(\eta)(\theta)} \cup \Delta^{q(\eta)(\theta)} \cup \{(\kappa, \delta, \alpha)\} & \text{if } \eta = \kappa \wedge \theta = \delta \\ \Delta^{p(\eta)(\theta)} \cup \Delta^{q(\eta)(\theta)} & \text{if } \eta \in \operatorname{supp}(r), \theta \in \operatorname{supp}(r(\eta)), (\eta, \theta) \neq (\kappa, \delta) \end{cases}$

Furthermore, $r(\kappa)(\delta)(\kappa, \delta, \alpha) = p(\kappa)(\delta)(\kappa, \delta, \alpha) \cup \{\rho\}$ (note that $(\kappa, \delta, \alpha) \notin \Delta^{q(\kappa)(\delta)}$ by its choice) and $\forall \eta \in \operatorname{supp}(r)\ \forall \theta \in \operatorname{supp}(r(\eta))\ \forall (\eta, \theta, \mu) \in \Delta^{r(\eta)(\theta)}$ $[(\eta, \theta, \mu) \neq (\kappa, \delta, \alpha) \to r(\eta)(\theta)(\eta, \theta, \mu) = p(\eta)(\theta)(\eta, \theta, \mu) \cup q(\eta)(\theta)(\eta, \theta, \mu)]$. Now r extends both p (by $(*)$) and q and $r \Vdash \rho \in \dot{X}$ (as $r \leq q$) and $r \Vdash \rho \in \dot{A}^\kappa_{\delta,\alpha}$ contradicting that $r \Vdash \dot{B} \cap \dot{A}^\kappa_{\delta,\alpha} \subseteq \gamma$ (as $r \leq p$ and $\rho > \gamma$). ☐

Remark 4.7. One can show by a counting nice names argument that in Theorem 4.6, also $V^{\mathbb{P}(C)} \models \forall \kappa \in C\ [\mathfrak{c}_\kappa = \max(C(\kappa))]$ holds.

Now we start with the exclusion of values. In order to do this we will replace the closed sets $C(\kappa)$ by κ-Blass spectra $B(\kappa)$. We first give a lemma.

Lemma 4.8.

(1) Let λ be a regular cardinal. If $\beta \leq \alpha$ are two ordinals, then $\mathbb{Q}_{\mathcal{I}_{\lambda,\beta}} \lessdot \mathbb{Q}_{\mathcal{I}_{\lambda,\alpha}}$.

(2) Let λ be a regular cardinal. If X is an index set and $C, D\colon X \to \text{Card}$ such that $\forall x \in X\ [C(x) \leq D(x)]$, then $\prod^{<\lambda}_{\xi \in X} \mathbb{Q}_{\mathcal{I}_{\lambda,C(\xi)}} \lessdot \prod^{<\lambda}_{\xi \in X} \mathbb{Q}_{\mathcal{I}_{\lambda,D(\xi)}}$.

(3) If C is a set of regular cardinals, and for each $\lambda \in C$, $C_\lambda, D_\lambda\colon X_\lambda \to \text{Card}$ are two functions on some index set X_λ such that $\forall x \in X_\lambda\ [C_\lambda(x) \leq D_\lambda(x)]$, then the Easton supported product $\prod_{\lambda \in C} \prod^{<\lambda}_{\xi \in X_\lambda} \mathbb{Q}_{\mathcal{I}_{\lambda,C_\lambda(\xi)}}$ is a complete suborder of the Easton supported product $\prod_{\lambda \in C} \prod^{<\lambda}_{\xi \in X_\lambda} \mathbb{Q}_{\mathcal{I}_{\lambda,D_\lambda(\xi)}}$.

