THE CONSISTENCY OF $t = \omega_1 < h = \omega_2$

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1. Preliminaries

In this section we systemize some well known definitions which will be used throughout the talk.

**Definition 1.** Suppose $E$ and $F$ are maximal almost disjoint families. We say that $E$ is a refinement of $F$ if and only if for every $x \in E$ there is $y \in F$ such that $x \subseteq^* y$.

In the following consider the partial order $([\omega]^{\omega}, \subseteq^*)$ consisting of infinite subsets of $\omega$ with extension relation almost-inclusion. That is if $A, B \in [\omega]^{\omega}$ then $A \leq B$ if and only if $A \subseteq^* B$. Note that in this setting $t$ is the greatest cardinal $\kappa$ such that $[\omega]^{\omega}$ is $\kappa$-closed.

**Definition 2.** The *distributivity cardinal* $h$ is defined as the least cardinal $\kappa$ such that forcing with $[\omega]^{\omega}$ adds a new real $h: \kappa \to V$ (where $V$ denotes the ground model as usual). Equivalently, $h$ is the least cardinal such that any collection of less than $\kappa$-many maximal almost disjoint families have a common refinement.

The above remark implies $t \leq h$ and so we have the following inequalities

$$p \leq t \leq h \leq s.$$ 

**Remark 1.** Certainly every tower has the strong finite intersection property and has no pseudo-intersection, which establishes the first inequality. To obtain that $h \leq s$ consider a splitting family $A = \{a_\alpha : \alpha \in s\}$ and let $G$ be a $[\omega]^{\omega}$-generic filter. Then in $V[G]$ define $f: s \to 2$ as follows:

$$f(\alpha) = 1 \text{ iff } a_\alpha \in G.$$ 

Consider any $a \in [\omega]^{\omega}$ as a condition in the associated partial order. Since the family $A$ is splitting, there is an $\alpha \in s$ such that both

$$a \cap a_\alpha \text{ and } a \cap a_\alpha^c$$

are infinite. But then $a$ does not decide $\dot{f}(\alpha)$ and so $f$ is a new function $s \to V$. Here $\dot{f}$ is an $[\omega]^{\omega}$-name for the function $f$.

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Recall also that the following:

Definition 3. The Mathias forcing notion $\mathbb{P}$ consists of all pairs $(s, A) \in [\omega]^{<\omega} \times [\omega]^\omega$ where $(t, B) \leq (s, A)$ (that is $(t, B)$ is stronger than $(s, A)$) if and only if $t$ end-extends $s$, $B \subseteq A$ and $t - s \subseteq B$.

Lemma 1. There is a two stage iteration $Q * \dot{R}$ of a countably closed forcing notion $Q$ and a $\sigma$-centered forcing notion $\dot{R}$ (that is $1 \Vdash_Q "\dot{R} \text{ is } \sigma \text{-centered}"$) such that the Mathias partial order $\mathbb{P}$ is densely embedded into $Q * \dot{R}$.

Proof. Let $Q = ([\omega]^\omega, \subseteq)$ and let $G$ be $Q$-generic filter. Then in $V[G]$ define $R$ to be the partial order consisting of all pairs $(s, A)$ in the Mathias partial order $\mathbb{P}$ for which the pure part $A$ belongs to $G$ with the extension relation inherited from $\mathbb{P}$ and let $\dot{R}$ be a $Q$-name for $R$. Then $Q$ is countably closed, $\Vdash_Q "\dot{R} \text{ is } \sigma \text{-centered}"$ and the mapping $(s, A) \mapsto (A, (s, A))$

is a dense embedding of $\mathbb{P}$ into $Q * \dot{R}$. □

We will refer to the above two-stage iteration as factored Mathias forcing.

