A CO-ANALYTIC COHEN-INDESTRUCTIBLE MAXIMAL COFINITARY GROUP

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Abstract. Assuming that every set is constructible, we find a Π^1_1 maximal cofinitary group of permutations of $\mathbb{N}$ which is indestructible by Cohen forcing. Thus we show that the existence of such groups is consistent with arbitrarily large continuum. Our method also gives a new proof, inspired by the forcing method, of Kastermans' result that there exists a Π^1_1 maximal cofinitary group in $\mathbf{L}$.

1. Introduction

(A) We denote the group of bijections (permutations) of $\mathbb{N}$ by S_∞, and its unit element by $\mathrm{id}_{\mathbb{N}}$. An element of S_∞ is cofinitary if and only if it has only finitely many fixed points, and $\mathcal{G}$ is called a *cofinitary group* precisely if (up to isomorphism) $\mathcal{G} \leq S_\infty$ and all elements of $\mathcal{G} \setminus \{\mathrm{id}_{\mathbb{N}}\}$ are cofinitary.

A cofinitary group is said to be *maximal* if and only if it is maximal under inclusion among cofinitary groups.

Maximal cofinitary groups (or short, *mcg*s) have long been studied under various aspects; see e.g. [3, 4, 1, 20, 21, 15]. A fair number of studies have been devoted to the possible sizes of mcgs; their relation to maximal almost disjoint (or *mad*) families, of which they are examples; as well as to inequalities relating $\mathfrak{a}_g$, i.e. the least size of a mcg, to other cardinal invariants of the continuum; see e.g. [22, 23, 8, 2, 14, 6]. Analogous questions about permutation groups on κ, where κ is an uncountable cardinal, have also been studied; see e.g. [5]. The isomorphism types of mcgs have been investigated in [12].

Finally, the line of research to which this paper belongs concerns the *definability* of mcgs.

(B) While the existence of mcgs follows from the axiom of choice, the question of whether a mcg can be *definable* has drawn considerable interest.

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It was shown by Truss [20] and Adeleke [1] that no mcg can be countable; this was improved by Kastermans' result [11, Theorem 10] that no mcg can be K_σ. On the other hand, Gao and Zhang [7] showed that assuming $\mathbf{V} = \mathbf{L}$, there is a mcg with a co-analytic generating set. This, too, was improved by Kastermans with the following theorem.

Theorem 1.1 ([11])**.** *If* $\mathbf{V} = \mathbf{L}$ *there is a* Π^1_1 *(i.e. effectively co-analytic) mcg.*

The previous theorem immediately raises the question of whether the existence if a Π^1_1 mcg is consistent with $\mathbf{V} \neq \mathbf{L}$, or even with the negation of the continuum hypothesis.

In this paper we answer these questions in the positive:

Theorem 1.2. *The existence of a* Π^1_1 *mcg is consistent with arbitrarily large continuum (assuming the consistency of ZFC).*

At the same time, we give a new proof of Kastermans' Theorem 1.1. This is worthwhile for several reasons: Firstly, our method shows that in $\mathbf{L}$, *any* countable cofinitary group is contained in a Π^1_1 mcg. Secondly, the 'coding technique' which ensures that the group is co-analytic, described in Definition 3.6, is much more straightforward than the one in [11]. Thirdly, this method seems open to a wider range of variation, allowing to construct mcgs with additional properties. An example of such a property is *Cohen-indestructibility*, which we now define.

For this, first observe that if $\mathcal{G}$ is a cofinitary group, then clearly it remains so in any extension of the universe.

Definition 1.3. Let $\mathcal{G}$ be a mcg and let $\mathbb{C}$ denote Cohen forcing. We say $\mathcal{G}$ is *Cohen-indestructible* if and only if $\Vdash_{\mathbb{C}} \check{\mathcal{G}}$ is maximal.

A Cohen-indestructible mcg was first obtained by Zhang [23]. The following is our main result; Theorem 1.2 is clearly a corollary.

Theorem 1.4. *If* $\mathbf{V} = \mathbf{L}$*, there is a* Π^1_1 *Cohen-indestructible mcg.*

To prove the theorem, we first find a forcing which, given a cofinitary group $\mathcal{G}$ and $z \in 2^{\mathbb{N}}$, adds a generic cofinitary group $\mathcal{G}'$ such that $\mathcal{G} \leq \mathcal{G}'$ and with the property that each element of $\mathcal{G}' \setminus \mathcal{G}$ lies above z in the Turing hierarchy. To find this forcing, we refine Zhang's forcing from [22] (also see [6] and [5] for variations).

We then use this to give a new proof of Kastermans' result Theorem 1.1, building our group from permutations which are generic over certain countable initial segments of $\mathbf{L}$. We use ideas from [5] to see that the group produced in this manner is Cohen-indestructible.

(C) The paper is structured as follows. In §2, we establish basic terminology. In particular, we establish a convenient shorthand notation for the path of a natural number under the action of an element of S_∞ on $\mathbb{N}$. In §3.1 we give a streamlined presentation of Zhang's forcing $\mathbb{Q}_\mathcal{G}$, in order to simplify the definition and discussion of our forcing $\mathbb{Q}^z_\mathcal{G}$, which follows in 3.2; the most important properties of $\mathbb{Q}^z_\mathcal{G}$ are collected in Theorem 3.16. In §4, we prove our main result, Theorem 1.4, in a slightly more general form (Theorem 4.2). We close in §5 by listing some questions which remain open.

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2. Notation and Preliminaries

We start by reviewing the necessary definitions and introduce convenient terminology, in particular the notion of a *path*.

(A) Since we build a generic element of S_∞ from finite approximations, we shall work with partial functions. We write $\operatorname{par}(\mathbb{N},\mathbb{N})$ for the set of partial functions from $\mathbb{N}$ to $\mathbb{N}$. For $a \in \operatorname{par}(\mathbb{N},\mathbb{N})$, when we write $a(n) = k$ it is clearly implied that $n \in \operatorname{dom}(a)$. We say *a(n) is defined* precisely when $n \in \operatorname{dom}(a)$ and *a(n) is undefined* otherwise. For the set of *fixed points of a* we write

$$\operatorname{fix}(a) = \{n \in \mathbb{N} : a(n) = n\}.$$

The set $\operatorname{par}(\mathbb{N},\mathbb{N})$ is naturally equipped with the operation of composition of partial functions

$$(fg)(n) = m \iff f(g(n)) = m,$$

making it an associative monoid.

Let $\mathcal{G}$ be an arbitrary group. By $\mathbb{F}(X)$ we denote the *free group with single generator* X. We identify the group $\mathcal{G} * \mathbb{F}(X)$, i.e. the free product of $\mathcal{G}$ and $\mathbb{F}(X)$, with the set $W_{\mathcal{G},X}$ of reduced words from the alphabet $\big(\mathcal{G} \setminus \{1_\mathcal{G}\}\big) \cup \{X, X^{-1}\}$, equipped with the familiar 'concatenate and reduce' operation (see e.g. [16, Normal Form Theorem]). The neutral element $1_\mathcal{G}$ is therefore identified with the empty word, which we denote by $\emptyset$.

By a *cyclic permutation* of a non-empty word $w = w_n \dots w_1$ we mean the result of reducing the word $w_{\sigma(n)} \dots w_{\sigma(1)}$, where σ is a cyclic permutation of $\{1, \dots, n\}$. By a subword of w we mean a contiguous subword $w_i \dots w_j$ for $n \geq i \geq j \geq 1$, or the empty word. Thus, e.g. ca is a subword of a cyclic permutation of abc. Of course, the empty word is both the only cyclic permutation and the only subword of itself.