Proof. (1) Recall the definition of $\mathbb{Q}_{\mathcal{I}_{\lambda,\alpha}}$ for an ordinal α $(> \lambda)$. It is known that this forcing can be decomposed in a two-step iteration as follows: Let $\beta \leq \alpha$ and let G be a $\mathbb{Q}_{\mathcal{I}_{\lambda,\beta}}$-generic over the ground model V and let $\mathcal{A} = \{A_i \colon i < \beta\}$ be the (maximal) almost disjoint family added by $\mathbb{Q}_{\mathcal{I}_{\lambda,\beta}}$. In $V[G]$ let $\mathbb{R}_{\mathcal{I}_{\lambda,\alpha\setminus\beta}}$ consist of pairs (p, H), where $p\colon \Delta^p \to [\lambda]^{<\lambda}$ such that $\Delta^p \in [\mathcal{I}_{\lambda,\alpha\setminus\beta}]^{<\lambda}$, $H \in [\beta]^{<\lambda}$ with $(p, H) \leq (q, K)$ iff $p \leq_{\mathbb{Q}_{\mathcal{I}_{\lambda,\alpha}}} q$, $K \subseteq H$ and for every $j \in \Delta^q$ and $i \in K$, $p(\lambda, \alpha, j) \cap A_i \subseteq q(\lambda, \alpha, j) \cap A_i$ holds. Then $\mathbb{Q}_{\mathcal{I}_{\lambda,\alpha}} \simeq \mathbb{Q}_{\mathcal{I}_{\lambda,\beta}} * \dot{\mathbb{R}}_{\mathcal{I}_{\lambda,\alpha\setminus\beta}}$.

(2) We make a similar observation for products. Let λ be a regular cardinal and let C and D be functions on the same index set X as in the assumption of (2). Then $\prod^{<\lambda}_{\xi \in X} \mathbb{Q}_{\mathcal{I}_{\lambda,C(\xi)}}$ is a complete suborder of $\prod^{<\lambda}_{\xi \in X} \mathbb{Q}_{\mathcal{I}_{\lambda,D(\xi)}}$, as the later can be decomposed as follows: Let G be a $\prod^{<\lambda}_{\xi \in X} \mathbb{Q}_{\mathcal{I}_{\lambda,C(\xi)}}$-generic over V. In $V[G]$ consider the product $P' := \prod^{<\lambda}_{i \in X} \mathbb{R}_{\mathcal{I}_{\lambda,D(i)\setminus C(i)}}$. Then $\prod^{<\lambda}_{\xi \in X} \mathbb{Q}_{\mathcal{I}_{\lambda,D(\xi)}} \simeq \prod^{<\lambda}_{\xi \in X} \mathbb{Q}_{\mathcal{I}_{\lambda,C(\xi)}} * \dot{P}'$.

(3) Finally, if we have a set C of regular cardinals and for each $\lambda \in C$ two closed sets of cardinals C_λ and D_λ as in the assumption of (3). Then we have that the Easton supported product $\prod_{\lambda \in C} \prod^{<\lambda}_{\xi \in X_\lambda} \mathbb{Q}_{\mathcal{I}_{\lambda,C_\lambda(\xi)}} =: P$ is a complete suborder of the Easton supported product

$\prod_{\lambda\in C}\prod^{<\lambda}_{\xi\in X_\lambda}\mathbb{Q}_{\mathcal{I}_{\lambda,D_\lambda(\xi)}} =: Q$. Let G be a P-generic over V. In $V[G]$ consider the Easton supported product $P' := \prod_{\lambda\in C}\prod^{<\lambda}_{\xi\in X_\lambda}\mathbb{R}_{\mathcal{I}_{\lambda,D_\lambda(\xi)\setminus C_\lambda(\xi)}}$. Then $Q \simeq P * \dot{P}'$. □

Theorem 4.9. (GCH) *Let C be the class of successors of regular cardinals together with $\aleph_0$ and $\{B(\kappa)\colon \kappa\in C\}$ be a family of κ-Blass spectra. Let $\mathbb{P}(C)$ be as in Definition 4.2. Then,*

$$V^{\mathbb{P}(C)} \vDash \forall\kappa\in C\ [\mathfrak{sp}(\mathfrak{a}_\kappa) = B(\kappa)].$$

Proof. First, the positive requirement, i.e. the requirement that in the generic extension there is for each $\kappa\in C$ and $\delta\in B(\kappa)$ a κ-mad family of size δ, is done by Theorem 4.6.