Theorem 1 (CH). Let $\mathbb{P}_{\omega_2}$ be $\omega_2$-stage iteration of Mathias forcing, or factored Mathias forcing. That is for every $\alpha < \omega_2$, we have that $1_\alpha \Vdash "Q_\alpha \text{ is Mathias forcing}"$ or respectively for every $\alpha < \omega_2$, $\alpha$-even $1_\alpha \Vdash "Q_\alpha * Q_{\alpha+1} \text{ is factored Mathias forcing}"$. Suppose that $\mathbb{P}_{\omega_2}$ satisfies the following conditions:

1. $\mathbb{P}_{\omega_2}$ is $\aleph_2$-c.c.
2. For every $p \in \mathbb{P}_{\omega_2}$ the support of $p$ is bounded.
3. For every $\alpha < \omega_2$ $V^{\mathbb{P}_\alpha} \models CH$.
4. $\mathbb{P}_{\omega_2}$ preserves $\omega_1$.

Then $V^{\mathbb{P}_{\omega_2}} \models \aleph = \omega_2$.

Proof. Let $\langle E_\gamma : \gamma \in \omega_1 \rangle$ be a collection of $\omega_1 \mathbb{P}_{\omega_2}$-names for maximal almost disjoint families and let $p \in \mathbb{P}_{\omega_2}$. We can assume that for every $\gamma < \omega_1$

$p \Vdash |E_\gamma| = \aleph_2$

and fix sequences of $\mathbb{P}_{\omega_2}$-names for infinite subsets of $\omega$ such that for every $\gamma < \omega_1$

$p \Vdash E_\gamma = \langle x_{\xi,\gamma} : \xi \in \omega_2 \rangle$. 
We will show that there is a $\mathbb{P}_{\omega_2}$-name $\dot{x}$ for an infinite subset of $\omega$ such that for all $\gamma < \omega_1$

$$p \Vdash \dot{x} \text{ is almost contained in an element of } E_\gamma.$$  

**Claim.** For every sentence $\phi$ in the forcing language of $\mathbb{P}_{\omega_2}$ there is an $\alpha < \omega_2$ such that if $q \in \mathbb{P}_{\omega_2}$ and $q$ decides $\phi$ then $q \upharpoonright \alpha$ decides $\phi$.

**Proof.** Fix a maximal antichain of conditions deciding $\phi$. Then since $\mathbb{P}_{\omega_2}$ is $\aleph_2$-c.c. $|A| \leq \aleph_1$. Furthermore the support of every condition is bounded which implies that there is an $\alpha < \omega_2$ such that

$$\bigcup \{ \text{support}(a) : a \in A \} \subseteq \alpha.$$  

Then certainly, for every $q$ which decides $\phi$, $q \upharpoonright \alpha$ decides $\phi$. \hfill \Box

**Claim.** There is a function $f : \omega_2 \to \omega_2$ such that for every $\beta < \omega_2$, every $\gamma < \omega_1$, and every $\mathbb{P}_\beta$-name such that $p \Vdash \dot{y} \in [\omega]^\omega$, we have

$$p \Vdash (\exists \xi < g(\beta) | \dot{y} \cap \dot{x}_{\xi\gamma} | = \aleph_0).$$  

**Proof.** Let $\beta < \omega_2$. Fix any $\gamma < \omega_1$. Then $p \Vdash "E_\gamma \text{ is mad}"$. Let $\dot{y}$ be a $\mathbb{P}_\beta$-term such that $p \Vdash \dot{y} \in [\omega]^\omega$. Then

$$p \Vdash \exists \xi < \omega_2 (| \dot{y} \cap \dot{x}_{\xi\gamma} | = \aleph_0).$$  

Fix a maximal antichain $A_\gamma(\dot{y})$ below $p$ such that for every $q \in A_\gamma(\dot{y})$ there is $\xi_q \in \omega_2$ such that

$$q \Vdash | \dot{y} \cap \dot{x}_{\xi_q} | = \aleph_0.$$
Then $|A_\gamma(\dot{y})| \leq \aleph_1$ and so there is $\alpha_\gamma(\dot{y}) < \omega_2$ such that
\[
\bigcup \{ \text{support}(a) : a \in A_\gamma(\dot{y}) \} \subseteq \alpha_\gamma(\dot{y}).
\]