We call a group homomorphism $\rho\colon \mathcal{G} \to S_\infty$ a *cofinitary representation of* $\mathcal{G}$ if and only if all elements of $\operatorname{ran}(\rho)$ are cofinitary. Clearly, if ρ is injective, we may identify $\mathcal{G}$ with the cofinitary subgroup $\operatorname{ran}(\mathcal{G}) \leq S_\infty$.

For the remainder of this section, assume $\mathcal{G} \leq S_\infty$. Choosing an arbitrary $s \in \operatorname{par}(\mathbb{N},\mathbb{N})$ gives rise to a unique homomorphism of monoids

$$\rho\colon \mathcal{G} * \mathbb{F}(X) \to \operatorname{par}(\mathbb{N},\mathbb{N})$$

such that $\rho(X) = s$ and ρ is the identity on $\mathcal{G}$. It can be defined by induction on the length of words in the obvious way. Let's denote this homomorphism by $\rho_{\mathcal{G},s}$, departing from [6] (where it is precisely the map $w \mapsto e_w(s)$). Its image is the compositional closure $\langle \mathcal{G}, s\rangle$ of $\mathcal{G} \cup \{s\}$ in $\operatorname{par}(\mathbb{N},\mathbb{N})$.

Convention 2.1. Whenever $\mathcal{G}$ can be inferred from the context, we adopt the convention to denote $\rho_{\mathcal{G},s}(w)$ by $w[s]$, for any $w \in W_{\mathcal{G},X}$ (we "substitute s for X in w"; as e.g. in [22, 11]).

Observe that slightly awkwardly, by this convention, $\emptyset[s] = \mathrm{id}_{\mathbb{N}}$ for any $s \in \operatorname{par}(\mathbb{N},\mathbb{N})$.

(B) In the remainder of this section, we define the notion of a *path*, which will be extremely useful in the next section. Fix $s \in \operatorname{par}(\mathbb{N},\mathbb{N})$. Say $w \in W_{\mathcal{G},X}$, and in reduced form

$$w = a_n \dots a_1.$$

We define the *path under* (w, s) *of* m (also called the (w, s)-*path of* m) to be the following sequence of natural numbers:

$$\operatorname{path}(w, s, m) = \langle m_i \colon i \in \alpha\rangle,$$

where $m_0 = m$ and for $l, i \in \mathbb{N}$ such that $0 < i \leq n$,

$$m_{nl+i} = a_i \dots a_1 w^l[s](m_0)$$

and $\alpha \in \omega+1$ is maximal such that all of these expressions are defined. That is, we simply iterate applying all the letters of w as they appear from right to left, and record the outcome until we reach an undefined expression

We can represent such a path e.g. as follows:

$$\dots m_{n+2} \xleftarrow{a_2} m_{n+1} \xleftarrow{a_1} m_n \xleftarrow{a_n} \dots \xleftarrow{a_2} m_1 \xleftarrow{a_1} m_0,$$

or more simply, we shall represent it as $\langle \dots, m_n, \dots, m_1, m_0\rangle$.

Supposing k is least such that $a_i[s](m_k)$ is undefined, for some i, we say that the path *terminates after* $\alpha - 1 = k$ *steps with last value* m_k. We shall also use the phrase *the (w,s)-path terminates before (an occurrence of) the letter a_i* in this situation.

If on the other hand $a_i[s](m_k)$ is defined, we use the phrase that the letter a_i *occurs*, or is *applied (to m_k)*, at step $k+1$ in the path (although strictly speaking, it is $a_i[s]$ that is applied).

Sometimes we are interested in the path merely as a set, rather than as a sequence; so let

$$\operatorname{use}(w,s,m) = \{m_i \colon i < \alpha\}.$$

(C) Of course, we identify $\mathbb{N}$ and ω, but prefer to denote this set as $\mathbb{N}$ in the context of permutations. We denote by $|A|$ the cardinality of A, for any set A. We do not regard it as compulsory to decorate names in the forcing language with dots and checks as in [9]; we shall nevertheless freely use such decorations occasionally, with the goal of aiding the reader.

3. Coding into a generic group extension

Fix, for this section, a cofinitary group $\mathcal{G} \leq S_\infty$. We want to enlarge it by $\sigma^* \in S_\infty$, such that $\langle \mathcal{G}, \sigma^* \rangle$ is cofinitary. This can be done using a forcing invented by Zhang [22], which has proven extremely valuable in applications (see [2, 23, 8, 24, 14, 7, 13]).

In Section 3.2, we introduce a new forcing $\mathbb{Q}^z_{\mathcal{G}}$, such that in addition to the above, every element of $\langle \mathcal{G}, \sigma^* \rangle$ not already in $\mathcal{G}$ 'codes' a given, fixed $z \in 2^{\mathbb{N}}$, in a certain sense.

Before we introduce this new forcing notion, we define our own version of Zhang's forcing, $\mathbb{Q}_{\mathcal{G}}$ in Section 3.1, differing slightly from [22]. We then analyze carefully how paths behave when conditions in $\mathbb{Q}_{\mathcal{G}}$ are extended, facilitating the treatment of $\mathbb{Q}^z_{\mathcal{G}}$.

Note that in the case of countable $\mathcal{G}$, Zhang's $\mathbb{Q}_{\mathcal{G}}$ from [22], our version of $\mathbb{Q}_{\mathcal{G}}$ described in §3.1, and the forcing $\mathbb{Q}^z_{\mathcal{G}}$ are all countable, i.e. particular presentations of Cohen forcing.

3.1. **Zhang's forcing, revisited.**

We now turn to our definition of the forcing to add a generic cofinitary representation of $\mathcal{G} * \mathbb{F}(X)$.

Definition 3.1 (The forcing $\mathbb{Q}_{\mathcal{G}}$)**.**

(a) Conditions of $\mathbb{Q}_{\mathcal{G}}$ are pairs $p = (s^p, F^p)$, where $s \in \operatorname{par}(\mathbb{N},\mathbb{N})$ is injective and $F^p \subseteq W_{\mathcal{G},X} \setminus \mathcal{G}$ is finite.

(b) $(s^q, F^q) \leq_{\mathbb{Q}_{\mathcal{G}}} (s^p, F^p)$ if and only if $s^q \supseteq s^p$, $F^q \supseteq F^p$ and for all $w \in F^p$, if $m \in \operatorname{fix}(w[s^q])$, then there is a non-empty subword w' of w such that $\operatorname{use}(w, s^q, m) \cap \operatorname{fix}(w'[s^p]) \neq \emptyset$.

For any condition $p \in \mathbb{Q}_{\mathcal{G}}$ we write (s^p, F^p) if we want to refer to the components of that condition.

If G is $(\mathbf{V}, \mathbb{Q}_{\mathcal{G}})$-generic, letting

$$\sigma_G = \bigcup_{p \in G} s^p,$$

we have $\sigma_G \in S_\infty$ and $\langle \mathcal{G}, \sigma_G \rangle$ is a cofinitary group which is isomorphic to $\mathcal{G} * \mathbb{F}(X)$ via $\rho_{\mathcal{G},\sigma_G}$.

Note that in (b) above, we demand that if $q \leq_{\mathbb{Q}_{\mathcal{G}}} p$ and s^q gives rise to a new fixed point of $w \in F^p$, then the (w, s^q)-path of that fixed point must meet a certain finite set of numbers, where this set depends only on p. We will see below in Lemma 3.11 and 3.13 that this guarantees that $\langle \mathcal{G}, \sigma_G \rangle$ is cofinitary.

