MAD families, splitting families and large continuum

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- con($\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$)
- In 1984 S. Shelah obtained the above consistency using an almost ${}^\omega\omega$- bounding version of Mathias forcing, in which the pure Mathias condition is supplied with additional structure in the form of a finite logarithmic measure.
- The countable support iteration of proper almost ${}^\omega\omega$-bounding posets is weakly bounding, which implies that in such extensions the ground model reals remain an unbounded family.

- A modification of the preceding argument produces the consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{a} = \mathfrak{s} = \aleph_2$.

- con($\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$)
- Obtain a ccc suborder of Shalah's poset, which behaves sufficiently similarly to the larger forcing notion.

Theorem (V. F., J. Steprāns)

Let κ be a regular, uncountable cardinal, $\forall \mu < \kappa (2^\mu \leq \kappa)$, cov$(\mathcal{M}) = \kappa$ and let $\mathcal{H}$ be an unbounded directed family of size κ. Then there is an ultrafilter $\mathcal{U}_\mathcal{H}$ on ω such that the relativized Mathias poset $\mathbb{M}(\mathcal{U}_\mathcal{H})$, preserves the unboundedness of $\mathcal{H}$.

- If $\mathcal{H}$ is an unbounded family, such that every countable subfamily of $\mathcal{H}$ is dominated by a element of the family, then in order to preserve the unboundedness of $\mathcal{H}$ in finite support iterated forcing construction, it is sufficient to preserve the family unbounded at each successor stage of the iteration.
- If $\mathcal{H}$ is unbounded and $\mathbb{P}$ is a poset of size smaller than the cardinality of $\mathcal{H}$, then $\mathcal{H}$ remains unbounded in $V^{\mathbb{P}}$.

- Add κ many Hechler reals to a model of GCH to obtain a directed unbounded family $\mathcal{H}$ of size κ.
- Proceed with a finite support iteration of length κ^+ alternating $\mathbb{C}_\kappa$, $\mathbb{M}(\mathcal{U}_\mathcal{H})$ and restricted Hechler forcing.
- An appropriate bookkeeping function will guarantee that in the final generic extension there are no unbounded families of size $< \kappa$ and so $\mathcal{H}$ will remain a witness to $\mathfrak{b} = \kappa$.
- Since cofinally often we add reals not split by the ground model reals, $\mathfrak{s} = \kappa^+$ in the final generic extension.

Theorem (V. F., J. Steprāns)

Let κ *be a regular uncountable cardinal. Then there is a ccc generic extension in which* $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{c} = \kappa^+$.

Theorem (J. Brendle)

Let κ be a regular uncountable cardinal. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{a} = \mathfrak{c} = \kappa^+$.

The iteration techniques of the last two models can be combined to produce the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \kappa^+$.

Theorem (V.F., J. Steprāns)

Assume CH. There is a countably closed, $\aleph_2$-c.c. poset $\mathbb{P}$ which adds a $\mathbb{C}_{\omega_2}$-name for an ultrafilter $\mathcal{U}$ such that in $V^{\mathbb{P}\times\mathbb{C}_{\omega_2}}$ the relativized Mathias poset $\mathbb{M}(\mathcal{U})$ preserves the unboundedness of all families of Cohen reals of size ω_1.

- How to iterate $(\mathbb{P} \times \mathbb{C}(\omega_2)) \times \mathbb{M}(\mathcal{U})$?
- How to force an entire forcing construction with the desired properties?

The method of matrix iteration was introduced by S. Shelah and A. Blass in their work on the ultrafilter and dominating number. Using this technique they establish the consistency of $\mathfrak{u} = \kappa < \mathfrak{d} = \lambda$ for $\kappa < \lambda$ arbitrary regular uncountable cardinals.

These are systems of finite support iterations $\langle\langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda\rangle\rangle$ such that:

- For all $\alpha \leq \kappa$, $\langle\langle \mathbb{P}_{\alpha,\zeta} : \zeta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \zeta < \lambda\rangle\rangle$ is a finite support iteration of ccc posets.
- For all $\alpha_1 \leq \alpha_2$ and $\zeta \leq \lambda$, $\mathbb{P}_{\alpha_1,\zeta}$ is a complete suborder of $\mathbb{P}_{\alpha_2,\zeta}$.