Second, the negative requirement is verified: Fix $\kappa\in C$. We show that there is no κ-mad family of size $\lambda\notin B(\kappa)$ in the final extension. Note that $\mathbb{P}(C)\cong\mathbb{P}(C^-_\kappa)\times\mathbb{P}(C^+_\kappa)$ and $\mathbb{P}(C^+_\kappa)$ is κ^+-closed, hence does not add new κ-reals. So, by considering $V^{\mathbb{P}(C^+_\kappa)}$ as the ground model, it is sufficient to show

$$V^{\mathbb{P}(C^-_\kappa)} \vDash \text{“there are no } \kappa\text{-mad families of size } \lambda\text{”}. \tag{1}$$

For this, we show that

$$V^{\mathbb{P}'(C^-_\kappa)} \vDash \text{“there are no } \kappa\text{-mad families of size } \lambda\text{”}. \tag{2}$$

for a suitable $\mathbb{P}(C^-_\kappa) \lessdot \mathbb{P}'(C^-_\kappa)$. By use of Lemma 4.1, we will argue that (2) implies (1). Also note that $\mathbb{P}(C^+_\kappa)$ preserves GCH at and below κ (as $\mathbb{P}(C^+_\kappa)$ is κ^+-closed and does not add new sequences of length $\leq\kappa$).

Let λ' be greater than λ, $\max(B(\kappa))$ and $\max(B(\bar\kappa))$ for every $\bar\kappa\in C\cap\kappa$. In $\mathbb{P}(C^-_\kappa)$ replace $\mathbb{Q}^{\bar\kappa}_\xi$ ($\bar\kappa\in C\cap\kappa$, $\xi\in B(\bar\kappa)$) by $\mathbb{Q}^{\bar\kappa}_{\lambda'}$. This gives us $\mathbb{P}'(C^-_\kappa)$. Now we have to verify (2).

Let $\lambda\notin B(\kappa)$. Define μ to be $\max(B(\kappa)\cap\lambda)$. Note that $\mathrm{cof}(\mu)>\kappa$ (by Definition 2.1(2)) and $|B(\kappa)|\leq\mu$ (by Definition 2.1(3)).

Suppose by way of contradiction that $\dot{\mathcal{A}} = \{\dot a_\alpha\colon \alpha<\lambda\}$ is forced by the maximal element in $\mathbb{P}'(C^-_\kappa)$ to be a κ-mad family of size λ in $V^{\mathbb{P}'(C^-_\kappa)}$. We may assume that each $\dot a_\alpha$ is a nice name.

We identify a nice name $\dot x$ for a κ-real with κ-many maximal antichains $\{A_\alpha(\dot x)\}_{\alpha<\kappa}$ each of cardinality κ, such that the conditions in $A_\alpha(\dot x)$ decide “$\check\alpha\in\dot x$”. We refer to $\Delta(\dot x) = \bigcup_{\alpha\in\kappa} A_\alpha(\dot x)$ as the *set of conditions involved in* $\dot x$. The set

$$J(\dot x) = \bigcup_{p\in\Delta(\dot x)}\ \bigcup_{\xi\in\mathrm{supp}(p)}\ \bigcup_{\beta\in\mathrm{supp}(p(\xi))} \Delta^{p(\xi)(\beta)}$$

is called the *support of* $\dot x$.

For each $\alpha\in\lambda$ let J_α be the support of $\dot a_\alpha$.

