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Maximal Cofinitary Groups Revisited

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Let κ be an arbitrary regular infinite cardinal and let $\mathcal{C}$ denote the set of κ-maximal cofinitary groups. We show that if GCH holds and C is a closed set of cardinals such that

1. $\kappa^+ \in C, \forall \nu \in C(\nu \geq \kappa^+)$,
2. if $|C| \geq \kappa^+$ then $[\kappa^+, |C|] \subseteq C$,
3. $\forall \nu \in C(\text{cof}(\nu) \leq \kappa \to \nu^+ \in C)$,

then there is a generic extension in which cofinalities have not been changed and such that $C = \{|\mathcal{G}| : \mathcal{G} \in \mathcal{C}\}$. The theorem generalizes a result of Brendle, Spinas and Zhang (see [4]) regarding the possible sizes of maximal cofinitary groups.

Our techniques easily modify to provide analogous results for the spectra of maximal κ-almost disjoint families in $[\kappa]^\kappa$, maximal families of κ-almost disjoint permutations on κ and maximal families of κ-almost disjoint functions in ${}^\kappa\kappa$. In addition we construct a κ-Cohen indestructible κ-maximal cofinitary group and so establish the consistency of $\mathfrak{a}_g(\kappa) < \mathfrak{d}(\kappa)$, which for $\kappa = \omega$ is due to Yi Zhang (see [10]).

1 Introduction

We will be interested in higher analogues of maximal almost disjoint families and maximal cofinitary groups. Throughout the paper, κ denotes a regular infinite cardinal. A subset $\mathcal{A} \subseteq [\kappa]^\kappa$ is said to be κ-a.d. if for all distinct $a, b \in \mathcal{A}$ $|a \cap b| < \kappa$. Similarly to the ω case, a κ-a.d. family of size $\geq \kappa$ is said to be maximal if it is maximal with respect to inclusion. We denote by $S(\kappa)$ the group of all permutations on κ. A subgroup $\mathcal{G}$ of $S(\kappa)$ is said to be *κ-cofinitary* if each of its non-identity elements has less than κ-many fixed points. A κ-cofinitary group is said to be *a κ-maximal cofinitary group* (abbreviated κ-mcg), if it is maximal among the κ-cofinitary groups under inclusion. Let $C_\kappa(\text{mad}) = \{|\mathcal{A}| : \mathcal{A} \text{ is a } \kappa\text{-mad family}\}$ and $C_\kappa(\text{mcg}) = \{|\mathcal{G}| : \mathcal{G} \text{ is a } \kappa\text{-mcg}\}$. We refer to $C_\kappa(\text{mad})$ and $C_k(\text{mcg})$ denote the *spectrum* of κ-mad families and κ-mcg respectively. Recall that $\mathfrak{a}(\kappa)$ denotes the minimal size of a κ-maximal almost disjoint family and $\mathfrak{a}_g(\kappa)$ denotes the minimal size of a κ-maximal cofinitary group. Thus in particular $\mathfrak{a}(\kappa) = \min C_\kappa(\text{mad})$ and $\mathfrak{a}_g(\kappa) = \min C_\kappa(\text{mcg})$.

The spectra of maximal almost disjoint families and maximal cofinitary groups on ω, $C_\omega(\text{mad})$ and $C_\omega(\text{mcg})$, have been studied by various authors. It is consistent that for every uncountable cardinal $\lambda \leq \mathfrak{c}$ there is a maximal almost disjoint family of cardinality λ (see [5, Theorem 3.2]). Furthermore, A. Blass showed in [1] that if GCH holds and C is a closed set of uncountable cardinals such that

1. $\aleph_1 \in C, \forall \nu \in C(\nu \geq \aleph_1)$,
2. if $|C| \geq \aleph_1$ then $[\aleph_1, |C|] \subseteq C$ and
3. $\forall \lambda(\lambda \in C \wedge \text{cof}(\lambda) = \omega \to \lambda^+ \in C)$,

then there is a ccc generic extension in which $C_\omega(\text{mad}) = C$. Brendle, Spinas and Zhang obtain an analogue of this result regarding maximal cofinitary groups (see [4]): whenever GCH holds and C is as above, then there is a ccc generic extension in which $C_\omega(\text{mcg}) = C$. We generalize these results to κ-mad families and κ-maximal cofinitary groups. Our main result states the following:

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Theorem 1.1 *(GCH) Let κ be a regular infinite cardinal and let C be a closed set of cardinals such that*

1. $\kappa^+ \in C$, $\forall \nu \in C(\nu \geq \kappa^+)$,
2. *if* $|C| \geq \kappa^+$ *then* $[\kappa^+, |C|] \subseteq C$,
3. $\forall \nu \in C(cof(\nu) \leq \kappa \to \nu^+ \in C)$.

Then there is a generic extension in which cofinalities have not been changed and such that $C = C_\kappa(mcg)$.

The result relies on one hand on a forcing notion which adds a κ-maximal cofinitary group of desired cardinality (see Theorem 2.15). This poset appears a natural generalization of a forcing notion introduced in [6] which adds a mcg (on ω) of desired cardinality. Our poset is a product-like forcing notion, which is $<\kappa$-closed and κ^+-Knaster (e.g. any family of κ^+-many distinct conditions, contains a subfamily of size κ^+ whose elements are pairwise compatible). Of particular interest for us are the combinatorial properties corresponding to Lemmas 2.11, 2.14 and 2.16. On the other hand in order to exclude cardinals outside of the chosen set C from the spectrum of the κ-maximal cofinitary groups, we develop a generalization to Blass's notion of a $\mathbf{\Pi^0_2}$-definable and OD($\mathbb{R}$)-definable cardinals. Furthermore our techniques, can be easily modified and applied to the study of various close relatives of the κ-maximal cofinitary groups. Let $\mathcal{C}$ denote either of the following sets: the set of κ-maximal cofinitary groups, the set of κ-maximal almost disjoint families, the set of κ-almost disjoint permutations on ${}^\kappa\kappa$, the set of κ-almost disjoint functions on ${}^\kappa\kappa$. Our results can be summarized as follows:

Theorem 1.2 *(GCH) Let C be a set of cardinals as in Theorem 1.1. Then there is a generic extension in which cofinalities have not been changed and such that $C = \{\mathcal{B} : \mathcal{B} \in \mathcal{C}\}$.*

Of interest remains the questions to what extent the restrictions on the set C above are necessary. Recently, S. Shelah and O. Spinas (see [9]) showed that the requirements $\aleph_1 \in C$ and $\forall \lambda \in C(\mathrm{cof}(\lambda) = \omega \to \lambda^+ \in C)$ in Blass's theorem from [1] are not necessary. An analogous weakening on the requirements which we impose on the spectrum of κ-maximal cofinitary groups (as well as on the spectrum of some of their κ-relatives) and more generally determining an optimal set of conditions for such sets of admissible values remains of interest. There are still many open questions regarding the possible sizes of κ-mad families and κ-maximal cofinitary groups. For example, it is known that consistently $\mathrm{cof}(\mathfrak{a}) = \omega$ and $\mathrm{cof}(\mathfrak{a}_g) = \omega$ (see [3] and [6] respectively) and so consistently $\mathrm{cof}(\min C_\omega(\mathrm{mad})) = \omega$ and $\mathrm{cof}(\min C_\omega(\mathrm{mcg})) = \omega$. However for κ uncountable regular cardinal the following questions remain open:

1. Is it consistent that $\mathrm{cof}(\mathfrak{a}(\kappa)) = \kappa$?
2. Is it consistent that $\mathrm{cof}(\mathfrak{a}_g(\kappa) = \kappa$?

In addition, we study some preservation properties of κ-maximal cofinitary groups. We show that:

Theorem 1.3 *(GCH) There is a κ-Cohen indestructible, κ-maximal cofinitary groups.*

Thus we generalize Y. Zhang's result on the existence of Cohen indestructible maximal cofinitary groups. Consequently, we obtain the relative consistency of $\mathfrak{a}_g(\kappa) < \mathfrak{d}(\kappa)$ (see Theorem 4.6). Furthermore, we show that if for some regular cardinal $\lambda \geq \kappa^{++}$ we add λ many κ-Cohen reals to a model of GCH, in the resulting extension every κ-maximal cofinitary group is either of size κ^+ or of size $2^\kappa = \lambda$ (see Theorem 5.1).

2 Adding κ-maximal cofinitary groups

We present a generalization of the poset developed in [6]: the original poset adds a maximal cofinitary group of desired size, while our generalized version adds a κ-maximal cofinitary group of desired cardinality. We will follow the notation of [6]. Thus in particular for A an index set, W_A denotes the set of all reduced words on the alphabet $\langle a^i : a \in A, i \in \{-1, 1\}\rangle$ and $\widehat{W_A}$ the subset of all words which are either power of a singleton, or start and end with a different letter. The elements of $\widehat{W_A}$ are referred to as *good words*. Given a mapping $\rho : A \to S(\kappa)$, let $\hat{\rho}$ denote the canonical extension of ρ to a group homomorphism between the free group $\mathbb{F}_A$ on A and $S(\kappa)$. We say that ρ induces *a κ-cofinitary representation* if the image of $\hat{\rho}$ is a κ-cofinitary subgroup of $S(\kappa)$. Whenever A is a set, $s \subseteq A \times \kappa \times \kappa$ and $a \in A$, we denote by $s_a = \{(\alpha, \beta) : (a, \alpha, \beta) \in s\}$. For a word $w \in W_A$, define the relation $e_w[s] \subseteq \kappa \times \kappa$ recursively by stipulating that

- if $w = a$ for some $a \in A$ then $(\alpha, \beta) \in e_w[s]$ iff $(\alpha, \beta) \in s_a$,
- if $w = a^{-1}$ for some $a \in A$, then $(\alpha, \beta) \in e_w[s]$ iff $(\beta, \alpha) \in s_a$, and
- if $w = a^i u$ for some $u \in W_A$, $a \in A$ and $i \in \{1, -1\}$ without cancelation, then $(\alpha, \beta) \in e_w[s]$ iff $(\exists\gamma) e_{a^i}[s](\gamma, \beta) \wedge e_u[s](\alpha, \gamma)$.

$e_w[s]$ is referred to *the evaluation of w given s*.