Thus for all $\alpha_1 \leq \alpha_2$, $\zeta_1 \leq \zeta_2$ we have $\mathbb{P}_{\alpha_1,\zeta_1} \lessdot \mathbb{P}_{\alpha_2,\zeta_2}$.

Theorem (Brendle, F., 2011)

Let $\kappa < \lambda$ be arbitrary regular uncountable cardinals. Then there is a ccc generic extension in which $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

Theorem (Brenlde, F., 2011)

Let μ be a measurable cardinal, $\kappa < \lambda$ regular such that $\mu < \kappa$. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$.

For γ an ordinal, $\mathbb{P}_\gamma$ is the poset of all finite partial functions $p : \gamma \times \omega \to 2$ such that $\mathrm{dom}(p) = F_p \times n_p$ where $F_p \in [\gamma]^{<\omega}$, $n_p \in \omega$. The order is given by $q \leq p$ if $p \subseteq q$ and $|q^{-1}(1) \cap F^p \times \{i\}| \leq 1$ for all $i \in n_q \backslash n_p$.

Let G be a $\mathbb{P}_\gamma$-generic filter and for $\delta \in \gamma$ let $A_\alpha = \{i : \exists p \in G(p(\alpha, i) = 1)\}$. Then

- $\{A_\alpha : \alpha \in \gamma\}$ is an a.d. family (maximal for $\gamma \geq \omega_1$),
- if $p \in \mathbb{P}_\gamma$ then for all $\alpha \in F_p(p \Vdash \dot{A}_\alpha \restriction n_p = p \restriction \{\alpha\} \times n_p)$,
- for all $\alpha, \beta \in F_p(p \Vdash \dot{A}_\alpha \cap \dot{A}_\beta \subseteq n_p)$.

Let $\gamma < \delta$, G a $\mathbb{P}_\gamma$-generic filter. In $V[G]$, let $\mathbb{P}_{[\gamma,\delta)}$ consist of all (p, H) such that $p \in \mathbb{P}_\delta$ with $F_p \in [\delta\backslash\gamma]^{<\omega}$ and $H \in [\gamma]^{<\omega}$. The order is given by $(q, K) \leq (p, H)$ if $q \leq_{\mathbb{P}_\delta} p$, $H \subseteq K$ and for all $\alpha \in F_p$, $\beta \in H$, $i \in n_q \backslash n_p$ if $i \in A_\beta$, then $q(\alpha, i) = 0$.

- That is for all $\alpha \in F_p, \beta \in H$, $p \Vdash \dot{A}_\alpha \cap \check{A}_\beta \subseteq n_p$.
- $\mathbb{P}_\delta$ is forcing equivalent to $\mathbb{P}_\gamma * \mathbb{P}_{[\gamma,\delta)}$.

Property $\star$

Let $M \subseteq N$, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq [\omega]^\omega \cap M$, $A \in N \cap [\omega]^\omega$. Then $(\star^{M,N}_{\mathcal{B},A})$ holds if for every $h : \omega \times [\gamma]^{<\omega} \to \omega$, $h \in M$ and $m \in \omega$ there are $n \geq m$, $F \in [\gamma]^{<\omega}$ such that $[n, h(n,F)) \backslash \bigcup_{\alpha \in F} B_\alpha \subseteq A$.

Lemma A

If $G_{\gamma+1}$ is $\mathbb{P}_{\gamma+1}$-generic, $G_\gamma = G_{\gamma+1} \cap \mathbb{P}_\gamma$, $\mathcal{A}_\gamma = \{A_\alpha\}_{\alpha<\gamma}$, where $A_\alpha = \{i : \exists p \in G(p(\alpha, i) = 1)\}$. Then $(\star^{V[G_\gamma],V[G_{\gamma+1}]}_{\mathcal{A}_\gamma,A_\gamma})$ holds.

Lemma B

Let $(\star^{M,N}_{\mathcal{B},A})$ hold, where $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma}$, let $\mathcal{I}(\mathcal{B})$ be the ideal generated by $\mathcal{B}$ and the finite sets and let $B \in M \cap [\omega]^\omega$, $B \notin \mathcal{I}(\mathcal{B})$. Then $|A \cap B| = \aleph_0$.

