INDEPENDENT FAMILIES AND COMPACT PARTITIONS

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ABSTRACT. Using a forcing of A. Miller, we prove that it is consistent that the independence number is ω_1 and there is no partition of size ω_1 of 2^ω into closed sets. Moreover, we establish the consistency of $\mathfrak{a} = \mathfrak{i} = \omega_1 < \mathfrak{u} = \mathfrak{a}_T = \omega_2$.

1. INTRODUCTION

One of the oldest and more fundamental questions regarding the theory of cardinal invariants of the continuum is the following question of Jerry Vaughan:

Problem 1 (Vaughan [48]). Is the inequality $\mathfrak{i} < \mathfrak{a}$ consistent?[1]

Not only this problem is interesting since it involves two fundamental objects in infinite combinatorics (maximal independent families and MAD families), but a positive solution to the problem will most likely require to develop new ideas and forcing techniques. The reason is that the most common forcing methods do not seem to help with the problem:

(1) *Finite support iteration of ccc forcings of length a regular cardinal over a model of CH.* This approach can not work since in the models obtained in this way, the size of the continuum is equal to $cov(\mathcal{M})$ and it is known that $cov(\mathcal{M}) \leq \mathfrak{i}$.[2]
(2) *Countable support iteration of definable proper forcings of length ω_2 over a model of CH.* It follows by the results of M. Džamonja, M. Hrušák and J. Moore in [41] that in all of this models the equality $\mathfrak{b} = \mathfrak{a}$ will hold, so in particular we will have that $\mathfrak{a} \leq \mathfrak{i}$.
(3) *Countable support iteration of non-definable proper forcings of length ω_2 over a model of CH.* This approach could work, however a model of $\mathfrak{i} < \mathfrak{a}$ obtained by this method will also be a model of $\omega_1 = \mathfrak{d} < \mathfrak{a}$, solving the problem of Roitman, which is consider to be one of the hardest problems on the theory of cardinal invariants.

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[1] The definitions of the undefined notions can be consulted in the next section.

[2] S. Shelah proved that $\mathfrak{d} \leq \mathfrak{i}$ (see the appendix of [48]). This result was improved by B. Balcar, F. Hernandez-Hernández and M. Hrušák in [2] where they proved that $cof(\mathcal{M}) \leq \mathfrak{i}$.

(4) *Forcing with ultrapowers and iterating along a template.* The method of forcing with ultrapowers and iterating along a template was introduced by S. Shelah in [45] to build models of $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$. This is a very powerful method that has been very useful and has been successfully applied to this day. Unfortunately, it seems that all forcings obtained using this method, tend to increase $\mathfrak{i}$ for the same reason they increase $\mathfrak{a}$. To learn more about this powerful method, see [9], [10], [8], [22], [36], [18], [16].

(5) *Short finite support iterations over models of* MA. Performing a finite support iteration of length ω_1 over a model of MA (for example) is a powerful method to add "small witnesses" of some cardinal invariants while keeping others large. Models obtained in this way are often called "dual models" (see [13] for several interesting results and applications of this methods). In [2] a dual model was constructed to add a small maximal independent family in order to build a model of $\mathfrak{i} < non(\mathcal{N})$. Unfortunately, it is not clear how one could avoid adding a small MAD family with this method. Moreover, it seems likely that the principle $\diamondsuit_{\mathfrak{d}}$ of M. Hrušák will hold in this models[3] (see [26]).

In principle, it could be possible to construct a model of $\mathfrak{i} < \mathfrak{a}$ using matrix iterations (see [7], [11] and [37] to learn more about this method), but one would need to be very careful in order to avoid problems like in the points 1 and 5 above.

In order to gain more insight into the problem of Vaughan, we can try to compare $\mathfrak{i}$ with other variants of the almost disjointness number. Several relatives of $\mathfrak{a}$ have been introduced and studied in the literature. In this article, we want to compare $\mathfrak{i}$ with the following cardinal invariant introduced by A. Miller in [40]:

Definition. Define $\mathfrak{a}_T$ as the smallest size of a partition of ω^ω into compact sets.

It is well-known that the Baire space ω^ω is not σ-compact (see [31]), which implies that $\mathfrak{a}_T$ is uncountable. Furthermore, $\mathfrak{d}$ is the least size of a family of compact sets covering ω^ω (see [3]), so it follows that $\mathfrak{d} \leq \mathfrak{a}_T$. It is known that the compact subspaces of the Baire space are in correspondence with the finitely branching subtrees of $\omega^{<\omega}$. Using this correspondence and König's lemma, it is easy to prove that $\mathfrak{a}_T$ is equal to least size of a maximal AD family of finitely branching subtrees of $\omega^{<\omega}$. M. Džamonja, M. Hrušák and J. Moore proved that $\diamondsuit_{\mathfrak{d}}$ implies that $\mathfrak{a}_T = \omega_1$ (see Theorem 7.6 of [41]), since $\diamondsuit_{\mathfrak{d}}$ holds in most of the natural models of $\mathfrak{d} = \omega_1$ (see [41] and [26] for a precise formulation of this statement), $\mathfrak{a}_T = \omega_1$ also holds in this models. To learn more about $\mathfrak{a}_T$, the reader may consult [40], [42], [47] and [27].

In the current article we obtain the following result. [4] (see Theorem 6).

Theorem. It is relatively consistent that $\mathfrak{i} < \mathfrak{a}_T$.

In order to achieve this, we will use a forcing introduced by A. Miller. In [40] for every partition of ω^ω into compact sets $\mathcal{K}$, Miller defined a forcing $\mathbb{P}(\mathcal{K})$ that destroys $\mathcal{K}$ (i.e. $\mathcal{K}$ no longer covers

[3] $\diamondsuit_{\mathfrak{d}}$ is the following principle: There is a family $\{d_\alpha \mid \alpha \in \omega_1\}$ such that $d_\alpha : \alpha \longrightarrow \omega$ and for every $F : \omega_1 \longrightarrow \omega_1$ there is $\alpha \geq \omega$ such that $F \restriction \alpha \leq^* d_\alpha$. In [26] it was proved that $\diamondsuit_{\mathfrak{d}}$ implies both $\mathfrak{d} = \omega_1$ and $\mathfrak{a} = \omega_1$.

[4] For an alternative proof see [21].

ω^ω after forcing with $\mathbb{P}(\mathcal{K})$). He proved (essentially) that this forcing is proper and has the Laver property. Some years later, O. Spinas proved that the forcing is ω^ω-bounding (see [47]). By combining this results, it follows that every partition of ω^ω into compact sets can be destroyed with a proper forcing that has the Sacks property, so the inequality $cof(\mathcal{N}) < \mathfrak{a}_T$ is consistent (in particular it is consistent that $\mathfrak{d} < \mathfrak{a}_T$, which used to be an open question). In Proposition 4.1.31 of [50], J. Zapletal proved that $\mathbb{P}(\mathcal{K})$ is forcing equivalent to the quotient of the Borel sets of ω^ω modulo a σ-ideal generated by closed sets. In this way, the forcing $\mathbb{P}(\mathcal{K})$ falls into the scope of the theory developed in [50] and [49]. In [24], M. Hrušák, O. Téllez and the third author proved that $\mathbb{P}(\mathcal{K})$ preserves tight MAD families (see [30], [28],[17], [1] to learn more about tight MAD families).