As is pointed out in [22, p. 42f.], one cannot replace (b) by the simpler

$(b)'$ $(s^q, F^q) \leq_{\mathbb{Q}_{\mathcal{G}}} (s^p, F^p)$ if and only if $s^q \supseteq s^p$, $F^q \supseteq F^p$ and for all $w \in F^p$, if $m \in \operatorname{fix}(w[s^q])$, then $m \in \operatorname{fix}(w[s^p])$.

For with this simpler definition, supposing $n \in \operatorname{fix}(g)$, any condition p such that $X^{-1}gX \in F^p$ and $n \notin \operatorname{ran}(s^p)$ cannot be extended to any q so that $n \in \operatorname{ran}(s^q)$. Similar examples abound; (b) is formulated to pinpoint the problem.

In a previous paper [6] by two of the present authors, allowing only so-called 'good words' in F^p made it possible to define $\leq_{\mathbb{Q}_{\mathcal{G}}}$ as in $(b)'$. Here, we define $\leq_{\mathbb{Q}_{\mathcal{G}}}$ differently from [6] and also slightly differently from [22]. This allows the coding to apply to arbitrary words (only subject to the obvious constraint that they not be from $\mathcal{G}$), while at the same time simplifying the proofs of the Extension Lemmas (see below).

We now prove increasingly stronger versions of the Domain Extension Lemma, culminating in a crucial lemma concerning the length of certain paths (Lemma 3.4). This will considerably clean up the presentation when we deal with the more complicated forcing $\mathbb{Q}^z_{\mathcal{G}}$.

The following is implicit in [22]; for the convenience of the reader, we include a new, very short proof.

Lemma 3.2 (Contingent Domain Extension for $\mathbb{Q}_{\mathcal{G}}$)**.** *Let $s \in \operatorname{par}(\mathbb{N}, \mathbb{N})$ and $w \in W_{\mathcal{G},X}$ be arbitrary, and suppose $n \in \mathbb{N}$ is such that $n \notin \operatorname{fix}(w'[s])$ for any non-empty subword w' of w. Then for a cofinite set of n', letting $s' = s \cup \{(n, n')\}$, we have that s' is injective and* $\operatorname{fix}(w[s']) = \operatorname{fix}(w[s])$.

Proof. Let W^* be the set of subwords of cyclic permutations of w and pick n' arbitrary such that

$$\begin{aligned}&n' \notin \bigcup \{\, \mathrm{fix}(w'[s]) \colon w' \in W^* \setminus \{\emptyset\}\},\\ &n' \notin \bigcup \{w'[s]^i(n) \colon i \in \{-1,1\}, w' \in W^*\}, \text{ and}\\ &n' \notin \mathrm{ran}(s).\end{aligned} \tag{3.1}$$

Clearly, (3.1) ensures that s' is injective.

Assume towards a contradiction that $m_0 \in \mathrm{fix}(w[s']) \setminus \mathrm{fix}(w[s])$. Write the beginning of the (w, s')-path of m_0 as

$$\ldots m_{k(3)} \xleftarrow{w''} m_{k(2)} \xleftarrow{X^j} m_{k(1)} \xleftarrow{w'} m_{k(0)} = m_0 \tag{3.2}$$

where $w', w'' \in W_{\mathcal{G},X}$ are the maximal subwords of w such that from $m_{k(0)}$ to $m_{k(1)}$ and $m_{k(2)}$ to $m_{k(3)}$, the path contains no application of X to n or of X^{-1} to n'; moreover, we ask that $m_{k(1)} = n$ when $j = 1$, $m_{k(1)} = n'$ when $j = -1$. Clearly, we must allow either of w', w'' to be empty. This is well-defined as m_0 is not a fixed point of $w[s]$.

Obviously, it makes no difference whether we use s' or s in the following:

$$\begin{aligned}w'[s](m_{k(0)}) &= w'[s'](m_{k(0)}) = m_{k(1)},\\ w''[s](m_{k(2)}) &= w'[s'](m_{k(0)}) = m_{k(3)}.\end{aligned}$$

First, we show that in fact $w = w''X^j w'$. Towards a contradition, assume not. Then by maximaliy of w'', the path has the form

$$\ldots \xleftarrow{X^{j'}} m_{k(3)} \xleftarrow{w''} m_{k(2)} \xleftarrow{X^j} m_{k(1)} \xleftarrow{w'} m_0$$

We make some simple observations:

1. $m_{k(2)} = m_{k(3)}$; for otherwise, $n' = (w'')^i[s](n)$ for some $i \in \{-1,1\}$, contradicting the choice of n'.
2. Thus, $w'' \neq \emptyset$, since on one side of w'' we have X and on the other X^{-1} and w is in reduced form.
3. As $n' \notin \mathrm{fix}(w''[s])$, we have that $m_{k(2)} = m_{k(3)} = n$.
4. So $n \in \mathrm{fix}(w''[s])$, contradicting the hypothesis of the lemma.

Thus, $w = w''X^j w'$ and $m_0 = m_{k(3)}$. We infer that $n' = (w'w'')^{-j}[s](n)$, again contradicting the choice of n'. □

This puts us in the position to give a short proof of Lemmas 2.2 and 2.3 from [22]:

Lemma 3.3 (Domain Extension for $\mathbb{Q}_{\mathcal{G}}$)**.** *For any $n \in \mathbb{N}$, the set of q such that $n \in \mathrm{dom}(s^q)$, is dense in $\mathbb{Q}$.*

Proof. Fix $p \in \mathbb{Q}$; we shall find a stronger condition $q \in \mathbb{Q}$ such that $n \in \operatorname{dom}(s^q)$. Analogously to the previous proof, let F^* consist of all words which are a subword of a cyclic permutation of a word in F^p, and let n' be arbitrary such that

$$
\begin{aligned}
&n' \notin \bigcup\{\operatorname{fix}(w[s])\colon w \in F^* \setminus \{\mathrm{id}_{\mathbb{N}}\}\},\\
&n' \notin \bigcup\big\{w[s]^i(n)\colon i \in \{-1,1\}, w \in F^* \cup \{\emptyset\}\big\}, \text{ and}\\
&n' \notin \operatorname{ran}(s).
\end{aligned}
\tag{3.3}
$$

Note that (3.3) excludes only finitely many possible values for n'. Define $s' = s \cup \{(n,n')\}$ and $q = (s', F^p)$. As in the proof of Lemma 3.2, s is injective.

Given $w \in F^p$ and supposing $m_0 \in \operatorname{fix}(w[s']) \setminus \operatorname{fix}(w[s])$, the proof of the previous lemma shows that there is a subword w' of w such that $n \in \operatorname{fix}(w'[s])$. As moreover, n shows up in the (w,s')-path of m_0, we have

$$
\operatorname{use}(w,s',m) \cap \operatorname{fix}(w'[s]) \neq \emptyset.
$$

Thus, $q \leq_{\mathbb{Q}_{\mathcal{G}}} p$. □

A crucial observation for the following discussion of $\mathbb{Q}^z_{\mathcal{G}}$ is that when extending the domain of s^p for a given condition p, we have fine control over the length of paths that result from this extension.

