FROM CREATURE FORCING TO BOOLEAN ULTRAPOWERS

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Abstract. We will consider four cardinal characteristics of the continuum, $\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{s}$ and discuss how their study has prompted the development of some of the most powerful forcing techniques: creature forcing, coherent systems of iterations, Shelah's method of template iterations and the method of boolean ultrapowers.

1. Introduction

The emergence of the subject of set theory can be traced back to the late nineteenth century, the advances of real analysis and the work of Georg Cantor on the trigonometric series representation of a function. In 1871, Cantor proved that if two trigonometric series converge to the same point except on finitely many points, then they converge to the same point everywhere. He soon generalized his theorem to an infinite set of exceptional points. However, this set of exceptional points was not arbitrary. It was subject to the requirement that for some $n \in \mathbb{N}$, its n-th derived set was finite. These developments were quickly followed by Cantor's proof that the set of natural numbers, $\mathbb{N}$, can not be put in bijective correspondence with the set of real numbers, $\mathbb{R}$, and the *continuum hypothesis*, which is the hypothesis that every infinite set of reals is either in bijective correspondence with $\mathbb{R}$ or with $\mathbb{N}$. The emerging necessity of comparing various sizes or infinities was soon answered by the appearance of Cantor's cardinal numbers and their cardinal arithmetic. The continuum hypothesis, abbreviated CH, can now be formulated as the claim that the cardinality of the real line is the first uncountable cardinal.

The cardinality of $\mathbb{R}$, denoted $\mathfrak{c}$, is in fact the very first cardinal characteristic of the real line. More generally, the *cardinal characteristics* of the real line are usually defined as the minimal size of a set of reals, which is characterized by a certain property. For example, consider the minimal cardinality of a family of meager sets, which covers the real line and denote this minimal size by $\mathrm{cov}(\mathcal{M})$. We refer to this cardinal characteristics as the *covering number of the meager ideal.* Since the countable union of meager sets is meager, by Baire category theorem $\aleph_0 < \mathrm{cov}(\mathcal{M})$. On the other hand, the family of all singletons clearly covers

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$\mathbb{R}$ and so $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{c}$. The problem of determining the cardinality of the real line was, and maybe still is, one of the major driving forces behind the development of set theory. It took almost a century to show that the usual axioms of set theory, i.e. the axiomatic system ZFC, do not determine the value of $\mathfrak{c}$. Already in 1939, Kurt Gödel established the *consistency of CH with ZFC*, by showing that it holds in his *Constructible Universe* (see [21]). It was not before the appearance of Cohen's *method of forcing* in 1962, that the consistency of the *negation of CH* with ZFC was obtained (see [12], [13]). Thus, with Cohen's result the *independence* of CH from ZFC was established. The method of forcing is a general method for obtaining *relative consistency* results, excellent expositions of which can be found in [25], or [24]. Since its appearance, the method has found broad applications to the study of the topological, measure theoretic and combinatorial properties the real line. Among others, it was used to show the independence of the Suslin hypothesis, as well as the independence of the Whitehead problem: while in the Constructible Universe every Whitehead group is free, it is consistent that there exists a non-free Whitehead group. Regarding the covering number of the meager ideal, the method of forcing can be used to show that each of the following is relatively consistent with ZFC: $\aleph_0 < \mathrm{cov}(\mathcal{M}) < \mathfrak{c}$, as well as $\aleph_1 < \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

In this article, we will focus on four combinatorial cardinal characteristics of the real line: *the bounding, the dominating, the almost disjointness and the splitting* numbers, denoted by $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{a}$, ans $\mathfrak{s}$ respectively. Apart from establishing their ZFC relations, we will make an overview of those developments of the method of forcing, which were triggered by the study of the independence of the characteristics in each of the pairs: $\{\mathfrak{a}, \mathfrak{d}\}$, $\{\mathfrak{a}, \mathfrak{s}\}$, $\{\mathfrak{b}, \mathfrak{s}\}$. Among those are some of the most interesting and powerful forcing techniques: creature posets; matrix iterations, and more generally coherent systems of iterations; Shelah's method of template iterations and its development from a method of iterating Suslin posets to a more general method permitting the iteration of Mathias-Prikry posets. Finally, we will briefly discuss the method of boolean ultrapowers and conclude with two open problems, which are central to the current development of the area.