Claim 2.1 *Let $s \subseteq A \times \kappa \times \kappa$ be such that s_a is a partial injection for all a. Then for every $w \in W_A$ the relation $e_w[s]$ is a partial injection.*

Whenever A and B are disjoint sets, $\rho : B \to S(\kappa)$, $w \in W_{A\cup B}$ and $s \subseteq A \times \kappa \times \kappa$, we define $(\alpha, \beta) \in e_w[s, \rho]$ iff $(\alpha, \beta) \in e_w[s \cup \{(b, \gamma, \delta) : \rho(b)(\gamma) = \delta\}]$. As in the ω-case, if s_a is a partial injection for $a \in A$ then $e_w[s, \rho]$ is also a partial injection. It is referred to as the *evaluation of w given s and ρ*. By definition $e_\emptyset[s, \rho]$ is the identity on $S(\kappa)$.

Definition 2.2 Let A and B be disjoint sets and let $\rho : B \to S(\kappa)$ be a function inducing a κ-cofinitary representation. The forcing notion $\mathbb{Q}^\kappa_{A,\rho}$ consists of all pairs

$$(s, F) \in [A \times \kappa \times \kappa]^{<\kappa} \times [\widehat{W}_{A\cup B}]^{<\kappa}$$

such that s_a is injective for every $a \in A$. The extension relation states that $(s, F) \leq_{\mathbb{Q}_{A,\rho}} (t, E)$ if $s \supseteq t$, $F \supseteq E$ and for all $\alpha \in \kappa$ and $w \in E$, if $e_w[s, \rho](\alpha) = \alpha$ then already $e_w[t, \rho](\alpha)$ is defined and $e_w[t, \rho](\alpha) = \alpha$. In case $B = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$.

Our goal is to show that if G is $\mathbb{Q}^\kappa_{A,\rho}$-generic, then the mapping $\rho_G : A \cup B \to S(\kappa)$, which is defined by $\rho_G \restriction B = \rho$ and $\rho_G(a) = \bigcup\{s_a : \exists F (s, F) \in G\}$ for every $a \in A$, induces a κ-cofinitary representation of $A \cup B$ which extends ρ. Note that the above poset is clearly $< \kappa$-closed. In analogy with the Knaster property, we will say that a poset $\mathbb{P}$ has the *κ-Knaster property*, if in every collection of κ-many conditions from $\mathbb{P}$ there are κ many which are pairwise compatible. The poset $\mathbb{Q}^\kappa_{A,\rho}$ is in fact κ^+-Knaster (see below).

Before proceeding with the proof of this fact, we fix some notation: whenever $p = (s, F) \in \mathbb{Q}^\kappa_{A,\rho}$ we denote by $\mathrm{oc}_A(s) = \{a \in A : \exists \alpha, \beta (a, \alpha, \beta) \in s\}$, $oc_A(F)$ the set of letters from A which appear in words from the set F and $oc_A(p) = \mathrm{oc}_A(s) \cup \mathrm{oc}_A(F)$. For a word $w \in W_{A\cup B}$ denote by $\mathrm{oc}_A(w)$ the set of all letters from A occurring in w. Also, whenever $A_0 \subseteq A \cup B$ and $p = (s, F) \in \mathbb{Q}^\kappa_{A,\rho}$ let $p \restriction A_0 = (s \cap (A_0 \times \kappa \times \kappa), F)$ and let $p \Vert\!\restriction A_0 = (s \cap (A_0 \times \kappa \times \kappa), F \cap \widehat{W}_{A_0})$. Note that $\rho \restriction A_0 \cap B : A_0 \cap B \to S_\kappa(\kappa)$ still induces a κ-cofinitary representation. Thus $p \Vert\!\restriction A_0$ is a condition in $\mathbb{Q}^\kappa_{\rho\restriction A_0 \cap B}$ while $p \restriction A_0$ is not necessarily a condition in $\mathbb{Q}^\kappa_{\rho\restriction A_0\cap B}$.

Lemma 2.3 *Let $\kappa^{<\kappa} = \kappa$. Then $\mathbb{Q}^\kappa_{A,\rho}$ is κ^+-Knaster.*

P r o o f. Consider any family $\{p_\alpha\}_{\alpha<\kappa^+}$ of κ^+-many conditions in $\mathbb{Q}^\kappa_{A,\rho}$. Let $p_\alpha = (s_\alpha, F_\alpha)$ for all α. By the Δ-system lemma, there is an index set I_0 of size κ^+ and a set Δ_0 of size $< \kappa$, such that $\{\mathrm{oc}_A(s_\alpha)\}_{\alpha\in I_0}$ form a Δ-system with root Δ_0. Similarly, there is an index set $I_1 \subseteq I_0$ of size κ^+ and a set Δ_1 of size $< \kappa$ such that $\{\mathrm{oc}_a(p_\alpha)\}_{\alpha\in I_1}$ form a Δ-system with root Δ_1. In particular $\Delta_0 \subseteq \Delta_1$. Since $|\Delta_1| < \kappa$, there are only κ-many choices for $s_\alpha \restriction \Delta_0 \times \kappa \times \kappa$ and so for some $I_3 \subseteq I_2$ of size κ^+ and $t \subseteq \Delta_1 \times \kappa \times \kappa$ we have that $s_\alpha \restriction \Delta_1 \times \kappa \times \kappa = t$ whenever $\alpha \in I_3$. Note that $\mathrm{oc}_A(t)$ must in fact be Δ_0.

Take any $\alpha \neq \beta$ from I_3. We claim that $q = (s_\alpha \cup s_\beta, F_\alpha \cup F_\beta)$ is a common extension of (s_α, F_α), (s_β, F_β). Note that $\mathrm{oc}_A(s_\alpha) \cap \mathrm{oc}_A(F_\beta) \subseteq \Delta_1$. However $s_\alpha \restriction \Delta_1 \times \kappa \times \kappa = t$, $t \subseteq s_\beta$ and so for every word $w \in F_\beta$ we have that $e_w[s_\alpha \cup s_\beta, \rho] = e_w[s_\beta, \rho]$. This implies that $q \leq (s_\beta, F_\beta)$. To see $q \leq (s_\alpha, F_\alpha)$ proceed analogously. □

We will need the following Lemma. Whenever $f : \kappa \to \kappa$ is a (partial) function, we denote by $\mathrm{fix}(f)$ the set of all fixed points of f.

Lemma 2.4 *Let A and B be disjoint sets, $\rho : B \to S(\kappa)$. Let $w \in W_{A\cup B}$ and $s \subseteq A \times \kappa \times \kappa$ be such that s_a is a partial injection for all $a \in A$. Suppose $w = uv$ without cancelation for some $u, v \in W_{A\cup B}$. Then $\alpha \in \mathrm{dom}(e_w[s, \rho])$ if and only if $\alpha \in \mathrm{dom}(e_v[s, \rho])$ and $e_v[s, \rho](\alpha) \in \mathrm{dom}(e_u[s, \rho])$. If moreover $w \in \widehat{W}_{A\cup B}$ then $\alpha \in \mathrm{fix}(e_w[s, \rho])$ if and only if $e_v[s, \rho](\alpha) \in \mathrm{fix}(e_{vu}[s, \rho])$. In particular, $\mathrm{fix}(e_w[s, \rho])$ and $\mathrm{fix}(e_{vu}[s, \rho])$ have the same cardinality.*

Following the notation of [6] we say that a word w in $W_{A\cup B}$ is *a-good of rank* $j \geq 1$, where $a \in A$, $j \in \omega$ if it is of the form

$$w = a^{k_j} u_j a^{k_{j-1}} u_{j-1} \cdots a^{k_1} u_1$$

where for all $i : 1 \leq i \leq j$, $\mathrm{oc}_A(u_i) \subseteq A \setminus \{a\}$ and k_i is a non-zero integer.

Lemma 2.5 *Let $s \in [A \times \kappa \times \kappa]^{<\kappa}$ be such that s_a is a partial injection for all $a \in A$. Let $a \in A$, and let $w \in W_{A\cup B}$ be a-good. Then for any $\alpha \in \kappa \setminus \mathrm{dom}(s_a)$ and $C \in [\kappa]^{<\kappa}$ for all but $<\kappa$-many β we have that*

$$(\forall \gamma \in \kappa) e_w[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C \iff e_w[s, \rho](\gamma){\downarrow} \wedge e_w[s, \rho](\gamma) \in C$$

P r o o f. By induction on the rank j. Let w be an a-good word of rank 1, $w = a^{k_1} u_1$.

Assume first $k_1 > 0$. Then pick $\beta \notin \mathrm{dom}(a)$ and $\beta \notin C$. Suppose $e_w[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C$ but $e_w[s, \rho](\gamma){\uparrow}$. Then there is some $0 < i < k_1$ such that $e_{a^i u_1}[s, \rho](\gamma) = \alpha$. If $i < k_1 - 1$ then $e_{a^{i+1} u_1}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma){\uparrow}$, so we must have $i = k_1 - 1$. But then $e_w[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) = \beta \notin C$, a contradiction.

Assume then $k_1 < 0$. Pick $\beta \notin \mathrm{ran}(e_{a^i u_1}[s, \rho])$ for all $k_1 \leq i < 0$. If $e_w[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C$ but $e_w[s, \rho](\gamma){\uparrow}$, then there is some $k_1 < i < 0$ such that $e_{a^i u_1}[s, \rho](\gamma){\downarrow}$ but $e_{a^{i-1} u_1}[s, \rho](\gamma){\uparrow}$. Since $e_{a^i u_1}[s, \rho](\gamma) \neq m$, it follows that $e_{a^{i-1} u_1}[s \cup \{(a, \alpha, \beta)\}, \rho]{\uparrow}$, a contradiction.