In order to prove that iterating forcings of the form $\mathbb{P}(\mathcal{K})$ keeps the independence number small, we will use the notion of *selective independent family*, which was introduced by S. Shelah in order to build a model of $\mathfrak{i} < \mathfrak{u}$ (see [44]). Selective independent families are families with very strong combinatorial properties, which resemble the combinatorial features of Ramsey ultrafilters. Studying the similarities and differences between selective independent families and Ramsey ultrafilters seems to be a very interesting line of research. The theory of selective independent families has been further extended by V. Fischer and D. Montoya in [19]. In [14] D. Chodounský, V. Fischer and J. Grebík used selective independent families to build a model of $\mathfrak{f} < \mathfrak{u}$. In his Ph.D. thesis, [43] Perron obtained more results regarding selective independent families. In [21] the second and fourth authors obtain a different proof of the fact that $\mathbb{P}(\mathcal{K})$ preserves selective independent families, show that $\mathbb{P}(\mathcal{K})$ preserves P-points and building on preservation theorems from [44] and [24] obtain the consistency of each of the following: $\mathfrak{a} = \mathfrak{i} = \mathfrak{u} = \omega_1 < \mathfrak{a}_T = \omega_1$ and $\mathfrak{i} = \omega_1 < \mathfrak{u} = \mathfrak{a}_T = \omega_2$.

In the current paper we show in addition that Shelah's poset $\mathbb{Q}_{\mathcal{I}}$ for destroying the maximality of a given maximal ideal from [44] strongly preserves tight MAD families and so establish (see Corollary 2):

Theorem. It is relatively consistent that $\mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{u} = \mathfrak{a}_T = \omega_2$.

2. Preliminaries

Recall that $\mathcal{B} \subseteq P(\omega)$[5] is an almost disjoint family if every element of $\mathcal{A}$ is infinite and for every distinct $A, B \in \mathcal{B}$, $A \cap B =^* \emptyset$. A family $\mathcal{B} \subseteq P(\omega)$ is an independent family if for every distinct $A_0, \ldots, A_n \in \mathcal{A}$ and $h : \{A_0, \ldots, A_n\} \longrightarrow 2$, $\bigcap\limits_{i \leq n} A_i^{f(A_i)}$ is infinite where $A_i^0 = A^i$ and $A_i^1 = \omega \backslash A^i$. To learn more about independent families, the reader may consult [6], [19], [20] and [12].

Now we recall the definitions of some cardinal invariants.

- $\mathfrak{c}$ is the cardinality of 2^ω.
- $\mathfrak{i}$ is the least size of a maximal independent family.
- $\mathfrak{b}$ is the least size of an unbounded family in $(\omega, \leq^*)$.
- $\mathfrak{d}$ is the least size of a dominating family in $(\omega, \leq^*)$

[5] $P(\omega)$ is the power set of ω.

- $cov(\mathcal{M})$ is the least size of a family of meager sets which union is 2^ω.
- $cof(\mathcal{M})$ is the least size of a family $\mathcal{X}$ of meager sets in 2^ω such that every meager set in 2^ω is contained in some element of $\mathcal{X}$.
- $\mathfrak{a}$ is the least size of an infinite maximal almost disjoint family.

To learn more about cardinal invariants, the reader may consult [6]. The Sacks forcing $\mathbb{S}$ consists of all perfect trees in $2^{<\omega}$ ordered by inclusion. That is, $p \in \mathbb{S}$ if an only if

(1) $p \subseteq 2^{<\omega}$,
(2) $\forall \sigma \in p\, \forall \tau \in 2^{<\omega}\big(\tau \subseteq \sigma \to \tau \in p\big)$,
(3) $\forall \sigma \in p\, \exists \tau, \tau' \in p\big(\sigma \subseteq \tau \wedge \sigma \subseteq \tau' \wedge \tau \not\subseteq \tau' \wedge \tau' \not\subseteq \tau\big)$.

Regarding this forcing, let us fix some notation. If $p \in \mathbb{S}$ and $\sigma \in p$ we let $p(\sigma) := \{\tau \in p \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$, and call σ a splitting node if $\sigma^\frown i \in p$ for each $i \in 2$. Let $split(p) := \{\sigma \in p \mid \sigma \text{ is a splitting node }\}$. For each $n \in \omega$ let $split_n(p) := \{\sigma \in split(p) \mid |\{\tau \in split(p) \mid \tau \subsetneq \sigma\}| = n\}$ and $stem(p)$ the unique element in $split_0(p)$. Finally, for $p \in \mathbb{S}$ let $[p] = \{f \in 2^\omega \mid \forall n \in \omega(\ f|_n \in p\ )\}$. To learn more about Sacks forcing, the reader may read [5], [4], [29], [23], [38], [15] or [51].

Definition 1. (Miller partition forcing) Let $\mathcal{K} \subseteq P(2^\omega)$ be an uncountable partition of 2^ω into closed sets and let $\mathbb{P}(\mathcal{K}) := \{p \in \mathbb{S} \mid \text{for every } K \in \mathbb{P},\ K \cap [p] \text{ is nowhere dense in } [p]\}$ ordered by reversed inclusion.

This forcing destroys the partition $\mathcal{K}$ in the following way; if G is a $\mathbb{P}(\mathcal{K})$-generic filter, then $r_{gen} := \bigcup \bigcap G$ is an element of 2^ω which does not belong to to the interpretation in $V[G]$ of any element of $\mathcal{K}$. So in $V[G]$, $\mathcal{K}$ is no longer a partition of 2^ω. Thus, if we start with a model of CH and define $\mathbb{PM}$ as the resulting model after forcing with a countable support iteration of length ω_2 of all forcing notions of the form $\mathbb{P}(\mathcal{K})$ with $\mathcal{K}$ ranging over all uncountable partitions in closed sets of 2^ω in all intermediate models, then $\mathbb{PM}$ will not have any uncountable partition in closed sets of 2^ω of size less than ω_2.

A. Miller defined and used $\mathbb{PM}$ in [40] to show that $cov(\mathcal{M}) = \omega_1$ does not imply that that $\mathfrak{a}_T = \omega_1$. In that same paper, he proved that $\mathbb{P}(\mathcal{K})$ has the Laver property and essentially proved that it is proper. Later, O. Spinas showed in [47] that $\mathbb{P}(\mathcal{K})$ is ω^ω-bounding.

Notice that if $\mathcal{K}$ is the partition of 2^ω into singletons, then $\mathbb{P}(\mathcal{K}) = \mathbb{S}$. Actually, it can be seen that if $\mathcal{K}$ is an analytic subset of $K(2^\omega)$ (The space of non empty closed subsets of 2^ω equiped with the Vietoris topology), then $\mathbb{P}(\mathcal{K})$ is actually forcing equivalent to the Sacks forcing $\mathbb{S}$.

Theorem 1. Let $\mathcal{K} \subseteq K(2^\omega)$ be an uncountable analytic partition of 2^ω. $\mathbb{P}(\mathcal{K})$ is forcing equivalent to the Sacks forcing $\mathbb{S}$.