Lemma 3.4 (Lengths of paths)**.** *Fix $w \in W_{\mathcal{G},X}$ and $m \in \mathbb{N}$. Moreover, let $s \in \operatorname{par}(\mathbb{N},\mathbb{N})$ and $n \in \mathbb{N} \setminus \operatorname{dom}(s)$ be given.*

Then for cofinitely many $n' \in \mathbb{N}$, if we let $s' = s \cup \{(n,n')\}$ the following obtains:

1. *If the (w,s)-path of m terminates with last value n before an occurrence of X, the (w,s')-path of m contains exactly one more application of X than does the (w,s)-path.*
2. *If the (w,s)-path of m does not terminate with last value n before an occurrence of X, $\operatorname{path}(w,s',m) = \operatorname{path}(w,s,m)$.*

Proof. Let $E = \operatorname{dom}(s) \cup \operatorname{ran}(s) \cup \{n\} \cup \{m\}$ and W^* be the set of subwords of cyclic permutations of w. Suppose n' is arbitrary such that

$$
\begin{aligned}
&n' \notin \bigcup\big\{\operatorname{fix}(w'[s])\colon w' \in W^* \setminus \{\emptyset\}\big\} \text{ and}\\
&n' \notin \bigcup\big\{w'[s]^i[E]\colon i \in \{-1,1\}, w' \in W^*\big\}.
\end{aligned}
\tag{3.4}
$$

For Case 1 of the lemma, suppose that the (w,s)-path of m terminates after k steps with last value $m_k = n$ before an occurrence of X. In the (w,s')-path we have on the contrary that $m_{k+1} = n'$.

If the next letter to be applied to $m_{k+1} = n'$ at step $k+2$ in this path is X, the path terminates here, after $k+1$ steps with last value n', since $n' \notin \mathrm{dom}(s')$ by (3.4).

If the next letter to be applied to $m_{k+1} = n'$ at step $k+2$ in this path is $g \in \mathcal{G} \setminus \{\mathrm{id}_\mathbb{N}\}$ the path terminates as otherwise, $n' \in g^{-1}[\mathrm{dom}(s') \cup \mathrm{ran}(s')]$, which implies $n' \in g^{-1}[E]$ or $n' \in \mathrm{fix}(g)$, contradicting (3.4).

In Case 2 of the lemma, show $\mathrm{path}(w, s', m) = \mathrm{path}(w, s, m)$. Suppose that the (w, s)-path of m terminates after k steps with last value m_k before an occurrence of X^j, $j \in \{-1, 1\}$. If $j = 1$, $m_k \neq n$ by assumption; and as the (w, s)-path terminates with m_k, we have $m_k \notin \mathrm{dom}(s) \cup \{n\}$, so (w, s')-path terminates as well.

So assume towards a contradiction that $j = -1$ and m_{k+1} in the (w, s')-path of m is defined. As the (w, s)-path terminates with m_k, before an occurrence of X^{-1}, while (w, s')-path does not terminate, $m_k = n'$. Thus, $n' \in w'[s](m)$ for a subword w' of w, contradicting (3.4). □

Remark 3.5. The requirements in (3.4) were chosen to be easy-to-state rather than minimal (and so were those in (3.1) and (3.3)).

All other proofs regarding $\mathbb{Q}_\mathcal{G}$ will be omitted (but note that they can be easily inferred from their counterparts for $\mathbb{Q}^z_\mathcal{G}$ in the next section).

3.2. **Coding into a generic cofinitary group extension.** Our next goal is to define, given $z \in 2^\mathbb{N}$, a forcing $\mathbb{Q}^z_\mathcal{G}$ such that whenever G is $(\mathbf{V}, \mathbb{Q}^z_\mathcal{G})$-generic, the following holds: There exists $\sigma_G \in S_\infty$ such that for each $\sigma \in \langle \mathcal{G}, \sigma_G \rangle \setminus \mathcal{G}$, we have $z \leq_T \sigma$.

First, we describe the algorithm by which z is computed from an element of $\langle \mathcal{G}, \sigma_G \rangle \setminus \mathcal{G}$. Since our forcing uses finite approximations to σ_G, we define the coding for elements of $\mathrm{par}(\mathbb{N}, \mathbb{N})$.

Definition 3.6 (Coding)**.**

(1) We say that $\sigma \in \mathrm{par}(\mathbb{N}, \mathbb{N})$ *codes* a finite string $t \in 2^l$ *with parameter* $m \in \mathbb{N}$ if and only if
$$(\forall k < l)\ \sigma^k(m) \equiv t(k) \bmod 2.$$
We say it *exactly* codes t (with parameter m) if and only if it codes t and in addition, $\sigma^l(m)$ is undefined.

(2) We say that $\sigma \in \mathrm{par}(\mathbb{N}, \mathbb{N})$ *codes* $z \in 2^\mathbb{N}$ *with parameter* m if and only if
$$(\forall k \in \mathbb{N})\ \sigma^k(m) \equiv z(k) \bmod 2.$$

For the rest of this section, fix an arbitrary $z \in 2^\mathbb{N}$. Now we can define the forcing.

Definition 3.7 (Definition of $\mathbb{Q} = \mathbb{Q}^z_{\mathcal{G}}$)**.**

(A) Conditions of $\mathbb{Q}$ are triples $p = (s^p, F^p, \bar{m}^p)$ s.t.
  (1) $(s^p, F^p) \in \mathbb{Q}_{\mathcal{G}}$
  (2) $\bar{m}^p$ is a partial function from F^p to $\mathbb{N}$.
  (3) For any $w \in \mathrm{dom}(\bar{m}^p)$ there is $l \in \omega$ such that $w[s^p]$ exactly codes $z \restriction l$ with parameter $\bar{m}^p(w)$.
  (4) If $w, w' \in \mathrm{dom}(\bar{m}^p)$ and $w \neq w'$,

$$\mathrm{use}(w, s^p, \bar{m}^p(w)) \cap \mathrm{use}(w', s^p, \bar{m}^p(w')) = \emptyset$$

(B) $(s^q, F^q, \bar{m}^q) \leq (s^p, F^p, \bar{m}^p)$ if and only if both
  (1) $(s^q, F^q) \leq_{\mathbb{Q}_{\mathcal{G}}} (s^p, F^p)$,
  (2) $\bar{m}^q$ extends $\bar{m}^p$ as a function.

For any condition $p \in \mathbb{Q}$ we write $(s^p, F^p, \bar{m}^p)$ if we want to refer to the components of that condition.

Note that (A3) ensures that for any $p \in \mathbb{Q}$ and $w \in \mathrm{dom}(\bar{m}^p)$, the path under (w, s^p) of $\bar{m}^p(w)$ is finite (although other paths may be eventually periodic and thus infinite). Also note that by (A4), $|\mathcal{G}|$ is collapsed to ω by $\mathbb{Q}$ whenever $\mathcal{G}$ is uncountable in the ground model.

For a $(\mathbf{V}, \mathbb{Q})$-generic G, as in the previous section we let

$$\sigma_G = \bigcup_{p \in G} s^p.$$

We now show in a series of lemmas that $\langle \mathcal{G}, \sigma_G \rangle$ is a cofinitary group which is isomorphic to $\mathcal{G} * \mathbb{F}(X)$ and for any $\tau \in \langle \mathcal{G}, \sigma_G \rangle \setminus \mathcal{G}$, τ codes z.

We begin with a Lemma showing that σ_G is forced by $\mathbb{Q}$ to be totally defined on $\mathbb{N}$.

Lemma 3.8 (Domain Extension)**.**

1. *For any $n \in \mathbb{N}$, the set of q such that $n \in \mathrm{dom}(s^q)$, is dense in $\mathbb{Q}$.*
2. *In fact, suppose $p \in \mathbb{Q}$ and $n \in \mathbb{N}$ are such that for some $w^* \in W_{\mathcal{G},X}$ and $m^* \in \mathbb{N}$, n is the last value of* $\mathrm{path}(w^*, s^p, m^*)$ *and this path terminates before an occurrence of X. Then one can find $q \in \mathbb{Q}$ such that $q \leq p$ and* $\mathrm{path}(w^*, s^q, m^*)$ *contains exactly one more application of X, and no further application of X^{-1}, than does* $\mathrm{path}(w^*, s^p, m^*)$.

