# HIGHER INDEPENDENCE

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Abstract. We study higher analogues of the classical independence number on $\omega$. For $\kappa$ regular uncountable, we denote by $i(\kappa)$ the minimal size of a maximal $\kappa$-independent family. We establish ZFC relations between $i(\kappa)$ and the standard higher analogues of some of the classical cardinal characteristics, e.g. $\mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$ and $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$.

For $\kappa$ measurable, assuming that $2^\kappa = \kappa^+$ we construct a maximal $\kappa$-independent family which remains maximal after the $\kappa$-support product of $\lambda$ many copies of $\kappa$-Sacks forcing. Thus, we show the consistency of $\kappa^+ = \mathfrak{d}(\kappa) = \mathfrak{i}(\kappa) < 2^\kappa$. We conclude the paper with interesting open questions and discuss difficulties regarding other natural approaches to higher independence.

## 1. Introduction

A family $\mathcal{A}$ contained in $[\omega]^\omega$ is said to be independent if for every two finite disjoint subfamilies $\mathcal{B}$ and $\mathcal{C}$ the set $\bigcap \mathcal{B} \setminus \bigcup \mathcal{C}$ is infinite. We refer to such sets as boolean combinations. The least size of a maximal (under inclusion) independent family is denoted $\mathfrak{i}$. For an excellent introduction to the subject of cardinal characteristics of the continuum and definition of various characteristics we refer the reader to [2].

The past decade has seen an increased volume of work regarding natural higher analogues for uncountable cardinals $\kappa$ of the classical cardinal characteristics. However, even though we already have a comparatively rich literature in this area there is very little known about analogues of the notion of independence. Even in the classical, countable setting, the independence number, and the notion of independence in general, do not seem to be that well-studied. Among the many open questions surrounding independence are the consistency of $\mathrm{cof}(\mathfrak{i}) = \omega$ and the consistency of $\mathfrak{i} < \mathfrak{a}$. A difficulty in the study of the classical invariant $\mathfrak{i}$ is the fact that there are very few available techniques, which allow to generically adjoin maximal independent families of desired size. More recent study of such techniques can be found in [8] and [9]. Another problem in the study of the higher independence number is the fact that it is not *a priori* clear what the natural generalization of the classical independence number should be. Given an uncountable cardinal $\kappa$[1] one may consider subfamilies $\mathcal{A}$ of $[\kappa]^\kappa$ which have the property that every boolean combination

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[1]In the paper, we only study the case in which $\kappa$ is regular.

generated by strictly less than $\kappa$ many elements of $\mathcal{A}$ is unbounded. That is, one may require that for every two disjoint subfamilies $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$, such that $|\mathcal{B}| < \kappa$ and $|\mathcal{C}| < \kappa$, the boolean combination $\bigcap \mathcal{B} \setminus \bigcup \mathcal{C}$ is unbounded. We refer to such families as strongly independent. A the major problem presenting itself in the study of this notion of strong independence on $\kappa$ is the very existence of maximal, under inclusion, strongly independent families. Results regarding these families, together with a number of interesting open questions are included in the last section of the paper. An earlier study of the notion of strong independence can be found in [14], where it is shown that the existence of a maximal strongly-$\omega_1$-independent family is equiconsistent with the existence of a measurable.

A more restrictive approach towards the generalization of the classical notion of independence, which proves to be more fruitful though, is the requirement that for a given family $\mathcal{A} \subseteq [\kappa]^\kappa$ only the finitely generated boolean combinations are unbounded. That is, given a family $\mathcal{A} \subseteq [\kappa]^\kappa$ we say that $\mathcal{A}$ is $\kappa$-independent if for every two disjoint finite subfamilies $\mathcal{B}$ and $\mathcal{C}$ contained in $\mathcal{A}$, the set $\bigcap \mathcal{B} \setminus \bigcup \mathcal{C}$ is unbounded.[2] The existence of a maximal under inclusion $\kappa$-independent family is provided by the Axiom of Choice and thus given an uncountable regular cardinal $\kappa$, one can define the higher independence number, denoted $\mathfrak{i}(\kappa)$, to be the minimal size of a maximal $\kappa$-independent family. A standard diagonalization argument going over all boolean combinations, shows that $\kappa^+ \leq \mathfrak{i}(\kappa)$. Classical examples of independent families of cardinality $2^\omega$ do generalize into the uncountable and provide the existence of $\kappa$-independent families and so of maximal $\kappa$-independent families of cardinality $2^\kappa$ (see Lemma 7). An example of a strongly $\kappa$-independent family of cardinality $2^\kappa$, under some additional hypothesis on $\kappa$, is provided in Lemma 57.

One of the main breakthroughs in the study of the classical independence number is the consistency of $\mathfrak{i} < \mathfrak{u}$, established in 1992 by S. Shelah (see [17]). The consistency proof carries a somewhat hidden construction of a Sacks indestructible maximal independent family, that is a maximal independent family which remains maximal after the countable support product and countable support iterations of Sacks forcing. For more recently studies of Sacks indestructible maximal independent families see [7, 8, 15]. In this paper, we prove:

**Theorem 1.** *Let $\kappa$ be a measurable cardinal and let $2^\kappa = \kappa^+$. Then there is a maximal $\kappa$-independent family, which remains maximal after the $\kappa$-support product of $\lambda$-many copies of $\kappa$-Sacks forcing.*

The existence of this indestructible maximal $\kappa$-independent family is closely related to the properties of a normal measure $\mathcal{U}$ on $\kappa$. With the indestructible family $\mathcal{A}$, we associate a $\kappa^+$-complete filter $\mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ which is properly contained in $\mathcal{U}$ and its elements meet every boolean combination on an unbounded set.

In the classical case, the analogous of Theorem 1 follows among others from a selectivity property of the filter $\mathrm{fil}(\mathcal{A})$, in our case, the analogous filter does not have the exact same properties, but its properties are still enough to get the result. Lemma 25 corresponds to one of the crucial steps in our proof.

[2] Clearly every strongly independent family is independent.

Finally, the existence of a $\kappa$-mad family, which remains maximal after arbitrarily long $\kappa$-supported product of $\kappa$-Sacks reals is a straightforward generalization of the classical case. Moreover, if $\mathfrak{d}(\kappa) = \kappa^+$ then $\mathfrak{a}(\kappa) = \kappa^+$ (see [4] and [16]). Thus our result leads to the following statement:

**Theorem 2.** *Let $\kappa$ be a measurable cardinal and $2^\kappa = \kappa^+$. Then there is a cardinal preserving generic extension in which*

$$\mathfrak{a}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{i}(\kappa) = \kappa^+ < 2^\kappa.$$

One of the very interesting open questions regarding the classical independence number is the consistency of $\mathfrak{i} < \mathfrak{a}$. As a very partial result towards this question we obtain the following:

**Corollary 3.** Let $\kappa$ be regular uncountable. If $\mathfrak{i}(\kappa) = \kappa^+$ then $\mathfrak{a}(\kappa) = \kappa^+$.

*Structure of the paper:* In Section 2 we define a notion of independence at $\kappa$, for $\kappa$ arbitrary infinite cardinal and define the cardinal number $\mathfrak{i}(\kappa)$ for $\kappa$ regular uncountable. In section 3, given a measurable cardinal $\kappa$, witnessed by a normal measure $\mathcal{U}$ and working under the hypothesis that $2^\kappa = \kappa^+$, we define a $\kappa^+$ closed poset $\mathbb{P}_\mathcal{U}$ which adjoins a maximal $\kappa$-independent family, which we denote $\mathcal{A}_G$.[3] In section 4 we study the properties of an ideal on $\kappa$, to which we refer as density ideal and denote $\mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$, which is contained in the dual ideal of $\mathcal{U}$ and which naturally captures crucial properties of the independent family $\mathcal{A}_G$. In section 5.1, we show that the dual filter of this ideal, denoted $\mathrm{fil}_{<\omega,\kappa}(\mathcal{A}_G)$ is both a $\kappa$-P-set, which means that every subfamily of cardinality $\kappa$ of $\mathrm{fil}_{<\omega,\kappa}(\mathcal{A}_G)$ has a pseduointersection in the filter (see Lemma 23). In Section 6 we show that the family $\mathcal{A}_G$ is densely maximal in a natural sense and characterize dense maximality in terms of properties of the density ideal. Section 7 introduces the concepts of preprocessed conditions and outer hulls necessary for the proofs in the last section. In Section 8 we prove our main theorem, by showing that the densely maximal $\kappa$-independent family $\mathcal{A}_G$ remains maximal after the $\kappa$-support product of $\lambda$ many copies of $\kappa$-Sacks forcing. We conclude the paper with some open questions and an appendix, discussing the notion of strong independence.

## 2. The Higher Independence Number

**Definition 4.** Let $\kappa$ be a regular uncountable cardinal and let $\mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ be the set of all finite partial functions with domain included in $\mathcal{A}$ and range the set $\{0,1\}$. For each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ let $\mathcal{A}^h = \bigcap\{\mathcal{A}^{h(A)} : A \in \mathrm{dom}(h)\}$ where $\mathcal{A}^{h(A)} = A$ if $h(A) = 0$ and $\mathcal{A}^{h(A)} = \kappa \backslash A$ if $h(A) = 1$. We refer to sets of the form $\mathcal{A}^h$ as boolean combinations.

With this we can state the definition of $\kappa$-independence.

**Definition 5.**

(1) A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is said to be $\kappa$-independent if for each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ the set $\mathcal{A}^h$ is unbounded. It is said to be a maximal $\kappa$-independent family if it is $\kappa$-independent and maximal under inclusion.

[3]Using the normal measure $\mathcal{U}$ and the hypothesis $2^\kappa = \kappa^+$ one can alternatively use the properties of the poset $\mathbb{P}_\mathcal{U}$ to construct a family $\mathcal{A}$ having all essential properties of $\mathcal{A}_G$ using a transfinite recursion of length $\kappa^+$.

(2) The least size of a maximal $\kappa$-independent family is denoted $\mathfrak{i}(\kappa)$.

**Remark 6.** For $\kappa = \omega$ the above notions coincides with the classical notions of independence on $[\omega]^\omega$ and $\mathfrak{i}(\kappa) = \mathfrak{i}$, where $\mathfrak{i}$ is the classical independence number.

**Lemma 7.** Let $\kappa$ be a regular infinite cardinal. Then

(1) Every $\kappa$-independent family is contained in a maximal $\kappa$-independent family.
(2) $\kappa^+ \leq \mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$
(3) There is a maximal $\kappa$-independent family of cardinality $2^\kappa$.
(4) $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$.

*Proof.* Since the increasing union of a collection of $\kappa$-independent families is $\kappa$-independent, by the Axiom of Choice every $\kappa$-independent family is contained in a maximal one. Note that if $\mathcal{A}$ is a maximal $\kappa$-independent family, then the set of boolean combinations $\{\mathcal{A}^h : h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})\}$ is not split and so $\mathfrak{r}(\kappa) \leq |\mathcal{A}|$. For a construction of a $\kappa$-independent family of cardinality $2^\kappa$, see [11, Theorem 4.2]. Finally, the proof that $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$ follows closely the proof of the classical case, i.e. $\mathfrak{d} \leq \mathfrak{i}$ (see [12]). □

One of the most interesting open questions, regarding the classical cardinal characteristics is the consistency of $\mathfrak{i} < \mathfrak{a}$. By the last item of the above theorem and the fact that if $\mathfrak{d}(\kappa) = \kappa^+$ implies that $\mathfrak{a}(\kappa) = \kappa^+$ (see [4] and [16]), we obtain the following:

**Corollary 8.** Let $\kappa$ be a regular uncountable cardinal. Then if $\mathfrak{i}(\kappa) = \kappa^+$ then $\mathfrak{a}(\kappa) = \kappa^+$.

## 3. Adjoining a maximal $\kappa$-independent family

Let $\kappa$ be a measurable cardinal and $\mathcal{U}$ a normal measure on $\kappa$.

**Definition 9.** Let $\mathbb{P}_\mathcal{U}$ be the poset of all pairs $(\mathcal{A}, A)$ where $\mathcal{A}$ is a $\kappa$-independent family of cardinality $\kappa$ and $A \in \mathcal{U}$ has the property that $\forall h \in \mathrm{FF}_{<\omega,\kappa}$ the set $\mathcal{A}^h \cap A$ is unbounded. The extension relation is defined as follows: $(\mathcal{A}_1, A_1) \leq (\mathcal{A}_0, A_0)$ if and only if $\mathcal{A}_1 \supseteq \mathcal{A}_0$ and $A_1 \subseteq^* A_0$.[4]

**Lemma 10.** Assume $2^\kappa = \kappa^+$. Then $\mathbb{P}_\mathcal{U}$ is $\kappa^+$-closed and $\kappa^{++}$-cc.

*Proof.* Let $\{(\mathcal{A}_i, A_i)\}_{i\in\kappa}$ be a decreasing sequence in $\mathbb{P}_\mathcal{U}$. We can assume that $\{A_i\}_{i\in\kappa}$ is strictly decreasing, i.e for each $i > j$ we have $A_j \subseteq A_i$. Then $\mathcal{A} = \bigcup_{i\in\kappa} \mathcal{A}_i$ is an independent family of cardinality $\kappa$ and the diagonal intersection $A' = \Delta_{i\in\kappa} A_i \in \mathcal{U}$.

