Template iterations and maximal cofinitary groups

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\begin{itemize}
  \item cofin(S_{\infty}) is the set of cofinitary permutations in S_{\infty}, i.e. permutations \( \sigma \in S_{\infty} \) which have finitely many fixed points.
  \item A mapping \( \rho : A \to S_{\infty} \) \textit{induces} a cofinitary representation of \( F_A \) if the canonical extension of \( \rho \) to a homomorphism \( \hat{\rho} : F_A \to S_{\infty} \) is such that \( \text{im}(\hat{\rho}) \subseteq \{I\} \cup \text{cofin}(S_{\infty}) \).
\end{itemize}
Forcing M.c.g.’s

Let $A, X$ be disjoint non-empty sets and let $\rho : X \to S_\infty$ induce a cofinitary representation. Then $Q_{A,\rho}$ is the poset of all $(s, F)$ where $s \subseteq A \times \omega \times \omega$ is finite, $s_a$ is a finite injection for all $a$ and $F \subseteq \check{W}_{A \cup X}$ is finite. Define $(s, F) \leq_{\mathbb{P}_{A,\rho}} (t, E)$ iff

- $s \supseteq t$, $F \supseteq E$ and,
- for all $n \in \omega$ and $w \in E$, if $e_w[s, \rho](n) = n$ then already $e_w[t, \rho](n) \downarrow$ and $e_w[t, \rho](n) = n$.

If $X = \emptyset$ then we write $Q_A$ for $Q_{A,\rho}$. If $A$ is clear from the context we just write $Q$. 
\( \mathbb{Q}_{A, \rho} \) is Knaster.

Let \( G \) be \( \mathbb{Q}_{A, \rho} \) generic and let \( \rho_G : A \cup X \to S_\infty \) be a mapping extending \( \rho \) and such that for all \( a \in A \)

\[
\rho_G(a) = \bigcup \{ s_a : (\exists F \in \hat{W}_{A \cup X}) (s, F) \in G \}.
\]

Then \( \rho_G \) induces a cofinitary representation of \( A \cup X \) extending \( \rho \).
Lemma: Complete Embeddings

Let $A_0 \cap A_1 = \emptyset$, $A = A_0 \cup A_1$ and let $G$ be $\mathbb{Q}_{A,\rho}$-generic. Then

- $\mathbb{Q}_{A_0,\rho}$ is a complete suborder of $\mathbb{Q}_{A,\rho}$,
- $H = G \cap \mathbb{Q}_{A_0,\rho}$ is $\mathbb{Q}_{A_0,\rho}$-generic, $K = \{(s|A_1, F) : (s, F) \in G\}$ is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$ and $\rho_G = (\rho_H)_K$. 
Theorem
Let $|A| > \kappa_0$ and $G$ be a $\mathbb{Q}_A,\rho$-generic over $V$. Then $\text{im}(\rho_G)$ is a maximal cofinitary group in $V[G]$.

Proof
Let $z \notin X \cup A$, where $\rho : X \to S_\infty$. Suppose there in $V[G]$ there is $\sigma \in \text{cofin}(S_\infty)$ such that $\rho'_G : A \cup X \cup \{z\} \to S_\infty$ defined by $\rho'_G|X \cup A = \rho_G$, $\rho'_G(z) = \sigma$ induces a cofinitary representation. Let $\dot{\sigma}$ be a name for $\sigma$. Then there is $A_0 \subseteq A$ countable so that $\dot{\sigma}$ is a $\mathbb{Q}_{A_0},\rho$-name and so $\sigma \in V[H]$, where $H = G \cap \mathbb{Q}_{A_0},\rho$. 
Let $a_1 \in A \setminus A_0$ and let $K$ be defined as in the previous Lemma. Note that for every $N \in \omega$

\[
D_{\sigma,N} = \{(s,F) \in Q_{A_1,\rho_H} : (\exists n \geq N)s_{a_1}(n) = \sigma(n)\}
\]

is dense in $Q_{A_1,\rho_H}$ and so in $V[H][K]$

\[
\exists^\infty n((\rho_H)_K(a_1)(n) = \sigma(n)).
\]

However $(\rho_H)_K = \rho_G$, which contradicts that $\rho'_G$ induces a cofinitary representation.
Lemma: Strong Embedding

Let $B, C \subseteq D$, $B \cap C = A$ be given set and $p \in \mathbb{Q}_{B, \rho}$. Then there is a condition $p_0 \in \mathbb{Q}_{A, \rho}$ such that whenever $q_0 \leq \mathbb{Q}_{C, \rho} p_0$, then $q_0$ is compatible in $\mathbb{Q}_{D, \rho}$ with $p$.

- We say that $\mathbb{Q}_{B, \rho}$ has the strong embedding property and $q_0$ is called a strong reduction of $p$.