Now let w be a-good of rank $j > 1$, and write $w = a^{k_j} u_j \bar{w}$, where $\bar{w}$ is a-good of rank $j - 1$. Let $C' = e_{u_j^{-1} a^{-k_j}}[s, \rho](C)$. By the inductive assumption there is $I_0 \subseteq \kappa$ such that $|\kappa \backslash I_0| < \kappa$ and for all $\beta \in I_0$,

$$(\forall \gamma \in \kappa) e_{\bar{w}}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C' \iff e_{\bar{w}}[s, \rho](\gamma){\downarrow} \wedge e_{\bar{w}}[s, \rho](\gamma) \in C'.$$

Let $I_1 \subseteq \kappa$ be of size κ such that $|\kappa \backslash I_1| < \kappa$ and for all $\beta \in I_1$,

$$\begin{aligned} &(\forall \gamma \in \kappa) e_{a^{k_j} u_j}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C \\ &\iff e_{a^{k_j} u_j}[s, \rho](\gamma){\downarrow} \wedge e_{a^{k_j} u_j}[s, \rho](\gamma) \in C. \end{aligned}$$

Then let $\beta \in I_1 \cap I_0$, and suppose $e_w[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C$. Then $e_{\bar{w}}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C'$ and so $e_{\bar{w}}[s, \rho](\gamma) \in C'$. It follows that

$$e_{a^{k_j} u_j}[s \cup \{(a, \alpha, \beta)\}, \rho](e_{\bar{w}}[s, \rho](\gamma)) \in C$$

and so we have $e_{a^{k_j} u_j}[s, \rho](e_{\bar{w}}[s, \rho](\gamma)) = e_w[s, \rho](\gamma) \in C$, as required. □

Lemma 2.6 *Let $(s, F) \in \mathbb{Q}^{\kappa}_{A,\rho}$, $a \in A$.*

1. *Let $\alpha \in \kappa \backslash \mathrm{dom}(s_a)$. Then there is $I = I_{a,\alpha}$ such that $|\kappa \backslash I| < \kappa$ and for all $\beta \in I$ we have that $(s \cup \{(a, \alpha, \beta)\}, F) \leq (s, F)$.*

2. *Let $\beta \in \kappa \backslash \mathrm{ran}(s_a)$. Then there is $J = J_{a,\beta}$ such that $|\kappa \backslash J| < \kappa$ and for all $\alpha \in J$ we have that $(s \cup \{(a, \alpha, \beta)\}, F) \leq (s, F)$.*

P r o o f. It is sufficient to obtain the claim for $F = \{w\}$. If w is a-good, then by the previous lemma there is a set $I \subseteq \kappa$ such that $|\kappa \backslash I| < \kappa$ and such that $\forall \gamma (e_w[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) \in C \iff e_s[s, \rho](\gamma) \in C)$, where $C = \{\delta : \delta \in \mathrm{fix}(e_w[s, \rho])\}$. Thus any $\beta \in I$ satisfies the claim. Thus suppose a is not good. Then $w = uva^k$ (without cancelation), where $a \notin \mathrm{oc}_A(u)$, v is a-good, and $k \in \mathbb{Z}$. Let $\bar{w} = va^k u$. Then $\bar{w}$ is a-good, and so there is $I \subseteq \kappa$ such that $|\kappa \backslash I| < \kappa$ and for all $\beta \in I$ we have that $(s \cup \{(a, \alpha, \beta)\}, \{\bar{w}\}) \leq_{\mathbb{Q}_{A,\rho}} (s, \{\bar{w}\})$.

We claim that for all $\beta \in I$ we have that $(s \cup \{(a, \alpha, \beta)\}, \{w\}) \leq (s, \{w\})$. Let $\beta \in I$ and $\gamma \in \kappa$ such that $e_w[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma) = \gamma$. Then by Lemma 2.4

$$e_{\bar{w}}[s \cup \{(a, \alpha, \beta)\}, \rho](e_{va^k}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma)) = e_{va^k}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma)$$

and since $\beta \in I$ we have

$$e_{\bar{w}}[s, \rho](e_{va^k}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma)) = e_{va^k}[s \cup \{(a, \alpha, \beta)\}, \rho](\gamma).$$

However $e_{(va^k)u}[s,\rho](e_{va^k}[s\cup\{(a,\alpha,\beta)\},\rho](\gamma))$ is equal by definition to

$$e_{va^k}[s,\rho](e_u[s,\rho](e_{va^k}[s\cup\{(a,\alpha,\beta)\},\rho](\gamma))$$

which is equal to $e_{va^k}[s,\rho](e_w[s\cup\{(a,\alpha,\beta)\},\rho](\gamma)) = e_{va^k}[s,\rho](\gamma)$. Then we obtain that $e_{va^k}[s,\rho](\gamma) = e_{va^k}[s\cup\{(a,\alpha,\beta)\},\rho](\gamma)$. Therefore

$$e_{(va^k)u}[s,\rho](e_{va^k}[s,\rho](\gamma)) = e_{va^k}[s,\rho](\gamma)$$

and so by Lemma 2.4 we obtain that $e_{uva^k}[s,\rho](\gamma)=\gamma$.

The proof of part (2) follows very closely the proof of [6, Lemma 2.7.(2)]. For completeness however we state it below. Let $(s,F)\in\mathbb{Q}^{\kappa}_{A,\rho}$, $a\in A$, and $\delta\notin\operatorname{ran}(s_a)$. We may assume that $F=\{w\}$. Define $\bar{s}\subseteq A\times\kappa\times\kappa$ by

$$(x,\alpha,\beta)\in\bar{s}\iff (x\neq a\wedge(x,\alpha,\beta)\in s)\vee(x=a\wedge(x,\beta,\alpha)\in s).$$

Let $\bar{w}$ be the word in which every occurrence of a is replaced with a^{-1}. Notice that $e_{\bar{w}}[\bar{s},\rho]=e_w[s,\rho]$, and that $\delta\notin\operatorname{dom}(\bar{s})$. By (1) above there is $I\subseteq\kappa$, $\kappa\backslash I$ of size $<\kappa$ such that $(\bar{s}\cup\{(a,\delta,\beta)\},\{\bar{w}\})\le(\bar{s},\{\bar{w}\})$ whenever $\beta\in I$, and so every $\beta\in I$ we have $(s\cup\{(a,\beta,\delta)\},\{w\})\le(s,\{w\})$. □

Corollary 2.7 *Let $w\in W_{A\cup B}$, and let $A_0=\operatorname{oc}_A(w)$. For any $(s,F)\in\mathbb{Q}^{\kappa}_{A,\rho}$ and sets C_0,C_1 in $[\kappa]^{<\kappa}$ there is $t\in[A_0\times\kappa\times\kappa]^{<\kappa}$ such that $(t\cup s,F)\le(s,F)$ and $\operatorname{dom}(e_w[s\cup t,\rho])\supset C_0$ and $\operatorname{ran}(e_w[s\cup t,\rho])\supset C_1$.*

Lemma 2.8 *Let $w\in\widehat{W}_{A\cup B}$ and suppose $(s,F)\Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_G](\alpha)=\alpha$ for some $\alpha\in\kappa$. Then $e_w[s,\rho](\alpha)$ is defined and $e_w[s,\rho](\alpha)=\alpha$.*

P r o o f. If G is a generic filter containing (s,F), then there is $(t,H)\in G$ such that $e_w[t,\rho](\alpha)=\alpha$. Without loss of generality (s,F) extends (t,H) and so by definition of the extension relation $e_w[s,\rho](\alpha)=\alpha$. □

As an immediate corollary we obtain the following:

Corollary 2.9 *Let $(s,F)\in\mathbb{Q}^{\kappa}_{A,\rho}$ and let w be a word in F. Then*

$$(s,F)\Vdash_{\mathbb{Q}^{\kappa}_{A,\rho}} \mathit{fix}(e_w[\rho_G])=\mathit{fix}(e_w[s,\rho]).$$

Proposition 2.10 *Let G be $\mathbb{Q}^{\kappa}_{A,\rho}$-generic. Then $\rho_G:A\cup B\to S(\kappa)$, where*

- *$\rho_G\restriction B=\rho$ and,*
- *for every $a\in A$, $\rho_G(a)=\bigcup\{s_a:\exists(s,F)\in G\}$,*

induces a cofinitary representation $\hat{\rho}_G:\mathbb{F}_{A\cup B}\to S(\kappa)$ extending $\hat{\rho}$.

P r o o f. For each $a\in A$ and $\alpha\in\kappa$, let $D_{a,\alpha}=\{(s,F)\in\mathbb{Q}^{\kappa}_{A,\rho}:(\exists\beta)(a,\alpha,\beta)\in s\}$ and let $R_{a,\alpha}=\{(s,F)\in\mathbb{Q}^{\kappa}_{A,\rho}:(\exists\beta)(a,\beta,\alpha)\in s\}$. For $w\in\widehat{W}_{A\cup B}$, let $D_w=\{(s,F)\in\mathbb{Q}^{\kappa}_{A,\rho}:w\in F\}$. Then D_w is easily seen to be dense, and $D_{a,\alpha}$ and $R_{a,\alpha}$ are dense by Lemma 2.6. Thus ρ_G is indeed a function $A\cup B\to S(\kappa)$. It remains to show that ρ_G induces a cofinitary representation. Let $w\in W_{A\cup B}$. There are $w'\in\widehat{W}_{A\cup B}$ and $u\in W_{A\cup B}$ such that $w=u^{-1}w'u$. Since $D_{w'}$ is dense, there is some condition $(s,F)\in G$ such that $w'\in F$. By the above corollary $\operatorname{fix}(e_{w'}[\rho_G])=\operatorname{fix}(e_{w'}[s,\rho])$, which is of cardinality $<\kappa$. Finally, $\operatorname{fix}(e_w[\rho_G])=e_u[\rho_G]^{-1}(\operatorname{fix}(e_{w'}[\rho_G]))$. Since $e_u[\rho_G]^{-1}$ is injective, we obtain that $\operatorname{fix}(e_w[\rho_G])$ is also of cardinality $<\kappa$. □

Lemma 2.11 *If $A_0\subseteq A$ then $\mathbb{Q}^{\kappa}_{A_0,\rho}$ is completely contained in $\mathbb{Q}^{\kappa}_{A,\rho}$.*

P r o o f. Let $A_1=A\setminus A_0$. Without loss of generality A_0 and A_1 are nonempty. Let $(s,F)\in\mathbb{Q}_{A,\rho}$. We will show that there is $t_0\in[A_0\times\kappa\times\kappa]^{<\kappa}$ such that $t_0\supseteq s\restriction A_0$ and whenever $(t,E)\le_{\mathbb{Q}_{A_0,\rho}}(t_0,F\cap\widehat{W}_{A_0\cup B})$ then $(s\cup t,F)\le_{\mathbb{Q}_{A,\rho}}(s,F)$. Then in particular $(t_0\cup s,F)\le_{\mathbb{Q}^{\kappa}_{A,\rho}}(s,F)$ and $(s\cup t,F\cup E)$ is a common extension of (s,F) and (t,E).