Lemma

Let $\mathbb{P}, \mathbb{Q}$ be partial orders, such that $\mathbb{P}$ is completely embedded into $\mathbb{Q}$. Let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion, $\dot{\mathbb{B}}$ a $\mathbb{Q}$-name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subseteq \dot{\mathbb{B}}$, and every maximal antichain of $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$ is a maximal antichain of $\dot{\mathbb{B}}$ in $V^{\mathbb{Q}}$. Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$.

Lemma C

Let $M \subseteq N$, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$ such that $(\star^{M,N}_{\mathcal{B},A})$. Let $\mathcal{U}$ be an ultrafilter in M. Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N such that

1. every maximal antichain of $\mathbb{M}_\mathcal{U}$ which belongs to M is a maximal antichain of $\mathbb{M}_\mathcal{V}$ in N,
2. $(\star^{M[G],N[G]}_{\mathcal{B},A})$ holds where G is $\mathbb{M}_\mathcal{V}$-generic over N (and thus, by (1), $\mathbb{M}_\mathcal{U}$-generic over M).

Lemma D

Let $M \subseteq N$, $\mathbb{P} \in M$ a poset such that $\mathbb{P} \subseteq M$, G a $\mathbb{P}$-generic filter over M, N. Let $\mathcal{B} = \{B_\alpha\}_{\alpha\in\gamma} \subseteq M \cap [\omega]^\omega$, $A \in N \cap [\omega]^\omega$ such that $(\star^{M,N}_{\mathcal{B},A})$ holds. Then $(\star^{M[G],N[G]}_{\mathcal{B},A})$ holds.

Lemma E

Let $\langle \mathbb{P}_{\ell,n}, \dot{\mathbb{Q}}_{\ell,n} : n \in \omega \rangle$, $\ell \in \{0,1\}$ be finite support iterations such that $\mathbb{P}_{0,n}$ is a complete suborder of $\mathbb{P}_{1,n}$ for all n. Let $V_{\ell,n} = V^{\mathbb{P}_{\ell,n}}$. Let $\mathcal{B} = \{A_\gamma\}_{\gamma<\alpha} \subseteq V_{0,0} \cap [\omega]^\omega$, $A \in V_{1,0} \cap [\omega]^\omega$. If $(\star_{\mathcal{B},A}^{V_{0,n},V_{1,n}})$ holds for all $n \in \omega$, then $(\star_{\mathcal{B},A}^{V_{0,\omega},V_{1,\omega}})$ holds.

Let $f : \{\eta < \lambda : \eta \equiv 1 \bmod 2\} \to \kappa$ be an onto mapping, such that for all $\alpha < \kappa$, $f^{-1}(\alpha)$ is cofinal in λ. Recursively define a system of finite support iterations

$$\langle\langle \mathbb{P}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\zeta} : \alpha \leq \kappa, \zeta < \lambda\rangle\rangle$$

as follows. For all α, ζ let $V_{\alpha,\zeta} = V^{\mathbb{P}_{\alpha,\zeta}}$.

(1) If $\zeta = 0$, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,0}$ is Hechler's poset for adding an a.d. family $\mathcal{A}_\alpha = \{A_\beta\}_{\beta<\alpha}$,

(2) If $\zeta = \eta + 1$, $\zeta \equiv 1 \mod 2$, then $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathbb{Q}}_{\alpha,\eta} = \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ where $\dot{\mathcal{U}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for an ultrafilter and for all $\alpha < \beta \leq \kappa$, $\Vdash_{\mathbb{P}_{\beta,\eta}} \dot{\mathcal{U}}_{\alpha,\eta} \subseteq \dot{\mathcal{U}}_{\beta,\eta}$,

(3) If $\zeta = \eta + 1$, $\zeta \equiv 0 \mod 2$, then if $\alpha \leq f(\eta)$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for the trivial forcing notion; if $\alpha > f(\eta)$ then $\dot{\mathbb{Q}}_{\alpha,\eta}$ is a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.

(4) If ζ is a limit, then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$.