Now, for each $i \in \kappa$, let $\{h_{i,j}\}_{j\in\kappa}$ enumerate $\mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$. Recursively we will define a set $A'' = \{k_{i,l,m}\}_{l,m<i;i<\kappa}$ which is a pseudo-intersection of $\{A_i\}_{i\in\kappa}$ and which meets every boolean combination $\mathcal{A}^h$ for $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ on an unbounded set. Then $A = A' \cup A''$ is an element of $\mathcal{U}$ and $(\mathcal{A}, A) \in \mathbb{P}_\mathcal{U}$ is a common extension of $\{(\mathcal{A}_i, A_i)\}_{i\in\kappa}$.

*Construction of $A''$:* At step $i$ pick $k_{i,m,l} \in A_i \cap \mathcal{A}_i^{h_{m,l}}$ for each $m, l < i$. Then in particular $k_{i,m,l} \in A_m$ for each $m \leq i$ and $k_{i,m,l} \in \mathcal{A}_m^{h_{m,l}}$ for each $m, l < i$. Take $A'' = \{k_{i,m,l}\}_{m,l<i;i<\kappa}$. Then $A''$ meets every boolean combination on an unbounded set and is a pseudo-intersection. Fix

[4] Throughout $A \subseteq^* B$ means $|A \backslash B| < \kappa$.

$\gamma \in \kappa$. Then for all $\xi$ such that $\xi > \gamma$ and all $m, l < \xi$ we have that $k_{\xi,l,m} \in A_\xi \subseteq A_\gamma$. Thus $A'' \backslash A_\gamma \subseteq \{k_{\xi,l,m}\}_{\xi<\gamma;l,m<\xi}$, which is a bounded set.

The poset has the $\kappa^{++}$-cc, because $|\mathbb{P}_\mathcal{U}| = \kappa^+$. Indeed, $\left|[[\kappa]^\kappa]^\kappa\right| = \kappa^+$. ☐

**Lemma 11.** If $(\mathcal{A}, A) \in \mathbb{P}_\mathcal{U}$, then there is $B \notin \mathcal{A}$ such that $B \subseteq A$ and $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$.

*Proof.* Let $\{h_i\}_{i\in\kappa}$ be a fixed enumeration of $\mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$. Since $\mathcal{A}^{h_0} \cap A$ is unbounded, we can find distinct $k_{0,0}$, $k_{0,1}$ in $\mathcal{A}^{h_0} \cap A$. Suppose we have defined $\{k_{i,j} : i \in \gamma, j \in 2\}$ distinct. Since $\mathcal{A}^{h_\gamma} \cap A$ is unbounded, we can find distinct $k_{\gamma,0}, k_{\gamma,1}$ in $(\mathcal{A}^{h_\gamma} \cap A)\backslash\{k_{i,j} : i \in \gamma, j \in 2\}$. Finally, take $B = \{k_{i,0}\}_{i\in\kappa}$. Clearly $B \subseteq A$ and $\mathcal{A} \cup \{B\}$ is independent. To verify the latter note that for each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ there are unboundedly many $h_i \supseteq h$. Then for unboundedly many $i \in \kappa$, $k_{i,0} \in \mathcal{A}^{h_i} \cap B \subseteq \mathcal{A}^h \cap B$ and $k_{i,1} \in \mathcal{A}^{h_i}\backslash B \subseteq \mathcal{A}^h\backslash B$. ☐

**Corollary 12.** Let $G$ be $\mathbb{P}_\mathcal{U}$-generic filter. Then $\mathcal{A}_G = \bigcup\{\mathcal{A} : \exists A \in \mathcal{U} \text{ with } (\mathcal{A}, A) \in G\}$ is a $\kappa$-maximal independent family.

*Proof.* Suppose $X \in [\kappa]^\kappa \setminus \mathcal{A}_G$ and $\mathcal{A}_G \cup \{X\}$ is independent. Take $(\mathcal{A}, A) \in G$ such that

$$(\mathcal{A}, A) \Vdash \text{“}\mathcal{A}_G \cup \{X\} \text{ is independent and } X \notin \mathcal{A}_G\text{”}.$$

Consider $(\mathcal{A}, A)$. Since $\mathbb{P}_\mathcal{U}$ is $\kappa^+$-closed, the set $X$ belongs to the ground model. Now, if for each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ the intersections $\mathcal{A}^h \cap X \cap A$ and $\mathcal{A}^h \cap A \cap X^c$ are unbounded, then $(\mathcal{A} \cup \{X\}, A) \leq (\mathcal{A}, A)$ and

$$(\mathcal{A} \cup \{X\}, A) \Vdash \text{“}X \in \mathcal{A}_G\text{“},$$

which is a contradiction. Therefore there is $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that either $\mathcal{A}^h \cap A \cap X$ or $\mathcal{A}^h \cap A \cap X^c$ is bounded. However, by Lemma 11, there is $B \notin \mathcal{A}$ such that $B \subseteq A$ and $(\mathcal{A} \cup \{B\}, A) \leq (\mathcal{A}, A)$. But then,

$$(\mathcal{A} \cup \{B\}, A) \Vdash \text{“}\exists h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_G) \text{ such that } \mathcal{A}^h_G \cap X \text{ or } \mathcal{A}^h_G \setminus X \text{ is bounded.”}$$

Therefore $(\mathcal{A} \cup \{B\}, A) \Vdash$ “$\mathcal{A}_G \cup \{X\}$ is not independent”, which is a contradiction. ☐

## 4. Density Ideal

**Definition 13.** Let $\mathcal{A}$ be an independent family. The density ideal $\mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ is the ideal of all $X \in \mathcal{U}^*$, where $\mathcal{U}^*$ is the dual ideal of $\mathcal{U}$, such that $\forall h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \cap X = \emptyset$.

**Lemma 14.**

(1) If $\mathcal{A}$ be an independent family, then $\mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ is an ideal.
(2) If $\mathcal{A}_0, \mathcal{A}_1$ are independent families such that $\mathcal{A}_0 \subseteq \mathcal{A}_1$, then $\mathrm{id}_{<\omega,\kappa}(\mathcal{A}_0) \subseteq \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_1)$.

*Proof.* To prove item (1) above consider any $X_0$ and $X_1$ in $\mathrm{id}_{<\omega,\kappa}(\mathcal{A})$. Fix any $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$. Then there is $h_0 \supseteq h$ such that $\mathcal{A}^{h_0} \cap X_0 = \emptyset$ and there is $h_1 \supseteq h_0$ such that $\mathcal{A}^{h_1} \cap X_1 = \emptyset$. But then $h_1 \supseteq h$ and $\mathcal{A}^{h_1} \cap (X_0 \cup X_1) = \emptyset$. Clearly, $\mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ is closed under subsets and thus $\mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ is an ideal.

To prove item (2) consider any $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_0)$. Let $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_1)$. Then $h' = h \restriction \mathcal{A}_0 \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_0)$ and by hypothesis there is $h_0$ in $\mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_0)$ extending $h'$ such that $\mathcal{A}_0^{h_0} \cap X = \emptyset$. Let $h_1 = h_0 \cup h \restriction (\mathcal{A}_1 \backslash \mathcal{A}_0)$. Then $\mathcal{A}_1^{h_1} \cap X \subseteq \mathcal{A}_0^{h_0} \cap X$ and so $\mathcal{A}_1^{h_1} \cap X = \emptyset$. □

**Remark 15.** Note that $\mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ is not necessarily $\kappa$-complete.

**Lemma 16.** $\Vdash_{\mathbb{P}_\mathcal{U}} \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G) = \bigcup\{\mathrm{id}_{<\omega,\kappa}(\mathcal{A}) : \exists A(\mathcal{A}, A) \in G\}$.

*Proof.* To see $\Vdash_{\mathbb{P}_\mathcal{U}} \bigcup\{\mathrm{id}_{<\omega,\kappa}(\mathcal{A}) : \exists A(\mathcal{A}, A) \in G\} \subseteq \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$ consider any $\mathbb{P}_\mathcal{U}$-generic filter $G$. In $V[G]$ we have $\mathcal{A}_G = \bigcup\{\mathcal{A} : \exists A(\mathcal{A}, A) \in G\}$. Now for all $(\mathcal{A}, A) \in G$, by Lemma 14.(2), $\mathrm{id}_{<\omega,\kappa}(\mathcal{A}) \subseteq \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$. Therefore $\bigcup\{\mathrm{id}_{<\omega,\kappa}(\mathcal{A}) : \exists A(\mathcal{A}, A) \in G\} \subseteq \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$.

The fact that $\Vdash_{\mathbb{P}_\mathcal{U}} \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G) \subseteq \bigcup\{\mathrm{id}_{<\omega,\kappa}(\mathcal{A}) : \exists A(\mathcal{A}, A) \in G\}$ follows from the $\kappa^+$-closure of $\mathbb{P}_\mathcal{U}$. Consider any $p = (\mathcal{A}, A) \in G$ and a $\mathbb{P}_\mathcal{U}$-name $\dot{X}$ for a subset of $\kappa$ such that $p \Vdash \dot{X} \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$. Fix $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$. Then

$$p \Vdash \exists h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_G)(h \subseteq h' \text{ and } \mathcal{A}_G^{h'} \cap X = \emptyset).$$

Thus there is $(\mathcal{A}', A') \leq (\mathcal{A}, A)$ such that $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}')$, $h' \supseteq h$ and $\mathcal{A}^{h'} \cap X = \emptyset$. Proceed inductively to construct a decreasing sequence $\{(\mathcal{A}_i, A_i)\}_{i\in\kappa}$ of conditions below $p$ such that if $\mathcal{A}_\kappa = \bigcup_{i\in\kappa} \mathcal{A}_i$ then for all $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_\kappa)$ there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_\kappa)$ extending $h$ and such that $\mathcal{A}^{h'} \cap X = \emptyset$. Thus $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_\kappa)$. By the $\kappa^+$-closure of $\mathbb{P}_\mathcal{U}$, there is $p' = (\mathcal{B}, B) \in \mathbb{P}_\mathcal{U}$ which is an extension of all $(\mathcal{A}_i, A_i)$. Thus $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{B})$, $p' \leq p$ and

$$p' \Vdash \dot{X} \in \bigcup\{\mathrm{id}_{<\omega,\kappa}(\mathcal{A}) : \exists A(\mathcal{A}, A) \in G\}.$$

□

**Lemma 17.** Let $(\mathcal{A}, A) \in \mathbb{P}_\mathcal{U}$ and let $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$. Then $(\mathcal{A}, A\backslash X) \in \mathbb{P}_\mathcal{U}$.

*Proof.* It is sufficient to show that for each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ the set $\mathcal{A}^h \cap (A\backslash X)$ is unbounded. Fix $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$. Since $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ there is $h' \supseteq h$, $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ extending $h$ such that $\mathcal{A}^{h'} \cap X = \emptyset$. Thus $\mathcal{A}^{h'} \subseteq \kappa\backslash X$. However

$$\mathcal{A}^{h'} \cap A = (\mathcal{A}^{h'} \cap A \cap X) \cup (\mathcal{A}^{h'} \cap A \cap X^c).$$

Thus $\mathcal{A}^{h'} \cap A = \mathcal{A}^{h'} \cap A \cap X^c$ is unbounded. Therefore $(\mathcal{A}, A\backslash X)$ is a condition. □

**Corollary 18.** Let $G$ be a $\mathbb{P}_\mathcal{U}$-generic filter. Then in $V[G]$ the density independence ideal $\mathrm{id}(\mathcal{A}_G)$ is generated by $\{\kappa\backslash A : \exists \mathcal{A}(\mathcal{A}, A) \in G\}$. That is

$$\Vdash_{\mathbb{P}_\mathcal{U}} \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G) =< \{\kappa\backslash A : \exists \mathcal{A}(\mathcal{A}, A) \in G\} > .$$

*Proof.* Let $G$ be a $\mathbb{P}_\mathcal{U}$-generic filter. By Lemma 16, $\mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G) = \bigcup\{\mathrm{id}_{<\omega,\kappa}(\mathcal{A}) : \exists A(\mathcal{A}, A) \in G\}$. Let $\mathcal{I}_G$ be the ideal generated by $\{\kappa\backslash A : \exists \mathcal{A}(\mathcal{A}, A) \in G\}$.

First we will show that $\mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G) \subseteq \mathcal{I}_G$. Let $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$. Thus there is $(\mathcal{A}, A) \in G$ such that $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$. However the set $D_X = \{(\mathcal{B}, B) \in \mathbb{P}_\mathcal{U} : X \cap B = \emptyset\}$ is dense below $(\mathcal{A}, A)$ (indeed, if $(\mathcal{B}, B) \leq (\mathcal{A}, A)$ then $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{B})$ and by Lemma 17 $(\mathcal{B}, B\backslash X) \leq (\mathcal{B}, B)$) and so there is $(\mathcal{B}, B) \in G$ such that $X \cap B = \emptyset$. That is $X \subseteq \kappa\backslash B$ and so $X \in \mathcal{I}_G$.