Let $\{w_i\}_{i\in\lambda}$ enumerate all words w in F such that $\mathrm{oc}_A(w)\cap A_1\neq\emptyset$. Then each word w_i may be written in the form $w_i=u_{i,k_i}v_{i,k_i}\cdots u_{i,1}v_{i,1}u_{i,0}$ where $u_{i,j}\in W_{A_0}$, $v_{i,j}\in W_{A_1}$, all words are nonempty except possibly u_{i,k_i} and $u_{i,0}$. Inductively we will construct an increasing sequence $\langle t^i\rangle_{i\in\lambda}$ such that $s\restriction A_0\times\kappa\times\kappa\subseteq t^0$, and $t_0=\bigcup_{i\in\lambda}t^i$ is the desired set. *Base case.* By repeated applications of Corollary 2.7 to (s,F) and the $u_{0,j}$ we can find $t^0\in[A_0\times\kappa\times\kappa]^{<\kappa}$ extending $s\restriction A_0\times\kappa\times\kappa$ such that

- $\mathrm{dom}(e_{u_{0,j}}[s\cup t^0,\rho])\supseteq\mathrm{ran}(e_{v_{0,j}}[s,\rho])$ for all $j\in k_0+1$,
- $\mathrm{ran}(e_{u_{0,j}}[s\cup t_0,\rho])\supseteq\mathrm{dom}(e_{v_{0,j+1}}[s,\rho])$ for all $j\in k_0$,

and satisfying $(s\cup t^0,F)\leq_{\mathbb{Q}^\kappa_{A,\rho}}(s,F)$. *Inductive step.* Suppose t^i has been defined. Just in the base case apply successively Corollary 2.7 to $(s\cup t^i,F)$ and the $u_{i+1,j}$'s to find $t^{i+1}\in[A_0\times\kappa\times\kappa]^{<\kappa}$ extending t^i such that

- $\mathrm{dom}(e_{u_{i+1,j}}[s\cup t^{i+1},\rho])\supseteq\mathrm{ran}(e_{v_{i+1,j}}[s,\rho])$ for all $j\in k_{i+1}+1$,
- $\mathrm{ran}(e_{u_{i+1,j}}[s\cup t^{i+1},\rho])\supseteq\mathrm{dom}(e_{v_{i+1,j+1}}[s,\rho])$ for all $j\in k_{i+1}$,

If i is a limit and t^j has been defined for all $j<i$, proceed as in the successor case with $t^i_0=\bigcup_{l<i}t^l$ (instead of t^i). With this the inductive construction is complete. Note in particular, that by construction $(s\cup t^0,F)\leq_{\mathbb{Q}^\kappa_{A,\rho}}(s,F)$ and for all $i<\lambda$, $(s\cup t^{i+1},F)\leq_{\mathbb{Q}^\kappa_{A,\rho}}(s\cup t^i,F)$. Since the poset $\mathbb{Q}^\kappa_{A,\rho}$ is $(<\kappa)$-closed, we obtain that $(s\cup t_0,F)\leq_{\mathbb{Q}^\kappa_{A,\rho}}(s,F)$.

Let $(t,E)\leq_{\mathbb{Q}_{A_0,\rho}}(t_0,F\cap\widehat{W}_{A_0\cup B})$. If $e_{w_i}[s\cup t,\rho](\alpha)$ is defined for some $\alpha\in\kappa$, then by definition of t_0 we must have that $e_{w_i}[s\cup t_0,\rho](\alpha)$ is defined. Therefore if $e_{w_i}[s\cup t,\rho](\alpha)=\alpha$ we have $e_{w_i}[s\cup t_0,\rho](\alpha)=\alpha$, and so since $(s\cup t_0,F)\leq_{\mathbb{Q}^\kappa_{A,\rho}}(s,F)$ it follows that $e_{w_i}[s,\rho](\alpha)=\alpha$. Thus $(s\cup t,F)\leq_{\mathbb{Q}_{A,\rho}}(s,F)$ as required. □

Remark 2.12 We refer to the condition $(t_0,F\cap\widehat{W}_{A_0\cup B})$ as *strong reduction* of (s,F) to $\mathbb{Q}^\kappa_{A_0,\rho}$.

Lemma 2.13 *Let* $A=A_0\cup A_1$. *If* $(t,E)\in\mathbb{Q}_{A_0,\rho}$ *and* $(t,E)\Vdash_{\mathbb{Q}^\kappa_{A_0,\rho}}(s_0,F_0)\leq_{\mathbb{Q}^\kappa_{A_1,\rho_{\dot G}}}(s_1,F_1)$ *then* $(t\cup s_0,F_0)\leq_{\mathbb{Q}^\kappa_{A,\rho}}(t\cup s_1,F_1)$.

P r o o f. Let $w\in F_1$ and suppose $e_w[t\cup s_0,\rho](\alpha)=\alpha$. If G is $\mathbb{Q}^\kappa_{A_0,\rho}$-generic such that $(t,E)\in G$ then in $V[G]$ we have $e_w[s_0,\rho_G](\alpha)=\alpha$, and so in $V[G]$ we have $e_w[s_1,\rho_G](\alpha)=\alpha$, from which it follows that $e_w[t\cup s_1,\rho](\alpha)=\alpha$. □

Lemma 2.14 *Let* G *be* $\mathbb{Q}^\kappa_{A,\rho}$*-generic over* V *and let* $A=A_0\cup A_1$, *where* A_0, A_1 *are non-empty and disjoint. Then* $H=G\cap\mathbb{Q}^\kappa_{A_0,\rho}$ *is* $\mathbb{Q}^\kappa_{A_0,\rho}$*-generic and* $K=\{p\restriction A_1:p\in G\}$ *is* $\mathbb{Q}^\kappa_{A_1,\rho_H}$*-generic over* $V[H]$. *Moreover* $\rho_G=(\rho_H)_K$.

P r o o f. Let D be a dense subset of $\mathbb{Q}^\kappa_{A_1,\rho_H}$ in $V[H]$. Then there are a condition $p_0\in H$ and a $\mathbb{Q}^\kappa_{A_0,\rho}$-name $\dot D$ such that

$$p_0\Vdash_{\mathbb{Q}^\kappa_{A_1,\rho_H}}\text{“}\dot D\text{ is a dense subset of }\mathbb{Q}^\kappa_{A_1,\rho_{\dot H}}\text{”}.$$

It is sufficient to show that the set

$$D'=\{q\in\mathbb{Q}^\kappa_{A,\rho}:q\parallel A_0\Vdash_{\mathbb{Q}^\kappa_{A_1,\rho_H}}q\restriction A_1\in\dot D\}$$

is dense in $\mathbb{Q}^\kappa_{A,\rho}$ below p_0. Let $p\leq_{\mathbb{Q}^\kappa_{A,\rho}}p_0$. Say $p=(s,F)$. Then there is a strong reduction $(t_0,F\cap\widehat{W}_{A_0\cup B})$ of p to $\mathbb{Q}^\kappa_{A_0,\rho}$. In particular $(t_0,\widehat{W}_{A_0\cup B})\leq_{\mathbb{Q}^\kappa_{A_0,\rho}}p\parallel A_0\leq_{\mathbb{Q}^\kappa_{A_0,\rho}}p_0$. Thus

$$(t_0,F\cap\widehat{W}_{A_0\cup B})\Vdash_{\mathbb{Q}^\kappa_{A_0,\rho}}\text{“}\dot D\text{ is dense”}.$$

Therefore

$$(t_0,F\cap\widehat{W}_{A_0\cup B})\Vdash\text{“‘}\exists\dot q\in\dot D\wedge\dot q\leq p\restriction A_1\text{”}.$$

Then using the fact that the poset is $(<\kappa)$-closed we can find an extension (t,E) of $(t_0, F\cap \widehat{W}_{A_0\cup B})$ and a pair $(s_1,F_1)\in[A_1\times\kappa\times\kappa]^{<\kappa}\times[\widehat{W}_{A_\cup B}]^{<\kappa}$ such that

$$(t,E)\Vdash_{\mathbb{Q}^\kappa_{A_0,\rho}} \text{“}(s_1,F_1)\in\dot{D}\wedge(s_1,F_1)\le_{\mathbb{Q}^\kappa_{A_1,\rho_{\dot{H}}}}(s\restriction A_1,F)=p\restriction A_1.$$

By the previous Lemma we obtain that $(t\cup s_1,F_1)\le_{\mathbb{Q}^\kappa_{A,\rho}}(t\cup s\restriction A_1,F)$. Since (t,E) extends the strong reduction $(t_0,F\cap\widehat{W}_{A_0\cup B})$ we have that $(t\cup s,F)\le(s,F)=p$. Note that $t\supseteq t_0\supseteq s\restriction A_0\times\kappa\times\kappa$ and so $(t\cup s\restriction A_1,F)=(t\cup s,F)$. Furthermore $q^*:=(t\cup s_1,F_1\cup E)\le(t\cup s_1,F_1)$ and so $q^*\le p$. Since $s_1\in[A_1\times\kappa\times\kappa]^{<\kappa}$ we have that $q^*\le(t,E)$, which implies that $q^*\Vert\restriction A_0\le(t,E)$. But then

$$q^*\Vert\restriction A_0\Vdash_{\mathbb{Q}^\kappa_{A_0,\rho}} q^*\restriction A_1=(s_1,F_1)\in\dot{D}.$$

Thus we found an extension q^* of p which is in D'. Since D' is dense below p_0 and $p_0\in G$, we obtain that $G\cap D'$ is non-empty, which implies that in $V[H]$ the intersection $K\cap D$ is non-empty. □

Theorem 2.15 *Suppose $\rho: B\to S(\kappa)$ induces a κ-cofinitary representation. If $|A|>\kappa$ and G is $\mathbb{Q}^\kappa_{A,\rho}$-generic over V, then $\operatorname{im}(\hat\rho_G)$ is a κ-maximal cofinitary group in $V[G]$ of cardinality $|A\cup B|$.*