Then $\alpha(\dot{y}) = \sup_{\gamma \in \omega_1} \alpha_\gamma(\dot{y})$ is also smaller than $\omega_2$. However $V^{P_\beta} \models CH$ and so we can define
\[
g(\beta) = \sup \{ \alpha_\gamma(\dot{y}) : \dot{y} \text{ is } P_\beta \text{-name s.t. } p \vdash_{\beta} \dot{y} \in [\omega]^\omega \}.
\]

Let $\alpha < \omega_2$ be such that $\cof(\alpha) = \omega_1$ and $\forall \beta < \alpha$, $f(\beta) < \alpha$ and $g(\beta) < \alpha$. Then the definition of $f$ implies that for every $\gamma < \omega_1$
\[
p \vdash (x_\xi : \xi < \alpha) \in V[G_\alpha]
\]
and furthermore the definition of $g$ implies that
\[
V[G_\alpha] \models \forall \gamma < \omega_1 (\langle x_\xi : \xi < \alpha \rangle \text{ is mad})
\]
since every real in $V[G_\alpha]$ appears in some $V[G_\beta]$ for $\beta < \alpha$. Really, suppose $\dot{x}$ is a $P_\alpha$-name for an infinite subset of $\omega$, which does not appear in $V[G_\beta]$ for any $\beta < \alpha$. Then in $V[G_\alpha]$ we can define a cofinal function $f : \omega \to \alpha$ as follows:
\[
f(n) = \gamma \text{ iff } \exists q \in G \Downarrow \gamma(q \text{ decides } "\dot{n} \in \dot{x}"),
\]
which is a contradiction since $V[G_\alpha]$ preserves $\omega_1$.

However, the Mathias generic real is almost contained in a member of every maximal almost disjoint family from the ground model and so if $g_\alpha$ is the $\alpha$-th Mathias real, then
\[
V[G] \models \forall \gamma < \omega_1 \exists x_\xi < \alpha (\text{range}(g_\alpha) \subseteq x_\xi).
\]

The following theorem is due to Baumgartner.

**Theorem 2.** Let $P$ be the Mathias partial order and let $\langle x_\alpha : \alpha < \kappa \rangle$ be a tower in $[\omega]^\omega$. Then $\langle x_\alpha : \alpha < \kappa \rangle$ remains a tower in $V^P$.

**Proof.** Suppose not. Then there is a $P$-generic extension $V[G]$ such that
\[
V[G] \models \exists x \in [\omega]^\omega \forall \alpha < \kappa (x \subseteq^* x_\alpha).
\]

Then there is a $P$-name for an infinite subset of $\omega$ and a condition $p = (s_0, A_0) \in G$ such that for every $\alpha < \kappa$
\[
(s_0, A_0) \vdash \dot{x} \subseteq^* x_\alpha.
\]

We can assume that the condition $(s_0, A_0)$ is pre-processed for $\dot{x}$. That is for every $k \in \omega$ and $t \leq (s_0, A_0)$ (that is $t$ end-extends $s_0$ and $t - s_0 \subseteq A_0$) if there is $C \subseteq A_0$ such that $(t, C) \vdash \dot{k} \in \dot{x}$ then there is
some \( m \in \omega \) such that \( (t, A_0 - m) \models k \in \check{x} \). Then we can define for every \( s \leq (s_0, A_0) \) the set
\[
F_s = \{ k : \exists C \subseteq A_0((s, C) \models k \in \check{x}) \} = \{ k : (\exists m)(s, A_0 - m) \models k \in \check{x} \}.
\]

Claim. There is \( (s, A) \leq (s_0, A_0) \) such that for every \( t \leq (s, A) \) the set \( F_s \) is finite.