2. Four cardinal characteristics and their ZFC relations

The results in this section are well-known and can be found in any expository presentation of the combinatorial cardinal characteristics of the real line, e.g. [4] or [23]. The following two notions, the notions of eventual dominance and almost containment, will be of particular importance for the upcoming discussion.

- For any two elements f, g in ${}^{\omega}\omega$, we say that f is *eventually dominated* by g, denoted $f <^* g$, if there is $n \in \omega$ such that for all $k \geq n$, $f(k) < g(k)$.
- For $A, B \in [\omega]^{\omega}$, we say that A is *almost contained* in B, denoted $A \subseteq^* B$ if $A \backslash B$ is a finite set.

Now, we can define two of the cardinal characteristics, which we will be of interested for our discussion:

- The *bounding number*, denoted $\mathfrak{b}$, is defined as the minimal size of an unbounded, with respect to the eventual dominance order, family in ${}^\omega\omega$. More precisely, $\mathcal{B} \subseteq {}^\omega\omega$ is said to be *unbounded*, if for every $f \in {}^\omega\omega$ there is $g \in \mathcal{B}$ such that g is not dominated by f. That is, $\exists^\infty n \in \omega(f(n) \leq g(n))$. Thus $\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is unbounded}\}$.
- The *dominating number*, denoted $\mathfrak{d}$, is defined as the minimal size of a dominating, with respect to the eventual dominance order, family in ${}^\omega\omega$. More precisely, $\mathcal{D} \subseteq {}^\omega\omega$ is said to be *dominating*, if for every $f \in {}^\omega\omega$ there is $g \in \mathcal{D}$ such that $f <^* g$. Thus $\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \text{ is dominating}\}$.

One of the first uses of the bounding number can be found in [29] (see also [32]). For a set $X \subseteq \mathbb{R}^n$ Rothberger defines X to have property λ, if each of its countable subsets is relative G_δ. Furthermore, he defines for a set X to have property λ', if $X \cup Y$ has property λ for each countable subset Y of $\mathbb{R}^n$. Then, he goes on to give a characterization of the sets with the property λ', which in contemporary terminology can be formulate as follows: *A set $X \subseteq \mathbb{R}^n$ has property λ' if and only if $|X| < \mathfrak{b}$.*

Lemma 2.1. $\omega_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$

Proof. Let $\mathcal{F} = \{f_n\}_{n\in\omega}$ be a countable family in ${}^\omega\omega$. Consider the function g, which diagonalizes $\mathcal{F}$, i.e. the function g defined by $g(k) = \max_{i\leq k} f_i(k) + 1$ for all k. Then g eventually dominates every member of $\mathcal{F}$ and so the minimal size of an unbounded family is strictly above $\aleph_0$. The fact that $\mathfrak{b} \leq \mathfrak{d}$ follows from the observation that every dominating family is unbounded. Furthermore, since the collection of all functions in ${}^\omega\omega$ is dominating, we clearly have $\mathfrak{d} \leq \mathfrak{c}$. □

The other two characteristics which will be for importance for our discussion are the almost disjointness and the splitting numbers.

Definition 2.2.

- A family $\mathcal{A} \subseteq [\omega]^\omega$ is said to be *almost disjoint* if for all $a, b \in \mathcal{A}$ such that $a \neq b$, the intersection $a \cap b$ is finite. An infinite almost disjoint family is said to be a *maximal almost disjoint* family, abbreviated m.a.d. family, if it is almost disjoint and maximal under inclusion. The minimal size of a maximal almost disjoint family is denoted $\mathfrak{a}$ and is referred to as the *almost disjointness number*.
- A family $\mathcal{S} \subseteq [\omega]^\omega$ is said to be *splitting* if for every $a \in [\omega]^\omega$ there is $s \in \mathcal{S}$ such that both $a \cap s$ and $a \cap (\omega \backslash s) = a \backslash s$ are infinite. The minimal cardinality of a splitting family is denoted $\mathfrak{s}$ and is referred to as the *splitting number*.