Furthermore the construction will satisfy the following two properties:

(a) $\forall\zeta \leq \lambda \forall\alpha < \beta \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$,

(b) $\forall\zeta \leq \lambda \forall\alpha < \kappa$ $(\star^{V_{\alpha,\zeta},V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha,A_\alpha})$ holds.

Proceed by recursion on ζ. For $\zeta = 0$, $\alpha \leq \kappa$ let $\mathbb{P}_{\alpha,0} = \mathbb{P}_\alpha$. Then clearly properties (*a*) and (*b*) above hold. Let $\zeta = \eta + 1$ be a successor ordinal and suppose $\forall \alpha \leq \kappa$, $\mathbb{P}_{\alpha,\eta}$ has been defined.

If $\zeta \equiv 1 \mod 2$ define $\dot{\mathbb{Q}}_{\alpha,\eta}$ by induction on $\alpha \leq \kappa$ as follows.

- *If* $\alpha = 0$, let $\dot{\mathcal{U}}_{0,\eta}$ be a $\mathbb{P}_{0,\eta}$-name for an ultrafilter, $\dot{\mathbb{Q}}_{0,\eta}$ a $\mathbb{P}_{0,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{0,\eta}}$ and let $\mathbb{P}_{0,\zeta} = \mathbb{P}_{0,\eta} * \dot{\mathbb{Q}}_{0,\eta}$.
- *If* $\alpha = \beta + 1$ and $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined, by the ind. hyp. and Lemma C there is a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}$, every maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ in $V_{\beta,\eta}$ is a maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and $(\star^{V_{\beta,\zeta}, V_{\beta+1,\zeta}}_{\mathcal{A}_\beta, A_\beta})$. holds. Let $\mathbb{P}_{\beta,\zeta} = \mathbb{P}_{\beta,\eta} * \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$. In particular $\mathbb{P}_{\beta,\zeta} \lessdot \mathbb{P}_{\alpha,\zeta}$.

- *If α is limit* and for all $\beta < \alpha$ $\dot{\mathcal{U}}_{\beta,\eta}$ has been defined (and so $\dot{\mathbb{Q}}_{\beta,\eta} = \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$) consider the following two cases.
  - If $\mathrm{cf}(\alpha) = \omega$, find a $\mathbb{P}_{\alpha,\eta}$-name $\dot{\mathcal{U}}_{\alpha,\eta}$ for an ultrafilter such that for all $\beta < \alpha$, $\Vdash_{\mathbb{P}_{\alpha,\eta}} \dot{\mathcal{U}}_{\beta,\eta} \subseteq \dot{\mathcal{U}}_{\alpha,\eta}$ and every maximal antichain of $\mathbb{M}_{\dot{\mathcal{U}}_{\beta,\eta}}$ from $V_{\beta,\eta}$ is a maximal antichain of $\mathbb{M}_{\mathcal{U}_{\alpha,\eta}}$ (in $V_{\alpha,\eta}$) and the relevant $\star$-property is preserved.
  - If $\mathrm{cf}(\alpha) > \omega$, then let $\dot{\mathcal{U}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\bigcup_{\beta<\alpha} \mathcal{U}_{\beta,\eta}$. Let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\eta}}$ and let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$.

If $\zeta \equiv 0 \mod 2$, then

- for all $\alpha \leq f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for the trivial poset
- for $\alpha > f(\eta)$ let $\dot{\mathbb{Q}}_{\alpha,\eta}$ be a $\mathbb{P}_{\alpha,\eta}$-name for $\mathbb{D}^{V_{f(\eta),\eta}}$.

Let $\mathbb{P}_{\alpha,\zeta} = \mathbb{P}_{\alpha,\eta} * \dot{\mathbb{Q}}_{\alpha,\eta}$. Note that for all $\alpha, \beta \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is a complete suborder of $\mathbb{P}_{\beta,\zeta}$ and $(\star^{V_{\alpha,\zeta},V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha,A_\alpha})$ holds for all α.