Proof. Let $p \in \mathbb{P}(\mathcal{K})$. It is enough to find $q \in \mathbb{P}(\mathcal{K})$ such that $q \leq p$ and $\{r \in \mathbb{P}(\mathcal{K}) \mid r \leq q\} = \{r \in \mathbb{S} \mid r \subseteq q\}$. To do this consider $f : F(2^\omega) \longrightarrow 2^\omega$ given by $f(A) = minA$, and let $X = \{K \cap [p] \mid K \in \mathcal{K}\} \setminus \{\emptyset\}$. Notice that X is an uncountable analytic subset of $F(2^\omega)$, $f|_X$ is inyective and $im(f|_X) \subseteq [p]$. This implies that there is $q \in \mathcal{S}$ such that $[q] \subseteq im(f_X)$. It is easy to see that $q \subseteq p$ and $|[q] \cap K| \leq 1$ for every $K \in \mathcal{A}$. Checking that q is as desired is straight forward. $\square$

Through out the rest of the paper, we will try to adapt fusion techniques of the Sacks forcing to the forcing $\mathbb{P}(\mathcal{K})$. The main problem is to know when does a fusion sequence belong to this forcing. This problem will be simplified by means of the following proposition.

Proposition 1. Let $p \in \mathbb{S}$. $p \in \mathbb{P}(\mathcal{K})$ if and only if there is a dense $D \subseteq [p]$ such that every two different elements of D belong to different elements of $\mathcal{K}$.

The following lemmas will be used to prove Theorem 3.

Lemma 1. Let $p \in \mathbb{P}(\mathcal{K})$ and let $\dot{f}$ be a $\mathbb{P}(\mathcal{K})$-name such that $p \Vdash \dot{f} \in 2^\omega$. Then there exists $q \leq p$, $g \in [q]$ and $h \in 2^\omega$ such that for every $m, n \in \omega$, if $g|_m \in split_n(q)$ then

$$q(g|_m) \Vdash h(n) = \dot{f}(n).$$

Proof. Recursively construct a sequence $\{q_n\}_{n\in\omega} \subseteq \mathbb{P}(\mathcal{K})$ such that:

(1) $q_0 \leq p$
(2) $\forall n \in \omega\big(q_{n+1} \leq q_n\big)$
(3) $\forall n \in \omega\big(stem(q_n) \subsetneq stem(q_{n+1})\big)$
(4) $\forall n \in \omega \exists i_n \in 2\big(q_n \Vdash \dot{f}(n) = i_n\big)$

This construction is straight forward.

Let $g := \bigcup_{n\in\omega} stem(q_n)$, and for every $n \in \omega$ define $s_n := |stem(q_n)|$. Now define

$$q = \bigcup_{n\in\omega} q_n\big(stem(q_n)^\frown(1 - g(s_n))\big),$$

and h such that $h(n) = i_n$ for every $n \in \omega$. Using Lemma 1 it is easy to see that $q \in \mathbb{P}(\mathcal{K})$. Moreover $q \leq p$, $g \in q$, for every $n \in \omega$ $g|_{s_n} \in split_n(q)$, and $q(g|_{s_n}) \leq q_n$. This last statement implies that for every $n \in \omega$:

$$q(g|_{s_n}) \Vdash h(n) = \dot{f}(n)$$

So we are done.

□

The following lemma can be deduced abstractly with Proposition 4.1.31 and Theorem 4.1.2 from the book [50]. For the convenience of the reader, we provide a direct proof.

Lemma 2. Let $p \in \mathbb{P}(\mathcal{K})$ and let $\dot{f}$ be a $\mathbb{P}(\mathcal{K})$-name such that $p \Vdash \dot{f} \in 2^\omega$. Then there exists $q \leq p$ and a continuous $H : [q] \longrightarrow 2^\omega$ such that

$$q \Vdash H(\dot{r}_{gen}) = \dot{f}$$

Proof. For $q \in \mathbb{P}(\mathcal{K})$, $g \in [q]$ and $h \in 2^\omega$, let us provisionally call the triplet (q, g, h) *good* if it satisfies the conclusion of Lemma 1. That is, for every $m, n \in \omega$, if $g|_m \in split_n(q)$ then $q(g|_m) \Vdash h(n) = \dot{f}(n)$. Notice that if a triplet (q, g, h) is good and $m \in \omega$ then the triplet $(q(g|_m), g, h)$ is also good.

To aim our goal, recursively construct a sequence $\{(q_\sigma, g_\sigma, h_\sigma)\}_{\sigma\in 2^{<\omega}}$ of good triplets and a sequence $\{T_\sigma\}_{\sigma\in 2^{<\omega}}$ of elements of $\mathcal{K}$, such that:

(a) $q_\emptyset \leq p$,
(b) $\forall \sigma, \tau \in 2^{<\omega}\big(\sigma \subseteq \tau \to q_\tau \leq q_\sigma\big)$.
(c) $\forall \sigma, \tau \in 2^n\big(\sigma \neq \tau \to [q_\sigma] \cap [q_\tau] = \emptyset\big)$.
(d) $\forall \sigma, \tau \in 2^n\big(\sigma \neq \tau \to [q_\sigma] \cap T_\tau = \emptyset\big)$.
(e) $\forall \sigma \in 2^{<\omega} \exists m \in \omega\big((q_{\sigma^\frown 0}, g_{\sigma^\frown 0}, h_{\sigma^\frown 0}) = (q_\sigma(g_\sigma|_m), g_\sigma, h_\sigma)\big)$
(f) $\forall \sigma \in 2^{<\omega}\big(g_\sigma \in T_\sigma\big)$
(g) $\forall \sigma \in 2^n\big(q_\sigma \Vdash f(n) = h_\sigma(n)\big)$

This can be easily achieved by applying repeatedly Lemma 1 and using the fact that each element of $\mathcal{K}$ is nowhere dense in every condition. Once the desired sequences are constructed, define $q = \bigcap_{n\in\omega} \bigcup_{\sigma\in 2^n} q_\sigma$. To see that $q \in \mathbb{P}(\mathcal{K})$ just notice that conditions (b) and (c) will assure that $q \in \mathcal{S}$, conditions (e) and (f) will assure that $g_\sigma \in [q]$ for each $\sigma \in 2^{<\omega}$, and condition (d), together with the fact that necessarily $\{g_\sigma\}_{\sigma\in 2^{<\omega}}$ is dense in q, will assure by Proposition 1 that q is in fact in $\mathbb{P}(\mathcal{K})$.

To finish, just note that conditions (c) and (f) assure that the function $H : [q] \longrightarrow 2^\omega$ given by

$$H(g)(n) = h_\sigma(n) \textit{ if and only if } \sigma \in 2^n \wedge g \in q_\sigma.$$

is well defined, and continuous. Moreover

$$q \Vdash H(\dot{r}_{gen}) = \dot{f}$$

So we are done. □

It can be seen that by slight modifications of the proof given above, one can show that $\mathbb{P}(\mathcal{K})$ has minimal real degree of constructibility, the Sacks property, and is Proper.