The existence of maximal almost disjoint families is an easy application of the Axiom of Choice, or equivalently Zorn's Lemma. It is also not difficult to construct a maximal almost disjoint family of size $\mathfrak{c}$ (see for example [4] or [23]). An interesting observation is the fact that the splitting number originally appeared as an algebraic characterization of sequential compactness. In [10], Booth shows that for every regular uncountable cardinal

λ, the space 2^λ is sequentially compact if and only if for every sequence $\langle a_\alpha : \alpha \in \lambda \rangle$ of infinite subsets of ω, there is $b \in [\omega]^\omega$ with the property that for every $\alpha \in \lambda$, $b \subseteq^* a_\alpha$ or $b \subseteq^* \omega \backslash a_\alpha$. In contemporary notation, Booth's result can be reformulated as the claim that 2^λ *is sequentially compact if and only if* $\lambda < \mathfrak{s}$.

Lemma 2.3. $\mathfrak{b} \leq \mathfrak{a}$

Proof. Consider an arbitrary maximal almost disjoint family $\mathcal{A} = \{x_\xi\}_{\xi \in \kappa}$. By finitely modifying the first ω members of the family, we can assume that they form a partition of ω. More precisely, take $x^* := \omega \backslash (\bigcup_{\xi \in \kappa} x_\xi)$, $x'_0 := x_0 \cup \{0\} \cup x^*$ and for each $n \geq 1$, $x'_n := (x_n \cup \{n\}) \backslash (\bigcup_{k \in n} x'_k)$. Then, since the elements of $\mathcal{A}$ are pairwise almost disjoint, each x'_n is infinite and clearly $\{x'_n\}_{n \in \omega}$ forms a partition of ω. It is also straightforward that $\mathcal{A} \backslash \{x_\xi\}_{\xi \in \omega} \cup \{x'_\xi\}_{\xi \in \omega}$ is a m.a.d. family.

Claim: There is a bijection $h : \omega \to \omega \times \omega$ such that $h^{-1}(\{n\} \times \omega) = x'_n$ for each n.

Proof: For each n, let g_n be the enumerating function of x'_n. Since $\{x'_n\}_{n \in \omega}$ forms a partition of ω, we can define

$$h(m) = (n, k) \text{ iff } m \in x'_n \text{ and } g_n(k) = m.$$

Clearly, h is as desired. □

For each $\xi \in \kappa$ and $k \in \omega$, define $f_\xi(k) := \max\{l : (k, l) \in h[x_\xi] \cap h[x'_k]\}$. Note that $f_\xi(k) = \max\{l : (k, l) \in h[x_\xi]\}$ and also, that since $x_\xi \cap x'_k$ is finite, the function $f_\xi(k)$ is well defined. However, the cardinality of $\mathcal{B} = \{f_\xi\}_{\xi \in \kappa}$ is smaller than $\mathfrak{b}$ and so there is a function f dominating all elements of $\mathcal{B}$. But, then $h^{-1}[\{(n, f(n))\}_{n \in \omega}]$ is a set, which is almost disjoint from every element of $\mathcal{A}$. □

It is not hard to see that the minimal size of a splitting family is strictly above $\aleph_0$ and that there is always a splitting family of size $\mathfrak{c}$ (see for examples [4]). Furthermore, we have the following:

Lemma 2.4. $\mathfrak{s} \leq \mathfrak{d}$

Proof. Note that if f is a strictly increasing function in ${}^\omega\omega$ with $f(0) > 0$, then f determines an interval partition of ω, given by $\{[f^n(0), f^{n+1}(0))\}_{n \in \omega}$, where $f^0(0) = 0$ and for each n, $f^{n+1}(0) = f(f^n(0))$. Then, for a strictly increasing function f the sets

$$\sigma^e_f = \bigcup \{[f^{2n}(0), f^{2n+1}(0)) : n \in \omega\} \text{ and } \sigma^o_f = \bigcup \{[f^{2n+1}(0), f^{2n+3}(0)) : n \in \omega\}$$

form a partition of ω into two infinite sets.

