The consistency of $b = \kappa < \mathfrak{s} = \kappa^+$

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Definition
A family $\mathcal{H} \subseteq \omega^\omega$ is unbounded, if there is no $g \in \omega^\omega$ which dominates all elements of $\mathcal{H}$. The bounding number $b$ is the minimal cardinality of an unbounded family.

Definition
A family $S \subseteq [\omega]^\omega$ is splitting, if for every $A \in [\omega]^\omega$ there is $B \in S$ such that both $A \cap B$ and $A \cap B^c$ are infinite. The splitting number $s$ is the minimal size of a splitting family.

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The consistency of $b = \kappa < s = \kappa^+$
$b = \omega_1 < s = \omega_2$

In 1984 S. Shelah showed the consistency of $b = \omega_1 < s = \omega_2$ using a proper, almost $\omega \omega$ bounding forcing notion of size continuum, which adds a real not split by the ground model reals.

$b = \kappa < s = \kappa^+$

Given an unbounded $<^*$-directed family $\mathcal{H}$ of size $\kappa$ we obtain a $\sigma$-centered suborder $\mathbb{P}_\mathcal{H}$ of Shelah’s poset, which preserves $\mathcal{H}$ unbounded and adds a real not split by $V \cap [\omega]^\omega$. 
The consistency of $b = \kappa < s = \kappa^+$

**Mathias forcing**

Logarithmic Measures
Induced Logarithmic Measure
Sufficient Condition for High Values
Shelah’s partial order

**Cardinal Characteristics**

Logarithmic Measures
Centered Families

\[ b = \kappa < s = \kappa^+ \]

\[ \mathfrak{M} \] adds a real not split by the ground model reals

If $G$ is $\mathfrak{M}$-generic, then $U_G = \bigcup \{ u : \exists A(u,A) \in G \}$ is an infinite set such that $\forall A \in V \cap [\omega]^\omega$, $U_G \subseteq^* A$ or $U_G \subseteq^* A^c$.

**Mathias forcing**

Logarithmic Measures
Induced Logarithmic Measure
Sufficient Condition for High Values
Shelah’s partial order

\[ b = \kappa < s = \kappa^+ \]

\[ \mathfrak{M} \] adds a dominating real

However, if $F_G$ is the enumerating function of $U_G$, then $F_G$ dominates all ground model reals.
Definition

- Let $s \subseteq \omega$. Then $h : [s]^{<\omega} \to \omega$ is called a logarithmic measure if $\forall A \in [s]^{<\omega}, \forall A_0, A_1$ such that $A = A_0 \cup A_1$, $h(A_i) \geq h(A) - 1$ for $i = 0$, or $i = 1$ unless $h(A) = 0$.
- If $s$ is finite, the pair $x = (s, h)$ is called a finite logarithmic measure. The value $h(s) = \|x\|$ is called the level of $x$ and $\text{int}(x)$ denotes $s$. 

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Definition

Let $P \subseteq [\omega]^{<\omega}$ be upwards closed family, which does not contain singletons. Then $P$ induces a logarithmic measure on $[\omega]^{<\omega}$ defined inductively as follows:

1. $h(e) \geq 0$ for every $e \in [\omega]^{<\omega}$
2. $h(e) > 0$ iff $e \in P$
3. for $\ell \geq 1$, $h(e) \geq \ell + 1$ iff whenever $e_0, e_1 \subseteq e$ are such that $e = e_0 \cup e_1$, then $h(e_0) \geq \ell$ or $h(e_1) \geq \ell$.

Then $h(e) = \max\{k : h(e) \geq k\}$. The elements of $P$ are called positive sets and $h$ is said to be induced by $P$. 
Example
Let $P = \{a \in [\omega]^{<\omega} : |a| \geq 2\}$. Then $h(a) = \min\{j : |a| \leq 2^j\}$ is the logarithmic measure induced by $P$, called *standard measure*.