Let $C_\kappa = \{h : \kappa \to 2 \mid |h| < \omega\}$ be the poset for adding κ Cohen reals ordered by reverse inclusion. For dense sets $C, D \subseteq C_\kappa$ we say that C *refines* D if for each $h \in C$ there exists $g \in D$ such that $g \subseteq h$.

Definition 2. Let V a model of ZFC and W an extension of V. We say that W is Cohen-preserving over V if for each dense $D \subseteq C_\omega$ in W there is a dense $C \in V$ such that C refines D. Additionally we say that a forcing $\mathcal{P}$ is Cohen-preserving if every generic extension via this forcing is Cohen-preserving.

The proof of the following propositions can be found in [14], but the first one was implicitly proved in [39].

Proposition 2. If a forcing notion has the Sacks property, then it is Cohen-preserving.

Proposition 3. Let $\mathcal{P}$ be a proper Cohen-preserving forcing notion and G a $\mathcal{P}$ generic filter. For each κ and each dense $D \subseteq C_\kappa$ in $V[G_\kappa]$, there exists a dense $C \in V$ refining D.

3. Selective Filters

Let $\mathcal{F}$ over ω. We say that $\mathcal{F}$ is a selective filter if and only if for every partition $\{X_i\}_{i\in\omega}$ of ω into elements of $\mathcal{F}^*$ (the dual ideal of $\mathcal{F}$), there exists $Y \in \mathcal{F}$ such that $|Y \cap X_i| \leq 1$ for each

$i \in \omega$. Selective filters have been shown to be useful to prove preservation theorems regarding $\mathfrak{i}$. See [44], [19] and [14].

The following game was introduced by C. Laflamme in [33].

Definition 3. Let $\mathcal{F}$ be a filter over ω. We define the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$ as follows. On the n turn, Player I will play some $U_n \in \mathcal{F}$ and Player II will respond with some $a_n \in U_n$. After ω turns, Player II wins if the sequence $\{a_n\}_{n\in\omega}$ belongs to $\mathcal{F}$. Otherwise, Player I wins.

It is not hard to prove that the Player II never has a winning strategy in this game. On the other hand, the following theorem was proved by C. Laflamme in [33] (see also [34]).

Theorem 2. Let $\mathcal{F}$ be a filter on ω. $\mathcal{F}$ is not selective if and only if Player I does have a winning strategy for the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$.

Lemma 3. Let $\mathcal{F}$ be a selective filter, $p \in \mathbb{S}$ and $H : [p] \longrightarrow P(\omega)$ be a continuous function such that for every $s \in p$, $\bigcup H\big[[p(s)]\big] \in \mathcal{F}$. Then there are $q \in \mathbb{S}$, and $Y \in \mathcal{F}$ such that $q \subseteq p$, and for every $f \in [q]$, $Y \subseteq H(f)$.

Proof. Before we start, define for every $q \in \mathcal{S}$ such that $q \subseteq p$, $L(q) := \bigcup H\big[[q]\big]$. Now consider the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$. Player I will play the following strategy at the same time that he constructs a sequence $\{t_\sigma\}_{\sigma\in 2^{<\omega}} \subseteq p$ such that:

(a) $\forall \sigma \in 2^{<\omega} \forall i \in 2 \big(t_\sigma \subsetneq t_{\sigma^\frown i}\big)$.

(b) $\forall \sigma, \tau \in 2^n \big(\sigma \neq \tau \rightarrow (t_\sigma$ and t_τ are incomparable $)\big)$.

On the first turn, Player I defines $t_\emptyset := \emptyset$ and plays $U_0 := L(p(t_\emptyset))$. As the rules dictate, Player II responds with some $a_0 \in U_0$.

As $a_0 \in \bigcup H\big[[p(t_\emptyset)]\big]$, there is $f \in [p(t_\emptyset)]$ such that $a_0 \in H(f)$. As H is continuous there is $k \in \omega$ such that for every $g \in [p(f|_k)]$, $a_0 \in H(g)$. Now Player I extends $f|_k$ to incomparable $t_0, t_1 \in p$ such that $t_\emptyset \subsetneq t_0, t_1$ and plays $U_1 := L(p(t_0)) \cap L(p(t_1))$. As rules dictate, Player II responds with some $a_1 \in U_1$.

In general, suppose that it is the $n+1$ turn, and that Player I has constructed $\{t_\sigma\}_{\sigma\in 2^{\leq n}}$. Moreover, suppose that for every $m \leq n$ he played $U_m := \bigcap\limits_{\sigma\in 2^{\leq m}} L(p(t_\sigma))$.

As $a_n \in \bigcap\limits_{\sigma\in 2^n} \big(\bigcup H\big[[p(t_\sigma)]\big]\big)$, for every $\sigma \in 2^n$ there is $f_\sigma \in [p(t_0)]$ such that $a_n \in H(f_\sigma)$. As H is continuous, there is $k \in \omega$ such that for every $\sigma \in 2^n$ and every $g \in [p(f_\sigma|_k)]$, $a_n \in H(g)$. Now Player I extends each $f_\sigma|_k$ to incomparable $t_{\sigma^\frown 0}, t_{\sigma^\frown 1} \in p$ such that $t_\sigma \subsetneq t_{\sigma^\frown 0}, t_{\sigma^\frown 1}$ and plays $U_{n+1} := \bigcap\limits_{\sigma\in 2^{\leq n+1}} L(p(t_\sigma))$. As rules dictate, Player II responds with some $a_{n+1} \in U_{n+1}$.

Since $\mathcal{F}$ is a selective filter, this is not a winning strategy for Player I, so there is a match where Player I plays by this strategy but Player II wins. Let $\{a_n\}_{n\in\omega}$ and $\{t_\sigma\}_{\sigma\in 2^{<\omega}}$ be the sequences associated to one of this matches and let $q := \{\tau \in p \mid \exists \sigma \in 2^\omega \big(\tau \subseteq t_\sigma\big)\}$. It is straightforward that q and $Y := \{a_n\}_{n\in\omega}$ are the objects we are looking for. $\square$

Theorem 3. Let $\mathcal{F}$ be a selective filter and G be a $\mathbb{P}(\mathcal{K})$-generic filter. In $V[G]$, for every $X \in P(\omega)$ one of the following statements occurs:

(a) There is $Y \in \mathcal{F} \cap V$ such that $Y \subseteq X$.
(b) There is $Z \in V$, such that $Z \notin \mathcal{F}$ and $X \subseteq Z$.

Proof. Let $\dot{X}$ be a name for a subset of ω, $p \in \mathbb{P}(\mathcal{K})$ and suppose that there is no condition below p that forces (b). By Lemma 2 we can suppose that there is a continuous $H : [p] \longrightarrow P(\omega)$ such that

$$p \Vdash H(\dot{r}_{gen}) = \dot{X}$$

For every $q \in \mathcal{S}$ such that $q \subseteq p$, let

$$L(q) = \bigcup F\big[[q]\big].$$

Notice that if such q is in $\mathbb{P}(\mathcal{K})$ then $L(q) \in \mathcal{F}$. This is because $q \Vdash X \subseteq L(q)$ and q does not force (b). Now, call $q \in \mathbb{S}$ (not necessarily in $\mathbb{P}(\mathcal{K})$) *special* if $q \subseteq p$ and for every $s \in q$ we have that

$$L(q(s)) \in \mathcal{F}.$$

We will divide the proof by cases.