The Theorem is a consequence of the following lemma:

Lemma 2.16 *Suppose $\rho: B\to S(\kappa)$ induces a κ-cofinitary representation $\hat\rho:\mathbb{F}_B\to S(\kappa)$ and that there is $b_0\in B$ such that $\rho(b_0)$ is not the identity permutation. Let $(s,F)\in\mathbb{Q}_{A,\rho\restriction B\setminus\{b_0\}}$ and let $a_0\in A$. Then there is $\Omega\in\kappa$ such that for all $\alpha\ge\Omega$*

$$(s\cup\{(a_0,\alpha,\rho(b_0)(\alpha))\},F)\le_{\mathbb{Q}_{A,\rho\restriction B\setminus\{b_0\}}}(s,F).$$

Proof. Let $\{w_i\}_{i<\lambda}$, where $\lambda<\kappa$, enumerate the words in F in which a_0 occur. Then we may write each word w_i on the form

$$w_i=u_{i,j_i}a_0^{k(i,j_i)}u_{i,j_i-1}a_0^{k(i,j_i-1)}\cdots u_{i,1}a_0^{k(i,1)}u_{i,0}$$

where $u_{i,m}\in W_{A\setminus\{a_0\}\cup B\setminus\{b_0\}}$ are non-$\emptyset$ whenever $m\notin\{j_i,0\}$. By Lemma 2.6 we may assume that for all $u_{i,m}$ with $\operatorname{dom}(e_{u_{i,m}}[s,\rho])$ and $\operatorname{ran}(e_{u_{i,m}}[s,\rho])$ of size $<\kappa$ that

- $\operatorname{dom}(e_{a_0^{k(i,m+1)}}[s,\rho])\supseteq\operatorname{ran}(e_{u_{i,m}}[s,\rho])$, and
- $\operatorname{ran}(e_{a_0^{k(i,m)}}[s,\rho])\supseteq\operatorname{dom}(e_{u_{i,m}}[s,\rho])$.

Let $\bar w_i$ be the word in which every occurrence of a_0 in w_i has been replaced by b_0. If $e_{\bar w_i}[\rho]$ is totally defined, then since ρ induces a κ-cofinitary representation there are less than κ many α's such that $e_{\bar w_i}[\rho](\alpha)=\alpha$. For each $\bar w_i$ with $e_{\bar w_i}[\rho]$ totally defined and $1\le m\le j_i$ let $\bar w_{i,m}=u_{i,m}b_0^{k(i,m)}\cdots u_{i,1}b_0^{k(i,1)}u_{i,0}$, and let

$$\begin{aligned}\Omega_i=\sup\{e_v[\rho](\alpha):&e_{\bar w_i}[\rho](\alpha)=\alpha\wedge v=b^{\operatorname{sign}(k(i,m)p}\bar w_{i,m}\wedge\\ &0\le p\le\operatorname{sign}(k(i,m))k(i,m)\wedge 0\le m\le j_i\}.\end{aligned}$$

Then let $\Omega\in\kappa$ be such that $\Omega\ge\max\{\Omega_i:i<\lambda\}$ and whenever $\alpha\ge\Omega$ we have that $\alpha\notin\operatorname{dom}(s_{a_0})$ and $\rho(b_0)(\alpha)\notin\operatorname{ran}(s_{a_0})$. Then for any $\alpha\ge\Omega$ we have that on the one hand, if $e_{\bar w_i}[\rho]$ is not everywhere defined, then $\operatorname{dom}(e_{w_i}[s,\rho])=\operatorname{dom}(e_{w_i}[s\cup\{(a_0,\alpha,\rho(b_0)(\alpha))\},\rho])$, while if $e_{\bar w_i}[\rho]$ is everywhere defined then necessarily $e_{w_i}[s\cup\{(a_0,\alpha,\rho(b_0)(\alpha))\},\rho](\beta)=\beta$ only if $e_{w_i}[s,\rho](\beta)=\beta$. □

Proof of Theorem 2.15. Let G be $\mathbb{Q}^\kappa_{A,\rho}$-generic. Suppose that $\operatorname{im}\hat\rho_G$ is not a κ-maximal cofinitary group. Then for some $c\notin A\cup B$ and $(<\kappa)$-cofinitary permutation σ in $V[G]$ we can extend ρ_G to a κ-cofinitary representation $\rho'_G: A\cup B\cup\{c\}\to S(\kappa)$ by defining $\rho'_G(c)=\sigma$. However the poset is κ^+-c.c. and so there is some subset A_0 of A, which is of size κ such that $\sigma\in V[H]$ where $H=G\cap\mathbb{Q}_{A_0,\rho}$. Pick any $a\in A\setminus A_0$. Then in $V[H]$ we have that for every $\Omega\in\kappa$ the set

$$D_{\sigma,\Omega}=\{(s,F)\in\mathbb{Q}_{A\setminus A_0,\rho_H}:\exists\alpha>\Omega(s_a(\alpha)=\sigma(\alpha))\}$$

is dense in $\mathbb{Q}_{A\setminus A_0,\rho_H}$. Consequently in $V[G]$ we have that $\sigma(\alpha)=(\rho_H)_K(a)(\alpha)$ for κ-many α's, which is a contradiction. □

3 The spectrum of generalized cofinitary groups

Throughout the paper κ denotes an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. The Baire space κ^κ consists of all functions $\eta : \kappa \to \kappa$. The basic open sets are all sets of the form

$$U_\sigma := \{\eta : \eta \text{ extends } \sigma\},$$

$\sigma \in \kappa^{<\kappa}$. The Borel sets for κ^κ are obtained by closing the basic open sets under complements and unions of size κ. Then we can speak of generalized $\Sigma^0_n(\kappa)$ and $\Pi^0_n(\kappa)$ classes, as well as the bold face analogues $\mathbf{\Sigma^0_n}(\kappa)$, $\mathbf{\Pi^0_n}(\kappa)$. By $OD(\kappa^\kappa)$ we denote the class of relations which are ordinal-definable from a function in ${}^\kappa\kappa$. In general by $\Gamma(\kappa)$ we will denote a point-class in the sense of this generalized descriptive set theory (see [7]).

Blass's notion of easily definable cardinal invariants of the continuum (see [1]) easily transfers to the generalized invariants and so we obtain the following definition.

Definition 3.1 Let κ be a regular infinite cardinal. An uncountable cardinal $\lambda > \kappa$ is a $\Gamma(\kappa)$-*characteristic*, if there is a family of λ sets each in $\Gamma(\kappa)$ such that ${}^\kappa\kappa$ is covered by the family, but not by any subfamily of cardinality $< \lambda$. A cardinal $\lambda > \kappa$ is a *uniform* $\Gamma(\kappa)$-*characteristic* if there is a binary relation R on ${}^\kappa\kappa$ such that $R \in \Gamma(\kappa)$ and such that λ is the minimum cardinality of a family $\mathcal{X} \subseteq {}^\kappa\kappa$ such that for all $y \in {}^\kappa\kappa \exists x \in \mathcal{X}(R(x,y))$.

Using this generalized notion of $OD({}^\kappa\kappa)$ characteristic, as well as our poset for adding a κ-maximal cofinitary group of desired cardinality, we obtain the following generalization of [4, Theorem 3.2].

Theorem 3.2 *(GCH) Let κ be an infinite regular cardinal and let C be a closed set of cardinals such that*

1. $\kappa^+ \in C$, $\forall \nu \in C(\nu \geq \kappa^+)$,
2. *if* $|C| \geq \kappa^+$ *then* $[\kappa^+, |C|] \subseteq C$,
3. $\forall \nu \in C(\mathit{cof}(\nu) \leq \kappa \to \nu^+ \in C)$.

Then there is a generic extension in which cofinalities (and cardinalities) have not been changed and such that for every $\nu \in C$ there is a κ-maximal cofinitary group of size ν, while for every $\nu \notin C$ there are no κ-maximal cofinitary groups of size ν.

We will be occupied with the proof of this theorem until the end of the section. For each $\xi \in C$, let $I_\xi = \{(\gamma, \xi) : \gamma < \xi\}$ and let $I = \bigcup_{\xi \in C} I_\xi$. Let $\mathbb{P}$ be the product of all posets $\mathbb{Q}^\kappa_{I_\xi}$ for $\xi \in C$ with supports of size $< \kappa$.

Lemma 3.3 $\mathbb{P}$ *is* $< \kappa$*-closed and* κ^+*-Knaster.*

Proof. It is clear that $\mathbb{P}$ is $< \kappa$-closed. We will show that $\mathbb{P}$ is κ^+-Knaster. Let $\{p_\alpha\}_{\alpha<\kappa^+}$ be given conditions. We have to show that there is a subfamily of size κ^+ which consists of pairwise compatible conditions. Without loss of generality, $\{\mathrm{supt}(p_\alpha)\}_{\alpha<\kappa^+}$ form a Δ-system with some root R_0, which is of size $< \kappa$. Consider the set $\{\prod_{\xi\in R_0} \mathrm{oc}_A(p_\alpha)(\xi)\}_{\alpha<\kappa^+}$. This is a collection of κ^+-many sets, each of size $< \kappa$ and so by the Δ-system lemma they form a Δ-system with root Δ. Note that $\Delta = \prod_{\xi\in R_0} \Delta_\xi$. For every α let $p_\alpha(\xi) = (s^{\alpha,\xi}, F^{\alpha,\xi})$. Then the sets $\{\prod_{\xi\in R_0} s^{\alpha,\xi}\restriction\Delta_\xi \times \kappa \times \kappa\}_{\alpha<\kappa^+}$ must coincide on a set of size κ^+, since $|\prod_{\xi\in R_0}(\Delta_\xi \times \kappa \times \kappa)| = \kappa$. Thus there is some $t = \prod_{\xi\in R_0} t^\xi$ such that for all $\xi \in R_0$ for all $\alpha < \kappa^+$ we have that $s^{\alpha,\xi}\restriction\Delta \times \kappa \times \kappa = t_\xi$. Note that if $b \in \mathrm{oc}_{A^\xi}(s^{\alpha,\xi}) \cap \mathrm{oc}_{A^\xi}(F^{\beta,\xi})$ then $b \in \Delta_\xi$. This implies that $(s^\alpha \cup s^\beta, F^\alpha \cup F^\beta) = \prod_{\xi\in R_0}(s^{\alpha,\xi} \cup s^{\beta,\xi}, F^{\alpha,\xi} \cup F^{\beta,\xi})$ is a common extension of $p_\alpha\restriction R_0$ and $p_\beta\restriction R_0$. Indeed. Fix $\xi \in R_0$ and $w \in F^{\beta,\xi}$. Suppose $e_w[s^{\alpha,\xi} \cup s^{\beta,\xi}, \rho_\xi](n) = n$. If $\mathrm{oc}_{A^\xi}(w) \subseteq \mathrm{oc}_{A^\xi}(s^{\beta,\xi})$ then we are done. If there is $b \in \mathrm{oc}_{A^\xi}(s^{\alpha,\xi}) \cap \mathrm{oc}_{A^\xi}(F^{\beta,\xi})$, then b is an element of Δ_ξ and so $e_w[s^{\alpha,\xi} \cup s^{\beta,\xi}, \rho_\xi] = e_w[s^{\alpha,\xi}\restriction\Delta_\xi \cup s^{\beta,\xi}, F^{\beta,\xi}] = e_w[t \cup s^{\beta,\xi}, F^{\beta,\xi}] = e_w[s^{\beta,\xi}, F^{\beta,\xi}]$. □