Equipped with the above partitions, we will show that every dominating family $\mathcal{D} \subseteq {}^\omega\omega$ gives in a natural way rise to a splitting family of the same cardinality. Fix an arbitrary dominating family $\mathcal{D}$. Without loss of generality, the elements of D are strictly increasing and for each $f \in \mathcal{D}$, $f(0) > 0$. Let $S_\mathcal{D} := \{\sigma_f : f \in \mathcal{D}\}$.

Claim: The family $S_\mathcal{D}$ is splitting.

Proof: For each $x \in [\omega]^\omega$, let f_x denote its enumerating function. Since $\mathcal{D}$ is dominating, for each $x \in [\omega]^\omega$, there is $f \in \mathcal{D}$ such that $f_x <^* f$, i.e. there is n_0 such that for each $k \geq n_0$, $f_x(k) < f(k)$, and so in particular for each $k \geq n_0$, $k \leq f_x(k) < f(k)$. Observe also that since both f and f_x are strictly increasing, $k \leq f^k(0)$ and $k \leq f_x(k)$ for all k.

Now, for $k \geq n_0$ we have

$$f^k(0) \leq f_x(f^k(0)) < f(f^k(0)) = f^{k+1}(0).$$

Therefore for all k, $f_x(f^k(0)) \in [f^k(0), f^{k+1}(0))$, which implies that both $x \cap \sigma_f^e$ and $x \cap \sigma_f^o$ are infinite. Clearly $\sigma_f^o = \omega \backslash \sigma_f^e$ and so σ_f^o splits x. $\square$

3. Creature posets, coherent Systems and template iterations

Our understanding of many of the combinatorial properties of the real line, which are not provable from ZFC, is often heavily dependent on our richness, or lack thereof, of forcing techniques. In this context, the four cardinal characteristics, we have chosen to consider, play an interesting and important role. Apart from the inequalities proved in Lemmas 2.1, 2.3, 2.4 and the fact that each of them takes values between $\aleph_1$ and $\mathfrak{c}$, and of course the fact that $\aleph_1 \leq \mathfrak{c}$, there are no other ZFC-provable inequalities between any two distinct elements of $\{\aleph_1, \mathfrak{c}, \mathfrak{b}, \mathfrak{a}, \mathfrak{d}, \mathfrak{s}\}$. To establish the lack of such further dependencies, we rely on the method of forcing. For example, to show that there is no ZFC proof of say $\mathfrak{b} \leq \mathfrak{s}$, we show that the negation of this statement, i.e. the strict inequality $\mathfrak{s} < \mathfrak{b}$, is relatively consistent with ZFC. In general, the task of establishing the independence between the cardinal characteristics in each of the pairs $\{\mathfrak{a}, \mathfrak{d}\}$, $\{\mathfrak{a}, \mathfrak{s}\}$ and $\{\mathfrak{b}, \mathfrak{s}\}$ is highly non-trivial and has brought the development of some of the most interesting forcing techniques, techniques which have already found applications far beyond the problems they were initially introduced for.

The relative consistency of $\mathfrak{s} < \mathfrak{b}$ was obtained by Baumgartner and Dordal, [3], in their study of what is now known as the Hechler model. Since $\mathfrak{b} \leq \mathfrak{a}$, in the same model $\mathfrak{s} < \mathfrak{a}$. Their consistency proof gives one of the first uses of a *rank argument* and presents an innovative for its time method of showing that a splitting family from the ground model remains splitting in the final generic extension. The consistencies of $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \aleph_2$ and $\mathfrak{b} = \aleph_1 < \mathfrak{a} = \aleph_2$ are due to Shelah (see [30]). To obtain the desired inequalities, he introduced a powerful forcing technique, known as *creature forcing*. His construction comes with a single drawback: the original creature posets are proper and so they can not be used to provide models in which $\mathfrak{c} > \aleph_2$.