Proof. Suppose the claim is not true. That is for every \( (s, A) \leq (s_0, A_0) \) there is \( t \leq (s, A) \) such that \( F_t \) is infinite. However there are only countably many \( F_t \)'s and so there is some \( \alpha < \kappa \) such that for every \( t \leq (s, A) \) such that \( F_t \) is infinite, \( F_t \not\subseteq \check{x}_\alpha \). Otherwise, for every \( \beta < \kappa \) there is an infinite \( F_t \) such that \( F_t \subseteq \check{x}_\beta \). However if \( F \) is an infinite subset of \( \omega \) such that \( F \subseteq^* F_t \) for every infinite \( F_t \), then \( F \) is a pseudo-intersection of \( \langle x_\alpha : \alpha < \kappa \rangle \) which belongs to the ground model which is a contradiction to \( \langle x_\alpha : \alpha < \kappa \rangle \) being a tower. Since \( (s_0, A_0) \models k \in \check{x} \subseteq^* x_\alpha \), there is an extension \( (s, A) \in G \) and \( j \in \omega \) such that
\[
(s, A) \models \check{x} - j \subseteq x_\alpha.
\]
By assumption there is \( t \leq (s, A) \) such that \( F_t \) is infinite. But then there is \( k \in F_t - x_\alpha - j \) and so by definition of \( F_t \) there is some \( m \in \omega \) such that \( (t, A_0 - m) \models k \in \check{x} \). However \( (t, A - m) \) extends both \( (s, A) \) and \( (t, A_0 - m) \) and so
\[
(t, A - m) \models (k \in \check{x} - j) \land (\check{x} - j \subseteq x_\alpha)
\]
which is a contradiction since \( k \not\in x_\alpha \). \( \square \)

Furthermore we have the following property.

Claim. Suppose \( (s_0, A_0) \) is a condition in \( \mathbb{P} \) such that for every \( s \leq (s_0, A_0) \) \( F_s \) is finite. Then there is \( B \subseteq A_0 \) such that for every \( t \leq (s_0, B) \)
\[
(s_0, B) \models \check{F}_t \subseteq \check{x}.
\]

Proof. We will construct the set \( B \) inductively. Suppose we have defined \( b_0 < b_1 < \cdots < b_{n-1} \) and a set \( B_n \subseteq A_0 \) such that \( b_0 > \max s_0, b_{n-1} < \min B_n \) and such that for every \( t \) which end-extends \( s_0 \) and such that \( t \setminus s_0 \subseteq \{ b_0, \ldots, b_{n-1} \} \), \( (t, B_n) \models \check{F}_t \subseteq \check{x} \). Let \( b_n = \min B_n \). Consider any \( t \) which end-extends \( s_0 \) such that \( t \setminus s_0 \subseteq \{ b_i \}_{i=0}^{n-1} \). Then \( F_{t - b_n} \) is finite and for every \( k \in T_{t - b_n} \) there is \( n_k \in \omega \) such that
\[
(t \setminus b_n, A_0 - n_k) \models k \in \check{x}
\]
and since \( B_n \subseteq A \) this implies that
\[
(t \setminus b_n, B_n - n_k) \models k \in \check{x}.
\]
Let $n_t = \max\{n_t^k : k \in F_t - b_n\}$. Then if $m$ is the maximum of all such $n_t$’s the set $B_n - m$ has the property that for every $t$ which end-extends $s_0$ and such that $t \setminus s_0 \subseteq \{b_i\}_{i \leq n}$

$$t \setminus b_n, B_n - m \models \hat{F}_{t - b_n} \subseteq \hat{x}.$$  

Let $B_{n+1} = B_n - m$. With this the inductive construction is complete. The set $B = \cap\{\{b_0, \ldots, b_{n-1}\} \cup B_n\} = \{b_i\}_{i \in \omega}$ has the desired properties. \qed

Thus we can assume that the chosen condition $(s_0, A_0)$ has the properties that for every $s \leq (s_0, A_0)$, $F_s$ is finite and there is $m \in \omega$ such that $(s, A_0 - m) \models F_s \subseteq \hat{x}$. Inductively, we will obtain an infinite subset $A$ of $A_0$ such that for every $s \leq (s_0, A)$ one of the following two conditions holds:

1. $\forall a \in A - (|s| + 1)F_{s - a} = F_s$.
2. $(\exists \alpha < \kappa)(\forall j \in \omega)(\exists m_j \in \omega)(\forall a \in A - m_j)F_{s - a} - x_{\alpha} - j \neq \emptyset$.