Theorem 3.4 *In $V^{\mathbb{P}}$ there is a κ-maximal cofinitary group of size ξ for all $\xi \in C$.*

Proof. For each $\xi \in C$ let $\mathcal{G}_\xi$ be the maximal cofinitary group added by the poset $\mathbb{Q}^\kappa_{I_\xi}$. Let $\xi_0 \in C$ be arbitrary. We will show that $\mathcal{G}_{\xi_0}$ remains maximal in $V^{\mathbb{P}}$. Suppose not. Thus there is a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name for a κ-cofinitary permutation τ such that $p \Vdash_{\mathbb{P}} \langle$"$\mathrm{im}(\hat{\rho}_{\xi_0}) \cup \{\dot{\tau}\}\rangle$ is a κ-cofin. group", where we have identified the cofinitary representation induced by $\mathbb{Q}^\kappa_{\xi_0}$ with its $\mathbb{P}$-name. We can assume that τ has a nice name and furthermore since $\mathbb{P}$ is κ^+-Knaster there are κ-many antichains $\{B_\alpha\}_{\alpha\in\kappa}$ each of size κ, such that for every $b \in B_\alpha$ there is $\beta_b \in \kappa$ with $b \Vdash_{\mathbb{P}} \dot\tau(\alpha) = \check\beta_b$. For each $b \in B_\alpha$ let $K_{\alpha,b}$ denote the support of b. Then the set

$$C' = [(\bigcup_{\alpha\in\kappa, b\in B_\alpha} K_{\alpha,b}) \cup \text{supt}(p)]\backslash\{\xi_0\}$$

is of size at most κ. Let $A_{\xi_0} = [\bigcup_{\alpha\in\kappa,b\in B_\alpha} \text{oc}(b(\xi_0))] \cup \text{oc}(p(\xi_0))$. That is A_{ξ_0} is the collection of all letters from I_{ξ_0} occurring in τ and p. Then A_{ξ_0} is of size at most κ and since C_{ξ_0} is of size $\xi_0 > \kappa$, there is some $a \in I_{\xi_0}\backslash A_{\xi_0}$.

Let $\bar{\mathbb{P}} = \prod_{\xi\in C'} \mathbb{Q}^\kappa_\xi$ with supports of size $< \kappa$ and $\bar{\mathbb{Q}} = \mathbb{Q}^\kappa_{A_{\xi_0}}$. Note that $\mathbb{Q}^\kappa_{A_{\xi_0}}$ is a complete suborder of $Q^\kappa_{I_{\xi_0}}$. Also p is a condition in $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$ and τ is a $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$-name for a κ-cofinitary permutation. Furthermore

$$p \Vdash_{\bar{\mathbb{P}}\times\bar{\mathbb{Q}}} \text{“}\langle \text{im}(\hat\rho_{\xi_0}) \cup \{\dot\tau\}\rangle \text{ is a } \kappa\text{-cofin. group”}.$$

Then as a corollary to Lemma 2.16 we obtain that if G is $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$ generic and $p \in G$, then in $V[G]$ we have that

$$\Vdash_{\mathbb{Q}^\kappa_{I_{\xi_0}\backslash A_{\xi_0},\rho_{A_{\xi_0}}}} \text{“}\forall\Omega < \kappa\exists\beta > \Omega(\rho_{I_{\xi_0}\backslash A_{\xi_0}}(a)(\beta) = \tau(\beta))\text{”}.$$

However

$$(\bar{\mathbb{P}} \times \mathbb{Q}^\kappa_{A_{\xi_0}}) * \mathbb{Q}^\kappa_{I_{\xi_0}\backslash A_{\xi_0},\rho_{A_{\xi_0}}} = \bar{\mathbb{P}} \times (\mathbb{Q}^\kappa_{A_{\xi_0}} * \mathbb{Q}^\kappa_{I_{\xi_0}\backslash A_{\xi_0},\rho_{A_{\xi_0}}}) = \bar{\mathbb{P}} \times \mathbb{Q}^\kappa_{I_{\xi_0}}.$$

Therefore $p \Vdash_{\bar{\mathbb{P}}\times\mathbb{Q}^\kappa_{I_{\xi_0}}}$ “$\forall\Omega < \kappa\exists\beta > \Omega(\rho_{\xi_0}(a)(\beta) = \dot\tau(\beta))$”, which is a contradiction. □

It remains to show that in $V^{\mathbb{P}}$ there are no κ-maximal cofinitary groups of size λ, whenever $\lambda \notin C$. In fact we will show that for every $\lambda \notin C$, λ is not $OD({}^\kappa\kappa)$ definable. Fix any $\lambda > \kappa^+$ such that $\lambda \notin C$. Suppose in $V[G]$ there is a λ-sequence of $OD({}^\kappa\kappa)$ sets X_α which cover ${}^\kappa\kappa$. Then we can fix sequences $\{u_\alpha\}_{\alpha\in\lambda}$ and $\{\Theta_\alpha\}_{\alpha\in\lambda}$ of functions in ${}^\kappa\kappa$ and ordinals respectively, such that X_α is the Θ_αth set ordinal definable from u_α in some standard well-order of $OD(u_\alpha)$.

Let μ be the largest element of C below λ. Then $\mu \geq \kappa^+$ and furthermore $cof(\mu) \geq \kappa^+$. By GCH in the ground model V we obtain that $\mu^\kappa = \mu$. It is sufficient to show that there is an index set $M \subseteq \lambda$ of size μ such that the family $\{X_\alpha\}_{\alpha\in M}$ covers ${}^\kappa\kappa$. The set M will be obtained as the union of a recursively definable sequence $\langle M_\gamma\rangle_{\gamma\in\kappa^+}$, where $|M_\gamma| \leq \mu$ for all γ. Whenever γ is a limit, M_γ is defined as the union of M_δ for $\delta < \gamma$ and M_0 is the empty set. As in the ω-case, the non-trivial part of the construction is the successor step.

For each $\alpha \in \lambda$ choose a subset J_α of $I = \bigcup_{\xi\in C} I_\xi$ of size κ such that for every p which is involved either in $\dot u_\alpha$ or in $\dot\Theta_\alpha$ and each ξ in the support of p we have that $\text{oc}(p(\xi)) \subseteq J_\alpha$. Let

$$S = \bigcup\{I_\gamma : \gamma \in \mu \cap C\} \cup \bigcup\{J_\alpha : \alpha \in \lambda\}.$$

Then $|S| = \lambda$.

Now suppose $K \subseteq S$ is of cardinality μ and $\bigcup_{\gamma\in\mu\cap C} I_\gamma \subseteq K$. Following [1], we will call a subset J of I such that $|J| = \kappa$ *a K-support* for the name $\dot x$ of a function in ${}^\kappa\kappa$ if for every condition p involved in $\dot x$ and every ξ in the support of p we have that $\text{oc}(p(\xi)) \subseteq J$ and if $J \cap I_\gamma\backslash K$ is nonempty then it is of size κ. Since every $\gamma \in C\backslash(\mu \cup \{\mu\})$ is strictly greater than λ, we have $|I_\gamma\backslash S| = |I_\gamma\backslash K| = \gamma$. Thus whenever we are given a K as above and a name for a function in ${}^\kappa\kappa$, we can assume that it has a K-support.

Let $\mathcal{G}$ be the group of those permutations of I that map each I_γ into itself and that fixes all members of K. Then $\mathcal{G}$ acts as a group of automorphisms on the notion of forcing $\mathbb{P}$ by sending each p to a condition $g(p)$ where $g(p)$ is defined as follows. Fix $p \in \mathbb{P}$ and $\xi \in \text{supt}(p)$. Let $p(\xi) = (s^\xi, F^\xi)$ where $s^\xi \in [I_\xi \times \kappa \times \kappa]^{<\kappa}$, $F^\xi \in [W_{I_\xi}]^{<\kappa}$. Then let $\text{supt}(g(p)) = \text{supt}(p)$. For $\xi \in \text{supt}(p)$, let $g(p(\xi)) = (g(s^\xi), g(F^\xi))$ where $\text{oc}(g(s^\xi)) = g(\text{oc}(s^\xi))$ and for every $(\alpha, \xi) \in \text{oc}(g(s^\xi)) = g(\text{oc}(s^\xi))$ if $(\alpha_0, \xi) \mapsto (\alpha, \xi)$ then $[g(s^\xi)]_{(\alpha,\xi]} = s^\xi_{(\alpha_0,\xi)}$. Furthermore for a word $w \in F^\xi$ define $g(w)$ to be the word obtained by substituting every appearance of a letter $a = (\alpha, \xi)$ in w with $g(\alpha, \xi)$. Then let $g(F^\xi)$ be the set of all $g(w)$ for $w \in F^\xi$. With this the automorphism action of $\mathcal{G}$ on $\mathbb{P}$ is defined. Note that each automorphism g preserves not only maximal antichains, but also the forcing relation.