To show that $\mathcal{I}_G \subseteq \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$, consider any $X \in \mathcal{I}_G$. Then there is a finite set of conditions $\{(\mathcal{A}_i, A_i)\}_{i\in n}$ in $G$ such that $X \subseteq \bigcup_{i\in n} \kappa\backslash A_i = \kappa\backslash \bigcap_{i\in n} A_i$. Note that $(\mathcal{B}, B) \in G$, where $(\mathcal{B}, B) = (\bigcup_{i\in n} \mathcal{A}_i, \bigcap_{i\in n} A_i)$. Thus $X \subseteq \kappa\backslash B$. Fix any $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_G)$. Then there is $(\mathcal{C}, C) \in G$ such that $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{C})$. Take $(\mathcal{E}, E) \in G$ which is a common extension of $(\mathcal{B}, B)$ and $(\mathcal{C}, C)$. Then $(\mathcal{E}, E) \leq (\mathcal{C}, B)$ and so in particular $(\mathcal{C}, B) \in G$. However the set $H_B = \{(\mathcal{C}', C') : \exists Y \in \mathcal{C}'(Y \subseteq B)\}$ is dense below $(\mathcal{C}, B)$ (apply Lemma 11) and so there is $(\mathcal{C}', C') \in G$ such that for some $Y \in \mathcal{C}'$, $Y \subseteq B$. Then $h' = h \cup \{(Y, 0)\} \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_G)$ and $\mathcal{A}_G^{h'} \cap X = \emptyset$. Thus $X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$. □

## 5. Density Filter

**Remark 19.** Let $G$ be $\mathbb{P}_{\mathcal{U}}$-generic, let $\mathcal{F}_G = \{A : \exists \mathcal{A}$ such that $(\mathcal{A}, A) \in G\}$ and let $\mathrm{fil}_{<\omega,\kappa}(\mathcal{A}_G)$ be the dual filter of $\mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$. By Corollary 18, $\mathrm{fil}_{<\omega,G}(\mathcal{A}_G)$ is generated by $\mathcal{F}_G$.

**Lemma 20.** Let $(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$, $Y \in [\kappa]^\kappa$ and $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$. Then there is $h^* \supseteq h$ in $\mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ and $B \subseteq A$ such that $(\mathcal{A}, B) \leq (\mathcal{A}, A)$ and $\mathcal{A}^{h^*} \cap B$ is contained either in $Y$, or in $\kappa\backslash Y$.

*Proof.* If there is $h'$ extending $h$ such that $\mathcal{A}^{h'} \cap A \cap Y$ is bounded, then $\mathcal{A}^{h'} \cap A =^* \mathcal{A}^{h'} \cap A \cap (\kappa\backslash Y)$ and so for all $h'' \supseteq h'$ the set $\mathcal{A}^{h''} \cap A \cap (\kappa\backslash Y)$ is unbounded. Then take $B = (\mathcal{A}^{h'} \cap A \cap (\kappa\backslash Y)) \cup (A\backslash \mathcal{A}^{h'})$. Then $B =^* A$ and so $B \in \mathcal{U}$, $(\mathcal{A}, B)$ is as desired.

If there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap A \cap (\kappa\backslash Y)$ is bounded, then $\mathcal{A}^{h'} \cap A =^* \mathcal{A}^{h'} \cap A \cap Y$ and so for all $h'' \supseteq h'$ the set $\mathcal{A}^{h''} \cap A \cap Y$ is unbounded. Then take $B = (\mathcal{A}^{h'} \cap A \cap Y) \cup (A\backslash \mathcal{A}^{h'})$. Then $B =^* A$ and so $B \in \mathcal{U}$, and the condition $(\mathcal{A}, B)$ is as desired.

Suppose, none of the above two cases holds. Thus for every $h' \supseteq h$, the sets $\mathcal{A}^{h'} \cap A \cap Y$ and $\mathcal{A}^{h'} \cap A \cap (\kappa\backslash Y)$ are unbounded. Then each of the sets $B_0 = (\mathcal{A}^h \cap A \cap Y) \cup (A\backslash \mathcal{A}^h)$ and $B_1 = (\mathcal{A}^h \cap A \cap (\kappa\backslash Y)) \cup (A\backslash \mathcal{A}^h)$ meets every boolean combination $\mathcal{A}^{h'}$ for $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ on an unbounded set. Thus if $A\backslash \mathcal{A}^h \in \mathcal{U}$, both $B$ and $B'$ are as desired. Suppose $A\backslash \mathcal{A}^h \notin \mathcal{U}$. Then $A \cap \mathcal{A}^h \in \mathcal{U}$ and so either $A \cap \mathcal{A}^h \cap Y$ or $A \cap \mathcal{A}^h \cap (\kappa\backslash Y)$ is in the normal measure. We can chose appropriately. □

**Corollary 21.** Let $\mathcal{E} = \{Y, \kappa\backslash Y\}$ be a partition. Then the set of $(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$ such that for each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ there is $h' \supseteq h$ in $\mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ with the property that $\mathcal{A}^{h'}$ is either contained in $Y$, or in $\kappa\backslash Y$ is dense in $\mathbb{P}_{\mathcal{U}}$.

*Proof.* Consider an arbitrary $(\mathcal{A}, A) \in \mathbb{P}_{\mathcal{U}}$. Fix $h_0 \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$. Then there is $A_0 \subseteq A$ such that $(\mathcal{A}, A_0) \leq (\mathcal{A}, A)$ and there is $h_1 \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ extending $h_0$ and $B \subseteq A$ such that $\mathcal{A}^{h_1} \cap B$ is contained either in $Y$, or in $\kappa\backslash Y$. However, by Lemma 11 there is $B_0 \subseteq B$ such that $(\mathcal{A} \cup \{B_0\}, B) \leq (\mathcal{A}, B)$. Then extend $h_1$ to $h_1' = h_1 \cup \{(B_0, 0)\}$ and note that $h_1' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_1)$, where $\mathcal{A}_1 = \mathcal{A} \cup \{B_0\}$, and that $\mathcal{A}_1^{h_1'}$ is either contained in $Y$ or in $\kappa\backslash Y$. Proceed inductively and use the fact that $\mathbb{P}_{\mathcal{U}}$ is $\kappa^+$-closed. □

**Definition 22.** Let $\mathcal{F} \subseteq [\kappa]^\kappa$. We say that $\mathcal{F}$ is a $\kappa$-P-set if every $\mathcal{H} \subseteq \mathcal{F}$ of cardinality $\leq \kappa$ has a pseudo-intersection in $\mathcal{F}$.

**Lemma 23.** Let $G$ be a $\mathbb{P}_{\mathcal{U}}$-generic filter. Then $\mathcal{F}_G$ is a $\kappa$-P-set.

*Proof.* Suppose $\mathcal{F}_G$ is not a $\kappa$-P-set. Thus there is $p \in \mathbb{P}_\mathcal{U}$ such that

$$p \Vdash \exists \mathcal{H} \in [\mathcal{F}_G]^\kappa \text{ s.t. } \forall F \in \mathcal{F}_G \exists H \in \mathcal{H}(F \not\subseteq^* H).$$

Fix $G$ a $\mathbb{P}_\mathcal{U}$-generic filter such that $p \in G$. Since $\mathbb{P}_\mathcal{U}$ is $\kappa^+$-closed, we can find $\mathcal{H}' = \{A_i\}_{i\in\kappa}$ in the ground model witnessing the above property. For each $i \in \kappa$, let $\mathcal{A}_i$ be such that $(\mathcal{A}_i, A_i) \in G$. We can assume that $\tau = \{(\mathcal{A}_i, A_i)\}_{i\in\kappa}$ is decreasing and that $(\mathcal{A}_0, A_0) \leq p$. Now, take $q = (\mathcal{A}, A)$ in $\mathbb{P}_\mathcal{U}$ to be a common lower bound of $\tau$. Then $q \leq p$ and $q$ forces that $A$ is a pseudo-intersection of $\mathcal{H}'$, which is a contradiction. □

5.1. **Increasing Functions and the Density Filter.** If $E \subseteq \kappa$ is an unbounded set and $\alpha \in \kappa$ let $s_E(\alpha) = \min\{\beta \in E : \beta > \alpha\}$.

**Lemma 24.** Let $f \in V \cap {}^\omega\omega$ be strictly increasing and let $(\mathcal{A}, A) \in \mathbb{P}_\mathcal{U}$. Then there is $A^* \subseteq A$ such that $(\mathcal{A}, A^*) \leq (\mathcal{A}, A)$ and if $\{a(i)\}_{i<\kappa}$ is the increasing enumeration of $A^*$ then $f(a(i)) < a(i+1)$ for all $i$.

*Proof.* Let $C_f = \{\xi < \kappa : \forall \zeta < \xi(f(\zeta) < \xi)\}$. Thus $C_f$ is a club and so $C_f \in \mathcal{U}$. Then $E = A \cap C_f \in \mathcal{U}$. Let $\{h_i\}_{i<\kappa}$ be an enumeration of the elements of $\mathrm{FF}_{<\omega}(\mathcal{A})$ such that each element occurs unboundedly often. The set $A^*$ will be constructed as the union of an increasing sequence $\{B_\xi\}_{\xi<\kappa}$ of subsets of $A$.

Let $B_0 = \emptyset$. If $\mathcal{A}^{h_0} \cap E \neq \emptyset$, let $a_0 = \min \mathcal{A}^{h_0} \cap E$. Otherwise, take $a_0 = \min \mathcal{A}^{h_0} \cap A$. Let

$$B_1 = \{a_0\} \cup (E \cap s_E(f(a_0))) \cup \{s_E(f(a_0))\}.$$

Suppose we have defined $B_\xi$. If $(\mathcal{A}^{h_{\xi+1}} \cap E)\backslash(B_\xi \cup \{\sup B_\xi\}) \neq \emptyset$, let $a_{\xi+1} = \min \mathcal{A}^{h_{\xi+1}} \cap E \backslash (B_\xi \cup \{\sup B_\xi\})$. Otherwise, let $a_{\xi+1} = \min\{a \in \mathcal{A}^{h_{\xi+1}} \cap A : a > \sup B_\xi\}$. Let

$$B_{\xi+1} = B_\xi \cup \{a_{\xi+1}\} \cup (E \cap s_E(f(a_{\xi+1}))) \cup \{s_E(f(a_\xi))\}.$$

Now, suppose $\xi$ is a limit and for all $\zeta < \xi$, the set $B_\zeta$ has been defined. Take $B^*_\xi = \bigcup_{\zeta<\xi} B_\zeta$. If $(\mathcal{A}^{h_\xi} \cap E)\backslash(B^*_\xi \cup \{\sup B^*_\xi\}) \neq \emptyset$, let $a_\xi = \min(\mathcal{A}^{h_\xi} \cap E)\backslash(B^*_\xi \cup \{\sup B^*_\xi\}) \neq \emptyset$. Otherwise, let $a_\xi = \min\{a \in \mathcal{A}^{h_\xi} \cap A : a > \sup B^*_\xi\}$. Let

$$B_\xi = B^*_\xi \cup \{a_\xi\} \cup E \cap s_E(f(a_\xi)) \cup \{s_E(f(a_\xi))\}.$$

Finally, take $A^* = \bigcup_{\xi<\kappa} B_\xi$. Then $A^*$ meets every boolean combination of $\mathcal{A}$ on an unbounded set (witnessed by the $a_\xi$'s), $A^* \subseteq A$ and since $\{a_\xi\}_{\xi<\kappa}$ is unbounded in $\kappa$ and

$$E \cap s_E(f(a_\xi)) \subseteq B_\xi \subseteq A^*$$

for each $\xi$, we also have that $E \subseteq A^*$. Let $b < a$ be elements of $A^*$. If $a \in E$, then by definition of $C_f$ we have that $f(b) < a$. If $a \notin E$ and $a = a_{\xi+1}$ for some $\xi$, then

$$a_\xi < f(a_\xi) < s_E(f(a_\xi)) < a_{\xi+1}$$

by construction. If $\xi$ is a limit, $a = a_\xi$ and $f(a_\zeta) < s_E(f(a_\zeta)) < a_x i$ for each $\zeta < \xi$ again by construction. Since $b = a_\zeta$ for some $\zeta < \xi$, $f(b) < a$. □

**Corollary 25.** Let $G$ be $\mathbb{P}_\mathcal{U}$-generic, $f \in V \cap {}^\kappa\kappa$ be strictly increasing. Then there is $A \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A}_G)$ such that if $\{a(i)\}_{i\in\kappa}$ is the increasing enumeration of $A$ then $f(a(i)) < a(i)$ for all $i$.

*Proof.* Since $\mathrm{fil}_{<\omega,\kappa}(\mathcal{A}_G)$ is generated by $\mathcal{F}_G$ (the set of second coordinated of elements of the generic filter $G$) and the previous lemma. □

## 6. Dense Maximality

**Definition 26.** An independent family $\mathcal{A}$ is said to be densely maximal if for every $X \in [\kappa]^\kappa \backslash \mathcal{A}$ and every $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ extending $h$ such that either $\mathcal{A}^{h'} \cap X = \emptyset$ or $\mathcal{A}^{h'} \cap (\kappa \backslash X) = \emptyset$.

**Lemma 27.** Let $\mathcal{A}$ be an independent family. Then $\mathcal{A}$ is densely maximal if and only if

$(*)$ $\forall h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}) \forall X \subseteq \mathcal{A}^h$ either there is $B \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash X \subseteq B$, or there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash X$.