Again, suppose we have defined $\{a_0, \ldots, a_{n-1}\}$ and $A_n \subseteq A_0$ such that for every $s$ which end-extends $s_0$ and such that $s - s_0 \subseteq \{a_i\}_{i \leq n}$ the corresponding two conditions above hold ($A$ substituted by $A_n$). Let $a_n = \min A_n$. Then successively consider all end-extensions $s$ of $s_0$ such that $s - s_0 \subseteq \{a_i\}_{i \leq n}$ and define a set $A_{s,n}$ which is contained in $A_{s',n}$ for every $s'$ considered prior to $s$ and $A_n$ as follows.

If $B^* = \bigcup\{F_{s - a} : a \in A_n\}$ is finite, then for every $k \in B^*$ either the set $B_k = \{a \in A_n : k \in A_n\}$ is finite or it is infinite. If $B_k$ is finite than we can remove the corresponding $a$’s from $A_n$ (note also that in this case $k$ does not belong to $F_s$). If $B_k$ is infinite, then for every $b \in B_k$ (by inductive hypothesis) we have $(s - b, B_k) \models \hat{k} \in \hat{x}$ and so $(s, B_k) \models \hat{k} \in \hat{x}$ which implies that $k \in F_s$.

If $B^* = \bigcup\{F_{s - a} : a \in A_n\}$ is infinite, then let $\alpha < \kappa$ be such that $B^* \not\subseteq x_\alpha$. Define $A_{s,n}$ so that if $a$ is the $j$-th element of $A_{s,n}$ then there is $k \geq j$ such that $k \in F_{s - a} - x_\alpha - j$.

Then define $A_{n+1}$ to be the intersection of all such $A_{s,n}$’s. Finally, let $A = \{a_n\}_{n \in \omega}$. Then $A \subseteq A_0$ and for every $s \leq (s_0, A)$ one to the two conditions above hold. Again since there are only countably many $s \leq (s_0, A)$ we can choose an $\alpha < \kappa$ such that $\alpha$ is greater than all $\beta$’s associated to finite sequences $s \leq (s_0, A)$ by part (ii) of the above two conditions. Than since $(s_0, A)$ extends $(s, A)$

$$(s_0, A) \models \hat{x} \subseteq x_\alpha$$

and so there is some $(s, B) \leq (s_0, A)$ and $j \in \omega$ such that

$$(s, B) \models \hat{x} - j \subseteq x_\alpha.$$
If for every \( b \in B \), \( F_{s \upharpoonright b} = F_s \), then \( (s, B) \models \hat{x} \subseteq \hat{F}_x \) which is a contradiction, since \( F_s \) is finite. Otherwise, we can find \( b \in B \) such that there is \( k \in F_{s \upharpoonright b} - x_\alpha - j \). Then there is some \( m \in \omega \) such that

\[
(s \upharpoonright b, B - m) \models \hat{k} \in \hat{x}
\]

which is a contradiction since \((s \upharpoonright b, B - m)\) is an extension of \((s, B)\) and so we would obtain \((s \upharpoonright b, B - m) \models k \in x_\alpha\), which is not possible. \(\Box\)

2. Mixed-support iteration of factored Mathias forcing

We will begin with a well known definition of iterated forcing:

**Definition 4.** A partial order \( P_\kappa \) is a \( \kappa \)-stage iteration if and only if \( P_\kappa \) is a set of \( \kappa \)-sequences and there is a sequence \( \langle Q_\alpha : \alpha < \kappa \rangle \) such that \( P_\alpha = \{ p \upharpoonright \alpha : p \in P_\kappa \} \) for all \( \alpha < \kappa \), the following holds:

1. \( (\forall \alpha < \kappa) P_\alpha \) is an \( \alpha \)-stage iteration, with stages \( \langle Q_\beta : \beta < \alpha \rangle \).
   Let \( \models P_\alpha \) denote forcing with \( P_\alpha \).
2. \( (\forall \alpha < \kappa) \models P_\alpha \) is a partial order”.
3. \( (\forall p \in P_\kappa) (\forall \alpha < \kappa) \models P_\alpha p_\alpha \in Q_\alpha \) and there is \( r \in P_{\alpha + 1} \) where \( r \upharpoonright \alpha = p \upharpoonright \alpha \) and \( r(\alpha) = \hat{q} \).
4. \( \forall p, q \in P_\kappa (p \leq q) \) if and only is \( (\forall \alpha < \kappa) p \upharpoonright \alpha \models p(\alpha) \leq q(\alpha) \).
5. \( (\forall \beta < \alpha \leq \kappa) (\forall p \in P_\alpha, q \in P_\beta) \) if \( q \leq p \upharpoonright \beta \) then \( q \wedge p \in P_\alpha \), where \( q \wedge p(\gamma) = q(\gamma) \) for all \( \gamma < \beta \) and \( q \wedge p(\gamma) = p(\gamma) \) for \( \gamma \geq \beta \).
6. The trivial condition \( 1 \in P_\kappa \), where for every \( \alpha < \kappa \), \( 1(\alpha) \) is forced to be the trivial condition in \( Q_\alpha \).

For limit \( \alpha \leq \kappa \) we have to specify how \( P_\alpha \) is constructed from

\[
\{ p : \text{dom}(p) = \alpha \land (\forall \beta < \alpha) p \upharpoonright \beta \in P_\beta \}.
\]

Usually we require \( P_\alpha \) to consists of all conditions for which

\[
\text{support}(p) = \{ \beta \in \text{dom}(p) : p(\beta) \neq 1 \}
\]

is finite or countable. Then we refer to the iteration \( P_\kappa \) as \textit{finite} respectively \textit{countable support iteration}.

In particular, we will be interested in mixed support iteration:

**Definition 5.** For \( \kappa \) any ordinal, let \( P_\kappa \) be an iterated forcing construction such that for every \( \alpha < \kappa \) either \( \models P_\alpha \) ”\( Q_\alpha \) is \( \sigma \) -centered” or \( \models P_\alpha \) ”\( Q_\alpha \) is countably closed”. We thus speak about \( \sigma \)-centered stages and countably closed stages. For \( p \in P_\kappa \) let

\[
\text{Fsupport}(p) = \{ \alpha < \kappa : \alpha \text{ is a } \sigma \text{ centered stage} \}.
\]
Then \( \mathbb{P}_\kappa \) is the finite/countable support iteration of the \( Q_\alpha \) is for every \( p \in \mathbb{P}_\kappa \), support \((p)\) is countable, \( \text{Fsupport}(p) \) is finite and (\( \forall \alpha < \kappa \) \( \vdash \kappa
\)) \( p(\alpha) \in Q_\alpha \).

**Definition 6.** We say that \( p \) is a direct extension of \( q \), denoted \( p \leq_D q \) if \( p \leq q \) and for all \( \sigma \)-centered stages \( \alpha < \kappa \), \( p \vdash_\alpha p(\alpha) = q(\alpha) \).

Similarly, we say \( p \) is a \( C \)-extension of \( q \), denoted \( p \leq_C q \) if \( p \leq q \) and for all countably closed stages \( \alpha < \kappa \), \( p \vdash_\alpha p(\alpha) = q(\alpha) \).

**Remark.** Similarly, we say \( p \) is a \( C \)-extension of \( q \), denoted \( p \leq_C q \) if \( p \leq q \) and for all countably closed stages \( \alpha < \kappa \), \( p \vdash_\alpha p(\alpha) = q(\alpha) \).

**Lemma 2.** Let \( \{p_n\}_{n \in \omega} \) be a sequence in \( \mathbb{P}_\kappa \) such that for every \( n \) \( p_{n+1} \leq_D p_n \). Then there is a condition \( p \in \mathbb{P}_\kappa \) such that \( p \leq_D p_n \) for all \( n \).