- If $C = A$, $B = D$ then the above gives in particular that $\mathbb{Q}_{A, \rho}$ is a complete suborder of $\mathbb{Q}_{B, \rho}$. 
Definition: \( \mathbb{L} \)
\( \mathbb{L} \) consists of pairs \((\sigma, \phi)\) such that \( \sigma \in <\omega(<\omega[\omega]), \phi \in \omega(<\omega[\omega]) \) such that \( \sigma \subseteq \phi, \forall i < |\sigma|(|\sigma(i)| = i) \) and \( \forall i \in \omega(|\phi(i)| \leq |\sigma|). \)

The extension relation is defined as follows: \((\sigma, \phi) \leq (\tau, \psi)\) if and only if \( \sigma \) end-extends \( \tau \) and \( \forall i \in \omega (\psi(i) \subseteq \phi(i)). \)

- A slalom is a function \( \phi : \omega \rightarrow [\omega]^<\omega \) such that \( \forall n \in \omega(|\phi(n)| \leq n) \). A slalom localizes a real \( f \in \omega \omega \) if there is \( m \in \omega \) such that \( \forall n \geq m(f(n) \in \phi(n)). \)

- \( \mathbb{L} \) adds a slalom which localizes all ground model reals.
add(\mathcal{N}) is the least cardinality of a family \( F \subseteq \omega^\omega \) such that no slalom localizes all members of \( F \).

cof(\mathcal{N}) is the least cardinality of a family \( \Phi \) of slaloms such that every real is localized by some \( \phi \in \Phi \).

\( a_g \geq \text{non}(\mathcal{M}) \).

In our intended forcing construction cofinally often we will force with the partial order \( \mathbb{L} \), which using the above characterization will provide a lower bound for \( a_g \).
Definition: $\sigma$-Suslin

Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion, whose conditions can be written in the form $(s, f)$ where $s \in <\omega \omega$ and $f \in \omega \omega$. We will say that $\mathbb{S}$ is $n$-Suslin if whenever $(s, f) \leq_{\mathbb{S}} (t, g)$ and $(t, h)$ is a condition in $\mathbb{S}$ such that

$$h|n \cdot |s| = g|n \cdot |s|$$

then $(s, f)$ and $(t, h)$ are compatible. A forcing notion is called $\sigma$-Suslin, if it is $n$-Suslin for some $n$.
▶ If $S$ is $n$-Suslin and $m \geq n$, then $S$ is also $m$-Suslin.
▶ Every $\sigma$-Suslin forcing notion is $\sigma$-linked and so has the Knaster property.
▶ Hechler forcing $\mathbb{H}$ is 1-Suslin, localization $\mathbb{L}$ is 2-Suslin.
Definition: Nice name for a real
Let $\mathcal{B}$ be a partial order and $y \in \mathcal{B}$. For each $n \geq 1$ let $\mathcal{B}_n$ be a maximal antichain below $y$. We will say that the set $\{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ is a nice name for a real below $y$ if

1. whenever $n \geq 1$, $b \in \mathcal{B}_n$ then $s(b) \in {}^n\omega$

2. whenever $m > n \geq 1$, $b \in \mathcal{B}_n$, $b' \in \mathcal{B}_m$ and $b, b'$ are compatible, then $s(b)$ is an initial segment of $s(b')$.

We can assume that all names for reals are nice and abusing notation we will write $\dot{f} = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \in \omega}$. 
Lemma: Canonical Projection of a name for a real

Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$, $y \in \mathbb{B}$ and $x$ a reduction of $y$ to $\mathbb{A}$. Let $\hat{f} = \{(b, s(b))\}_{b \in \mathbb{B}, n \geq 1}$ be a nice name for a real below $y$. Then there is $\hat{g} = \{(a, s(a))\}_{a \in \mathbb{A}, n \geq 1}$, a $\mathbb{A}$-nice name for a real below $x$, such that for all $n \geq 1$, for all $a \in \mathbb{A}_n$, there is $b \in \mathbb{B}_n$ such that $a$ is a reduction of $b$ and $s(a) = s(b)$.

Whenever $\hat{f}, \hat{g}$ are as above, we will say that $\hat{g}$ is a canonical projection of $\hat{f}$ below $x$. 
Definition: Good Suslin

Let $\mathbb{S}$ be a Suslin forcing notion, whose conditions can be written in the form $(s, f)$ where $s \in <\omega \omega$, $f \in \omega \omega$. Then $\mathbb{S}$ is said to be good if whenever $A$ is a complete suborder of $\mathbb{B}$, $x \in A$ is a reduction of $y \in \mathbb{B}$ and $\dot{f}$ is a nice name for a real below $y$ such that $y \Vdash_{\mathbb{B}} (\check{s}, \dot{f}) \in \mathbb{S}$ for some $s \in <\omega \omega$, there is a canonical projection $\dot{g}$ of $\dot{f}$ below $x$ such that $x \Vdash (\check{s}, \dot{g}) \in \mathbb{S}$. 
$\mathcal{D}$ and $\mathcal{L}$ are good $\sigma$-Suslin forcing notions.
Let $(L, \leq)$ be a linearly ordered set, $x \in L$. Then
$L_x := \{y \in L : y < x\}$.