*Proof.* Suppose $\mathcal{A}$ satisfies property $(*)$. Let $X \in [\kappa]^\kappa$, $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ and consider $Y = X \cap \mathcal{A}^h$. Apply property $(*)$. If there is $B \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash X \subseteq B$, then $\mathcal{A}^h \backslash X \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$. Then there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap (\mathcal{A}^h \backslash X) = \mathcal{A}^{h'} \backslash X = \emptyset$. If there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash X$, then $\mathcal{A}^{h'} \cap X = \emptyset$. Thus $\mathcal{A}$ is densely maximal.

Now suppose $\mathcal{A}$ is densely maximal. Fix $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that $X \subseteq \mathcal{A}^h$. We will show that $\mathcal{A}$ satisfies property $(*)$. Suppose, there is no $B \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash X \subseteq B$. Thus in particular $\mathcal{A}^h \backslash X \notin \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ and so there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that for all $h'' \supseteq h'$ the set $\mathcal{A}^{h''} \cap (\mathcal{A}^h \backslash X) \neq \emptyset$. If $h$ and $h'$ are incompatible as conditions in $\mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$, then $\mathcal{A}^{h'} \cap (\mathcal{A}^h \backslash X) = \emptyset$, which is a contradiction. Therefore $h$ and $h'$ are compatible. Without loss of generality, $h' \supseteq h$ (otherwise pass to a common extension of $h$ and $h'$). Thus $h$ has an extension, namely $h'$, such that for all $h'' \supseteq h'$ the set $\mathcal{A}^{h''} \backslash X$ is non-empty. Apply the fact that $\mathcal{A}$ is densely maximal to $\mathcal{A}^{h'}$ and $X$. Thus, there is $h'' \supseteq h'$ such that $\mathcal{A}^{h''} \cap X = \emptyset$. Therefore $\mathcal{A}^{h''} \subseteq \mathcal{A}^{h'} \backslash X \subseteq \mathcal{A}^h \backslash X$, which completes the proof of property $(*)$. □

**Lemma 28.** Let $G$ be $\mathbb{P}_{\mathcal{U}}$-generic. Then in $V_0 = V[G]$ the family $\mathcal{A}_G := \bigcup \{\mathcal{A} : \exists A (\mathcal{A}, A) \in G\}$ is densely maximal.

*Proof.* It is sufficient to show that $\mathcal{A}_G$ satisfies property $(*)$ from Lemma 27. Thus, fix $h$ and $X$ as in $(*)$. Suppose there is no $B \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$ such that $\mathcal{A}_G^h \backslash X \subseteq B$. Then, in particular $\mathcal{A}_G^h \backslash X \notin \mathrm{id}_{<\omega,\kappa}(\mathcal{A}_G)$ and so there is $h_0 \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_G)$ such that for all $h_1 \supseteq h_0$ the set $\mathcal{A}^{h_1} \cap (\mathcal{A}^h \backslash X) \neq \emptyset$ (by definition of the density ideal). Consider the partition

$$\mathcal{E} = \{\mathcal{A}^h \backslash X, \kappa \backslash (\mathcal{A}^h \backslash X)\}$$

and the set $\mathcal{A}^{h_0}$. By Corollary 21 there is $h_1 \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}_G)$ extending $h_0$ such that $\mathcal{A}^{h_1}$ is contained in one element of $\mathcal{E}$. However, if $\mathcal{A}^{h_1} \subseteq \kappa \backslash (\mathcal{A}^h \backslash X)$, then $\mathcal{A}^{h_1} \cap (\mathcal{A}^h \backslash X) = \emptyset$, which is a contradiction to the choice of $h_0$. Thus $\mathcal{A}^{h_1} \subseteq \mathcal{A}^h \backslash X$ and so $\mathcal{A}^{h_1} \cap X = \emptyset$. □

## 7. Preprocessed Conditions and Outer hulls

Throughout this section we work under the assumption of GCH (at least $2^\kappa = \kappa^+$ and $2^{<\kappa} = \kappa$) and $\kappa$ measurable. Thus in particular $\kappa$ is strongly inaccessible. We will work with the generalization of Sacks forcing and its products to the uncountable, both of which were first studied by Kanamori [13].

Recall that $p \subseteq 2^{<\kappa}$ is a *tree* if it is closed under initial segments. That is, $u \in p$ and $v \subseteq u$ imply $v \in p$. Whenever $p$ is a tree, $t, r \in p$ and $t$ is a proper initial segment or equal to $r$, we write $t \trianglelefteq r$ and $r \trianglerighteq t$. A node $u \in p$ *splits* in $p$ if both $u^\frown 0$ and $u^\frown 1$ belong to $p$. Given a tree $p$, we denote by $\text{split}(p)$ the set of splitting nodes of $p$.

**Definition 29.** For strongly inaccessible $\kappa$, the $\kappa$-Sacks forcing, denoted $\mathbb{S}_\kappa$, is the poset consisting of sub-trees $p$ of $2^{<\kappa}$ such that:

1. for each $u \in p$ there is $t \in p$ such that $u \trianglelefteq t$ and $t$ splits in $p$ ($t$ is said to be a splitting extension of $u$);
2. for any $\alpha < \kappa$, if $(u_\beta : \beta < \alpha)$ is a sequence of nodes in $p$ such that $\beta < \gamma < \alpha \to u_\beta \subseteq u_\gamma$, then $\bigcup\{u_\beta : \beta < \alpha\} \in p$;
3. if $\delta < \kappa$ is a limit ordinal, $u \in 2^\delta$ and for arbitrarily large $\beta < \delta$ the node $u \restriction \beta$ splits in $p$, then $u$ splits in $p$.

The extension relation on $\mathbb{S}_\kappa$ is defined by $p \leq q$ if and only if $p \subseteq q$.

As in the countable case we define the $\text{stem}(p)$ where $p$ is a condition in $\mathbb{S}_\kappa$ as the unique splitting node that is comparable with all elements in $p$. By recursion on $\kappa$ define:

**Definition 30** (The $\alpha$-th *splitting level* of $p$)**.** Given $p \in \mathbb{S}_\kappa$ let

- $\text{split}_0(p) = \text{stem}(p)$,
- $\text{split}_{\alpha+1}(p) = \{\text{stem}(p_{u^\frown i}) : u \in \text{split}_\alpha(p) \text{ and } i \in 2\}$,
- for $\delta < \kappa$ is a limit ordinal, $\text{split}_\delta(p) = \{s \in p : s \text{ is a limit of nodes in } \bigcup_{\alpha<\delta} \text{split}_\alpha(p)\}$.

We refer to $\text{split}_\alpha(p)$ as the $\alpha$-th splitting level of $p$. Moreover for $t \in \text{split}(p)$, let $\text{sl}(t, p) = \alpha$ where $t \in \text{split}_\alpha(p)$.

Using this splitting levels we define the *fusion orderings* $\leq_\alpha$ on $\mathbb{S}_\kappa$: Given $q$ and $p$ in $\mathbb{S}_\kappa$ let $q \leq_\alpha p$ if and only if $q \leq p$ and $\text{split}_\alpha(p) = \text{split}_\alpha(q)$.

**Definition 31.** A *fusion sequence* $(p_\alpha : \alpha < \kappa) \subseteq \mathbb{S}_\kappa$ is sequence of conditions in $\mathbb{S}_\kappa$ such that $p_{\alpha+1} \leq_\alpha p_\alpha$ for all $\alpha < \kappa$ and whenever $\delta < \kappa$ is a limit, then $p_\delta \leq_\alpha p_\alpha$ for all $\alpha < \delta$.

For any regular uncountable cardinal $\lambda$ we denote by $\mathbb{S}_\kappa^\lambda$ the $\kappa$-support product of $\lambda$ many copies of $\mathbb{S}_\kappa$. Moreover:

**Definition 32** (Product fusion, Definition 1.7 in [13])**.**

- If $(p_\alpha : \alpha < \beta) \subseteq \mathbb{S}_\kappa^\lambda$, we define a condition $p = \bigwedge_{\alpha<\beta} p_\alpha$ with $\text{dom}(p) = \bigcup_{\alpha<\beta} \text{dom}(p_\alpha)$ and for every $\gamma \in \text{dom}(p)$, $p(\gamma) = \bigcap\{p_\alpha(\gamma) : \gamma \in \text{dom}(p_\alpha)\}$. Note that in the case $p \restriction \gamma \notin \mathbb{S}_\kappa^\gamma$ for $\gamma \in \text{dom}(p)$ or $|\text{dom}(p)| > \kappa$ then $p$ is left undefined.
- If $p, q \in \mathbb{S}_\kappa^\lambda$, $\alpha < \kappa$ and $F \subseteq \text{dom}(q)$ with $|F| \leq \kappa$, we say $p \leq_{F,\alpha} q$ if and only if $p \leq q$ and for every $\beta \in F$, $p(\beta) \leq_\alpha q(\beta)$.

**Lemma 33** (Generalized fusion [13])**.** Suppose $(p_\alpha : \alpha < \kappa) \subseteq \mathbb{S}_\kappa^\lambda$ and $F_\alpha \subseteq \lambda$ have the following properties:

(1) $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$ and $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$ when $\delta$ is a limit ordinal $<\kappa$.

(2) $|F_\alpha|< \kappa$, $F_\alpha \subseteq F_{\alpha+1}$, $F_\delta = \bigcup_{\alpha<\delta} F_\alpha$ for limit $\delta < \kappa$ and $\bigcup_{\alpha<\kappa} F_\alpha = \bigcup_{\alpha<\kappa} \mathrm{dom}(p_\alpha)$.

Then $p = \bigwedge_{\alpha<\kappa} p_\alpha \in \mathbb{S}^\lambda_\kappa$ and we refer to $(p_\alpha, F_\alpha : \alpha < \kappa)$ as a generalized fusion sequence.

**Definition 34.**

- Let $p \in \mathbb{S}^\lambda_\kappa$. By $\mathrm{supp}_\alpha(p)$ we denote the set of the first $\alpha$-many elements of $\mathrm{supp}(p)$.
- Given a condition $p \in \mathbb{S}^\lambda_\kappa$, $\alpha < \kappa$ and $F \subseteq \mathrm{supp}_\alpha(p)$ let $\Lambda^F_\alpha(p) = \prod_{i\in F} \mathrm{split}_\alpha(p(i))$. That is
$$\Lambda^F_\alpha(p) = \{\bar\sigma = (\sigma_i)_{i\in F} : \sigma_i \in \mathrm{split}_\alpha(p(i))\}.$$
- For all $\bar\sigma \in \Lambda^F_\alpha(p)$ let $p_{\bar\sigma} \leq p$ be defined as follows: $\mathrm{supp}(p) = \mathrm{supp}(p_{\bar\sigma})$ and
$$p_{\bar\sigma}(i) = \begin{cases} (p(i))_{\sigma_i} & \text{if } i \in F \\ p(i) & \text{otherwise} \end{cases}$$
- Given $h \in {}^F2$ and $\bar\sigma \in \Lambda^F_\alpha(p)$, let $p^h_{\bar\sigma}$ be defined as follows: $\mathrm{supp}(p^h_{\bar\sigma}) = \mathrm{supp}(p)$ and
$$p^h_{\bar\sigma}(i) = \begin{cases} (p(i))_{\sigma_i^\frown h(i)} & \text{if } i \in F \\ p(i) & \text{otherwise} \end{cases}$$

7.1. **Preprocessed conditions.**

**Definition 35.**

(1) Let $\dot X$ be a $\mathbb{S}_\kappa$-name for a subset of $\kappa$. We say that $p \in \mathbb{S}_\kappa$ is preprocessed for $\dot X$ if for all $\alpha \in \kappa$ and all $t \in \mathrm{split}_\alpha(p)$ there is $x_t \in {}^\alpha 2$ such that $p_t \Vdash \chi_{\dot X} \restriction \alpha = \check x_t$.

(2) Let $\dot X$ be a $\mathbb{S}^\lambda_\kappa$ name for a subset of $\kappa$. We say that $p \in \mathbb{S}^\lambda_\kappa$ is preprocessed for $\dot X$ if for all $\alpha \in \kappa$, $F \subseteq \mathrm{supp}_\alpha(p)$ and $\bar\sigma \in \Lambda^F_\alpha(p)$ there is $x_{\bar\sigma} \in {}^\alpha 2$ such that $p_{\bar\sigma} \Vdash \chi_{\dot X} \restriction \alpha = \check x_{\bar\sigma}$.

**Remark 36.**

(1) Note that if $p \in \mathbb{S}_\kappa$ is preprocessed for $\dot X$, then for each $\alpha \in \kappa$ there is $x_\alpha \subseteq {}^\alpha 2$ such that $p \Vdash \chi_{\dot X} \restriction \alpha \in \check x_\alpha$. Indeed, take $x_\alpha = \bigcup\{x_t : t \in \mathrm{split}_\alpha(p)\}$ where $x_t$ is defined as in the definition above.

(2) Similarly, if $p \in \mathbb{S}^\lambda_\kappa$ is preprocessed for $\dot X$, then for each $\alpha < \kappa$ there is $x_\alpha \subseteq {}^\alpha 2$ such that $p \Vdash \chi_{\dot X} \restriction \alpha \in \check x_\alpha$. Just take $x_\alpha = \bigcup\{x_{\bar\sigma} : \bar\sigma \in \Lambda^{\mathrm{supp}(p)\cap\alpha}_\alpha(p)\}$ where $x_{\bar\sigma}$ is defined as above.