While our only use of Part 2 of the lemma is to simplify the proof that $\rho_{\mathcal{G},\sigma_G}$ is forced to be injective (Lemma 3.15), a fact which is not needed for the main result of this paper, its proof presents no additional burden.

Before we prove the lemma, to avoid repetition, we introduce the following terminology: For $w \in W_{\mathcal{G},X}$ and $j \in \{-1, 1\}$, call an occurrence of X^j in w *critical* if there is no occurrence of X or X^{-1} in w to its left. Otherwise, we call it an *uncritical occurrence*. Clearly, it is through a critical occurrence of X (resp. X^{-1}) in some word in $\operatorname{dom}(\bar{m})$ that the coding requirements from (A3) restrict our possibilities to extend $\operatorname{dom}(s^p)$ (resp. $\operatorname{ran}(s^p)$).

Proof of Lemma 3.8. Let $p \in \mathbb{Q}$, $w^* \in W_{\mathcal{G},X}$, and $m^*, n \in \mathbb{N}$ be as in the statement of the lemma and suppose $n \notin \operatorname{dom}(s^p)$. We will find n' such that for $s' = s^p \cup \{(n, n')\}$, $q = (s', F^p, \bar{m}^p)$ is a condition stronger than p.

Write s for s^p and $\bar{m}$ for $\bar{m}^p$. Let

$$E = \operatorname{ran}(s) \cup \operatorname{dom}(s) \cup \{n\} \cup \operatorname{ran}(\bar{m}) \cup \{m^*\},$$

and let F^* consist of all words which are a subword of a cyclic permutation of a word in $F^p \cup \{w^*\}$. The first requirement we make is that n' be chosen such that

$$\begin{aligned} &n' \notin \bigcup \{\operatorname{fix}(w[s]) \colon w \in F^* \setminus \{\emptyset\}\} \text{ and} \\ &n' \notin \bigcup \big\{g^{-1} w^i[s][E] \colon i \in \{-1, 1\}, w \in F^*, g \in F^* \cap \mathcal{G}\big\}. \end{aligned} \tag{3.5}$$

Note that (3.5) excludes only finitely many possible values for n'. The taking of preimages under $g \in F^* \cap \mathcal{G}$ in (3.5) serves the sole purpose of ensuring that the following obtains:

$$g(n') \notin \bigcup \{\operatorname{use}(w, s, \bar{m}(w)) \colon w \in \operatorname{dom}(\bar{m})\} \tag{3.6}$$

for $g = \mathrm{id}_{\mathbb{N}}$ as well as for all $g \in \mathcal{G} \setminus \{\mathrm{id}_{\mathbb{N}}\}$ occuring in a word from F^p.

If for some $w \in \operatorname{dom}(\bar{m})$, n appears in the (w, s)-path of $\bar{m}(w)$ before a critical occurrence of X, we must make an additional requirement. So fix such w, and note that there is no other $w' \in \operatorname{dom}(\bar{m})$ in whose path n appears. Let l be such that $w[s]$ exactly codes $z \restriction l$ with parameter $\bar{m}(w)$. Further, suppose $w = gXw'$, where $w' \in W_{\mathcal{G},X}$ and we allow $g \in \mathcal{G}$ to be $\mathrm{id}_{\mathbb{N}}$ but no cancellation in Xw'. Now in addition to (3.5), require that $g(n')$ be even if $z(l) = 0$ and odd if $z(l) = 1$.

To see that q is a condition, we verify (A3) and (A4). For (A3), towards a contradiction, let $m = \bar{m}(w)$ and assume $w^{l+1}[s'](m) \not\equiv z(l) \pmod 2$. As $w[s]$ exactly codes an initial segment of z, the (w, s')-path is longer than the (w, s)-path of m. Thus, by choice of n' and the proof of Lemma 3.4, we have that Case 1 in the statement of Lemma 3.4 holds: The (w, s)-path of m terminates with last value n before an occurrence of X, and the (w, s')-path of m continues for exactly one or two more steps, as follows:

$$w^{l+1}[s'](m) \xleftarrow{g} n' \xleftarrow{X} n \longleftarrow \dots,$$

where we allow $g \in \mathcal{G}$ to be $\mathrm{id}_{\mathbb{N}}$. Thus, the occurrence of X in the above is critical; but then n' was chosen so that $w^{l+1}[s'](m) = g(n') \equiv z(l) \pmod 2$, in contradiction to the assumption.

We have seen that if (w, s)-path and the (w, s')-path of $\bar{m}(w)$ differ for $w \in \mathrm{dom}(\bar{m})$, then the latter must terminate with n. Thus, by (A4), at most one such path acquires new values, and these were seen to be n' and possibly $g(n')$, where $g \in \mathcal{G}$ occurs in a word in $\mathrm{dom}(\bar{m})$. By (3.6), requirement (A4) holds of q, allowing us to conclude that q is a condition in $\mathbb{Q}$.

We end the proof of Part 1 of the lemma by quoting the proof of the Domain Extension Lemma for $\mathbb{Q}_{\mathcal{G}}$ to conclude that $q \leq p$.

For Part 2 of the lemma, note that by the proof of Lemma 3.4, indeed the (w^*, s)-path of m^* contains exactly one more application of X, and no further application of X^{-1}, than does the (w^*, s')-path of m^*. $\square$

Remark 3.9. Again, the requirements in (3.5) are by no means minimal.

The next lemma shows that $\mathbb{Q}$ forces σ_G to be onto $\mathbb{N}$.

Lemma 3.10 (Range Extension)**.**

1. *For any $n \in \mathbb{N}$, the set of q such that $n \in \mathrm{ran}(s^q)$, is dense in $\mathbb{Q}$.*
2. *In fact, suppose $p \in \mathbb{Q}$ and $n \in \mathbb{N}$ are such for some $w^* \in W_{\mathcal{G},X}$ and $m^* \in \mathbb{N}$, n is the last value of* $\mathrm{path}(w^*, s^p, m^*)$ *and this path terminates before an occurrence of X^{-1}. Then one can find $q \in \mathbb{Q}$ such that $q \leq p$ and* $\mathrm{path}(w^*, s^q, \bar{m}^*)$ *contains exactly one more application of X^{-1}, and no further application of X, than does* $\mathrm{path}(w^*, s^p, \bar{m}^*)$.

Proof. The lemma is entirely symmetrical to the Domain Extension Lemma. By symmetry, the proofs of Lemmas 3.2, 3.3, 3.4 and 3.8 can easily be adapted. $\square$

By the previous two lemmas, $\Vdash_{\mathbb{Q}} \sigma_{\dot{G}} \in S_\infty$. By the next two lemmas, $\mathbb{Q}$ forces that for all $w \in W_{\mathcal{G},X}$, $w[\sigma_{\dot{G}}]$ codes z, as promised:

Lemma 3.11. *For any $w \in W_{\mathcal{G},X}$, the set of q such that $w \in F^q$ is dense in $\mathbb{Q}$.*

Proof. Simply observe that $(s^p, F^p \cup \{w\}, \bar{m}^p)$ is a condition in $\mathbb{Q}^z_{\mathcal{G}}$. $\square$

Lemma 3.12 (Generic Coding)**.** *If $p \in \mathbb{Q}$, $w \in W_{\mathcal{G},X} \setminus \mathcal{G}$ and $l \in \mathbb{N}$, there is $q \leq p$ such that $w \in \mathrm{dom}(\bar{m})$ and q exactly codes $z \restriction l$ with parameter $\bar{m}(w)$.*

Proof. Fix p, w and l as above. We may assume $w \in \mathrm{dom}(\bar{m}^p)$; otherwise, find an $n' \in \mathbb{N}$ such that (3.5) holds and replace F^p by $F^p \cup \{w\}$ and $\bar{m}^p$ by $\bar{m}^p \cup \{(w, n')\}$ in p and call the result p'. By (3.5), (A4) is satisfied for p' and by the argument in the proof of Lemma 3.4, the $(w, s^{p'})$-path of n' will terminate before the right-most application of X or X^{-1} in w. As $s^p = s^{p'}$, this suffices to show p' is a condition below p.