Let θ be large enough that $\mathbb{P}'(C^-_\kappa)\in H(\theta)$ and $V\vDash \mathrm{cof}(\theta) > |\mathbb{P}'(C^-_\kappa)|$. Let $\mathcal{M}\preceq H(\theta)$ be an elementary submodel such that $|M|=\mu$, $\mu\subseteq M$, $M^\kappa\subseteq M$, $C^-_\kappa\subseteq M$, $B(\kappa)\subseteq M$, $\lambda'\in M$, $\mathbb{P}'(C^-_\kappa)\in M$ and M contains all other relevant parameters.

Let $\bar\alpha\in\lambda\setminus M$. Fix a permutation of the index set $\mathcal{I} = \bigcup_{\xi\in C^-_\kappa}\bigcup_{\beta\in B(\xi)}\mathcal{I}_{\xi,\beta}$ which fixes $\mathcal{I}_{\kappa,\beta}$ for $\beta\leq\mu$, and for $\xi\neq\kappa\vee\beta>\mu$ maps the $\leq\kappa$-sized set $J_{\bar\alpha}\cap\mathcal{I}_{\xi,\beta}\setminus M$ into $(\mathcal{I}_{\xi,\beta}\setminus\bigcup_{i<\lambda}J_i)\cap M$ (otherwise fixing elements of $\mathcal{I}_{\xi,\beta}$). Such a permutation of the index set exists, because if $\beta>\mu$, then $\beta>\lambda$ as well. Consequently $|\bigcup_{i<\lambda}J_i| = \lambda*\kappa=\lambda$, and $|\mathcal{I}_{\kappa,\beta}\setminus\bigcup_{i<\lambda}J_i| = \beta>\kappa$ holds in

$H(\theta)$ and by elementarity also in $\mathcal{M}$. The same holds if $\xi \neq \kappa$, because we enlarged the index set to λ', i.e. $|\mathcal{I}_{\xi,\lambda'} \setminus \bigcup_{i<\lambda} J_i| = \lambda' > \kappa$. This permutation of the index set $\mathcal{I}$ induces an automorphism $\pi : \mathbb{P}'(C_\kappa^-) \to \mathbb{P}'(C_\kappa^-)$ of the poset. As names are defined recursively, $\pi \in \mathrm{Aut}(\mathbb{P}'(C_\kappa^-))$ induces a map $\pi^*\colon V^{(\mathbb{P}'(C_\kappa^-))} \to V^{(\mathbb{P}'(C_\kappa^-))}$ (where $V^{(\mathbb{P}'(C_\kappa^-))}$ denotes the class of all $\mathbb{P}'(C_\kappa^-)$-names) by $\pi^*(\tau) = \{\langle \pi^*(\sigma), \pi(p)\rangle \colon \langle \sigma, p\rangle \in \tau\}$. The automorphism π preserves antichains and the forcing relation. And as $\dot{a}_{\bar{\alpha}}$ is supposed to be a nice name, and any antichain of $\mathbb{P}'(C_\kappa^-)$ is of size $\leq \kappa$ (by the κ^+-c.c. of $\mathbb{P}'(C_\kappa^-)$) and M is closed w.r.t. κ-sequences, we have $\pi^*(\dot{a}_{\bar{\alpha}}) \in M$.

Let G be a generic filter. Then $\pi''(G)$ is a generic filter. It is well-known that $\mathcal{M}[\pi''(G)] \preceq ((H(\theta))^{V[\pi''(G)]}, \in)$ (see [11, Theorem III.2.11.]). As $\dot{\mathcal{A}}$ is forced to be κ-mad, we have

$$\Vdash_{\pi(\mathbb{P}'(C_\kappa^-))} \forall x \in {}^\kappa\kappa\ \exists \beta < \lambda\ [|x \cap \dot{a}_\beta| = \kappa].$$

We can relativize the statement to $H(\theta)$, so

$$\Vdash_{\pi(\mathbb{P}'(C_\kappa^-))} \forall x \in {}^\kappa\kappa \cap H(\theta)\ \exists \beta < \lambda \cap H(\theta)\ [|x \cap \dot{a}_\beta| = \kappa].$$