If $L_0 \subseteq L$ and $A \subseteq L$, then the $L_0$-closure of $A$, $\text{cl}_{L_0}(A)$, is the smallest set $B \supseteq A$ such that if $x \in B$ then $L_x \cap L_0 \subseteq B$. 
Definition: Template

A template is a tuple $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ where $L = L_0 \cup L_1$, $L_0 \cap L_1 = \emptyset$, $(L, \leq)$ is a linear order, $\mathcal{I} \subseteq \mathcal{P}(L)$, such that

- $\mathcal{I}$ is closed under finite intersections and unions, $\emptyset, L \in \mathcal{I}$.
- If $x, y \in L$, $y \in L_1$ and $x < y$ then $\exists A \in \mathcal{I}(A \subseteq L_y \land x \in A)$.
- If $A \in \mathcal{I}$, $x \in L_1 \setminus A$, then $A \cap L_x \in \mathcal{I}$.
- $\{A \cap L_1 : A \in \mathcal{I}\}$ is well-founded when ordered by inclusion.
- All $A \in \mathcal{I}$ are $L_0$-closed.
Define $D_p : \mathcal{I} \to \mathbb{ON}$ by letting $D_p(A) = 0$ for $A \subseteq L_0$ and

$$D_p(A) = \sup\{D_p(B) + 1 : B \in \mathcal{I} \land B \cap L_1 \subset A \cap L_1\}.$$ 

Let $R_k(\mathcal{T}) = D_p(L)$.

For $A \subseteq L$ let

$$\mathcal{T}_A = ((A, \leq), \mathcal{I}|A, L_0 \cap A, L_1 \cap A),$$

where $\mathcal{I}|A = \{A \cap B : B \in \mathcal{I}\}$. If $A \in \mathcal{I}$ then $R_k(\mathcal{T}_A) = D_p(A)$.

For $x \in L$ let $\mathcal{I}_x = \{B \in \mathcal{I} : B \subseteq L_x\}$. 
Definition: Iterating good $\sigma$-Suslin posets along a template and adding m.c.g.

Let $Q = Q_{L_0}$ the poset adding a m.c.g. with $L_0$-generators, $S$ good $\sigma$-Suslin. $P(T, Q, S)$ is defined recursively:

If $Rk(T) = 0$, then $P(T, Q, S) = Q_{L_0}$. Let $P(T, Q, S)$ be defined for all templates of rank $< \kappa$. Let $Rk(T) = \kappa$ and for all $B \in I(Dp(B) < \kappa)$ let $P_B = P(T_B, Q, S)$. Then

$P(T, Q, S)$ consists of all $P = (p, F^p)$ where $p$ is a finite partial function with $\text{dom}(p) \subseteq L$, $(p \upharpoonright L_0, F^p) \in Q$ and if $x_p \overset{\text{def}}{=} \max\{\text{dom}(p) \cap L_1\}$ is defined then $\exists B \in I_{x_p}$ such that $P \upharpoonright L_{x_p} = (p \upharpoonright L_{x_p}, F^p) \in P_B$, $p(x_p) = (\check{s}_x^p, \check{f}_x^p)$, where $s_x^p \in <\omega$, $\check{f}_x^p$ is a $P_B$ name for a real and $(P \upharpoonright L_{x_p}, p(x_p)) \in P_B \ast \dot{S}$. 
Define $Q \leq_P P$ iff $\text{dom}(p) \subseteq \text{dom}(q)$, $(q\restriction L_0, F^q) \leq_Q (p\restriction L_0, F^p)$, and if $x_p$ is defined then either

- $x_p < x_q$ and $\exists B \in \mathcal{I}_{x_q}$ such that $P\restriction L_{x_q}, Q\restriction L_{x_q} \in \mathbb{P}_B$ and $Q\restriction L_{x_q} \leq_P P\restriction L_{x_q}$, or

- $x_p = x_q$ and $\exists B \in \mathcal{I}_{x_q}$ witnessing $P, Q \in \mathbb{P}$, and such that

$$((Q\restriction L_{x_q}, q(x_q)) \leq_{\mathbb{P}_B \ast S} (P\restriction L_{x_p}, p(x_p))).$$
Completeness of Embeddings Lemma

Let $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$, let $\mathcal{Q} = \mathcal{Q}_{L_0}$ be the poset for adding m.c.g. with $L_0$-generators, $\mathbb{S}$ be good $\sigma$-Suslin.