In particular, if J is a support of a name $\dot{x}$, then $g(J)$ is a support of the name $g(\dot{x})$. If in addition g fixes all members of J, then it also fixes the name $\dot{x}$.

Just as in the ω-case (see [1]), if J is a support then its $\mathcal{G}$-orbit is determined by $J \cap K$ and $\bar{J} = \{\gamma \in C : J \cap I_\gamma - K \neq \emptyset\}$. That is, if J' is another support with $J' \cap K = J \cap K$ and $\bar{J}' = \bar{J}$, then there is $g \in \mathcal{G}$ with $g(J) = J'$. Since $J \cap K$ is of size $\leq \kappa$ and $|K| = \mu = \mu^\kappa$, there are only μ possibilities for $J \cap K$. Now consider $[C]^{\leq\kappa}$. If $[\kappa^+, |C|] \neq \emptyset$, then $[\kappa^+, |C|] \subseteq C$. Thus in this case $|C| \leq \mu$. If $[\kappa^+, |C|] = \emptyset$, i.e. $|C| \leq \kappa$, then since $\mu \geq \kappa^+$ we have again $|C| \leq \mu$. Therefore we have no more than $\mu^\kappa = \mu$ many possibilities for $\bar{J} \in [C]^{\leq\kappa}$ and so there are only μ many orbits of supports. Now for each $\mathcal{G}$-orbit of supports, fix a member J such that $J \cap S = J \cap K$. Those representatives will be referred to as standard supports. Note that for each fixed support J there are only $\kappa^\kappa = \kappa^+$ (where we used GCH in V) many names. Since $\mu \geq \kappa^+$, we obtain that there are only μ-many names that have standard supports.

Now for each name $\dot{x}$ with a standard support, fix a set $A = A(\dot{x}) \in [\lambda]^{\leq\kappa} \cap V$ such that $\mathbb{P}$ forces "$(\exists\alpha \in \check{A})\dot{x} \in \dot{X}_\alpha$". Let

$$B = \bigcup\{A(\dot{x}) : \dot{x} \text{ has a standard support}\}.$$

Then $|B| \leq \mu$.

We will proceed with the successor step in the inductive definition of $\langle M_\sigma\rangle_{\sigma<\kappa^+}$. Let

$$K_\sigma = \bigcup_{\alpha\in M_\sigma} J_\alpha \cup \bigcup_{\gamma\leq\mu\cap C} I_\gamma.$$

Then $|K_\sigma| = \mu$. Let $M_{\sigma+1}$ be obtained from K_σ in the same way that B was obtained from K above. Then $|M_{\sigma+1}| \leq \mu$. Note also that the K_σ's do form a monotone increasing sequence. Define $M = \bigcup_{\sigma\in\kappa^+} M_\sigma$ and $K = \bigcup_{\sigma\in\kappa^+} K_\sigma$. We will show that for every $\mathbb{P}$-name $\dot{x}$ for a function in ${}^\kappa\kappa$, $\mathbb{P}$ forces that "$(\exists\alpha \in M)\dot{x} \in \dot{X}_\alpha$".

Thus fix a $\mathbb{P}$-name $\dot{x}$ for a function in ${}^\kappa\kappa$ and let J be a subset of I of size κ such that for every condition p involved in $\dot{x}$ and every ξ in the support of p the set $\mathrm{oc}(p(\xi))$ is contained in J. Fix $\sigma < \kappa^+$ such that $J \cap K \subseteq K_\sigma$. For each $\gamma \in C$ such that $J \cap I_\gamma - K_\sigma \neq \emptyset$, we have that $\gamma > \lambda (> \mu)$. Then in particular $I_\gamma - K$ is of size λ. Thus enlarging J is necessary we can assume that it is a K_σ-support and $J \cap K \subseteq K_\sigma$. Consider the group of all permutations of I which fix K_σ and map each I_γ to itself. By the above discussion there is a permutation $g \in \mathcal{G}$ such that $g(J)$ is a K_σ-standard support. Then neither J nor $g(J)$ meets $K_{\sigma+1} - K_\sigma$. For J this follows, since $J \cap K \subseteq K_\sigma$ and for $g(J)$ since $g(J) \cap (S - K_\sigma) = \emptyset$, and clearly $K_{\sigma+1} \subseteq S$. Then there is a permutation h which agrees with g on J and with the identity map on $K_{\sigma+1} - K_\sigma$. In particular $h(J) = g(J)$ is standard and h leaves $K_{\sigma+1}$ pointwise fixed.

Since $h(\dot{x})$ has standard support $h(J)$, it is one of the μ names for which we chose a set $A = A(h(\dot{x}))$ to include in $M_{\sigma+1}$. Thus $\Vdash_{\mathbb{P}}$ "$(\exists\alpha \in \check{A})h(\dot{x}) \in \dot{X}_\alpha$", which implies that

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{A}[h(\dot{x}) \text{ is in the } \dot{\Theta}_\alpha \text{th set ordinal-definable from } \dot{u}_\alpha]\text{”}.$$

However $A \subseteq M_{\sigma+1}$ and so for any $\alpha \in A$ we have that $J_\alpha \subseteq K_{\sigma+1}$ and so h fixes J_α pointwise. But this implies that h fixes the names $\dot{\Theta}_\alpha$ and $\dot{u}_\alpha$. Therefore

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{A}[h(\dot{x}) \text{ is in the } h(\dot{\Theta}_\alpha)\text{th set ordinal-definable from } h(\dot{u}_\alpha)]\text{”}.$$

Since, h is an automorphism of $\mathbb{P}$ which preserves the forcing relation, we obtain that

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{A}[\dot{x} \text{ is in the } \dot{\Theta}_\alpha \text{th set ordinal-definable from } \dot{u}_\alpha]\text{”}.$$

Using the fact that $M_{\sigma+1} \subseteq M$ we obtain that

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{M}(\dot{x} \in \dot{X}_\alpha)\text{”},$$

which completes the proof that λ is not $OD({}^\kappa\kappa)$-definable.

4 κ-Cohen indestructible κ-maximal cofinitary group

Following standard notation, $\text{Fn}_{<\kappa}(\kappa,\kappa)$ denotes the κ-Cohen poset, e.g. the poset of all partial functions from κ to κ of cardinality $<\kappa$ with extension relation superset.

Theorem 4.1 *(GCH) There is a κ-Cohen indestructible κ-maximal cofinitary group.*

Proof. Let $\{\langle p_\xi, \dot\tau_\xi\rangle : \kappa \le \xi < \kappa^+, \xi \in \text{Succ}(\kappa^+)\}$ enumerate all pairs $\langle p, \tau\rangle$ where $p \in \text{Fn}_{<\kappa}(\kappa,\kappa)$ and τ is a $\text{Fn}_{<\kappa}(\kappa,\kappa)$-name for a cofinitary permutation. Recursively we will construct a family $\{\rho_\xi\}_{\kappa\le\xi<\kappa^+}$ of cofinitary representations such that

1. for all ξ, $\rho_\xi : \xi \to S(\kappa)$,
2. for all $\eta < \xi$ $\rho_\eta = \rho_\xi \restriction \eta$, and
3. $\bigcup_{\kappa\le\xi<\kappa^+} \rho_\xi : \kappa^+ \to S(\kappa)$ induces a cofinitary representation $\hat\rho$ such that $\text{im}(\hat\rho)$ is a κ-maximal cofinitary group, which is $\text{Fn}_{<\kappa}(\kappa,\kappa)$-indestructible.

Let ρ_κ be a cofinitary representation of κ given by $\mathbb{Q}^\kappa_\kappa$ (here the index set is simply the cardinal κ). Suppose for all $\xi : \kappa \le \xi < \eta$, ρ_ξ has been defined.

Case 1. Suppose η is a successor, i.e. $\eta = \xi + 1$. Consider the pair $\langle p_\xi, \dot\tau_\xi\rangle$. If

$$p_\xi \Vdash_{\text{Fn}_{<\kappa}(\kappa,\kappa)} \langle \text{im}(\hat\rho_\xi) \cup \{\dot\tau_\xi\}\rangle \text{ is a } \kappa\text{-cofin. group}$$

proceed as follows.

Let $q \le p_\xi$. Then $q \Vdash_{\text{Fn}_{<\kappa}(\kappa,\kappa)} \langle \text{im}(\hat\rho_\xi) \cup \{\dot\tau_\xi\}\rangle$ is a cofin. group, and so if G is $\text{Fn}_{<\kappa}(\kappa,\kappa)$-generic and $q \in G$, then in $V[G]$ for every $\Omega \in \kappa$ the set

$$D_{\dot\tau_\xi[G],\Omega} = \{(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi} : \exists\alpha \le \Omega(s(\alpha) = \tau_\xi[G](\alpha))\}$$

is dense. Thus for every $\Omega \in \kappa$ and every $(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi}$ there are $q' \le_{\text{Fn}_{<\kappa}(\kappa,\kappa)} q$, $\alpha > \Omega$ and $(s',F') \le (s,F)$ such that $q' \Vdash_{\text{Fn}_{<\kappa}(\kappa,\kappa)} \check{s}'(\alpha) = \dot\tau_\xi(\alpha)$. Therefore the set

$$D^q_\Omega = \{(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi} : \exists\alpha > \Omega \exists q' \le q(q\prime \Vdash s(\alpha) = \dot\tau_\xi(\alpha))\}$$

is dense in $\mathbb{Q}_{\{\xi\},\rho_\xi}$.

Now let $G \subseteq \mathbb{Q}^\kappa_{\{\xi\},\rho_\xi}$ be a filter meeting the dense sets $D^{\text{domain}}_\alpha = \{(s,F) : \alpha \in \text{dom}(s)\}$, $D^{\text{range}}_\alpha = \{(s,F) : \alpha \in \text{range}(s)\}$, $D_w = \{(s,F) : w \in F\}$, and D^q_Ω where $\alpha, \Omega \in \kappa$, $q \le_{\text{Fn}_{<\kappa}(\kappa,\kappa)} p_\xi$ and $w \in \widehat{W}_{\{\xi\}\cup\xi}$. Note that since these are only κ many dense sets and the forcing notion $\mathbb{Q}^\kappa_{\{\xi\},\rho_\xi}$ is $<\kappa$-closed such a filter G exists. Then we have that the mapping

Claim 4.2 $\rho_{\xi+1} : \xi+1 \to S(\kappa)$ *where* $\rho_{\xi+1}\restriction\xi = \rho_\xi$, $\rho_{\xi+1}(\xi) = \bigcup\{s : \exists F(s,F) \in G\}$ *induces a κ-cofinitary representation extending ρ_ξ.*

Furthermore,

Claim 4.3 $p_\xi \Vdash_{Fn_{<\kappa}(\kappa,\kappa)}$ "$\forall\Omega \in \kappa\exists\alpha > \Omega(\tau_\xi(\alpha) = \rho_{\xi+1}(\xi)(\alpha))$".