It took more than a decade to overcome this difficulty and establish the consistency of $\mathfrak{b} = \kappa < \mathfrak{a} = \kappa^+$ (see [6]) for κ arbitrary regular uncountable cardinal. Almost another decade was necessary before the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ for κ arbitrary regular, uncountable cardinal, was obtained (see [18]). For the forcing specialist, it might be interesting to know, that each of those last two result was only possible, because of the existence of a a *ccc poset*, which has all crucial properties of a *proper, non-ccc counterpart* from [30]. Furthermore, the results involve the construction of a special (ultra)filter: given an unbounded, directed family $\mathcal{H}$ in ${}^\omega\omega$ with $|\mathcal{H}| = \mathfrak{c}$, there is an (ultra)filter $\mathcal{U}$ such that the associated Mathis-Prikry poset, $\mathbb{M}(\mathcal{U})$ preserves the unboundedness of $\mathcal{H}$. These special filters are clearly a special form of what is now know as *Canjar filters* (see [11] and [22]).[1]

The task of obtaining a larger spread between $\mathfrak{b}$ and $\mathfrak{s}$ proved to be quite difficult. The main challenge, being the problem of generically adjoining an unslit real, while preserving the unboundedness of a family $\mathcal{H}$ from the ground model, whose cardinality is *much smaller* than the size of the continuum. The consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$ for $\kappa < \lambda$ arbitrary regular uncountable cardinals was obtained in [9]. Furthermore, the presented proof is one of the few places in the literature, where a solution to the above problem can be found.[2] The generic extension in which the above constellation is realized is obtained via a *matrix iteration*. Recall that a matrix iteration is a system of κ-many finite support iterations of ccc posets, $\langle \mathbb{P}_{\alpha,\beta} : \beta \leq \lambda\rangle$, here $\alpha \in \kappa$, with the following property: if V is the ground model and $V_{\gamma,\delta}$ is the generic extension of V obtained via $\mathbb{P}_{\gamma,\delta}$, then whenever $\gamma_1 \leq \gamma_2$ and $\delta_1 \leq \delta_2$, the poset $\mathbb{P}_{\gamma_1,\delta_1}$ is a complete suborder of $\mathbb{P}_{\gamma_2,\delta_2}$ and so V_{γ_2,δ_2} is a generic extension of V_{γ_1,δ_1}. Such two-dimensional systems of generic extensions allow a much finer analysis of the interplay between unboundedness and splitting. Indeed, assume in addition that the ground model $V(= V_{0,0})$ satisfies the Generalized Continuum Hypothesis (GCH) and that $V_{\kappa,0}$ is obtained by adjoining a family $\mathcal{C}$ of κ-many Cohen reals over V. Inductively along a column of the intended matrix, say β for $\kappa < |\beta| \leq \lambda$, one can construct an ultrafiler $\mathcal{U}_{\kappa,\beta}$ in $V_{\kappa,\beta}$ such that forcing with the Mathias-Prikry poset $\mathbb{M}(\mathcal{U}_{\kappa,\beta})$ over $V_{\kappa,\beta}$ preserves the family $\mathcal{C}$ unbounded. Note that for β sufficiently large, $V_{\kappa,\beta} \vDash |\mathcal{C}| = \kappa < \mathfrak{c}$. Thus in particular, the Mathias-Prikry poset $\mathbb{M}(\mathcal{U}_{\kappa,\beta})$ solves the above problem and the ultrafilter $\mathcal{U}_{\kappa,\beta}$ and can be viewed as a strongly Canjar filter.

The paper [9] provides the second appearance in the literature of the idea of a matrix iteration (the first being the original appearance of this idea in [5]). Among others, the consistency proof of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$ offers a new method of preserving a maximal almost disjoint family along a matrix iteration. Since then, matrix iterations have been applied to the study of the characteristics of measure and category [26] and [16].[3] The technique has been recently generalized not only to three dimensional systems of finite support iterations, but even more generally to arbitrary *coherent systems of iterations* as defined in [16]. Furthermore, matrix iterations have been used to answer one of the long standing open questions in the area, namely to show that the splitting number can be singular (see [14]).

The independence of $\mathfrak{a}$ and $\mathfrak{d}$ marked the appearance of one of the most interesting and intricate forcing methods. The consistency of $\mathfrak{a} < \mathfrak{d}$ is not difficult to obtain: in a model of CH one can inductively construct a maximal almost disjoint family, which remains maximal in the Cohen extension of the same model (for a proof see for example [25]). It remains

[1]For the case $\mathfrak{b} < \mathfrak{s}$, see also [15].