But $\mathcal{M}[\pi''(G)] \preceq ((H(\theta))^{V[\pi''(G)]}, \in)$ and $M \cap Ord = M[\pi''(G)] \cap Ord$, so

$$\Vdash_{\pi(\mathbb{P}'(C_\kappa^-))} \forall x \in {}^\kappa\kappa \cap M\ \exists \beta < \lambda \cap M\ [|x \cap \dot{a}_\beta| = \kappa].$$

As $\pi^*(\dot{a}_{\bar{\alpha}})$ was in $M \subseteq \mathcal{M}[\pi''(G)]$, we have

$$\Vdash_{\pi(\mathbb{P}'(C_\kappa^-))} \exists \beta < \lambda \cap M\ [|\pi^*(\dot{a}_{\bar{\alpha}}) \cap \dot{a}_\beta| = \kappa].$$

However $\pi^*(\dot{a}_\beta) = \dot{a}_\beta$ for ordinals $\beta \in M$ as the permutation π fixes the ordinals mentioned in $\dot{a}_\beta$ for $\beta \in M$. Therefore we have

$$\Vdash_{\pi(\mathbb{P}'(C_\kappa^-))} \exists \beta < \lambda \cap M\ [|\pi^*(\dot{a}_{\bar{\alpha}}) \cap \pi^*(\dot{a}_\beta)| = \kappa]$$

and by applying π^{-1} we have

$$\Vdash_{\mathbb{P}'(C_\kappa^-)} \exists \beta < \lambda \cap M\ [|\dot{a}_{\bar{\alpha}} \cap \dot{a}_\beta| = \kappa],$$

contradicting the κ-madness of $\dot{\mathcal{A}}$ in the generic extension and verifying (2).

However, (2) implies (1): If $\mathbb{P}(C_\kappa^-)$ did add a κ-mad family of an undesired size, this κ-mad family would be preserved, by Lemma 4.1, in the extension by $\mathbb{P}'(C_\kappa^-)$ since the quotient of $\mathbb{P}'(C_\kappa^-)$ over $\mathbb{P}(C_\kappa^-)$ is κ-c.c (here we use that κ is the successor of a regular cardinal or equal to $\aleph_0$). However we showed that there is no κ-mad family of an undesired size in the extension by $\mathbb{P}'(C_\kappa^-)$. □

5. Questions

We conclude the paper, with some remaining open questions. It still remains of interest, if the result in Theorem 4.9 still holds if the assumption of being successor of a regular for elements of the intended spectrum at κ is omitted. More precisely one can ask:

Question 5.1. Let C be a class of regular cardinals and $\{B(\kappa) \colon \kappa \in C\}$ be a family of κ-Blass spectra. Is there a cardinal-preserving forcing extension satisfying $\forall \kappa \in C\ [\mathfrak{sp}(\mathfrak{a}_\kappa) = B(\kappa)]$?

It is still open which sets of cardinals can be realized as the spectrum of $\aleph_0$-madness. Not all of the requirements given by the notion of a Blass-spectrum are in general necessary (see [12]), and in fact giving a characterization of those sets which can be realized as $\mathfrak{sp}(\mathfrak{a})$ remains open:

Question 5.2. When can a set of cardinals be realized as $\mathfrak{sp}(\mathfrak{a})$ in a cardinal preserving extension?

Finally, concerning Theorems 3.10 and 3.13 one can ask:

Question 5.3. Is $\mathfrak{a}_\kappa = \kappa^+ = \mathfrak{b}_\kappa < \mathfrak{d}_\kappa = \mathfrak{c}_\kappa$ or $\mathfrak{sp}(\mathfrak{a}_\kappa) = \{\kappa^+, 2^\kappa\}$ consistent globally in the presence of large cardinals?

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