Lemma
Let $A \subseteq \omega$ does not contain a set of measure $\geq \ell + 1$ for some $\ell \in \omega$. Then there are $A_0, A_1$ such that $A = A_0 \cup A_1$ and none of $A_0, A_1$ contain a set of measure $\geq \ell$. 
Lemma
Let $P \subseteq [\omega]^{<\omega}$ be upwards closed family, which does not contain singletons and let $h$ be induced by $P$. Then if for every $n \in \omega$ and partition $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is $j \in n$ such that $A_j$ contains a positive set, then for every $k \in \omega$, for every $n \in \omega$ and partition $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is $j \in n$ such that $A_j$ contains a set of measure $\geq k$. 
Definition

Let $Q$ be the set of all pairs $(u, T)$ where $u$ is a finite subset of $\omega$ and $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a sequence of logarithmic measures such that

1. $\max u < \min s_0$
2. $\max s_i < \min s_{i+1}$ for all $i \in \omega$
3. $\langle h_i(s_i) : i \in \omega \rangle$ is unbounded.

Also $\text{int}(T) = \bigcup \{s_i : i \in \omega\}$. If $u = \emptyset$, then $(\emptyset, T)$ is a pure condition and is denoted by $T$. Note that if $(u, T)$ is Shelah’s condition, then $(u, \text{int}(T))$ is Mathias.
We say \((u_2, T_2) \leq (u_1, T_1)\), where \(T_\ell = \langle (s_\ell^i, h_\ell^i) : i \in \omega \rangle\) for \(\ell = 1, 2\), if the following conditions hold:

1. \(u_2\) is an end-extension of \(u_1\) and \(u_2 \setminus u_1 \subseteq \text{int}(T_1)\)
2. \(\text{int}(T_2) \subseteq \text{int}(T_1)\) and furthermore there is an infinite sequence \(\langle B_i : i \in \omega \rangle\) of finite subsets of \(\omega\) such that \(\max u_2 < \min s_1^j\) for \(j = \min B_0\), \(\max(B_i) < \min(B_{i+1})\) and \(s_2^i \subseteq \bigcup\{s_1^j : j \in B_i\}\).
3. for every subset \(e\) of \(s_2^i\) such that \(h_2^i(e) > 0\) there is \(j \in B_i\) such that \(h_1^j(e \cap s_1^j) > 0\).

If \(u_1 = u_2\), then \((u_2, T_2)\) is a pure extension of \((u_1, T_1)\).
Extensions in $Q$:\quad $T_1 \leq T_2$

$T_2$

$T_1$

$B_0 = \{0, 1\}$

$B_1 = \{4\}$

$B_2 = \{5, 6\}$

Amalgamation

Restriction

Amalgamation and restriction

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The consistency of $b = \kappa < s = \kappa^+$
The consistency of $b = \kappa < s = \kappa^+$
Definition

Let $\mathcal{F}$ be family of pure conditions. Then $Q(\mathcal{F})$ is the suborder of $Q$ of all $(u, T) \in Q$ such that $\exists R \in \mathcal{F}(R \leq T)$.

- If $C$ is centered, then $Q(C)$ is $\sigma$-centered.
- Let $p, q \in Q(C)$. Then $p \not\perp_Q q$ iff $p \not\perp_{Q(C)} q$.
- If $C \subseteq Q(C')$ then $C'$ is said to extend $C$.
- If $T \not\perp C$ and $\omega = A_0 \cup \cdots \cup A_{n-1}$, then $\exists j \in n$ and $R \leq T(R \not\perp C)$ such that $\text{int}(R) \subseteq A_j$. 

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The consistency of $b = \kappa < s = \kappa^+$
\[ \mathbb{P}_H = Q(C_H) \]

The forcing notion \( \mathbb{P}_H \) is of the form \( Q(C_H) \). Starting with an arbitrary pure condition \( T \) and \( C_0 = \{ T \setminus v : v \in \omega^{<\omega} \} \), we will obtain a sequence \( \langle C_{\alpha} : \alpha < \kappa^+ \rangle \) of centered families such that \( \forall \alpha < \beta (C_{\alpha} \subseteq Q(C_{\beta})) \) and \( C_H = \bigcup_{\alpha \in \kappa} C_{\alpha} \).

\[ C_{\alpha} \subseteq Q(C_{\alpha+1}) \]

At successor stages, we will use three distinct countable forcing notions each of which adds a single pure condition with desired combinatorial properties.
Definition
Let $Q_{\text{fin}}$ be the poset of all $\bar{r} = \langle r_0, \ldots, r_n \rangle$ of finite measure such that $\forall i \in n$, $\max \text{int}(r_i) < \min \text{int}(r_{i+1})$ and $\| r_i \| < \| r_{i+1} \|$ with extension relation end-extension.