**Lemma 37.**

(1) Let $p \in \mathbb{S}_\kappa$ and let $\dot X$ be a $\mathbb{S}_\kappa$-name for a subset of $\kappa$. Then there is $q \leq p$ such that $q$ is preprocessed for $\dot X$.

(2) Let $p \in \mathbb{S}^\lambda_\kappa$ and let $\dot X$ be a $\mathbb{S}^\lambda_\kappa$-name for a subset of $\kappa$. Then there is $q \leq p$ such that $q$ is preprocessed for $\dot X$.

*Proof.* (1) We build a fusion sequence $\langle q_\alpha : \alpha < \kappa\rangle$ below $p$ such that for all $\alpha > 0$ and all $t \in \mathrm{split}_\alpha(q_\alpha)$ there is $x_t \in {}^\alpha 2$ such that $(q_\alpha)_t \Vdash \chi_{\dot X} \restriction \alpha = \check x_t$. Start with $q_0 = p$. Consider $t \in \mathrm{split}_0(p)$, i.e. $t = \mathrm{stem}(p)$. For each $i \in \{0, 1\}$ there is $w_{t,i} \leq p_{t^\frown i}$ and $x(t, i) \in \{0, 1\}$ such that $w_{t,i} \Vdash \chi_{\dot{X}}(0) = \check{x}(t,i)$. Note that $\mathrm{stem}(w_{t,i}) \trianglerighteq \mathrm{stem}(p_{t^\frown i}) \trianglerighteq t^\frown i$. Define $q_1 = w^0_{t,0} \cup w^0_{t,1}$. Then $q_1 \leq_0 q_0$ and for all $s \in \mathrm{split}_1(q_1)$ there is $x_s \in {}^1 2$ such that $(q_1)_s \Vdash \chi_{\dot{X}} \restriction 1 = \check{x}_s$. Indeed, if $s \in \mathrm{split}_1(q_1)$ then $s \trianglerighteq t^\frown i$ for $i \in \{0,1\}$ and so $(q_1)_s = w_{t,i}$. Thus $(q_1)_s \Vdash \chi_{\dot{X}} \restriction 1 = x_s$ where $x_s = (0, x(s,i))$.

Now, suppose $q_\alpha$ has been defined and $\forall t \in \mathrm{split}_\alpha(q_\alpha)$ there is $x_t \in {}^\alpha 2$ such that $(q_\alpha)_t \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_t$. For each $t \in \mathrm{split}_\alpha(q_\alpha)$ and each $i \in \{0,1\}$ find $w_{t,i} \leq (q_\alpha)_{t^\frown i}$ and $x(t,i) \in \{0,1\}$ such that $w^\alpha_{t,i} \Vdash \chi_{\dot{X}}(\alpha) = \check{x}(t,i)$. Then, take $q_{\alpha+1} = \bigcup\{w_{t,i} : t \in \mathrm{split}_\alpha(q_\alpha), i \in \{0,1\}\}$. Then $\mathrm{split}_\alpha(q_{\alpha+1}) = \mathrm{split}_\alpha(q_\alpha)$ and so $q_{\alpha+1} \leq_\alpha q_\alpha$. Moreover, for all $t \in \mathrm{split}_{\alpha+1}(q_{\alpha+1})$ there is $x_t \in {}^{\alpha+1}2$ such that $(q_{\alpha+1})_t \Vdash \chi_{\dot{X}} \restriction \alpha+1 = \check{x}_t$. Indeed. Fix $t \in \mathrm{split}_{\alpha+1}(q_{\alpha+1})$. Thus $r^\frown i \trianglelefteq t$ for some $r \in \mathrm{split}_\alpha(q_{\alpha+1}) = \mathrm{split}_\alpha(q_\alpha)$ for some $i \in \{0,1\}$. By Inductive Hypothesis $(q_\alpha)_r \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_r$ for some $x_r \in {}^\alpha 2$. However $t \trianglerighteq r^\frown i$ and so $(q_{\alpha+1})_t = w_{r,i} \leq (q_\alpha)_{r^\frown i} \leq (q_\alpha)_r$. Thus, $(q_{\alpha+1})_t \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_r$ and $\chi_{\dot{X}}(\alpha) = \check{x}(r,i)$. That is $(q_{\alpha+1})_t \Vdash \chi_{\dot{X}} \restriction \alpha+1 = \check{x}_t$ where $x_t = x_r \cup \{(\alpha, x(r,i))\}$.

It remains to consider the limit case. Suppose $\langle q_\beta : \beta < \alpha \rangle$ have been defined and for all $\beta < \alpha$ and all $t \in \mathrm{split}_\beta(q_\beta)$ there is $x_t \in {}^\beta 2$ such that $(q_\beta)_t \Vdash \chi_{\dot{X}} \restriction \beta = \check{x}_t$. Then take $q_\alpha = \wedge_{\beta<\alpha} q_\beta$. Note that if $t \in \mathrm{split}_\alpha(q_\alpha)$ then there is $\{\eta_\xi : \xi < \alpha\}$ unbounded in $\alpha$ such that $t \restriction \eta_\xi \in \mathrm{split}_\xi(q_\xi)$ and so by inductive hypothesis for some $x_{t\restriction\eta_\xi} \in {}^\xi 2$ we have $(q_\xi)_{t\restriction\eta_\xi} \Vdash \chi_{\dot{X}} \restriction \xi = \check{x}_{t\restriction\eta_\xi}$. Then for $x_t = \bigcup\{x_{t\restriction\eta_\xi} : \xi < \alpha\}$ we have $(q_\alpha)_t \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_t$.

(2) The argument for the product runs similarly as the above case. Likewise, we want to define a fusion sequence $\langle q_\alpha, F_\alpha : \alpha < \kappa \rangle \subseteq \mathbb{S}^\lambda_\kappa$ below $p$ such that for all $\alpha > 0$ and all $\bar{\sigma} \in \Lambda^{F_\alpha}_\alpha(q_\alpha) = \prod_{i\in F} \mathrm{split}_\alpha(p(i))$ there is $x_{\bar{\sigma}} \in {}^\alpha 2$ such that $(q_{\alpha+1})_{\bar{\sigma}} \Vdash \chi_{\dot{X}} \restriction \alpha = \check{x}_{\bar{\sigma}}$. Since similar arguments will be used in the upcoming results we give the proof in full detail. Start with $q_0 = p$ and $F_0 = \{\min(\mathrm{supp}(p))\}$. At limit stages $\alpha < \kappa$ construct $p_\alpha$ and $F_\alpha$ so that the conditions in Definition 32 are fulfilled.

Suppose $q_\alpha$ and $F_\alpha$ have been defined. Fix an enumeration $\{\gamma_l = (\bar{\sigma}_l, h_l) : l < \rho : l < \rho\}$ of all pairs of the form $(\bar{\sigma}, h)$ such that $\bar{\sigma} \in \Lambda^{F_\alpha}_\sigma(q_\alpha)$ and $h \in {}^{F_\alpha}2$. Note that the ordinals $\rho$ is $< \kappa$.

Inductively, we will construct a sequence $\{r^\alpha_l : l < \rho\}$ of conditions below $q_\alpha$ satisfying:

(1) $r^\alpha_0 = q_\alpha$.
(2) $r^\alpha_{l+1} \leq_\alpha r^\alpha_l$.
(3) $(r^\alpha_{l+1})^{h_l}_{\bar{\sigma}_l}$ forces a value $\check{x}(\alpha, l)$ for $\chi_{\dot{X}} \restriction \alpha$.
(4) For $l$ limit ordinal $r^\alpha_l = \bigwedge_{k<l} r^\alpha_k$.

It is enough to explain how the successor step is built: Suppose then that we have constructed $r^\alpha_l$ satisfying the conditions above and consider the pair $\gamma_l = (\bar{\sigma}, h)$ and find a condition $w_l \leq (r_l)^h_{\bar{\sigma}}$ forcing a value $x(\bar{\sigma}, l)$ for $\chi_{\dot{X}} \restriction \alpha$, $w_l$ is clearly not a condition that satisfies (2), so we build $r^\alpha_{l+1}$ as follows: $\mathrm{supp}(r^\alpha_{l+1}) = \mathrm{supp}(w_l)$ and

$$r^\alpha_{l+1}(i) = \begin{cases} (w_l(i)) \cup \{(q_\alpha(i))_{\tau^\frown j} : \tau \in \mathrm{split}_\alpha(r_l(i)) \setminus \sigma_i \text{ or } j = 1 - h(i)\} & \text{if } i \in F_\alpha \\ w_l(i) & \text{otherwise} \end{cases}$$

Note that this is now a condition satisfying the properties above. Finally, put $q_{\alpha+1} = \bigwedge_{l<\rho} r_l^\alpha$ and $F_{\alpha+1} = F_\alpha \cup \{$the first $\alpha$ many elements of $\mathrm{supp}(q_{\alpha+1} \backslash F_\alpha)\}$. Now, if $\bar\tau \in \Lambda^{F_\alpha}_{\alpha+1}(q_{\alpha+1})$ there is $x(\bar\tau) \in {}^\alpha 2$ such that $(q_{\alpha+1})_\tau \Vdash \chi_{\dot X} \restriction \alpha = \check x(\bar\tau)$. Indeed, if $\bar\tau \in \Lambda^{F_\alpha}_{\alpha+1}(q_{\alpha+1})$, then there is $\bar\sigma \in \Lambda^{F_\alpha}_\alpha(q_{\alpha+1})$ such that for each $i \in F_\alpha$ there is $j_i \in \{0,1\}$ so that $\sigma_i^\frown j \trianglelefteq \tau_i$. Since $\sigma_i \in \mathrm{split}_\alpha(q_{\alpha+1}(i)) = \mathrm{split}_\alpha(q_\alpha(i))$ using the induction hypothesis we get that $(q_\alpha)_{\bar\sigma} \Vdash \chi_{\dot X} \restriction \alpha = \check x(\bar\sigma)$. Also by the construction $(q_{\alpha+1})^h_{\bar\sigma} \Vdash \chi_{\dot X}(\alpha) = \check x(\bar\sigma, h)$ where $h(i) = j_i$, $(q_{\alpha+1})_{\bar\tau} \le (q_\alpha)_{\bar\sigma}$, $(q_{\alpha+1})_{\bar\tau} \le (q_{\alpha+1})^h_{\bar\sigma}$ and therefore $(q_{\alpha+1})_{\bar\tau} \Vdash \chi_{\dot X} \restriction (\alpha+1) = \check x_{\bar\tau}$ where $\check x_{\bar\tau} = \check x_{\bar\sigma} \cup \{\check x(\bar\sigma, h)\}$. □

### 7.2. **Outer hull.**

**Definition 38.**

- Let $p \in \mathbb{S}_\kappa$ and let $\dot X$ be a $\mathbb{S}_\kappa$-name for a subset of $\kappa$. For each $t \in \mathrm{split}_\alpha(p)$, we refer to the set $Y_t = \{\beta \in \kappa : p_t \not\Vdash \check\beta \notin \dot X\}$, as the outer hull of $\dot X$ below $p_t$. Moreover, if $q_{t,\beta} \le p_t$ and $q_{t,\beta} \Vdash \check\beta \in \dot X$, we say that $q_{t,\beta}$ is a witness for $\beta \in Y_t$.
- Let $p \in \mathbb{S}^\lambda_\kappa$ and let $\dot X$ be a $\mathbb{S}^\lambda_\kappa$-name for a subset of $\kappa$. For all $\alpha < \kappa$, $F \subseteq \mathrm{supp}(p) \cap \alpha$ and $\bar\sigma \in \Lambda^F_\alpha(p)$ we refer to the set $Y_{\bar\sigma} = \{\beta \in \kappa : p_{\bar\sigma} \not\Vdash \check\beta \notin \dot X\}$, as the outer hull of $\dot X$ below $p_{\bar\sigma}$. Moreover, if $q_{\bar\sigma,\beta} \le p_{\bar\sigma}$ and $q_{\bar\sigma,\beta} \Vdash \check\beta \in \dot X$, we say that $q_{\bar\sigma,\beta}$ is a witness for $\beta \in Y_{\bar\sigma}$.

**Remark 39.** Suppose $Y_t$ is the outer hull of $\dot X$ below $p_t$. If $p_t \Vdash \check\beta \in \dot X$, then $p_t$ is a witness to $\beta \in Y_t$ and so $p_t \Vdash \dot X \subseteq \check Y_t$. Analogously for $Y_{\bar\sigma}$.

**Lemma 40.**

(1) Let $p \in \mathbb{S}_\kappa$ be preprocessed for $\dot X$ and for each $t \in \mathrm{split}(p)$ let $Y_t$ be the outer hull of $\dot X$ below $p_t$. Then for each $\alpha < \kappa$, $t \in \mathrm{split}_\alpha(p)$ and $\beta \in Y_t$ there is a $r(t,\beta) \in p$ such that $t$ is an initial segment of $r(t,\beta)$ and $p_{r(t,\beta)} \Vdash \check\beta \in \dot X$. Moreover, for each $t \in \mathrm{split}(p)$ and each $\beta \in Y_t$ there is $r = r(t,\beta) \in \mathrm{split}_\beta(p)$ such that $p_r \Vdash \check\beta \in \dot X$.