So supposing $w \in \operatorname{dom}(\bar{m}^p)$, let m' be the last value of the (w, s^p)-path of $\bar{m}(w)$ and assume this path terminates before an occurrence of the letter X. By the Domain Extension Lemma, we may find $q \leq p$ such that $m' \in \operatorname{dom}(s^q)$ and the (w, s^q)-path at $\bar{m}(w)$ terminates either at the next step or after one further application of a letter in $\mathcal{G} \setminus \{\mathrm{id}_{\mathbb{N}}\}$.

If instead the (w, s^p)-path of $\bar{m}(w)$ terminates before an occurrence of the letter X^{-1}, argue similarly using the Range Extension Lemma.

Repeating the argument if necessary, we obtain a condition q such that s^q exactly codes $z \restriction l$. $\square$

By the next lemma, $\langle \mathcal{G}, \sigma_G \rangle$ is forced to be cofinitary. The reader may care to notice that the proofs of the remaining lemmas, up to Theorem 4.2, go through (sometimes in simpler form) for $\langle \mathbb{Q}_{\mathcal{G}}, \leq_{\mathcal{G}} \rangle$ (as was the case for Lemma 3.11).

Lemma 3.13. *For all $w \in W_{\mathcal{G},x}$, $\Vdash_{\mathbb{Q}} \operatorname{fix}(w[\sigma_{\dot{G}}])$ is finite.*

Proof. We shall show that whenever $p \in \mathbb{Q}$ satisfies $w \in F^p$, there is N such that $p \Vdash \operatorname{fix}(w[\sigma_{\dot{G}}]) \setminus \operatorname{fix}(w[s^p])$ has size at most N. Thus, the set of p which force $\operatorname{fix}(w[\sigma_{\dot{G}}])$ to be finite, is dense.

So fix p satisfying $w \in F^p$ and let $q \leq p$ be arbitrary. Consider

$$n \in \operatorname{fix}(w[s^q]) \setminus \operatorname{fix}(w[s^p]). \tag{3.7}$$

Then, letting $\langle m_k, \ldots, m_0 \rangle$ be the (w, s^q)-path of n, we have that for some $l \leq k$, $m_l \in \operatorname{fix}(u[s^p])$, for some subword u of a word in F^p. For each n satisfying (3.7), pick some such $l = l(n)$ and $u = u(n)$ and let $m(n) = m_l \in \operatorname{fix}(u[s^p])$. If we have

$$n, n' \in \operatorname{fix}(w[s^q]) \setminus \operatorname{fix}(w[s^p]),$$

such that $l(n) = l(n')$, $u(n) = u(n')$ and $m(n) = m(n')$, it must be that $n = n'$ (by injectivity).

Let N' be the number of triples (l, u, m) such that l is less than the length of w, u is a subword of w, and $m \in \operatorname{fix}(u[s^p])$. We have that no $q \leq p$ can force that $(w[\sigma_{\dot{G}}])$ has more than N' fixed points not already in $\operatorname{fix}(w[s^p])$. $\square$

The next lemma shows that our construction yields a group which is maximal with respect to permutations from the ground model. It is a special case of the $\mathbb{P}$-generic Hitting Lemma in §4.

Lemma 3.14 (Generic Hitting)**.** *For any $\tau \in S_\infty$ and $m \in \mathbb{N}$, the set of q such that there is $n \geq m$ with $s^q(n) = \tau(n)$, is dense.*

Proof. The proof is a warm-up for the $\mathbb{P}$-generic Hitting Lemma below. Let $p \in \mathbb{Q}$, $\tau \in S_\infty$ and $m \in \mathbb{N}$ be given. Find $n \in \mathbb{N}$ such that $n \geq m$, and such that letting $n' = \tau(n)$, n' satisfies the first requirement given in the proof of the Domain Extension Lemma, i.e. (3.5) with $s = s^p$, $E = \mathrm{dom}(s^p) \cup \mathrm{ran}(s^p) \cup \{n\}$ and F^* precisely as defined there.

To see this is possible, note that (3.5) holds for $n' = \tau(n)$ if and only if for $E' = \mathrm{dom}(s) \cup \mathrm{ran}(s)$,

$$\begin{aligned} & n \notin \bigcup \tau^{-1}\Big[\big\{ \mathrm{fix}(w[s]) \colon w \in F^* \setminus \{\emptyset\}\big\}\Big], \\ (3.8)\quad & n \notin \bigcup \tau^{-1}\Big[\big\{g^{-1}w'[s]^i[E'] \colon i \in \{-1,1\}, w' \in F^*, g \in F^* \cap \mathcal{G}\big\}\Big], \text{ and} \\ & n \notin \bigcup \big\{ \mathrm{fix}(\tau^{-1}g^{-1}w'[s]^i) \colon i \in \{-1,1\}, w' \in F^*, g \in F^* \cap \mathcal{G}\big\}. \end{aligned}$$

These requirements exclude only finitely many n, proving n as above can indeed be found.

By the proof of the Domain Extension Lemma, letting $s' = s^p \cup \{(n, \tau(n))\}$, $q = (s', F^p)$ is a condition stronger that p. □

For the sake of completeness we also show the following:

Lemma 3.15. *$\mathbb{Q}$ forces that $\rho_{\mathcal{G},\sigma_G}$ is injective.*

Proof. In fact, we show that for any $p \in \mathbb{Q}$, $w \in W_{\mathcal{G},X} \setminus \mathcal{G}$ and $\tau \in S_\infty$, we can find $q \leq p$ such that

$$q \Vdash_{\mathbb{Q}} w[\sigma_{\dot{G}}] \neq \check{\tau}.$$

By taking inverses, we can assume without loss of generality that w starts with X and ends with X^j for $j \in \{-1, 1\}$. Suppose w has length k.

Pick $n \notin \mathrm{dom}(s^p) \cup \mathrm{ran}(s^p)$, or in any case such that the (w, s^p)-path of n terminates before the kth step. If necessary, by repeatedly using Part 2 of the Domain extension Lemma or the Range extension Lemma, find $p' \leq p$ such that the $(w, s^{p'})$-path of n terminates after $k-1$ steps, before the first letter from the left in w, i.e. X. Let m be its last value. As $m \notin \mathrm{dom}(s^{p'})$, we may easily extend p' once more to obtain $q \in \mathbb{Q}$, $q \leq p'$ such that $s^q(m) \neq \tau(n)$, since the proof of the Domain Extension Lemma shows we can chose $s^q(m)$ arbitrarily in a cofinite subset of $\mathbb{N}$. □

We sum up the crucial properties of $\mathbb{Q}^z_{\mathcal{G}}$ in the following theorem:

Theorem 3.16. *Suppose $\mathcal{G} \leq S_\infty$ is cofinitary, $z \in 2^{\mathbb{N}}$ and M is a transitive $\in$-model satisfying the axiom of separation and such that $\{\mathcal{G}, z\} \subseteq M$ (whence $\mathbb{Q}^z_{\mathcal{G}} \in M$). For any $(M, \mathbb{Q}^z_{\mathcal{G}})$-generic filter G, letting*

$$\sigma_G = \bigcup_{p \in G} s^p$$

we have:

(I) $\sigma_G \in S_\infty$

(II) $\langle \mathcal{G}, \sigma_G \rangle$ *is a cofinitary group isomorphic to* $\mathcal{G} * \mathbb{F}(X)$.