Definition
Let $\bar{r} \in Q_{\text{fin}}$ and $T$ a pure condition. Then $\bar{r} \leq T$ if there is a pure condition $R \leq T$ such that $\bar{r} \subseteq R$. 

Definition
Let $T$ be a pure condition. Then $\mathbb{P}(T)$ is the suborder of $Q_{fin}$ of all finite sequences $\bar{r}$ extending $T$.

Lemma
Let $T \not\perp X$, $n \in \omega$. Then

$$D_T(X, n) = \{\bar{r} \in \mathbb{P}(T) : \exists r_j \in \bar{r}(r_j \leq X \text{ and } \|r_j\| \geq n)\}$$

is dense in $\mathbb{P}(T)$.
Corollary

Let $C \not\subseteq T$, let $G$ be $\mathbb{P}(T)$-generic filter. Then in $V[G]$

- $R_G = \cup G = \langle r_i : i \in \omega \rangle \leq T$.
- $\exists C'$ such that $C \cup \{R_G\} \subseteq Q(C')$, $|C| = |C'|$.

Proof.

Since $G \cap D_T(X, n) \neq \emptyset$ for all $X \in C$, $n \in \omega$, the set $I_X = \langle i : r_i \leq X \rangle$ is infinite and so $R_G \wedge X = \langle r_i : i \in I_X \rangle$ is a common extension of $R_G$ and $X$. If $X \leq Y$ then $I_X \subseteq I_Y$ and so $R_G \wedge X \leq R_G \wedge Y$. Then $C' = \{R_G \wedge X\}_{X \in C}$ is centered. \qed
Lemma

Let $\text{cov}(\mathcal{M}) = \kappa$, $C$ centered $|C| < \kappa$, $\dot{f}$ a good $Q(C)$-name for a real. Then there is $T = \langle r_i : i \in \omega \rangle$ of logarithmic measures of strictly increasing levels, such that

$\forall X \in C$ the set $J_X = \{i : r_i \leq X\}$ is infinite and

$\forall i \forall v \subseteq i \forall s \subseteq \text{int}(r_i)$ which is $r_i$-positive $\exists w \subseteq s \exists p \in A_i(\dot{f})$ such that $(v \cup w, T) \leq p$. 
The proof uses two countable forcing notions, the first of which produces a pure condition which is preprocessed for $\dot{f}$.

Under $\text{cov}(\mathcal{M}) = \kappa$ certain subfamilies of $[\omega]^{<\omega}$ induce logarithmic measures which take arbitrarily high values. The second forcing notions amalgamates such measures into the pure condition $T$. 
Theorem

Let $\text{cov}(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq \omega \omega$ unbounded, $<^*\text{-directed}$, $|\mathcal{H}| = \kappa$, $C$ centered, $|C| < \kappa$, $\dot{f}$ good $Q(C)$-name for a real. Then

$\exists C' \exists h \in \mathcal{H}$, such that $C'$ extends $C$, $|C'| = |C|$ and $\forall C''$ extending $C'$, $\models Q(C'')$ “$\check{h} \not<^* \check{f}$”.

Proof

Let $T = \langle r_i : i \in \omega \rangle$ satisfy the preceding Lemma for $C$ and $\dot{f}$. Then $\forall i \in \omega$ let $g(i)$ be the maximal $k$ such that there are $\nu \subseteq i$, $\omega \subseteq \text{int}(r_i)$, $p \in A_i(\dot{f})$ with $p \models \check{k} = \check{f}(i)$ and $(\nu \cup w, T) \leq p$. 
\[ \forall X \in C \ J_X = \{ i : r_i \leq X \} \text{ is infinite. Then } \forall n \in \omega \text{ let } F_X(n) = g(J_X(i + 1)) \text{ iff } n \in (J_X(i), J_X(i + 1)] \text{ where } J_X(n) \text{ is the } n\text{-th element of } J_X. \text{ Then } \forall X \in C \exists h_X \in H(h_X \not\leq^* F_X). \]