(2) Let $p \in \mathbb{S}^\lambda_\kappa$ be preprocessed for $\dot X$ and for each $\bar\sigma \in \Lambda^F_\alpha(p)$ let $Y_{\bar\sigma}$ be the outer hull of $\dot X$ below $p_{\bar\sigma}$. Then for each $\alpha < \kappa$, $\bar\sigma \in \Lambda^F_\alpha(p)$, $\beta \in Y_{\bar\sigma}$ and $i \in F$ there is a $r_i(\bar\sigma,\beta) \in p(i)$ such that $\sigma_i$ is an initial segment of $r_i(\bar\sigma,\beta)$ and $p_{\bar\tau} \Vdash \check\beta \in \dot X$ where $\bar\tau = (r_i(\bar\sigma,\beta) : i \in F)$. Moreover, for each $\bar\sigma \in \Lambda^F_\alpha(p)$, $\beta \in Y_{\bar\sigma}$ and $i \in F$ there is $r_i = r_i(\bar\sigma,\beta) \in \mathrm{split}_\beta(p)(i)$ such that $p_{\bar\tau} \Vdash \check\beta \in \dot X$ where $\bar\tau = (r_i(\bar\sigma,\beta) : i \in F)$.

*Proof.* (1) Fix $\alpha, t$ and $\beta \in Y_t$. Then there is $q \le p_t$ such that $q \Vdash \beta \in \dot X$. Take any $r = r(t,\beta) \in \mathrm{split}_\beta(q)$. Note that $t \trianglelefteq r$. There is $\beta' \ge \beta$ such that $r \in \mathrm{split}_{\beta'}(p)$. By Claim 37 there is $x_t \subseteq {}^{\beta'}2$ such that $p_r \Vdash \chi_{\dot X} \restriction \beta' \in \check x_t$. Let $y = \chi_{\dot X} \restriction \beta'$. Since $q_r \le p_r$ we must have that $y(\beta) = 1$. Thus $p_r \Vdash \beta \in \dot X$.

Moreover, if $r^* \trianglelefteq r$ and $r^* \in \mathrm{split}_\beta(p)$ then already $p_{r^*} \Vdash \check\beta \in \dot X$. In particular, if $\beta \le \mathrm{sl}(t,p)$ then in fact $p_t \Vdash \check\beta \in \dot X$.

(2) In the same way, fix now $\alpha, F$ and $\bar\sigma$. Then there is $q \le p_{\bar\sigma}$ such that $q \Vdash \beta \in \dot X$. For all $i \in F$ take any $r_i = r_i(\bar\sigma,\beta) \in \mathrm{split}_\beta(q(i))$. Note that $\sigma_i \trianglelefteq r_i$ for all $i \in F$. We can find $\beta' \ge \beta$ such that $r_i \in \mathrm{split}_{\beta'}(p(i))$ uniformly for all $i \in F$. Hence if $\bar\tau = (r_i(\bar\sigma,\beta) : i \in F)$, by Claim 37 there is $x_{\bar\tau} \in {}^{\beta'}2$ such that $p_{\bar\tau} \Vdash \chi_{\dot X} \restriction \beta' = \check x_{\bar\tau}$. Let $y = \chi_{\dot X} \restriction \beta'$. Since $q_{\bar\tau} \le p_{\bar\tau}$ we must have that $y(\beta) = 1$. Thus $p_{\bar\tau} \Vdash \beta \in \dot X$. The argument for the moreover part is the same as for part (1). □

**Definition 41.** For $p \in \mathbb{S}^\lambda_\kappa$ and $F \in [\mathrm{supp}(p)]^{<\kappa}$, let $\Lambda(F) = \bigcup_{\alpha<\kappa} \Lambda^F_\alpha$. For $\bar\sigma$ and $\bar\tau$ in $\Lambda(F)$ we say that $\bar\tau \trianglelefteq \bar\sigma$ iff for each $i \in F(\bar\tau(i) \trianglelefteq \bar\sigma(i))$.

**Corollary 42.**

(1) Let $\dot X$ be a $\mathbb{S}_\kappa$-name for an infinite subset of $\kappa$. If $p \in \mathbb{S}_\kappa$ is preprocessed for $\dot X$, $t \in \mathrm{split}(p)$ and $Y_t$ is the outer hull of $\dot X$ below $p_t$ then

$$Y_t = \{\beta < \kappa : \exists r \in \mathrm{split}(p) \text{ such that } t \trianglelefteq r \text{ or } r \trianglelefteq t \text{ and } p_r \Vdash \check\beta \in \dot X\}.$$

(2) Let $\dot X$ be a $\mathbb{S}^\lambda_\kappa$-name for an infinite subset of $\kappa$. If $p \in \mathbb{S}^\lambda_\kappa$ is preprocessed for $\dot X$, $F \in [\mathrm{supp}(p)]^{<\kappa}$ and $\sigma \in \Lambda(F)$, then

$$Y_{\bar\sigma} = \{\beta \in \kappa : \exists \bar\tau \in \Lambda(F) \text{ s.t. } \bar\tau \trianglelefteq \bar\sigma \text{ or } \bar r \trianglelefteq \bar\sigma \text{ and } p_{\bar\tau} \Vdash \check\beta \in \dot X\}.$$

## 8. $\kappa$-Sacks indestructibility

**Theorem 43.** *(GCH) Let $\kappa$ be a measurable cardinal, $\mathcal{U}$ a normal measure on $\kappa$ and let $G$ be $\mathbb{P}_\mathcal{U}$-generic filter over the ground model $V_0$. Let $\mathcal{A} = \mathcal{A}_G$ and $V = V_0[G]$. Then*

$$V^{\mathbb{S}_\kappa} \vDash \mathcal{A} \textit{ is a densely maximal independent family.}$$

*Proof.* Note that GCH holds in $V$ and $\kappa$ is inaccessible in $V$. By Lemma 28 the family $\mathcal{A}$ is densely maximal in $V$. To prove that $(\mathcal{A}$ is densely maximal$)^{V^{\mathbb{S}_\kappa}}$ we will show that in $V^{\mathbb{S}_\kappa}$, property $(*)$ of $\mathcal{A}$ from Lemma 27 holds. More precisely, we will show that in $V^{\mathbb{S}_\kappa}$ for each $X \subseteq \kappa$ and each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that $X \subseteq \mathcal{A}^h$ property $(*)_{X,h}$ holds, where

$(*)_{X,h}$ either $\exists B \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash X \subseteq B$ or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash X$.

Suppose not. Thus, there are $X \subseteq \kappa$, $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that $X \subseteq \mathcal{A}^h$ and $\neg(*)_{X,h}$. That is,

$$V^{\mathbb{S}_\kappa} \vDash X \subseteq \mathcal{A}^h \wedge \mathcal{A}^h \backslash X \notin \mathrm{id}_{<\omega,\kappa}(\mathcal{A}) \wedge \forall h' \supseteq h(\mathcal{A}^{h'} \cap X \neq \emptyset).$$

Let $\dot X$ be a $\mathbb{S}_\kappa$-name for $X$ in $V$ and let $p \in \mathbb{S}_\kappa$ force the above. By Lemma 37 we can assume that $p$ is preprocessed for $\dot X$ and by Corollary 42 that for each $t \in \mathrm{split}(p)$ for the outer hull $Y_t$ of $\dot X$ below $p_t$ is of the form $Y_t = \{\beta < \kappa : \exists r \in \mathrm{split}(p) \text{ such that } t \trianglelefteq r \text{ or } r \trianglelefteq t \text{ and } p_r \Vdash \check\beta \in \dot X\}$. Given $t \in \mathrm{split}(p)$ and $\beta \in Y_t$, let $r(t,\beta)$ be a witness to $\beta \in Y_t$ such that $t$ is comparable with $r(t,\beta)$ and $r(t,\beta)$ is of least splitting level. Define $H \in {}^\kappa\kappa \cap V$ as follows

$$H(\gamma) = \sup\{\gamma+1\} \cup \{\mathrm{sl}(r(t,\beta)) : t \in \mathrm{split}_\gamma(p), \beta \leq \gamma\},$$

where if $r(t,\beta)$ is not defined, i.e. $\beta \notin Y_t$, then we take $\mathrm{sl}(r(t,\beta)) = 0$.

**Claim 44.** Let $t \in \mathrm{split}(p)$. Then $Y_t \subseteq \mathcal{A}^h$.

*Proof.* Let $m \in Y_t$. Thus there is $q_{t,m} \leq p_t$ such that $q_{t,m} \Vdash \check m \in \dot X$. But $p \Vdash \dot X \subseteq \mathcal{A}^h$ and so $m$ must be an element of $\mathcal{A}^h$. □

Fix $t \in \mathrm{split}(p)$. Since $Y_t \subseteq \mathcal{A}^h$, by the dense maximality of $\mathcal{A}$ in $V$ either there is $B \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash Y_t \subseteq B$ or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash Y_t$. In the latter case, $\mathcal{A}^{h'} \cap Y_t = \emptyset$ and since $p \Vdash \dot X \subseteq \check Y_t$, we obtain that $p \Vdash \mathcal{A}^{h'} \cap \dot X = \emptyset$, contrary to the choice of $p$. Thus, we can assume that $\forall t \in \mathrm{split}(p) \exists B_t \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash Y_t \subseteq B_t$, $\mathcal{A}^h \backslash Y_t \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$. But then $Y_t \cup \kappa\backslash\mathcal{A}^h \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$. Now, since $\mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ is a $\kappa$-p-set, there is $C \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ such that $C \subseteq^* Y_t \cup \kappa\backslash\mathcal{A}^h$ for each $t \in \mathrm{split}(p)$. Thus in particular, $C \cap \mathcal{A}^h \subseteq^* Y_t$ for all $t \in \mathrm{split}(p)$ and so we can find a function $f \in V \cap {}^\kappa\kappa$ such that

$$\forall\alpha \in \kappa\ \forall t \in \mathrm{split}_{\alpha+1}(p)\ (C \cap \mathcal{A}^h)\backslash Y_t \subseteq f(\alpha).$$

Equivalently, for each $\alpha \in \kappa$,

$$(C \cap \mathcal{A}^h)\backslash f(\alpha) \subseteq \bigcap_{t\in\mathrm{split}_{\alpha+1}(p)} Y_t.$$

Moreover, we can assume that $H(\alpha) + 1 < f(\alpha)$ and that $f$ is strictly increasing.

By Corollary 25, there is $C^* \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ such that $\forall\alpha \in C^*\forall\gamma \in \alpha \cap C^*(f^2(\gamma) < \alpha)$. Now, let $C' = C \cap C^* \cap (f(1), \kappa)$. Thus $C' \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$. Let $\{k(\alpha) : \alpha < \kappa\}$ be an increasing enumeration of $C' \cap \mathcal{A}^h$. Since $C' \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ the latter set is indeed unbounded in $\kappa$. Recursively, we will define a fusion sequence $\tau = \langle q_\alpha : \alpha \in \kappa\rangle$ below $p$ such that if $q$ is the fusion of $\tau$ then $q \Vdash C' \cap \mathcal{A}^h \subseteq \dot{X}$. But then, since $q \Vdash \mathcal{A}^h\backslash\dot{X} \subseteq \mathcal{A}^h\backslash C'$ and $\mathcal{A}^h\backslash C' \subseteq \omega\backslash C' \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$, we obtain that $q \Vdash \mathcal{A}^h\backslash\dot{X} \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$, which is again a contradiction to the choice of $p$.

Here is the construction of $\tau$. Start with $q_0 = p$ and at limits take intersections. Consider $k(0)$ and put $r = \mathrm{stem}(p)$. Since

$$(C \cap \mathcal{A}^h)\backslash f(0) \subseteq \bigcap_{t\in\mathrm{split}_1(p)} Y_t,$$

for each $j \in \{0, 1\}$ and $p$ is preprocessed for $\dot{X}$ there is $r_j(t, k(0)) \in \mathrm{split}_{H(k(0))}(p)$ such that $p_{r_j(t,k(0))} \Vdash \check{k}(0) \in \dot{X}$. Let $q_1 = \bigcup\{p_{r_j(t,k(1))} : t \in \mathrm{split}_1(p), j \in \{0, 1\}\}$. Thus $q_1 \leq_0 q_0$ as we wanted.

Now, consider $k(1)$ and $t \in \mathrm{split}_1(q_1)$. First note that since $\mathrm{split}_1(q_1) = \mathrm{split}_\delta(p)$ for some $1 \leq \delta \leq H(k(0))$ and since $f^2(k(0)) < k(1)$ and $C\backslash Y_r \subseteq f(f(k(0)))$ for all $r \in \mathrm{split}_2(p)$ we get that $k(1) \in Y_r$ and we can use the fact that $p$ is preprocessed and repeat the argument above and find conditions $r_j(t, k(1)) \leq p_{t^\frown j}$ forcing $r_j(t, k(1)) \Vdash k(1) \in \dot{X}$ for all $j \in \{0, 1\}$. Put $q_2 = \bigcup\{r_j(t, k(1)) : t \in \mathrm{split}_1(q_1) \wedge j \in \{0, 1\}\}$, this is a condition $q_2 \leq_1 q_1$ and $q_2 \Vdash k(1) \in \dot{X}$.