Case 1 For every $s \in p$ there is an ordered pair (q, T) such that:

(1) $q \in \mathbb{S}$ *is special,*
(2) $T \in \mathcal{K}$,
(3) $[q] \subseteq [p(s)] \cap T$.

In this case, consider the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$. Player I will play by the following strategy at the same time that he recursively constructs sequences $\{q_\sigma\}_{\sigma \in 2^{<\omega}} \subseteq \mathbb{S}$, $\{s_\sigma\}_{\sigma \in 2^{<\omega}} \subseteq p$, and $\{T_\sigma\}_{\sigma \in 2^{<\omega}} \subseteq \mathcal{K}$ such that:

(a) $\forall \sigma \in 2^{<\omega}\big((q_\sigma, T_\sigma)$ satisfies *Case I* conditions for $s_\sigma\big)$.
(b) $\forall \sigma \in 2^{<\omega}\big(q_{\sigma^\frown 0} \subseteq q_\sigma\big)$.
(c) $\forall \sigma \in 2^{<\omega}\big(T_{\sigma^\frown 0} = T_\sigma\big)$.
(d) $\forall \sigma, \tau \in 2^n\big(\sigma \neq \tau \rightarrow T_\sigma \neq T_\tau\big)$.
(e) $\forall \sigma \in 2^{<\omega} \forall i \in 2\big(s_\sigma \subsetneq s_{\sigma^\frown i}\big)$.
(f) $\forall \sigma \in 2^{<\omega}\big(s_{\sigma^\frown 0} \in q_\sigma \wedge [p(s_{\sigma^\frown 1})] \cap T_\sigma = \emptyset\big)$.

On the first turn, Player I defines $s_\emptyset := \emptyset$, and $(q_\emptyset, T_\emptyset)$ that satisfies *Case I* conditions for $s_\emptyset$. Then he plays $U_0 := L(q_\emptyset)$. As the rules dictate, Player II responds with some $a_0 \in U_0$.

As $a_0 \in \bigcup H\big[[q_\emptyset]\big]$, there is some $f \in [q_\emptyset]$ such that $a_0 \in H(f)$. As H is continuous there is $k \in \omega$ such that for every $g \in [p(f|_k)]$, $a_0 \in H(g)$. Notice that since $(q_\emptyset, T_\emptyset)$ satisfies *Case I* conditions for $s_\emptyset$, then $f|_k$ is compatible with s_σ, and moreover we can extend $f|_k$ to incomparable $s_0, s_1 \in p$ such that $s_\emptyset \subsetneq s_0, s_1$, $s_0 \in q_\emptyset$, and $[p(s_1)] \cap T_\emptyset = \emptyset$. Now Player I defines $q_0 := q_\emptyset(s_0)$, $T_0 := T_\emptyset$, and (q_1, T_1) that satisfies *Case I* conditions for s_1. Then he plays $U_1 := L(q_0) \cap L(q_1)$. As the rules dictate, Player II responds with some $n_1 \in U_1$.

In general, suppose that is the $n+1$ Turn, and that Player I has constructed q_σ, s_σ, and T_σ for every $\sigma \in 2^{\leq n}$. Moreover, suppose that for every $m \leq n$ he has played $U_m = \bigcap_{\sigma \in 2^m} L(q_\sigma)$.

As $a_n \in \bigcap\limits_{\sigma\in 2^n} \left(\bigcup H[[q_\sigma]]\right)$, for every $\sigma \in 2^n$ there is $f_\sigma \in [q_\sigma]$ such that $an \in H(f_\sigma)$. As H is continuous, there is $k \in \omega$ such that for every for every $\sigma \in 2^n$ and every $g \in [p(f_\sigma|_k)]$, $a_n \in H(g)$. As all the pairs (q_σ, T_σ) satisfy *Case I* conditions for the respective s_σ, we have that $f_\sigma|_k$ is compatible with s_σ, and that $\bigcup\limits_{\sigma\in 2^n} T_\sigma \cap [p]$ is nowhere dense in $[p]$. Then we can extend each $f_\sigma|_k$ to incomparable $s_{\sigma^\frown 0}, s_{\sigma^\frown 1} \in p$ such that $s_\sigma \subsetneq s_{\sigma^\frown 0}, s_{\sigma^\frown 1}$, $s_{\sigma^\frown 0} \in q_\sigma$, and $p(s_{\sigma^\frown 1})] \cap T_\sigma = \emptyset$.

Now Player I defines $q_{\sigma^\frown 0} := q_\sigma(s_{\sigma^\frown 0})$, $T_{\sigma^\frown 0} := T_\sigma$, and $(q_{\sigma^\frown 1}, T_{\sigma^\frown 1})$ that satisfies *Case I* conditions for $s_{\sigma^\frown 1}$. Then he plays $U_{n+1} := \bigcap\limits_{\sigma\in 2^{n+1}} L(q_\sigma)$. As the rules dictate, Player II responds with some $a_{n+1} \in U_{n+1}$.

Since $\mathcal{F}$ is a selective filter, this is not a winning strategy for Player I, so there is a match where Player I plays by this strategy but Player II wins. Let $\{a_n\}_{n\in\omega}$, $\{q_\sigma\}_{\sigma\in 2^\omega}$, $\{s_\sigma\}_{\sigma\in 2^\omega}$, and $\{T_\sigma\}_{\sigma\in 2^\omega}$ be the sequences associated to one of these matches.

To finish this case, define $q := \{\tau \in p \mid \exists\sigma \in 2^\omega\big(\tau \subseteq s_\sigma\big)\}$. Moreover, if c_0 is the constant 0 function in 2^ω, and $\sigma \in 2^{<\omega}$ then $g_\sigma := \bigcup\{s_\tau \mid \tau \subseteq \sigma^\frown c_0\} \in T_\sigma$, and the set $Q := \{g_\sigma \mid \sigma \in 2^{<\omega}\}$ is dense in $[q]$. This implies by Lemma 1 that $q \in \mathbb{P}(\mathcal{K})$.Sine for every $n \in \omega$, every $\sigma \in 2^{n+1}$ and every $g \in [p(s_\sigma)]$, $a_n \in H(g)$, and every $g \in [q]$ satisfies this condition for some $\sigma \in 2^{n+1}$, we conclude that for every $g \in [q]$, $\{a_n\} \subseteq H(g)$. In particular we have that:

$$q \Vdash \{a_n\}_{n\in\omega} \subseteq \dot{X}.$$

Since $\{a_n\}_{n\in\omega} \in \mathcal{F}$, we are done.

Case 2 There is $s_0 \in p$ for which every ordered pair (q, T) does not satisfy one of the following conditions:

(1) $q \in \mathbb{S}$ is special,
(2) $T \in \mathcal{K}$,
(3) $[q] \subseteq [p(s_0)] \cap T$.

In this Case, we use Lemma 3 to find $q \in \mathcal{S}$, and $Y \in \mathcal{F}$ such that $q \subseteq p(s_0)$ and for every $f \in [q]$, $Y \subseteq H(f)$. Notice that q is special.