\[ \text{Let } h \in H \text{ dominate all } h_X\text{'s. Then } J = \{ i : g(i) < h(i) \} \text{ and } I_X = J_X \cap J \text{ are infinite. Let } R = \langle r_i \rangle_{i \in J}, \ R \land X = \langle r_i \rangle_{i \in I_X} \text{ and } C' = \{ R \land X \}_{X \in C}. \]
\[ \mathcal{J}_X = \{ i : \mathcal{U}_i \subseteq X \} \]

\[ J = \{ i : g(i) < h(i) \} \]

\[ \exists i \in J_X \ ( F_X(i) < h(i) ) \]

otherwise \( h \preceq^* F_X \)

Therefore \( I_X = \mathcal{J}_X \cap J \) is infinite

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∀C'' extending C' ⊩ Q(C'') ̸\!\!<^* \check{\hat{f}}

- Let C'' be centered, C' ⊆ Q(C''), a ∈ [ω]^{<ω}, k_0 ∈ ω and let (b, R') ∈ Q(C'') be an extension of (a, R). There is i ∈ J, i > k_0 such that b ⊆ i and s = int(R') ∩ int(r_i) is r_i-positive. Then ∃w ⊆ s ∃p ∈ A_i(\check{\hat{f}}) such that (b ∪ w, T) ≤ p.

- Therefore (b ∪ w, R') extends (b, R') and p. Let k ∈ ω be such that p ⊩ \check{\hat{f}}(i) = ˇk. Then by definition of g, k ≤ g(i) and since i ∈ J, g(i) < h(i). Thus (b ∪ w, R') ⊩ Q(C'') "\check{\hat{f}}(i) = ˇk ≤ ˇg(i) < ˇh(i)".
Lemma

Let $\text{cov}(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq \omega^\omega$ be an unbounded, directed family of cardinality $\kappa$ and let $\forall \lambda < \kappa (2^\lambda \leq \kappa)$. Then there is a centered family $C$, $|C| = \kappa$ such that $Q(C)$ preserves $\mathcal{H}$ unbounded and adds a real not split by $V \cap [\omega]^{\omega}$. 
Let $\mathcal{N} = \{\dot{f}_\alpha\}_{\alpha < \kappa}$ enumerate all $Q(C')$ names for functions in $\omega\omega$ where $|C'| < \kappa$. Let $\mathcal{A} = \{A_{\alpha+1}\}_{\alpha < \kappa}$ enumerate $V \cap [\omega]^{\omega}$. By induction of length $\kappa$ obtain a sequence $\langle C_\alpha : \alpha < \kappa \rangle$ such that $\forall \alpha < \beta C_\alpha \subseteq Q(C_\beta)$, $|C_\alpha| < \kappa$ as follows:

- Begin with any $T$ and $C_0 = \{T \setminus v : v \in [\omega]^{<\omega}\}$
- If $\alpha$ is a limit, let $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$
If $\alpha = \beta + 1$, let $\dot{g}_\alpha$ be the name with least index in $\mathcal{N} \setminus \{\dot{g}_{\gamma+1}\}_{\gamma < \beta}$ which is a $Q(C_\beta)$-name.

- If $\dot{g}_\alpha$ is good, let $C_\alpha$ extend $C_\beta$, $|C_\alpha| = |C_\beta|$ such that
  1. $\exists h_\alpha \in \mathcal{H} \forall C''$ extending $C_\alpha \models Q(C'')$ "$\dot{h}_\alpha \not<^* \dot{g}_\alpha$"  
  2. $\exists T_\alpha \in Q(C_\alpha)(\text{int}(T_\alpha) \subseteq A_\alpha$ or $\text{int}(T_\alpha) \subseteq A^c_\alpha$).