(III) For any word $w \in W_{\mathcal{G},X} \setminus \mathcal{G}$, *we have that* $w[\sigma_G]$ *codes* z *in the sense of definition 3.6, and thus* $z \leq_T w[\sigma_G]$.

(IV) For any $\tau \in \operatorname{cofin}(S_\infty) \cap M$ *such that* $\tau \notin \mathcal{G}$, *there is no cofinitary group* $\mathcal{G}'$ *such that*

$$\langle \mathcal{G}, \sigma_G \rangle \cup \{\tau\} \subseteq \mathcal{G}'.$$

Proof. The only fine point here is that we do not assume that M can define the forcing relation. We thus have to circumvent its use.

The Domain and Range Extension Lemmas can be seen as describing a countable family of dense subsets of $\mathbb{Q}$, and by separation each of these dense subsets of $\mathbb{Q}$ is an element of M. Thus G meets each of them, proving (I).

By analogous arguments, (II), (III) and (IV) are obtained using dense sets described in (the proofs of) Lemma 3.11, the Generic Coding Lemma 3.12, Lemma 3.13, Lemma 3.15, and the Hitting Lemma 3.2. □

4. A co-analytic Cohen-indestructible mcg

We now use the ideas from the previous section to prove the main results of this paper, Theorem 4.2 and Corollary 4.4 below. At the same time, we give a new proof of Kastermans' result that there is a Π^1_1 mcg in $\mathbf{L}$, based on the idea of finding generics over countable models.

We make crucial use of the following lemma, the proof of which draws inspiration from [5, Theorem 4.1]. The lemma implies that for any forcing $\mathbb{P} \in \mathbf{V}$, the product $\mathbb{P} \times \mathbb{Q}^z_{\mathcal{G}}$ forces that the generic group extension added by $\mathbb{Q}^z_{\mathcal{G}}$ is maximal with respect to $S_\infty \cap \mathbf{V}^{\mathbb{P}}$.

Lemma 4.1 ($\mathbb{P}$-generic hitting). *Let* $\mathcal{G}$, z *and* $\mathbb{Q} = \mathbb{Q}^z_{\mathcal{G}}$ *be as in Theorem 3.16. Let an arbitrary forcing* $\mathbb{P}$, *a* $\mathbb{P}$*-name* $\dot\tau$, *a condition* $(p,q) \in \mathbb{P} \times \mathbb{Q}$ *and* $k \in \omega$ *be given and suppose* $p \Vdash_{\mathbb{P}} \dot\tau \in S_\infty$.

Then there is $(p', q') \in \mathbb{P} \times \mathbb{Q}$ *such that* $(p', q') \leq_{\mathbb{P}\times\mathbb{Q}} (p, q)$ *and*

$$(p', q') \Vdash_{\mathbb{P}\times\mathbb{Q}} (\exists n \in \mathbb{N})\ n > k \wedge \sigma_{\dot G}(n) = \dot\tau(n).$$

Proof. Fix $\mathbb{P}$, $\dot\tau$ and $(p,q) \in \mathbb{P} \times \mathbb{Q}$ as in the statement of the lemma. Let G be $(\mathbb{P}, \mathbf{V})$-generic and such that $p \in G$.

Working in $\mathbf{V}[G]$, argue just as in the proof of the Generic Hitting Lemma: Find n such that for $s = s^q$, F^* equal to the set of subwords of cyclic permutations of a word in F^q, $E' = \operatorname{dom}(s^q) \cup \operatorname{ran}(s^q)$ and $\tau = \dot\tau[G]$, (3.8) holds. Letting $n' = \dot\tau[G]$, we can find $p' \in G$ extending p such that

$$p' \Vdash_{\mathbb{P}} \dot\tau(\check n) = \check n'.$$

Just as in the proof of the Generic Hitting Lemma, by choice of n and n', we have that for $E = E' \cup \{n\}$, (3.5) holds in $\mathbf{V}$.

By the Domain Extension Lemma we can extend q to $q' \in \mathbb{Q}$ such that

$$q' \Vdash_{\mathbb{Q}} \sigma_{\dot{G}}(\check{n}) = \check{n}',$$

and we are done. $\square$

We are now ready to prove the main theorem of this paper:

Theorem 4.2. *Assume $\mathbf{V} = \mathbf{L}$. Let $\mathcal{G}_0$ be any countable cofinitary group, and fix $c \in 2^{\mathbb{N}}$ such that $\mathcal{G}_0$ is $\Delta^1_1(c)$ as a subset of $\mathbb{N}^{\mathbb{N}}$. Then there is a Cohen-indestructible $\Pi^1_1(c)$ maximal cofinitary group which contains $\mathcal{G}_0$ as a subgroup.*

While for the appropriate choice of $\mathcal{G}_0$, our method will produce a group which is isomorphic to Kastermans' group from [11], these groups are not outright identical. In fact, we shall see our group is Cohen-indestructible, which is unlikely to be the case for Kastermans' group.

Our argument resembles Miller's classical construction of co-analytic sets given in [19].

To simplify matters, we adopt the following convention: Given $x \in 2^{\mathbb{N}}$, let $E_x \subseteq \omega^2$ be the binary relation defined by

$$m \, E_x \, n \iff x(2^m 3^n) = 0.$$

If it is the case that E_x is well-founded and extensional, we denote by M_x the unique transitive $\in$-model isomorphic to $\langle \omega, E_x \rangle$. Note also that for any countable transitive set M, we can find $x \in 2^{\mathbb{N}}$ such that $\langle M, \in \rangle = M_x$. Moreover, if $M \in \mathbf{L}_{\omega_1^y}$, for $y \in 2^{\mathbb{N}}$, then we may find such $x \in \mathbf{L}_{\omega_1^y}$.

Proof. Fix $\mathcal{G}_0$ as above. Since the argument relativizes to any parameter c, we may suppress c and assume that $\mathcal{G}_0$ is (lightface) Δ^1_1.

Work in $\mathbf{L}$. For each $\xi < \omega_1$ we shall define

- $\delta(\xi)$, a countable ordinal,
- $z_\xi \in 2^{\mathbb{N}} \cap \mathbf{L}_{\delta(\xi)+\omega}$,
- $\sigma_\xi \in S_\infty \cap \mathbf{L}_{\delta(\xi)+\omega}$.

We define these so that the following is satisfied for each $\xi < \omega_1$:

(i) $\delta(\xi)$ is the least ordinal δ above $\sup_{\nu<\xi} \delta(\nu)$ such that $\mathbf{L}_\delta$ projects to ω and $\mathcal{G}_0 \in \mathbf{L}_\delta$.

(ii) z_ξ is the unique code for the theory of $\mathbf{L}_{\delta(\xi)}$, obtained via the canonical definable surjection from ω onto $\mathbf{L}_{\delta(\xi)}$ (see, e.g. [10]).

(iii) Letting $\mathcal{G}_\xi = \{ \sigma_\nu \colon \nu < \xi\} \cup \mathcal{G}_0$, we have that $\mathcal{G}_\xi$ is cofinitary and $\mathcal{G}_\xi \in \mathbf{L}_{\delta(\xi)}$.

(iv) $\sigma_\xi = \sigma_G$, where G is the unique $(\mathbf{L}_{\delta(\xi)}, \mathbb{Q}^{z_\xi}_{\mathcal{G}_\xi})$-generic obtained by hitting dense subsets of $\mathbb{Q}^{z_\xi}_{\mathcal{G}_\xi}$ in the order in which they are enummerated by the canonical definable surjection from ω onto $\mathbf{L}_{\delta(\xi)}$.