Proof. Suppose not. Then there are $q \le p_\xi$ and $\Omega \in \kappa$ such that

$$q \Vdash_{\text{Fn}_{<\kappa}(\kappa,\kappa)} \text{“}\{\alpha : \dot\tau_\xi(\alpha) = \rho_{\xi+1}(\xi)(\alpha)\} \subseteq \check\Omega\text{”}.$$

Then let $(s,F) \in G \cap D^\Omega_q$. Then there are $\alpha > \Omega$ and $q' \le_{\text{Fn}_{<\kappa}(\kappa,\kappa)} q$ such that $q' \Vdash_{\text{Fn}_{<\kappa}(\kappa,\kappa)} \dot\tau_\xi(\alpha) = s(\alpha)$. It remains to observe that $\rho_{\xi+1}(\xi)(\alpha) = s(\alpha)$ and so we have reached a contradiction. □

Case 2. Suppose ξ is a limit. Then define $\rho_\xi := \bigcup_{\eta<\xi} \rho_\eta$.

Claim 4.4 $\rho_\xi : \xi \to S(\kappa)$ *induces a cofinitary representation.*

Proof. Let $w \in \mathbb{F}_\xi$. Then there is a good word $w' \in \widehat{W_\xi}$ such that for some $u \in W_\xi$ we have $w = u^{-1}w'u$. However in each of those words there are only finitely many letters involved and so there is $\eta < \kappa^+$ such that w, u, w' are in fact elements in W_η. Then $e_{w'}[\rho_\xi] = e_{w'}[\rho_\eta]$ and since by Inductive Hypothesis ρ_η induces a κ-cofinitary representation we have that the set of all fixed points of $e_{w'}[\rho_\xi]$ is of cardinality smaller than κ. However $|\text{fix}(e_w[\rho_\xi])| = |\text{fix}(e_{w'}[\rho_\xi])|$, which completes our argument. □

With this the inductive construction of the sequence $\langle \rho_\xi \rangle_{\kappa \leq \xi < \kappa^+}$ is complete. Let $\rho := \bigcup_{\kappa \leq \xi < \kappa^+} \rho_\xi$.

Claim 4.5 *im($\hat{\rho}$) is a κ-maximal cofinitary group which is κ-Cohen indestructible.*

Proof. Let G be $\text{Fn}_{<\kappa}(\kappa, \kappa)$-generic filter. Suppose $V[G] \vDash$ (im($\hat{\rho}$) is not a κ maximal cof. group). Then $V[G] \vDash \exists \tau(\langle \text{im}(\hat{\rho}) \cup \{\tau\}\rangle$ is a κ cofin. group). Therefore there is $p \in G$ and a $\text{Fn}_{<\kappa}(\kappa, \kappa)$-name for a cofinitary permutation $\dot{\tau}$ such that

$$p \Vdash_{\text{Fn}_{<\kappa}(\kappa,\kappa)} (\langle \text{im}(\hat{\rho}) \cup \{\dot{\tau}\}\rangle \text{ is a } \kappa \text{ cofin. group}).$$

Note that there is $\xi : \kappa \leq \xi < \kappa^+$, successor such that $\langle p, \tau \rangle = \langle p_\xi, \tau_\xi \rangle$. Then by our construction

$$p \Vdash \forall \Omega \exists \alpha > \Omega(\rho(\xi + 1)(\alpha) = \dot{\tau}(\alpha)),$$

which is a contradiction. □

This completes the proof of the theorem. □

Theorem 4.6 *(GCH) Let $\kappa^{++} \leq \lambda$ be regular uncountable cardinals and let $\mathbb{P} = Fn_{<\kappa}(\lambda \times \kappa, \kappa)$. Then $V^{\mathbb{P}} \vDash \mathfrak{a}_g(\kappa) < \mathfrak{d}(\kappa) = \mathfrak{c}(\kappa)$.*

Not that the case $\kappa = \omega$ of the above theorem is due to Yi Zhang.

5 Concluding Remarks

The usual isomorphism of names argument, which shows that in the Cohen extension each maximal cofinitary group is either of size $\aleph_1$ (assuming CH in the ground model) or of size continuum, easily lifts to the case of κ-Cohen forcing. Taking into consideration the existence of κ-Cohen indestructible κ-maximal cofinitary groups, we obtain the following:

Theorem 5.1 *(GCH) Let $\kappa^{++} \leq \lambda$ be regular uncountable cardinals and let $\mathbb{P} = Fn_{<\kappa}(\lambda \times \kappa, \kappa)$. Then in $V^{\mathbb{P}}$ every κ-maximal cofinitary group is either of size κ^+ or of size $2^\kappa = \lambda$.*

The techniques developed in the previous two sections can also be applied to some relatives of the $\mathfrak{a}_g$-number:

• Let $\mathfrak{a}_p(\kappa)$ denote the minimal size of a maximal family of κ-almost disjoint permutations on κ. Let A be a generating set and let $\bar{\mathbb{Q}}^\kappa_A$ denote the suborder of the poset $\mathbb{Q}^\kappa_A$ (defined in section 2), which consists of all pairs (s, F) where every word in F is of the form ab^{-1} for $a, b \in A$. Then $\bar{\mathbb{Q}}_A$ is $(< \kappa)$-closed and κ^+-Knaster and in case $|A| \geq \kappa^+$, it adds a maximal family of κ-a.d. permutations on κ.

• Let $\mathfrak{a}_e(\kappa)$ denote the minimal size of a maximal family of κ-a.d. functions on ${}^\kappa\kappa$. For A a generating set, let $\tilde{\mathbb{Q}}^\kappa_A$ be the poset of all pairs (s, F) where $s \subseteq A \times \kappa \times \kappa$ is of size $< \kappa$, s_a is a partial function for every a and $F \in [\widehat{W_A}]^{<\kappa}$ where each word in F is of size ab^{-1} for $a \neq b$ in the index set A. The extension relation of $\tilde{\mathbb{Q}}^\kappa_A$ is defined in the same way as the extension relation of $\mathbb{Q}^\kappa_A$. Then $\tilde{\mathbb{Q}}_A$ is $(< \kappa)$-closed and κ^+-Knaster and in case $|A| \geq \kappa^+$, it adds a maximal family of κ-a.d. functions on κ.

• Let $\mathfrak{a}(\kappa)$ denote the minimal size of a maximal κ-almost disjoint family in $[\kappa]^\kappa$. Let $\mathbb{D}^\kappa_A$ denote the poset of all pairs $(s, F) \in [A \times \kappa \times 2]^{<\kappa} \times [A]^{<\kappa}$ where for all $a \in A$, $s^p_a = \{(\alpha, \beta) : (a, \alpha, \beta) \in s\}$ is a $(< \kappa)$-partial function. The condition q $\mathbb{D}^\kappa_A$-extends the condition p, if $s^q \supset s^p$, $F^q \supset F^p$ and for all $a, b \in F^p(s^q_a \cap s^q_b \subseteq s^p_a \cap s^p_b)$. If $|A| \geq \kappa^+$ then $\mathbb{D}^\kappa_A$ adds a κ-maximal almost disjoint family of size κ.

As a straightforward modification of the argument presented in section 3, we obtain the following. Let $\mathcal{C}$ denote either of the following sets: set of all κ-maximal cofinitary groups, the set of κ-maximal almost disjoint families, the set of κ-almost disjoint permutations on ${}^\kappa\kappa$, the set on κ-almost disjoint functions on ${}^\kappa\kappa$. Then:

Theorem 5.2 *(GCH) Let κ be a regular uncountable cardinal and let C be a closed set of cardinals such that*

1. $\kappa^+ \in C$, $\forall \nu \in C(\nu \geq \kappa^+)$,
2. *if* $|C| \geq \kappa^+$ *then* $[\kappa^+, |C|] \subseteq C$ *and*
3. $\forall \nu \in C(cof(\nu) \leq \kappa \to \nu^+ \in C)$.

Then there is a generic extension in which cofinalities have not been changed and such that $C = \{|\mathcal{G}| : \mathcal{G} \in \mathcal{C}\}$.

References

[1] A. Blass *Simple cardinal characteristics of the continuum*, in "Set theory of the reals" (H. Judah, Ed.), Israel Mathematical Conference Proceedings, Vol. 6, pp. 63 - 90, Am. Math. Soc., Providence, 1993.
[2] J. Brendle *Mad families and iteration theory*, Contemporary mathematics **302**, 2002, p. 1-31.
[3] J. Brendle *The almost-disjointness number may have countable cofinality*, Trans. of the Amer. Math. Soc. 355, no. 7 (2003), 2633 - 2649.
[4] J. Brendle, O. Spinas, Y. Zhang *Uniformity of the Meager Ideal and Maximal Cofinitary Groups* Journal of Algebra 232 (2000) no. 1, 209 - 225.
[5] S. Hechler *Short complete nested sequences in* $\beta\mathbb{N}\backslash\mathbb{N}$ *and small maximal almost-disjoint families* General Topology and its Applications 2 (1972) 139 - 149.
[6] V. Fischer, A. Törnquist *Template Iterations and Maximal Cofinitary groups* Fundamenta Mathematicae, to appear.
[7] S. D. Friedman, *Invariant Descriptive Set Theory*, lecture notes, 2013.
[8] K. Kunen *Set theory. An introduction to independence proofs* Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam 1983.
[9] S. Shelah, O. Spinas *MAD spectra*, Journal of Symbolic Logic, to appear.
[10] Y. Zhang *Maximal Cofinitary Groups* Arch. Math. Logic. 39 (2000), no. 1, 41 - 42.
[11] Y. Zhang *Permutation groups and covering properties* J. London Math. Soc. (2) 63(2001), no. 1, 1-15.