- If $\dot{g}_\alpha$ is not good, let $C_\alpha$ extend $C_\beta$, $|C_\alpha| = |C_\beta|$ such that
  1. $\dot{g}_\alpha$ is not a $Q(C_\alpha)$-name,  
  2. $\exists T_\alpha \in Q(C_\alpha)(\text{int}(T_\alpha) \subseteq A_\alpha$ or $\text{int}(T_\alpha) \subseteq A^c_\alpha$).

Then let $C = \bigcup_{\alpha < \kappa} C_\alpha$. 
**H** is unbounded
If \( \dot{f} \) is a \( Q(C) \)-name, then \( \exists \beta \in \kappa \) such that \( \dot{f} \) is a good \( Q(C_\beta) \)-name and is the name with least index in \( \mathcal{N} \setminus \{ \dot{g}_{\gamma + 1} \}_{\gamma < \beta} \) which is a \( Q(C_\beta) \)-name. Then \( (\mathcal{H} \text{ is unbounded})^{V^{Q(C)}} \).

\( \exists \) a real not split by the ground model reals
Let \( G \) be \( Q(C) \)-generic. Then for every \( A \in V \cap [\omega]^\omega \) there is \( (u, T) \) in \( G \) such that \( \text{int}(T) \subseteq A \) or \( \text{int}(T) \subseteq A^c \). Note also that if \( U_G = \bigcup \{ u : \exists T(u, T) \in G \} \), then \( U_G \subseteq^* \text{int}(T) \) for all \( T \) such that \( \exists u(u, T) \in G \).
Theorem
Let \( \mathcal{H} \subseteq \omega \omega \) be unbounded family such that every countable subfamily of \( \mathcal{H} \) is dominated by an element of \( \mathcal{H} \) and let \( \langle P_\gamma : \gamma \leq \alpha \rangle \) be a finite support iteration of ccc forcing notions of length \( \alpha \), \( \text{cf}(\alpha) = \omega \) such that \( \forall \gamma < \alpha \ (\mathcal{H} \text{ is unbounded})^V_{P_\gamma} \). Then \( (\mathcal{H} \text{ is unbounded})^V_{P_\alpha} \).

Theorem
Let \( \mathcal{H} \subseteq \omega \omega \) be unbounded, directed family, \( |\mathcal{H}| = \kappa \). Then for every partial order \( P \) of size less than \( \kappa \), \( (\mathcal{H} \text{ is unbounded})^V_P \).
Theorem (GCH)

Let $\kappa$ be a regular uncountable cardinal. Then there is a ccc generic extension in which $b = \kappa < s = \kappa^+$. 
Add $\kappa$ Hechler reals to obtain a model $V$ of $b = c = \kappa$. Let $\mathcal{H} = V \cap \omega \omega$. Define a finite support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \kappa^+ \rangle$ such that $\forall \alpha < \kappa^+$

$$\models_{P_\alpha} \text{ "} \dot{Q}_\alpha \text{ is ccc and } |\dot{Q}_\alpha| \leq c \text{"}$$

as follows. If $\alpha$ is a limit, let $P_\alpha$ be the finite support iteration of $\langle P_\beta, \dot{Q}_\beta : \beta < \alpha \rangle$. If $\alpha = \beta + 1$ is a successor, then
Let $\dot{Q}_\beta$ be $P_\beta$-name for $C(\kappa)$ and $P_\alpha = P_\beta * \dot{Q}_\beta$. \exists C$ such that $Q(C)$ preserves $H$ unbounded and destroys $V^{P_\alpha} \cap [\omega]^{\omega}$ as a splitting family.

Let $\dot{Q}_\alpha$ be a $P_\alpha$ name for $Q(C)$ and $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$.

Let $A \subseteq V^{P_{\alpha+1}} \cap \omega_\omega$ be unbounded of size less than $\kappa$. Then let $\dot{Q}_{\alpha+1}$ be $P_{\alpha+1}$-name for $H(A)$; $P_{\alpha+2} = P_{\alpha+1} * \dot{Q}_{\alpha+1}$.

Then in $V^{P_{\kappa^+}}$ $H$ is unbounded and there are no splitting families of size less than $\kappa^+$. Using a suitable bookkeeping device one can guarantee that there are no unbounded families of size less than $\kappa$. 

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