Obtaining such a sequence is straight-forward, since (i) and (ii) determine $\langle(\delta(\xi), z_\xi) \colon \xi < \omega_1\rangle$, and assuming (iii) by induction, (iv) uniquely determines σ_ξ from $\delta(\xi)$, z_ξ and the sequence $\langle\sigma_\nu \colon \nu < \xi\rangle$. $\mathcal{G}_{\xi+1}$ is a cofinitary group by induction and Theorem 3.16.

Finally, we let

$$\mathcal{G} = \bigcup_{\xi<\omega_1} \mathcal{G}_\xi,$$

which is a cofinitary group by (iii) above.

Claim 4.3. *As a subset of* $\mathbb{N}^\mathbb{N}$, $\mathcal{G}$ *is* Π^1_1.

Proof of Claim. Let $\Psi(\vec{x})$ be the formula saying that for some ξ, $\vec{x}$ is a sequence

$$\vec{x} = \langle(\rho(\xi), z_\xi, \sigma_\xi) \colon \xi \leq \zeta\rangle,$$

such that for every $\xi < \zeta$, (i), (ii) and (iv) above hold. That is, $\Psi(\vec{x})$ holds if and only if $\vec{x}$ is an initial segment of our construction above.

Note that $\Psi(\vec{x})$ is absolute for all transitive models of a certain fragment of ZFC—say, Mathias' MW from [18]—satisfied by all initial segments of the $\mathbf{L}$-hierarchy of limit height.

Thus, membership in $\mathcal{G}$ is determined by a $\Sigma_1(\mathbf{H}(\omega_1))$ formula: $\sigma \in \mathcal{G}$ holds if and only if

(4.1) there exists a countable $\in$-model M of MW s.t. for some $\vec{x} \in M$,

$$M \vDash \text{“}\vec{x} = \langle(\delta_\zeta, z_\zeta, \tau_\zeta) \colon \zeta \leq \xi\rangle) \wedge \Psi(\vec{x})\text{” and } \sigma = (\tau_\xi)^M.$$

In fact, examining our construction of the sequence $\langle(\delta(\xi), z_\xi, \sigma_\xi) \colon \xi < \omega_1\rangle$, one finds that for $\sigma = \sigma_\xi$, we can take M in (4.1) to be $\mathbf{L}_{\delta(\xi)+\omega}$.

Let $\Phi(y, \sigma)$ be the formula expressing that E_y is well-founded and extensional, $M_y \models$ MW and for some $\vec{x} \in M_y$,

$$M_y \vDash \text{“}\vec{x} = \langle(\delta_\zeta, z_\zeta, \tau_\zeta) \colon \zeta \leq \xi\rangle) \wedge \Psi(\vec{x})\text{”}$$

with $\sigma = (\tau_\xi)^{M_y}$. We can take $\Phi(y, \sigma)$ to be a Π^1_1 formula.

Thus (4.1) is equivalent to

$$(\exists y \in 2^\mathbb{N})\ \Phi(y, \sigma).$$

We now make use of the fact that y as in the preceding formula can be found effectively in σ.

Since a well-order of length $\delta(\xi)$ is computable in z_ξ and $z_\xi \leq_T \sigma_\xi$, we have $\delta(\xi) < \omega_1^{\sigma_\xi}$. Thus, a $y \in 2^{\mathbb{N}}$ such that $M_y = \mathbf{L}_{\delta(\xi)+\omega}$ can be found inside $\mathbf{L}_{\omega_1^\sigma}$ when $\sigma = \sigma_\xi$. This gives us "$\Rightarrow$" in the following ("$\Leftarrow$" is obvious):

$$\sigma \in \mathcal{G} \iff (\exists y \in 2^{\mathbb{N}} \cap \mathbf{L}_{\omega_1^\sigma})\ \Phi(y, \sigma).$$

By Mansfield-Solovay [17, Corollary 4.19, p. 53], the right-hand side can be rendered as a Π^1_1 formula, proving the claim. $\square_{\text{Claim.}}$

Since any $\sigma \in S_\infty$ appears in some $\mathbf{L}_\delta(\xi)$, maximality of $\mathcal{G}$ follows from (IV) of Theorem 3.16, and (iv) above. In fact, we show the stronger statement that $\mathcal{G}$ is Cohen-indestructible:

Towards a contradiction, suppose we have a $\mathbb{C}$-name $\dot\tau$ and $p \in \mathbb{C}$ such that

$$p \Vdash_{\mathbb{C}} \langle \check{\mathcal{G}}, \dot\tau \rangle \text{ is cofinitary.}$$

We may assume that there is $\xi < \omega_1$ such that $\dot\tau \in L_{\delta(\xi)}$. In fact, we may assume that there is a $\Delta_0(\dot\tau)$ formula $\Psi(x, y, z)$ such that for all $p' \in \mathbb{C}$ below p and all $n, n' \in \mathbb{N}$,

$$p \Vdash_{\mathbb{C}} \dot\tau(\check n) = \check n'$$

is equivalent to $\Psi(p, n, k)$ (by choosing a 'nice' name).

We may also assume (by strengthening p if necessary) that there is N such that

$$p \Vdash_{\mathbb{C}} |\{n \in \mathbb{N} \colon \check\sigma_\xi(n) = \dot\tau(n)\}| = \check N. \tag{4.2}$$

By repeatedly using Lemma 4.1, the set D of q $\in \mathbb{Q}^{z_\xi}_{\mathcal{G}_\xi}$ such that for some $p' \in \mathbb{C}$ stronger than p and for some set $Z \subseteq \operatorname{dom}(s^q)$ of size $N+1$ we have

$$(\forall n \in Z)\ p' \Vdash_{\mathbb{C}} \dot\tau(\check n) = \check s^q(\check n) \tag{4.3}$$

is dense in $\mathbb{Q}^{z_\xi}_{\mathcal{G}_\xi}$. As (4.3) can be replaced by a $\Delta_0(\dot\tau)$ formula, $D \in \mathbf{L}_{\delta(\xi)}$. Thus, the generic which gave rise to σ_ξ meets D and we conclude that for some $p' \in \mathbb{C}$ stronger than p and for some set $Z \subseteq \mathbb{N}$ of size $N+1$ we have

$$(\forall n \in Z)\ p' \Vdash_{\mathbb{C}} \dot\tau(\check n) = \check\sigma_\xi(\check n),$$

contradicting (4.2); thus, $\mathcal{G}$ is Cohen-indestructible. $\square$

We obtain as an immediate corollary:

Corollary 4.4. *Theorem 1.2 holds: The existence of a* Π^1_1 *maximal cofinitary group is consistent with* $\neg$CH *(in fact, with arbitrarily large continuum), provided ZFC is consistent.*

5. Questions

Considering the many known models where some inequality holds between $\mathfrak{a}_g$ and another cardinal invariant of the continuum, the methods developed in the present paper suggest to consider the following definable analog:

For Γ an arbitrary pointclass, let $\mathfrak{a}_g(\Gamma)$ be the least cardinal κ such that there is a mcg $\mathcal{G} \in \Gamma$ of size κ.

Question 5.1. How does $\mathfrak{a}_g(\Pi^1_1)$ compare to other cardinal invariants of the continuum?

Lastly, we would like to mention the following long-standing open question in relation to definable mcgs:

Question 5.2. Can a mcg be Borel (equivalently, analytic)?

Note that it is still open whether ZFC rules out that a mcg be closed.

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