If ζ is a limit and for all $\eta < \zeta$, $\mathbb{P}_{\alpha,\eta}$, $\dot{\mathbb{Q}}_{\alpha,\eta}$ have been defined, let $\mathbb{P}_{\alpha,\zeta}$ be the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \dot{\mathbb{Q}}_{\alpha,\eta} : \eta < \zeta \rangle$. Then $\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\zeta}$ and by Lemma E $(\star^{V_{\alpha,\zeta}, V_{\alpha+1,\zeta}}_{\mathcal{A}_\alpha, A_\alpha})$ holds.

Lemma

For $\zeta \leq \lambda$:

1. for every $p \in \mathbb{P}_{\kappa,\zeta}$ there is $\alpha < \kappa$ such that p belongs to $\mathbb{P}_{\alpha,\zeta}$,
2. for every $\mathbb{P}_{\kappa,\zeta}$-name for a real $\dot{f}$ there is $\alpha < \kappa$ such that $\dot{f}$ is a $\mathbb{P}_{\alpha,\zeta}$-name.

Lemma

$V_{\kappa,\lambda} \vDash \mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

$\{A_\alpha\}_{\alpha\in\kappa}$ remains mad in $V_{\kappa,\lambda}$. Otherwise $\exists B \in V_{\kappa,\lambda} \cap [\omega]^\omega$ such that $\forall\alpha < \kappa(|B \cap A_\alpha| < \omega)$. However there is $\alpha < \kappa$ such that $B \in V_{\alpha,\lambda} \cap [\omega]^\omega$ and $B \notin \mathcal{I}(\mathcal{A}_\alpha)$. On the other hand $(\star^{V_{\alpha,\lambda},V_{\alpha+1,\lambda}}_{\mathcal{A}_\alpha,A_{\alpha+1}})$ and so $|B \cap A_{\alpha+1}| = \omega$ (Lemma B) which is a contradiction. Therefore $\mathfrak{a} \leq \kappa$.

Let $\mathcal{B} \subseteq V_{\kappa,\lambda} \cap {}^\omega\omega$ be of size $< \kappa$. Then there are $\alpha < \kappa$, $\zeta < \lambda$ such that $\mathcal{B} \subseteq V_{\alpha,\zeta}$. Since $\{\gamma : f(\gamma) = \alpha\}$ is cofinal in λ, there is $\zeta' > \zeta$ such that $f(\zeta') = \alpha$. Then $\mathbb{P}_{\alpha+1,\zeta'+1}$ adds a real dominating $V_{\alpha,\zeta'} \cap {}^\omega\omega$ (and so $V_{\alpha,\zeta} \cap {}^\omega\omega$ since $V_{\alpha,\zeta} \subseteq V_{\alpha,\zeta'}$). Thus $\mathcal{B}$ is not unbounded. Therefore $V_{\kappa,\lambda} \Vdash \mathfrak{b} \geq \kappa$.

However $\mathfrak{b} \leq \mathfrak{a}$ and so $V_{\kappa,\lambda} \Vdash \mathfrak{b} = \mathfrak{a} = \kappa$.

To see that $V_{\kappa,\lambda} \vDash \mathfrak{s} = \lambda$, note that if $S \subseteq V_{\kappa,\lambda} \cap [\omega]^\omega$ is a family of cardinality $< \lambda$, then there is $\zeta < \lambda$ such that $\zeta = \eta + 1$, $\zeta \equiv 1 \bmod 2$ and $S \subseteq V_{\kappa,\eta}$. Then $\mathbb{M}_{\mathcal{U}_{\kappa,\eta}}$ adds a real not split by S and so S is not splitting.

Theorem (Brendle, F., 2011)

Let $\kappa < \lambda$ be arbitrary regular uncountable cardinals. Then there is a ccc generic extension in which $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$.

Corollary

Let $\kappa < \lambda$ be arbitrary regular uncountable cardinals. Then there is a ccc generic extension in which $\mathfrak{a} = \kappa < \mathfrak{a}_g = \lambda$.

Proof:

Since $\mathfrak{s} \leq \mathfrak{a}_g$. □

- Is it relatively consistent that $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$?
- Is it relatively consistent that $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$?
- It is relatively consistent that $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$ without the assumption of a measurable?
- How about $\mathfrak{b} = \mathfrak{s} = \aleph_1 < \mathfrak{a} = \aleph_2$?

Thank you!