**Proof.** Construct \( p \) inductively. If \( \alpha \) is a limit and \( p \upharpoonright \beta \) is defined for every \( \beta < \alpha \), then \( p \upharpoonright \alpha \) is clear. At successor stage \( \alpha + 1 \) there are two cases. If \( \alpha \) is a countably closed stage and

\[
\begin{align*}
p \upharpoonright \alpha \models p_0(\alpha) &\geq p_1(\alpha) \geq \cdots \geq p_n(\alpha) \cdots
\end{align*}
\]

then since \( Q_\alpha \) is countably closed we can choose \( p(\alpha) \) to be a \( \mathbb{P}_\alpha \)-name for an element of \( Q_\alpha \) such that \( p \upharpoonright \alpha \models p(\alpha) \leq p_n(\alpha) \) for every \( n \). If \( \alpha \) is a \( \sigma \)-centered stage and

\[
\begin{align*}
p \upharpoonright \alpha \models p_0(\alpha) = p_1(\alpha) = \cdots = p_n(\alpha) = \ldots
\end{align*}
\]

then we can simply define \( p(\alpha) = p_0(\alpha) \). \( \square \)

**Lemma 3.** Let \( p \leq q \) in \( \mathbb{P}_\kappa \). Then there is \( r \in \mathbb{P}_\kappa \) such that \( p \leq_C r \leq_D q \).

**Proof.** The condition \( r \) is defined by induction on \( \alpha \). If \( \alpha \) is a limit and \( r \upharpoonright \beta \) is defined for every \( \beta < \alpha \) then \( r \upharpoonright \alpha \) is clear. So, consider successor stages \( \alpha + 1 \). If \( \alpha \) is a \( \sigma \)-centered stage, then define \( r(\alpha) = q(\alpha) \). If \( \alpha \) is a countably closed stage we define \( r(\alpha) \) to be a \( \mathbb{P}_\alpha \)-term as follows:

\[
\begin{align*}
(1) \text{ if } \overline{r} \leq p \upharpoonright \alpha, \text{ then } \overline{r} \upharpoonright r(\alpha) &\models p(\alpha) \\
(2) \text{ if } \overline{r} \perp p \upharpoonright \alpha, \text{ then } \overline{r} \upharpoonright r(\alpha) &\models q(\alpha).
\end{align*}
\]

With this the inductive construction is defined. It remains to verify that \( p \leq_C r \) and \( r \leq_D q \).

By induction on \( \alpha \) verify that \( p \upharpoonright \alpha \leq r \upharpoonright \alpha \) and for countably closed stages \( p \upharpoonright \alpha \models p(\alpha) = r(\alpha) \) (holds by definition of \( r(\alpha) \)), and for \( \sigma \)-centered stages \( p \upharpoonright \alpha \models p(\alpha) \leq q(\alpha) = r(\alpha) \) (again by definition of \( r(\alpha) \)).

Similarly, by induction on \( \alpha \) verify that \( r \upharpoonright \alpha \leq q \upharpoonright \alpha \) and for countably closed stages \( r \upharpoonright \alpha \models r(\alpha) \leq q(\alpha) \), and for \( \sigma \)-centered stages
Then $r \upharpoonright \alpha \models r(\alpha) = q(\alpha)$. The latter holds by definition of $r(\alpha)$, so it remains to verify the former. Consider any $\bar{r} \leq r \upharpoonright \alpha$. If $\bar{r} \leq p \upharpoonright \alpha$, then

$$\bar{r} \models r(\alpha) = p(\alpha) \land p(\alpha) \leq q(\alpha).$$

If $\bar{r} \not\models p \upharpoonright \alpha$, then again by definition of $r(alpha)$, $\bar{r} \models r(\alpha) = q(\alpha)$. Therefore every extension of $r \upharpoonright \alpha$ forces that $\uparrow r(\alpha) \leq q(\alpha)$ and so $r \upharpoonright \alpha \models r(\alpha) \leq q(\alpha)$. □

**Definition 7.** Suppose $\alpha$ is a $\sigma$-centered stage. Then in $V^{\mathbb{P}_\alpha}$ define a function $s : Q_\alpha \to \omega$ so that

$$\models_{\alpha} (\forall p, q \in Q_\alpha)(s(p) = s(q) \implies p \not\models q).$$