[2]Alternative construction, though restricted to cardinalities $\aleph_1$ for the unbounded family and cardinality $\aleph_2$ for the continuum, can be found in [19].

[3]An excellent exposition of the characteristics of measure and category can be found in [1] and [2].

to observe that the Cohen real is unbounded, which easily gives the desired inequality. However, the consistency of $\mathfrak{d} < \mathfrak{a}$ required a completely new idea. The result was obtained only after the appearance of Shelah's method of *iterations along a template* (see [31] and for more axiomatic approach [7]). Shelah's template model is a ccc generic extension in which $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{d} = \kappa < \mathfrak{a} = \lambda$. The requirement that $\mathfrak{b} = \mathfrak{d} > \aleph_1$ is necessary. These original technique allows the iteration of nicely definable, in fact Suslin, posets along a template. The fact that $\mathfrak{s} = \aleph_1$ in Shelah's extension is almost accidental: the preservation properties of the construction imply that a family of $\aleph_1$ Cohen reals, which are generically adjoined along the forcing construction, remains splitting in the final generic extension. Obtaining the same constellation with the additional requirement that $\mathfrak{s}$ is arbitrarily large required major developments in Shelah's template iteration techniques. On one hand is the appearance of a technique permitting the iteration of Mathias-Prikry posets along a template (see [27]) and on the other, the realization that a template can be characterized not only by its length, but also by a notion of a width (see [17]). In particular, equipped with the new notion of a width, we could mimic the original isomorphism-of-names argument typical for the template constructions from [31] to posets, which are defined as template iterations involving non-definable iterands and so, establish the relative consistency of $\aleph_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{d}$, which is the main result of [17].

Apart from the above advances in Shelah's technique of template iterations, there is one more direction which should be mentioned. In [8], Brendle modifies the original construction to completely embed into the template poset a forcing notion which generically adjoins a maximal almost disjoint family of arbitrary cardinality. The technique produced the first model in which the almost disjointness number is of countable cofinality. The modified construction of Brendle was axiomatized and further developed by Fischer and Törnquist in [20], who showed that the minimal size of a maximal cofinitary group can be of countable cofinality (see [20]).

4. Boolean Ultrapowers and beyond

There are many possible constellations of $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{s}\}$ whose consistency remain open and for which our current methods of obtaining relative consistency results seem to be inadequate or simply, of little help. Concerning the characteristics $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{s}$ there are two other ZFC-admissible constellations in which all three of those take distinct values, in addition to $\mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ which was discussed in section 3. These are: $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$ and $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$.

Using the technique of *boolean ultrapowers* Raghavan and Shelah obtain the consistency of $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$, however at the expense of assuming the existence of super compact cardinals (see [28]). The technique of boolean ultrapowers is comparatively new to the study of the cardinal characteristics of the continuum, nevertheless it can be fully expected to produce many interesting new result, as well as bring new insights into the area.

The consistency of $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$ *it still open.* One possible approach could be the further development of the theory of coherent systems of iterations, with the aim of introducing the iteration of appropriate Mathias-Prikry posets along a three dimensional coherent system. Note that, the existing three dimensional constructions (see [16]) allow the iteration only of nicely definable posets. Thus, the suggested approach reminds greatly the development of Shelah's template iteration theory: originally the constructions allowed only the iteration of nicely definable posets ([30]), while later the theory was successfully generalized to include non-definable (in fact for now, only Mathias-Prikry) iterands (see [27, 17]).

Another well-known admissible constellation, which seems to evade our existing methods, is the well-known Roitman's Problem: *Is it a ZFC theorem that* $\mathfrak{d} = \aleph_1$ *implies* $\mathfrak{a} = \aleph_1$*?* We could either hope to obtain a ZFC proof of this implication, or if not, then obtain the relative consistency of $\mathfrak{d} = \aleph_1 < \mathfrak{a}$. The problem remains one of the most interesting open questions in the field.

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