In general, suppose we have constructed $q_\alpha$ and consider $k(\alpha)$ and $t \in \mathrm{split}_\alpha(q_\alpha)$, then there is $\delta \geq \alpha$ so that $t \in \mathrm{split}_\delta(p)$ and $\delta \leq H(k(\alpha))$ and since $f^2(k(\alpha)) < k(\alpha+1)$ and $C\backslash Y_r \subseteq f(f(k(\alpha)))$ for all $r \in \mathrm{split}_{\alpha+1}(p)$ we get that $k(\alpha) \in Y_t$, so again use that $p$ is preprocessed and repeat the argument above to find conditions $r_j(t, k(\alpha)) \leq p_{t^\frown j}$ forcing $r_j(t, k(\alpha)) \Vdash k(\alpha) \in \dot{X}$ $j \in \{0, 1\}$. Put $q_{\alpha+1} = \bigcup\{r_j(t, k(\alpha)) : t \in \mathrm{split}_\alpha(q_1) \wedge j \in \{0, 1\}\}$, this is a condition $q_{\alpha+1} \leq_\alpha q_\alpha$ and $q_{\alpha+1} \Vdash k(\alpha) \in \dot{X}$.

□

**Theorem 45.** *The generic maximal independent family adjoined by $\mathbb{P}_\mathcal{U}$ over a model $V_0$ of GCH remains maximal after the $\kappa$-support product $\mathbb{S}_\kappa^\lambda$.*

*Proof.* The idea for this proof is not much different than the one for the successor step, however the fusion argument has to be handled more carefully. We give now an outline with the most important details.

As in the case above start with a densely maximal independent family $\mathcal{A}$ in $V = V^{\mathbb{P}_{\mathcal{U}}}$. To prove that

$$(\mathcal{A} \text{ is densely maximal independent family})^{V^{\mathbb{S}_\kappa^\lambda}}$$

we will show that in $V^{\mathbb{S}_\kappa^\lambda}$, property $(*)$ of $\mathcal{A}$ from Lemma 27 holds. Specifically, we show that in $V^{\mathbb{S}_\kappa^\lambda}$ for each $X \subseteq \kappa$ and each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that $X \subseteq \mathcal{A}^h$ property $(*)_{X,h}$ holds, where $(*)_{X,h}$ either $\exists B \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash X \subseteq B$ or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash X$. Suppose not. Thus, there are $X \subseteq \kappa$, $h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A})$ such that $X \subseteq \mathcal{A}^h$ and $\neg(*)_{X,h}$. That is,

$$V^{\mathbb{S}_\kappa^\lambda} \vDash X \subseteq \mathcal{A}^h \wedge \mathcal{A}^h \backslash X \notin \mathrm{id}_{<\omega,\kappa}(\mathcal{A}) \wedge \forall h' \supseteq h(\mathcal{A}^{h'} \cap X \neq \emptyset).$$

Let $\dot{X}$ be a $\mathbb{S}_\kappa^\lambda$-name for $X$ in $V$ and let $p \in \mathbb{S}_\kappa^\lambda$ force the above. Passing to a stronger condition if necessary we can assume that $p$ is preprocessed for $\dot{X}$.

For every $\alpha < \kappa$ and all $F \subseteq \mathrm{supp}_\alpha(p)$ consider the sets $\Lambda_\alpha^F(p)$. These have size $<\kappa$ because $|\mathrm{split}_\alpha(p(i))| \leq 2^{|\alpha|}$ and $|F| < \kappa$ (Note that there are at most $\kappa$-many $\Lambda_\alpha^F$'s). Also, for each $\bar{\sigma}$, consider the outer hull for $p$ and $\dot{X}$, $Y_{\bar{\sigma}}$. Note also that there are at most $\kappa$-many $Y_{\bar{\sigma}}$'s.

Given $\bar{\sigma} \in \Lambda_\alpha^F(p)$ and $\beta \in Y_{\bar{\sigma}}$, let $r(\bar{\sigma}, \beta)$ be a witness to $\beta \in Y_{\bar{\sigma}}$ such that for each $i \in F$ the stem of the condition $r(\bar{\sigma}, \beta)(i)$ is of minimal height. Define $H \in {}^\kappa\kappa \cap V$ as follows:

$$H(\gamma) = \sup\{\gamma + 1\} \cup \{\mathrm{sl}(\mathrm{stem}(r(\bar{\sigma}, \beta))) : \bar{\sigma} \in \Lambda_\gamma^F(p), \beta \leq \gamma, F \subseteq \mathrm{supp}_\alpha(p)\},$$

Analogously we get

**Claim 46.** For all $\alpha < \kappa$, $F \subseteq \mathrm{supp}_\alpha(p)$ and $\bar{\sigma} \in \Lambda_\alpha^F(p)$, $Y_{\bar{\sigma}} \subseteq \mathcal{A}^h$.

Again, following the argument above, we can assume that for all $\alpha < \kappa$ and all $\bar{\sigma} \in \Lambda_\alpha^F(p)$ there exists $B_{\bar{\sigma}} \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$ such that $\mathcal{A}^h \backslash Y_{\bar{\sigma}} \subseteq B_{\bar{\sigma}}$, $\mathcal{A}^h \backslash Y_{\bar{\sigma}} \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$. But then $Y_{\bar{\sigma}} \cup \kappa \backslash \mathcal{A}^h \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$. Now, since $\mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ is a $\kappa$-p-set, there is $C \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ such that $C \subseteq^* Y_{\bar{\sigma}} \cup \kappa \backslash \mathcal{A}^h$ for each $\bar{\sigma} \in \Lambda_\alpha^F$. Thus in particular, $C \cap \mathcal{A}^h \subseteq Y_{\bar{\sigma}}$ for each $\bar{\sigma} \in \Lambda_\alpha^F(p)$ and so we can find a function $f \in V \cap {}^\kappa\kappa$ such that,

$$\forall \alpha \in \kappa \ \forall F \subseteq \mathrm{supp}(p) \ \forall \bar{\sigma} \in \Lambda_{\alpha+1}^F(p) \ (C \cap (\mathcal{A}^h \backslash Y_{\bar{\sigma}}) \subseteq f(\alpha))$$

In other words, for each $\alpha \in \kappa$, $\bar{\sigma} \in \Lambda_\alpha^F(p)$ and $\beta \in C \cap \mathcal{A}^h$, if $\beta > f(\alpha)$ then $\beta \in Y_{\bar{\sigma}}$. Moreover, we can assume that $H(\alpha) \leq f(\alpha)$ and $\alpha + 2 < f(\alpha)$ for all $\alpha \in \kappa$.

By Corollary 25, there is $C^* \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ such that $\forall \alpha \in C^* \forall \gamma \in \alpha \cap C^*(f(\gamma) < \alpha)$. Now, let $C' = C \cap C^* \cap [f(0), \kappa)$. Thus $C' \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$. Let $\{k(\alpha) : \alpha \in \kappa\}$ be an increasing enumeration of $C' \cap \mathcal{A}^h$. Since $C' \in \mathrm{fil}_{<\omega,\kappa}(\mathcal{A})$ the latter set is indeed unbounded in $\kappa$. Recursively, we will define a fusion sequence $\tau = \langle q_\alpha, F_\alpha : \alpha \in \kappa \rangle$ below $p$ such that if $q$ is the fusion of $\tau$ then $q \Vdash C' \cap \mathcal{A}^h \subseteq \dot{X}$. But then, since $q \Vdash \mathcal{A}^h \backslash \dot{X} \subseteq \mathcal{A}^h \backslash C'$ and $\mathcal{A}^h \backslash C' \subseteq \omega \backslash C' \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$, we obtain that $q \Vdash \mathcal{A}^h \backslash \dot{X} \in \mathrm{id}_{<\omega,\kappa}(\mathcal{A})$, which is again a contradiction to the choice of $p$.

We finish the proof by showing how the construction of $\tau$ goes: Start with $q_0 = p$ and $F_0 = \{\min(\mathrm{supp}(p))\}$. For limit $\alpha < \kappa$ construct $p_\alpha$ and $F_\alpha$ so that conditions in Definition 32 are fulfilled. Consider $k(0)$ and put $\bar{\sigma} = \langle \mathrm{stem}(p(i)) : i \in F_0 \rangle$. Since for all $\bar{\tau} \in \Lambda_1^{F_0}(p)$, $C \setminus Y_{\bar{\sigma}} \subseteq f(0)$ and $k(0) \geq f(0)$ and $p$ is preprocessed for $\dot{X}$ we have that $k(0) \in Y_{\bar{\tau}}$ and for each $h \in^{F_0} 2$ there exists a condition $r_h(\bar{\sigma}, k(0)) \leq p^h_{\bar{\sigma}}$ such that $r_h(\bar{\sigma}, k(0)) \Vdash k(0) \in \dot{X}$. Recall that the condition $p^h_{\bar{\sigma}}$ is defined as follows:

$$p^h_{\bar{\sigma}}(i) = \begin{cases} (p(i))_{\sigma_i^\frown h(i)} & \text{if } i \in F \\ p(i) & \text{otherwise} \end{cases}$$

Then if $q_1 = \bigcup\{r_h(\bar{\sigma}, k(0)) : h \in {}^{F_0}2, \sigma \in \Lambda_0^{F_0}(p)\}$ and $F_1 = F_0 \cup \{\min\{\mathrm{supp}(q_1)\backslash F_0\}\}$ we have $q_1 \leq_{0,F_0} q_0$ and $q_1 \Vdash k(0) \in \dot{X}$ as we wanted.

In general, suppose we have constructed $q_\alpha$ and $F_\alpha$ as desired. Consider $k(\alpha)$ and the set $\Lambda^{F_\alpha}_{\alpha+1}(p_\alpha)$. Fix an enumeration $\{\gamma_l = (\bar{\sigma}_l, h_l) : l < \rho\}$ of all pairs of the form $(\bar{\sigma}, h)$ such that $\bar{\sigma} \in \Lambda^{F_\alpha}_{\sigma}(q_\alpha)$ and $h \in {}^{F_\alpha}2$. Note that the ordinal $\rho$ is $< \kappa$.

Inductively, we will construct a sequence $\{r^\alpha_l : l < \rho\}$ of conditions below $q_\alpha$ satisfying:

1. $r^\alpha_0 = q_\alpha$.
2. $r^\alpha_{l+1} \leq_\alpha r^\alpha_l$.
3. $(r^\alpha_{l+1})^{h_l}_{\bar{\sigma}_l} \Vdash \check{k}(\alpha) \in \dot{X}$.
4. For $l$ limit ordinal $r^\alpha_l = \bigwedge_{k<l} r^\alpha_k$.

It is enough to explain how the successor step is built: Suppose then that we have constructed $r^\alpha_l$ satisfying the conditions above and consider the pair $\gamma_l = (\bar{\sigma}, h)$. Notice that if $i \in F_\alpha$, $\mathrm{split}_\alpha(q_\alpha(i)) \subseteq \mathrm{split}_\delta(p(i))$ for some $\delta \leq H(k(\alpha))$ so without loss of generality we can assume that $\bar{\sigma}(i) \in \mathrm{split}_{\leq\delta}(q_\alpha(i))\}$ for all $i$. Also, since $f^2(k(\alpha)) < k(\alpha+1)$ and for all $\bar{\tau} \in \Lambda^{F_\alpha}_{\alpha+1}(p_\alpha)$, $C\backslash Y_{\bar{\tau}} \subseteq f(f(k(\alpha)))$ we get that $k(\alpha) \in Y_{\bar{\tau}}$. Thus, again we repeat the argument above using that $p$ is preprocessed for $\dot{X}$ and find a condition $w_h(\bar{\sigma}, k(\alpha)) \leq p^h_{\bar{\sigma}}$ forcing $w_h(\bar{\sigma}, k(\alpha)) \Vdash k(\alpha) \in \dot{X}$, moreover without loss of generality we can choose it so that $w_h(\bar{\sigma}, k(\alpha)) \leq (q_\alpha)^h_{\bar{\sigma}}$; $w_h(\bar{\sigma}, k(\alpha))$ is clearly not a condition that satisfies (2), so we build $r^\alpha_{l+1}$ as follows: $\mathrm{supp}(r^\alpha_{l+1}) = \mathrm{supp}(w_l)$ and

$$r^\alpha_{l+1}(i) = \begin{cases} (w_h(\bar{\sigma}, k(\alpha))(i) \cup \{(q_\alpha(i))_{\tau^\frown j} : \tau \in \mathrm{split}_\alpha(r_l(i)) \setminus \sigma_i \text{ or } j = 1 - h(i)\} & \text{if } i \in F_\alpha \\ w_h(\bar{\sigma}, k(\alpha))(i) & \text{otherwise} \end{cases}$$

Note that this is now a condition satisfying the properties above. Finally, put $q_{\alpha+1} = \bigwedge_{l<\rho} r^\alpha_l$ and $F_{\alpha+1} = F_\alpha \cup \{\text{the first } \alpha \text{ many elements of } \mathrm{supp}(q_{\alpha+1}\backslash F_\alpha)\}$. The construction is now complete. Indeed, to see that $q_{\alpha+1} \Vdash \check{k}(\alpha) \in \dot{X}$ notice that for all $\bar{\sigma} \in \Lambda_\alpha(q_{\alpha+1})$ for all $F \subseteq \mathrm{supp}_\alpha(q_{\alpha+1})$ and $h \in {}^F2$ then $(q_{\alpha+1})^h_{\bar{\sigma}} \Vdash \check{k}(\alpha) \in \dot{X}$. □

**Remark 47.** Note that $\kappa$ might cease to be measurable in $V^{\mathbb{S}^\lambda_\kappa}$ from the above theorem. For a preparation of the universe, which guarantees that $\kappa$ remains measurable see [10].