**Remark 3.** Abusing notation we will identify $s$ with its $\mathbb{P}_\alpha$-name $\dot{s}$.

**Definition 8.** Condition $p \in \mathbb{P}_\kappa$ is said to be determined if for all $\alpha \in \text{Fsupport}(p)$ there is $n \in \omega$ such that

$$p \upharpoonright \alpha \models s(p(\alpha)) = \check{n}.$$ 

**Lemma 4.** The set of determined conditions in $\mathbb{P}_\kappa$ is dense. Suppose determined conditions $q_1$ and $q_2$ are given with

$$\text{Fsupport}(q_1) = \text{Fsupport}(q_2)$$

and for all $\alpha$ in this finite support there is $n \in \omega$ such that $q_1 \upharpoonright \alpha \models s(q_1(\alpha)) = \check{n}$ and $q_2 \upharpoonright \alpha \models s(q_2(\alpha)) = \check{n}$. Suppose also that for some $p \in \mathbb{P}_\kappa$, $q_1 \leq p$ and $q_2 \leq_C p$. Then $q_1$ and $q_2$ are compatible.

**Proof.** Proceed by induction on $\kappa$. Let $p \in \mathbb{P}_\kappa$. If $\kappa$ is a limit, then there is $\alpha < \kappa$ such that $\text{Fsupport}(p) \subseteq \alpha$. Then by inductive hypothesis there is a determined $\bar{r} \leq p \upharpoonright \alpha$ and so $\bar{r} \land p^\alpha$ is a determined condition extending $p$. At successor $\sigma$-centered stages $\alpha + 1$, we can find determined $\bar{r} \leq p \upharpoonright \alpha$ such that for some $n \in \omega$ $\bar{r} \models s(p(\alpha)) = \check{n}$. Then $\bar{r} \land r^\alpha$ is a determined extension of $p$.

To obtain the second claim of the Lemma, we will define a common extension $r$ of $q_1$ and $q_2$ inductively. Suppose $\alpha$ is a limit and for every $\beta < \alpha$ we have defined $r \upharpoonright \beta$. Then let $r \upharpoonright \alpha = \bigcup_{\beta < \alpha} r \upharpoonright \beta$. At countably closed stages let $r(\alpha) = q_1(\alpha)$. Then

$$r \upharpoonright \alpha \models q_1(\alpha) = r(\alpha) \leq p(\alpha) = q_2(\alpha).$$

At $\sigma$-centered stages we have $r \upharpoonright \alpha \models s(q_1(\alpha)) = s(q_2(\alpha)) = \check{n}$ for some $n$. Therefore we can choose $r(\alpha)$ to be a $\mathbb{P}_\alpha$-name for a common extension of $q_1(\alpha)$ and $q_2(\alpha)$ and so

$$r \upharpoonright \alpha \models (r(\alpha) \leq q_1(\alpha)) \land (r(\alpha) \leq q_2(\alpha)).$$ □
Lemma 5. Let $\mathbb{P}_\alpha$ be a finite/countable support iteration with $\alpha$ a limit ordinal and let $\langle x_\xi : \xi < \lambda \rangle$ be a tower in $[\omega]^\omega$ for some regular $\lambda$. If there is an infinite $x \subseteq \omega$ in $V^{\mathbb{P}_\alpha}$ such that for all $\xi < \lambda$, $x \subseteq^* x_\xi$, then there is $\beta < \alpha$ and an infinite $y \subseteq \omega$ in $V^{\mathbb{P}_\beta}$ such that $\forall \xi < \lambda, y \subseteq^* x_\xi$.

Theorem 3 (CH). Let $\mathbb{P}_{\omega_2}$ be the $\omega_2$-stage finite/countable factored Mathias iteration. Then, in $V^{\mathbb{P}_{\omega_2}}$ we have $\mathfrak{h} = 2^{\aleph_0} = \aleph_2$ and there are no $\omega_2$-towers in $[\omega]^\omega$.

References