Suppose towards a contradiction that $q \notin \mathbb{P}(\mathcal{K})$. Since every element of $\mathcal{K}$ is closed, this means that there is some $T \in \mathcal{K}$ such that $T \cap [q]$ has nonempty interior in $[q]$. So let $\tau \in q$ such that $[q(\tau)] \subseteq T$. Then $(q(\tau), T)$ satisfies all three *Case I* conditions for s_0, but this is a contradiction. We conclude that $q \in \mathbb{P}(\mathcal{K})$.

To finish this case, just note that as before $q \Vdash Y \subseteq \dot{X}$. □

Suppose that $\mathcal{K}$ is the partition of 2^ω in singletons. In this particular case we have that $\mathbb{P}(\mathcal{K}) = \mathbb{S}$, thus *Case 1* of Theorem 3 never occurs. This means that Lemma 3 actually yields a complete proof of Theorem 3 for Sacks forcing.

Corollary 1. $\mathbb{P}(\mathcal{K})$ preserves selective ultrafilters.

4. Independent Families

Following the notation of [14], for an independent family $\mathcal{A}$ we let $\mathbf{C}_{\mathcal{A}}$ be set of all finite partial functions from $\mathcal{A}$ to 2 and order it by inclusion. For each $h \in \mathbf{C}_{\mathcal{A}}$ we put $\mathcal{A}^h = \bigcap\{A^{h(A)} \mid A \in dom(h)\}$ where $A^0 = \omega \backslash A$, and $A^1 = A$ for each $A \in \omega$. For each $h \in \mathbf{C}_{\mathcal{A}}$, and each $X \in \omega$ we will say that h hits X if $\mathcal{A}^h \subseteq^* X$. Additionally we will say that h reaps X if h hits either X or his complement.

Definition 4. Let $\mathcal{A}$ be an independent family. We say that $\mathcal{A}$ is dense if and only if for every $X \in \omega$ we have that

$$\{h \in \mathbf{C}_{\mathcal{A}} \mid h \textit{ reaps } X\}$$

is dense in $\mathbf{C}_{\mathcal{A}}$.

Notice that every dense independent family is maximal. This observation is the key to preserving maximal independent families.

Definition 5. Let $\mathcal{A}$ be an independent family. We define

$$\mathcal{F}_{\mathcal{A}} := \{X \in P(\omega) \mid \{h \in \mathbf{C}_{\mathcal{A}} \mid h \textit{ hits } X\} \textit{ is dense in } \mathbf{C}_{\mathcal{A}}\}.$$

Additionally, let $\mathcal{C}_{\mathcal{A}} := \{X \backslash \mathcal{A}^h \mid h \in \mathbf{C}_{\mathcal{A}}\}$.

It is easy to see that $\mathcal{F}_{\mathcal{A}}$ is a filter and that $\mathcal{C}_{\mathcal{A}} \subseteq P(\omega)\backslash F$. Moreover, the definition of $\mathcal{C}_{\mathcal{A}}$ is absolute. The following results are proved in [14].

Lemma 4 ([14])**.** Let $\mathcal{A} \in V$ be an independent system and let W be a Cohen-preserving extension of V. In W, $\mathcal{F}_{\mathcal{A}}$ is generated by $\mathcal{F}^V_{\mathcal{A}}$.

Lemma 5 ([14])**.** Let $\mathcal{A} \in V$ be a dense independent family, and W be a Cohen-preserving extension of V.In W, If $\mathcal{C}_{\mathcal{A}}$ is cofinal in $P(\omega)\backslash\langle\mathcal{F}^V_{\mathcal{A}}\rangle$[6] then $\mathcal{A}$ remains dense in W.

An independent family is called *selective* if it satisfies properties (1) and (2) of the following theorem.

Theorem 4 ([14])**.** Assume CH. There exists an independent family $\mathcal{A}$ such that:
(a) $\mathcal{A}$ is dense,
(b) $\mathcal{F}_{\mathcal{A}}$ is a selective filter.

In order to prove that $\mathfrak{i} = \omega_1$ in the model $\mathbb{PM}$, we will need the following preservation result due to S. Shelah.

Lemma 6 ([44], Lemma 3.2)**.** Let $\mathcal{F}$ and $\mathcal{H}$ be families of subsets of ω such that:

(1) $\mathcal{F}$ is a selective filter,
(2) $\mathcal{H} \subseteq P(\omega)\backslash\mathcal{F}$ is cofinal in $P(\omega)\backslash\mathcal{F}$ with respect to $\subseteq^*$.

[6]if $\mathcal{F}$ is a subset of $P(\omega)$ then $\langle\mathcal{F}\rangle$ is the filter generated by $\mathcal{F}$

If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ is a countable support iteration of ω^ω-bounding proper forcing notions such that for all $\alpha < \delta$,

$$\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \mathcal{H} \text{ is cofinal in } P(\omega) \backslash \langle \mathcal{F} \rangle.$$

then the same holds for δ.

Theorem 5. Let $\mathcal{A}$ be a selective independent family, and G be a $\mathbb{P}(\mathcal{K})$-generic filter. In $V[G]$, $\mathcal{A}$ is still selective independent.

Proof. Since $\mathbb{P}(\mathcal{A})$ is proper and has the Sacks property, $\langle \mathcal{F}^V_\mathcal{A} \rangle$ is a selective filter in $V[G]$, but by Lemma 4 we know that $\mathcal{F}^{V[G]}_\mathcal{A} = \langle \mathcal{F}^V_\mathcal{A} \rangle$. To show that $\mathcal{A}$ remains dense in $V[G]$, just notice that Theorem 3 implies that $P(\omega)^V \backslash \mathcal{F}^V_\mathcal{A}$ is cofinal in $P(\omega)^{V[G]} \backslash \langle \mathcal{F}^V_\mathcal{A} \rangle$, but by Hypothesis we know that $\mathbf{C}_\mathcal{A}$ is cofinal in $P(\omega)^V \backslash \mathcal{F}^V_\mathcal{A}$, so by Lemma 5 we are done. □

As a direct consequence of Lemma 6 and Theorem 5 we have the following.

Theorem 6. In $\mathbb{PM}$, $\omega_1 = \mathfrak{i} < \mathfrak{a}_T = \mathfrak{c} = \omega_2$.

5. The poset $\mathbb{Q}_\mathcal{I}$

For a maximal ideal $\mathcal{I}$ on ω, below $\mathbb{Q}_\mathcal{I}$ denotes the forcing notion introduced by S. Shelah in [44] for obtaining the consistency of $\mathfrak{i} < \mathfrak{u}$. In [44] it is shown that $\mathbb{Q}_\mathcal{I}$ is proper [44, Claim 1.13], ${}^\omega\omega$-bounding [44, Claim 1.12] and even has the Sacks property [44, Claim 1.12]. In the $\mathbb{Q}_\mathcal{I}$-generic extension, $\mathcal{I}$ is no longer a maximal ideal [44, Claim 1.5]. For completeness of the presentation we repeat below the definition and some of the key properties of $\mathbb{Q}_\mathcal{I}$.