Let $B \in \mathcal{I}$, $A \subset B$ be closed. Then $\mathbb{P}_B$ is a poset, $\mathbb{P}_A \subset \mathbb{P}_B$, every $P = (p, F^p) \in \mathbb{P}_B$ has a canonical reduction $P_0 = (p_0, F^{p_0}) \in \mathbb{P}_A$ such that

- $\text{dom}(p_0) = \text{dom}(p) \cap A$, $F^{p_0} = F^p$,
- $s^p_{\text{dom}(p_0)} = s^p_x$ for all $x \in \text{dom}(p_0) \cap L_1$
- $(p_0|_{L_0}, F^{p_0})$ is a strong $\mathbb{Q}_A$-reduction of $(p|_{L_0}, F^p)$

and whenever $D \in \mathcal{I}$, $B, C \subseteq D$, $C$ is closed, $C \cap B = A$ and $Q_0 \leq_{\mathbb{P}_C} P_0$, then $Q_0$ and $P$ are compatible in $\mathbb{P}_D$. 
If $A = C$, $D = B$ then $\mathbb{P}_A$ is a complete suborder of $\mathbb{P}_B$. 
Lemma

- \( P(\mathcal{T}, \mathcal{Q}, \mathcal{S}) \) is Knaster.
- Let \( x \in L_1, A \in \mathcal{I}_x \). Then the two-step iteration \( P_A \ast \mathcal{S} \) completely embeds into \( P \).
- For any \( p \in P(\mathcal{T}, \mathcal{Q}, \mathcal{S}) \) there is countable \( A \subseteq L \) such that \( p \in P_{\text{cl}(A)} \). If \( \tau \) is a \( P \)-name for a real then there is a countable \( A \subseteq L \) such that \( \tau \) is a \( P_{\text{cl}(A)} \)-name.
Lemma

Let $P = P(\mathcal{T}, Q_{L_0}, L)$ and let $\lambda_0$ be a regular uncountable cardinal such that $\lambda_0 \subseteq L_1$ (as an order), $\lambda_0$ is cofinal in $L$, and $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$. Then in $V^P$, $\text{non}(\mathcal{M}) = \lambda_0$ and so $\text{ag} \geq \lambda_0$. 
Proof

Let $G$ be $\mathbb{P}$-generic and let $\phi_\alpha$ be the slalom added in coordinate $\alpha < \lambda_0$. Since $\lambda_0$ is regular, uncountable and is cofinal in $L$, the family $\langle \phi_\alpha : \alpha < \mu \rangle$ localizes all reals $V[G]$ (indeed any real must appear in some $V[G \cap \mathbb{P}_{L_\alpha}]$ for some $\alpha < \lambda_0$.) Thus $\text{cof}(\mathcal{N}) \leq \lambda_0$. On the other hand, if $F \subseteq \omega^\omega$ is a family of size $< \lambda_0$ in $V[G]$, then there must be some $\alpha < \lambda_0$ such that all reals of $F$ already are in $V[G \cap \mathbb{P}_{L_\alpha}]$, and so $\phi_\alpha$ localizes all reals in $F$. Thus $\text{add}(\mathcal{N}) \geq \lambda_0$. Therefore $\text{non}(\mathcal{M}) = \lambda_0$ and so $\alpha_g \geq \mu$. 

$\square$
Lemma

Let $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathcal{Q}_{L_0}, L)$, $L$ of uncountable cofinality, $L_0$ cofinal in $L$. Then $\mathbb{P}$ adds a maximal cofinitary group of size $|L_0|$.
Assume \( CH \). Let \( \lambda = \bigcup_n \lambda_n \), where \( \lambda_n \) is a regular cardinal, \( \{\lambda_n\}_{n \in \omega} \) increasing and \( \lambda_0 \geq \aleph_2 \). Consider a template \( T = (L, \mathcal{I}) \) such that

- \( \lambda_0 \subseteq L_1 \), \( \lambda_0 \) is cofinal in \( L \), \( L_\alpha \in \mathcal{I} \) for all \( \alpha < \lambda_0 \).
- \( L \) has uncountable cofinality, \( L_0 \) is cofinal in \( L \).

Then in \( V^P \) for \( P = P(T, Q_{L_0}, L) \)

- \( \lambda_0 = \text{non}(\mathcal{M}) \), and so \( \lambda_0 \leq a_g \)
- there is a mcg of size \( \lambda \) and so \( a_g \leq \lambda \).
An isomorphism of names argument provides that in $V^P$ there are no mcg of size $< \lambda$ and so $V^P \models a_g = \lambda$. 
Theorem (V.F., A. Törnquist)

It is consistent with the usual axioms of set theory that the minimal size of a maximal cofinitary group is of countable cofinality.
Thank you!