## 9. Concluding Remarks and Questions

The use of the assumption $2^\kappa = \kappa^+$ played a crucial role in our construction of a densely maximal $\kappa$-independent family. Thus one may ask:

**Question 48.** Does ZFC imply the existence of a densely maximal $\kappa$-independent families?

Even though we are able to show both that consistently $\mathfrak{i}_f(\kappa) = \kappa^+ < 2^\kappa$ and $\kappa^+ < \mathfrak{i}_f(\kappa) = 2^\kappa$, the currently available techniques seem to be insufficient to answer the following:

**Question 49.** Let $\kappa$ be a regular uncountable cardinal. Is it consistent that $\kappa^+ < \mathfrak{i}(\kappa) < 2^\kappa$?

The analogous question in the countable can be answered to the positive with the use of the so called diagonalization filters (see [8]). A natural generalization of the notion of a diagonalization filter to the uncountable is given below:

**Definition 50.** Let $\mathcal{A}$ be a $\kappa$-independent family. A $\kappa$-complete filter $\mathcal{F}$ is said to be an $\kappa$-*diagonalization filter for* $\mathcal{A}$ if $\forall F \in \mathcal{F} \forall h \in \mathrm{FF}_{<\omega,\kappa}(\mathcal{A}) |F \cap \mathcal{A}^h| = \kappa$ and $\mathcal{F}$ is maximal with respect to the above property.

Moreover, as a straightforward generalization of the countable case (see [8]) one can show that:

**Lemma 51.** (see [8, Lemma 2]) Suppose $\mathcal{A}$ is a $\kappa$-independent family and $\mathcal{F}$ is a $\kappa$-diagonalization filter for $\mathcal{A}$. Let $\mathbb{M}^\kappa_\mathcal{F}$ be the generalized Mathias forcing relativized to the filter $\mathcal{F}$.[5] Let $G$ be a $\mathbb{M}^\kappa_\mathcal{F}$-generic filter and let $x_G = \bigcup\{a : \exists A (a, A) \in G\}$. Then $\mathcal{A} \cup \{x_G\}$ is $\kappa$-independent and moreover for each $Y \in ([\kappa]^\kappa \cap V) \backslash \mathcal{A}$ such that $\mathcal{A} \cup \{Y\}$ is $\kappa$-independent, the family $\mathcal{A} \cup \{x_G, Y\}$ is not $\kappa$-independent.

Even though an appropriate iteration of posets of the above form would produce a positive answer to Question 49, the following remains open:

**Question 52.** Given a $\kappa$-independent family $\mathcal{A}$ is there a $\kappa$-diagonalization filter for $\mathcal{A}$? The co-bounded filter does satisfy the characterization property in Definition 50, however the requirement for maximality is not straightforward to satisfy. Is there a large cardinal property which guarantees the existence of such maximal filter? Note that a diagonalization filter is never an ultrafilter.

Moreover of interest remain the following:

**Question 53.** Is it consistent that $\mathfrak{i}(\kappa) < \mathfrak{a}(\kappa)$?

Clearly, if the above is consistent then in the corresponding model, $\mathfrak{i}(\kappa) \geq \kappa^{++}$. One of the original questions, which motivated the work on this project is the evaluation of $\mathfrak{i}(\kappa)$ in the model from [5]. More precisely, we would like to know:

**Question 54.** Is it consistent that $\mathfrak{i}(\kappa) < \mathfrak{u}(\kappa)$?

The consistency of $\mathfrak{r} < \mathfrak{i}$ holds in the Miller model. However, products of the generalized Miller poset $\mathbb{MI}^\mathcal{U}_\kappa$, where $\mathcal{U}$ is a $\kappa$-complete normal ultrafilter on $\kappa$ add $\kappa$-Cohen reals (see [3, Theorem 85]) and so increase $\mathfrak{r}(\kappa)$. Even though $\mathbb{MI}^\mathcal{U}_\kappa$ has the generalized Laver property (see [3, Proposition 81]), it is open if the generalized Laver property is preserved under $\kappa$-support iterations. This leaves us with the following:

**Question 55.** Is it consistent that $\mathfrak{r}(\kappa) < \mathfrak{i}(\kappa)$?

[5]That is $\mathbb{M}^\kappa_\mathcal{F}$ consists of all pairs $(a, A) \in [\kappa]^{<\kappa} \times \mathcal{F}$ such that $\sup a < \min A$.

## 10. Appendix: Strong Independence

Another approach towards finding a higher analogues of independence for a given uncountable cardinal $\kappa$ is to consider boolean combinations generated by strictly less than $\kappa$ (not just finitely) many members of the family. More precisely one can give the following definition:

**Definition 56.** Let $\kappa$ be a regular uncountable cardinal, $\mathcal{A} \subseteq [\kappa]^\kappa$ of cardinality at least $\kappa$.

(1) Let $\mathrm{FF}_{<\kappa,\kappa}(\mathcal{A})$ be the set of partial functions $h : \mathcal{A} \to \{0,1\}$ with domain of cardinality strictly below $\kappa$ and for $h \in \mathrm{FF}_{<\kappa,\kappa}(\mathcal{A})$ let $\mathcal{A}^h = \bigcap\{\mathcal{A}^{h(A)} : A \in \mathrm{dom}(h)\}$ where $\mathcal{A}^{h(A)} = A$ if $h(A) = 0$ and $\mathcal{A}^{h(A)} = \kappa \backslash A$ if $h(A) = 1$.
(2) The family $\mathcal{A}$ is said to be strongly-$\kappa$-independent if for every $h \in \mathrm{FF}_{<\kappa,\kappa}(\mathcal{A})$ the boolean combination $\mathcal{A}^h$ is unbounded.
(3) The family $\mathcal{A}$ is said to be maximal strongly-$\kappa$-independent if it is strongly-$\kappa$-independent and is not properly contained in another strongly-$\kappa$-independent family.
(4) Suppose $\kappa$ is a regular uncountable cardinal for which maximal strongly-$\kappa$-independent families exists. With $\mathfrak{i}_s(\kappa)$ we denote the minimal size of a maximal strongly-$\kappa$-independent family.

Note that the increasing union of a countable sequence of strongly-$\kappa$-independent families is not necessarily strongly-$\kappa$-independent. Thus one can not apply Zorn's lemma to claim the existence of maximal strongly-$\kappa$-independent families. What we can say is the following:

**Theorem 57.** *Let $\kappa$ be a regular uncountable caridnal.*

*(1) For $\kappa$ strongly inaccessible, there is a strongly-$\kappa$-independent family of cardinality $2^\kappa$.*
*(2) If $\mathcal{A}$ is strongly-$\kappa$-independent and $|\mathcal{A}| < \mathfrak{r}(\kappa)$ then $\mathcal{A}$ is not maximal.*
*(3) Suppose $\mathfrak{d}(\kappa)$ is such that for every $\gamma < \mathfrak{d}(\kappa)$, $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$. If $\mathcal{A}$ is strongly-$\kappa$-independent and $|\mathcal{A}| < \mathfrak{d}(\kappa)$ then $\mathcal{A}$ is not maximal.*

*Proof.* We will prove (1). Let $\mathcal{C} = \{(\gamma, A) : \gamma < \kappa, A \subseteq \mathcal{P}(\gamma)\}$. Given $X \subseteq \kappa$ define $\mathcal{Y}_X = \{(\gamma, A) \in \mathcal{C} : X \cap \gamma \in A\}$. Then $\mathcal{Y}_X : X \subseteq \kappa\}$ is strongly-$\kappa$-independent. Indeed. Consider two disjoint subfamilies of $[\kappa]^\kappa$, each of size strictly smaller than $\kappa$, say $\{X_i\}_{i\in I_1}$ and $\{Z_j\}_{j\in I_2}$. Note that $(\gamma, A) \in \mathcal{X} = \bigcap_{i\in I_1} \mathcal{Y}_{X_i} \cap \bigcap_{j\in I_2}(\mathcal{C}\backslash\mathcal{Y}_{Z_j})$ if for all $i \in I_1$, $X_i \cap A \in A$ and for all $j \in I_2$, $Z_j \cap \gamma \notin A$. However, there are unboundedly many $\gamma \in \kappa$ such that

- $X_i \cap \gamma \neq X_{i'} \cap \gamma$ for $i \neq i'$ both in $I_1$, and
- $Z_j \cap \gamma \neq Z_{j'} \cap \gamma$ for $j \neq j'$ both in $I_2$, and
- $X_i \cap \gamma \neq Z_j \cap \gamma$ for all $i \in I_1$, $j \in I_2$.

It remains to observe that for each such $\gamma$, we have $(\gamma, A_\gamma) \in \mathcal{X}$, where $A_\gamma = \{X_i \cap \gamma : i \in I_1\}$.

To see part (2) note that if $|\mathcal{A}| < \mathfrak{r}(\kappa)$, then the set $\{\mathcal{A}^h : h \in \mathrm{FF}_{<\kappa,\kappa}(\mathcal{A})\}$ is split by some $X \in [\kappa]^\kappa$ and so $\mathcal{A} \cup \{X\}$ is strongly $\kappa$-independent which properly contains $\mathcal{A}$.

For a proof of part (3), see [5, Proposition 27]. □

**Corollary 58.** Thus, if $\mathfrak{i}_s(\kappa)$ is defined, then $\kappa^+ \leq \mathfrak{i}_s(\kappa) \leq 2^\kappa$. Moreover $\mathfrak{r}(\kappa) \leq \mathfrak{i}_s(\kappa)$ and if for every $\gamma < \mathfrak{d}(\kappa)$, $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$, then $\mathfrak{d}(\kappa) \leq \mathfrak{i}_s(\kappa)$.

**Question 59.**

(1) Is there a large cardinal property which implies the existence of a maximal strongly-$\kappa$-independent family?

(2) Given a strongly $\kappa$-independent family $\mathcal{A}$, is there a large cardinal property which implies the existence of a $\kappa$-diagonalization filter for $\mathcal{A}$?

(3) Suppose $\mathfrak{i}_s(\kappa)$ is defined. A family which is strongly-$\kappa$-independent is $\kappa$-independent. However a maximal strongly independent family is not necessarily maximal independent. Is there a ZFC relation between $\mathfrak{i}_s(\kappa)$ and $\mathfrak{i}(\kappa)$?

## References


[1] T. Baumhauer *The higher Cichoń diagram* Doctoral Dissertation, Technical University of Vienna (2018).
[2] A. Blass *Cardinal characteristics of the continuum* Handbook of Set Theory, Springer.
[3] J. Brendle, A. Brooke-Taylor, S. D. Friedman, D. Montoya *Cichón's diagram for uncountable cardinals* Israel Journal of Mathematics; doi.org/10.1007/s11856-018-1688-y,2018.
[4] A. Blass, T. Hyttinen, Y. Zhang *Mad families and their neighbours*, preprint.
[5] A. Brooke-Taylor, V. Fischer, S. D. Friedman, D. C. Montoya *Cardinal characteristics at $\kappa$ is a small $\mathfrak{u}(\kappa)$ model* APAL 168 (1), pp. 37–49, 2017.
[6] V. Fischer *Maximal cofinitary groups revisited*, Math. Log. Quart. 61 (2015), 367 – 379.
[7] V. Fischer, D. C. Montoya *Ideals of Independence* Arch. Math. Logic (2019) 58: 767, doi.org/10.1007/s00153-019-00669-8.
[8] V. Fischer, S. Shelah *The spectrum of independence* Arch. Math. Logic (2019); doi.org/10.1007/s00153-019-00665-y.
[9] V. Fischer, S. Shelah *More on the spectrum of independence* preprint.
[10] S. D. Friedman *Cardinal characteristics of the uncountable*, Lecture notes, Institute for Mathematics, Kurt Gödel Research Center, University of Vienna, 2014.
[11] S. Geschke *Almost Disjoint and Independent Families* RIMS Kokyuroku 1790 (2012), pp. 1-9.
[12] L. Halbeisen *Combinatorial Set Theory. With a gentle introduction to forcing.* Springer Monographs in Mathematics, Springer, London pp. xvi+453 (2012).
[13] A. Kanamori *Perfect-set forcing for uncountable cardinals* Annals of Mathematical Logic 19, 97 - 114 (1980).
[14] K. Kunnen *Maximal $\sigma$-independent families* Fundamenta Mathematicae 117 (1), 75 - 80 (1983).
[15] M. Perron *On the structure of independent families* Doctoral Dissertation, Ohio University (2017).
[16] D. Raghavan, S. Shelah *Two results on cardinal invariants at uncountable cardinals* arXiv preprint arXiv:1801.09369 (2018)
[17] S. Shelah *Con*($\mathfrak{u} > \mathfrak{i}$), Archive for Mathematical Logic 31.6 (1992), pp. 433-443.
[18] S. Shelah, *A parallel to the null ideal for inaccessible $\lambda$*, Archive for Mathematical Logic 56 (2017), pp. 319–383.



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