Definition 6. Let $\mathcal{I}$ be an ideal on ω.

(1) An equivalence relation E on a subset of ω is an $\mathcal{I}$-equivalence relation if $\operatorname{dom} E \in \mathcal{I}^*$ and each E-equivalence class is in $\mathcal{I}$.
(2) For $\mathcal{I}$-equivalence relations E_1, E_2, we denote $E_1 \leq_\mathcal{I} E_2$ if $\operatorname{dom} E_1 \subseteq \operatorname{dom} E_2$, and E_1-equivalence classes are unions of E_2-equivalence classes.
(3) Let $A \subseteq \omega$. A function g is A-n-determined if $g\colon {}^A\{0,1\} \to \{0,1\}$ and there is $w \subseteq A \cap (n+1)$ such that for any $\eta, \nu \in {}^A\{0,1\}$ with $\eta \upharpoonright w = \nu \upharpoonright w$ we have $g(\eta) = g(\nu)$.

For $i \in A$, by g_i we denote a function from ${}^A\{0,1\}$ to $\{0,1\}$ which maps $\eta \in {}^A\{0,1\}$ to $\eta(i)$.

Claim 1. Each A-n-determined function is equal to a function $\varphi(g_0, \ldots, g_n)$ which is obtained as a maximum, minimum, and complement (i.e., $1 - g_i$) of $g_0, \ldots, g_n, 0, 1$.

For an $\mathcal{I}$-equivalence relation E we denote $A = A(E) = \{x\colon x \in \operatorname{dom} E, x = \min[x]_E\}$.

Definition 7 (Set of conditions in $\mathbb{Q}_\mathcal{I}$)**.** Let $\mathcal{I}$ be an ideal on ω. We define a forcing notion $\mathbb{Q}_\mathcal{I}$:

$$p \in \mathbb{Q}_\mathcal{I} \text{ iff } p = (H, E) = (H^p, E^p) \text{ where}$$

(1) E is an $\mathcal{I}$-equivalence relation,
(2) H is a function with $\operatorname{dom} H = \omega$,
(3) a value $H(n)$ is an $A(E)$-n-determined function,
(4) if $n \in A(E)$ then $H(n) = g_n$,

(5) if $n \in \operatorname{dom} E \setminus A(E)$ and nEi for $i \in A(E)$ then $H(n)$ is g_i or $1 - g_i$.

For a condition $q \in \mathbb{Q}_{\mathcal{I}}$, let A^q be $A(E^q)$ in the following.

Definition 8. If $p, q \in \mathbb{Q}_{\mathcal{I}}$ with $A^p \subseteq A^q$ then we write $H^p(n) =^{**} H^q(n)$ if for each $\eta \in {}^{A^p}\{0,1\}$ we have $H^p(n)(\eta) = H^q(n)(\eta')$ where

$$\eta'(j) = \begin{cases} \eta(j) & j \in A^p, \\ H^p(j)(\eta) & j \in A^q \setminus A^p. \end{cases}$$

Definition 9 (The order of $\mathbb{Q}_{\mathcal{I}}$)**.** If $p, q \in \mathbb{Q}_{\mathcal{I}}$ then $p \leq q$ if

(1) $E^p \leq_{\mathcal{I}} E^q$,
(2) If $H^q(n) = g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = H^p(i)$,
(3) If $H^q(n) = 1 - g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = 1 - H^p(i)$,
(4) If $n \in \omega \setminus \operatorname{dom} E^q$ then $H^p(n) =^{**} H^q(n)$.

Finally, $p \leq_n q$ if $p \leq q$ and A^p contains the first n elements of A^q.

The following has been proven in [44]. Items (1) and (2) correspond to [44, Claim 1.7, (2)], item (3) is a straightforward modification of [44, Claim 1.8].

Claim 2. Let $p \in \mathbb{Q}_{\mathcal{I}}$. For an initial segment u of A^p, and $h\colon u \to \{0,1\}$, let $p^{[h]}$ be the pair $q = (H^q, E^q)$ defined by (i) and (ii) below:

(i) $E^q = E^p \restriction \bigcup\{[i]_{E^p} : i \in A^p \setminus u\}$.
(ii) If $H^p(n)$ is $\varphi(g_0, \ldots, g_n)$ then $H^q(n)$ is $\varphi(g_0, \ldots, g_i/h(i), \ldots, g_n)$, where the substitution is done just for $i \in u$.

Then we have:

(1) $p^{[h]}$ is a condition in $\mathbb{Q}_{\mathcal{I}}$ stronger than p.
(2) The set $\{p^{[h]} : h \in {}^u\{0,1\}\}$ is predense below p.
(3) If u is the set of first n elements of A^p, D a dense subset of $\mathbb{Q}_{\mathcal{I}}$ then there is $q \in \mathbb{Q}_{\mathcal{I}}$ such that $q \leq_n p$ and $q^{[h]} \in D$ for any $h \in {}^u\{0,1\}$.

Definition 10 (The game $\mathrm{GM}_{\mathcal{I}}(E)$)**.** $\mathrm{GM}_{\mathcal{I}}(E)$ is the following game. In the n-th move, the first player chooses an $\mathcal{I}$-equivalence relation $E^1_n \leq_{\mathcal{I}} E^2_{n-1}$ ($E^1_0 = E$), and the second player chooses an $\mathcal{I}$-equivalence relation $E^2_n \leq_{\mathcal{I}} E^1_n$. In the end, the second player wins if

$$\bigcup_{n>0} (\operatorname{dom} E^1_n \setminus \operatorname{dom} E^2_n) \in \mathcal{I}.$$

Otherwise, the first player wins.

Remark 2. If the second player wins in the game $\mathrm{GM}_{\mathcal{I}}(E)$, then the game is invariant to taking subsets. That is, the game is invariant to taking $\leq_{\mathcal{I}}$-extensions $\{E^{2,*}_n\}_{n\in\omega}$ with $\operatorname{dom}(E^{2,*}_n) \subseteq \operatorname{dom} E^2_n$.

The next lemma corresponds to [44, Claim 1.10, (1)]

Lemma 7. The game $\mathrm{GM}_{\mathcal{I}}(E)$ is not determined for a maximal ideal $\mathcal{I}$.

6. Tight MAD families

Tight MAD families were investigated in [35, 32, 24]. An AD family $\mathcal{A}$ is called tight if for every $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.

Preservation theorem for tight MAD family under countable support iteration of proper forcing notions was developed by O. Guzmán, M. Hrušák and O. Téllez [24].

Definition 11. Let $\mathcal{A}$ be a tight MAD family. A proper forcing $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$ if for every $p \in \mathbb{P}$, M a countable elementary submodel of $H(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathbb{P}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there is $q \leq p$ an $(M, \mathbb{P})$-generic condition such that

$$q \Vdash \text{“}(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega\text{”},$$

where $\dot{G}$ denotes the name of generic filter.

We restate Corollary 32 by O. Guzmán, M. Hrušák and O. Téllez [24] which is crucial for preserving MAD families in the forthcoming model.

Theorem 7 (O. Guzmán, M. Hrušák, O. Téllez)**.** Let $\mathcal{A}$ be a tight MAD family. If the sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ is a countable support iteration of proper posets such that

$$\mathbb{P}_\alpha \Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ strongly preserves the tightness of } \mathcal{A}\text{”},$$

then $\mathbb{P}_{\omega_2} \Vdash_\alpha$ “$\mathcal{A}$ is a tight MAD family”.

We need the following fact about the outer hulls observed in [24].

Lemma 8. Let $\mathcal{A}$ be an AD family, $\mathbb{P}$ a partial order, $\dot{B}$ a $\mathbb{P}$-name for a subset of ω and $p \in \mathbb{P}$ such that $p \Vdash$ “$\dot{B} \in \mathcal{I}(\mathcal{A})^+$”. Then the set $\{n : (\exists q \leq p)\ q \Vdash \text{“}n \in \dot{B}\text{”}\}$ is in $\mathcal{I}(\mathcal{A})^+$.

And now we are ready to show the main result of the paper.

Theorem 8. Let $\mathcal{A}$ be a tight MAD family, $\mathcal{I}$ being a maximal proper ideal on ω. The poset $\mathbb{Q}_\mathcal{I}$ strongly preserves the tightness of $\mathcal{A}$.

Proof. Let $p \in \mathbb{Q}_\mathcal{I}$, M a countable elementary submodel of $H(\kappa)$ such that $\mathcal{I}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. We fix an enumeration $\{D_n : n \in \omega\}$ of all open dense subsets of $\mathbb{Q}_\mathcal{I}$ that are in M, and an enumeration $\{\dot{Z}_n : n \in \omega\}$ of all $\mathbb{Q}_\mathcal{I}$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M with names repeating infinitely many times.

We define a strategy for the first player in the game $\mathrm{GM}_\mathcal{I}(E)$, which cannot be winning in all rounds.

We set $p_0 = q_0 = p$ and $u_0 = \emptyset$. We assume that the first player has chosen E^1_n, q_n, p_n, u_n, and the second one an E^2_n. We give instructions to choose E^1_{n+1}, q_{n+1}, p_{n+1}, u_{n+1}. We begin with q_{n+1}:

(1) $\operatorname{dom} E^{q_{n+1}} = \operatorname{dom} E^{p_n}$,
(2) $xE^{q_{n+1}}y$ iff one of the following holds:
  (i) $xE^2_n y$.

(ii) There is $k \in u_n$ with $x, y \in [k]_{E^{p_n}}$ and $x, y \notin \operatorname{dom} E^2_n$.

(iii) There are $k_0, k_1 \notin \bigcup\{[i]_{E^{p_n}} : i \in u_n\}$ with $x \in [k_0]_{E^{p_n}}$, $y \in [k_1]_{E^{p_n}}$ and $k_0, k_1 \notin \operatorname{dom} E^2_n$.

(3) $H^{q_{n+1}}$ is chosen such that:

(i) If $l \in \omega \setminus \operatorname{dom} E^{p_n}$ then $H^{q_{n+1}}(l) =^{**} H^{p_n}(l)$.

(ii) If $l \in \operatorname{dom} E^{p_n} \setminus A^{q_{n+1}}$, $H^{p_n}(l) = g_i$ then $H^{q_{n+1}}(l) = H^{q_{n+1}}(i)$.

(iii) If $l \in \operatorname{dom} E^{p_n} \setminus A^{q_{n+1}}$, $H^{p_n}(l) = 1 - g_i$ then $H^{q_{n+1}}(l) = 1 - H^{q_{n+1}}(i)$.

(iv) If $l \in A^{p_n} \setminus A^{q_{n+1}}$ then $H^{q_{n+1}}(l) =^{**} H^{p_n}(\min[l]_{E^{q_{n+1}}})$.

Note that for the already defined condition q_{n+1} we have $q_{n+1} \leq_n p_n$. Take $u_{n+1} = u_n \cup \{\min(A^{q_{n+1}} \setminus u_n)\}$. By Lemma 8, the set $D'_n = \{r \in \mathbb{Q}_{\mathcal{I}} : r \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n\text{”}\}$ is open dense below p (and also below q_{n+1}). Then $D'_n \cap D_n$ is dense below q_{n+1}. Therefore we can apply Lemma 7 to obtain $p_{n+1} \leq_{n+1} q_{n+1}$ such that for each $h \in {}^{u_{n+1}}\{0,1\}$, the condition $p^{[h]}_{n+1} \in D'_n \cap D_n \cap M$. In particular, if $h \in {}^{u_{n+1}}\{0,1\}$ then $p^{[h]}_{n+1} \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n \neq \emptyset\text{”}$ and $p^{[h]}_{n+1} \in D_n \cap M$. By Lemma 7 we have $p_{n+1} \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n \neq \emptyset\text{”}$. Finally, we set

$$E^1_{n+1} = E^{p_{n+1}} \restriction (\operatorname{dom} E^{p_{n+1}} \setminus \bigcup\{[i]_{E^{p_{n+1}}} : i \in u_{n+1}\}).$$

We define a fusion q of a sequence $\langle p_n : n \in \omega\rangle$. Relation E^q has $\operatorname{dom} E^q = \bigcap\{\operatorname{dom} E^{p_n} : n \in \omega\}$, and $xE^q y$ if for every n large enough, $xE^{p_n}y$. Function H^q is equal to H^{p_n} for large enough n. In order to guarantee $q \in \mathbb{Q}_{\mathcal{I}}$, it is necessary to choose a play with the first player using described strategy, but he looses. Thus the second player wins and by Remark 2, we can assume that $\min \operatorname{dom} E^2_n > \max u_{n+1}$. Consequently, $\operatorname{dom} E^{p_n} \setminus \operatorname{dom} E^2_n \subseteq \bigcup\{[k]_{E^{q_{n+1}}} : k \in u_{n+1}\}$, and thus $\operatorname{dom} E^q \in \mathcal{I}^*$. One can check that other properties for $q \in \mathbb{Q}_{\mathcal{I}}$ are satisfied by the definition of q.

Finally, condition q is $(M, \mathbb{Q}_{\mathcal{I}})$-generic, and $q \leq_n p_n$ for each n. Hence, we have $q \Vdash \text{“}(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega\text{”}$. □

7. Selective independence and tight madness

In the statement below, we put all of the above preservation results together.

Corollary 2. It is relatively consistent that $\operatorname{cof}(\mathcal{N}) = \mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2$.

Proof. Work over a model of CH. Let $\mathcal{A}_0$ be Shelah's selective independent family and let $\mathcal{A}_1$ be a tight mad family. Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ of posets such that for even α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{P}(\mathcal{K})$ for some uncountable partition of 2^ω into compact sets, for odd α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}_{\mathcal{I}}$ for some maximal ideal on ω, and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \mathfrak{u} = \omega_2$. The iteration $\mathbb{P}_{\omega_2}$ has the Sacks property and therefore $\operatorname{cof}(\mathcal{N}) = \omega_1$. By the indestructibility of selective independence the family $\mathcal{A}_0$ remains maximal independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. Moreover, by the preservation properties of tight MAD families, see [24], and the above preservation theorems, $\mathcal{A}_1$ is a witness to $\mathfrak{a} = \omega_1$ in the final model. □

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