PARTITION FORCING AND INDEPENDENT FAMILIES

JORGE A. CRUZ-CHAPITAL, VERA FISCHER, OSVALDO GUZMÁN, AND JAROSLAV ŠUPINA

Abstract. We show that Miller partition forcing preserves selective independent families and P-points, which implies the consistency of $\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} < \mathfrak{a}_T = \omega_2$. In addition, we show that Shelah's poset for destroying the maximality of a given maximal ideal preserves tight mad families and so, we establish the consistency of $\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{i} = \omega_1 < \mathfrak{u} = \mathfrak{a}_T = \omega_2$.

1. Introduction

One of the oldest questions regarding the theory of cardinal invariants of the continuum is the following question of Jerry Vaughan [50]: Is the inequality $\mathfrak{i} < \mathfrak{a}$ consistent?[1]

Not only this problem is interesting since it involves two fundamental objects in infinite combinatorics (maximal independent families and MAD families), but a positive solution to the problem will most likely require the development of new ideas and forcing techniques. [2] In order to gain more insight into the problem of Vaughan, we compare $\mathfrak{i}$ with the following cardinal invariant introduced by A. Miller in [42].

Definition. Define $\mathfrak{a}_T$ as the smallest size of a partition of ω^ω into compact sets.

It is well-known that the Baire space ω^ω is not σ-compact (see [32]), which implies that $\mathfrak{a}_T$ is uncountable. Furthermore, $\mathfrak{d}$ is the least size of a family of compact sets covering ω^ω (see [3]), so it follows that $\mathfrak{d} \leq \mathfrak{a}_T$. It is known that the compact subspaces of the Baire space are in correspondence with the finitely branching subtrees of $\omega^{<\omega}$. Using this correspondence and König's lemma, it is easy to prove that $\mathfrak{a}_T$ is equal to the least size of a maximal AD family of finitely branching subtrees of $\omega^{<\omega}$. M. Džamonja, M. Hrušák and J. Moore proved that $\diamondsuit_{\mathfrak{d}}$ implies that $\mathfrak{a}_T = \omega_1$ (see Theorem 7.6 of [43]). Thus, since $\diamondsuit_{\mathfrak{d}}$ holds in most of the natural models of $\mathfrak{d} = \omega_1$ (see [43] and [27] for a precise formulation of this statement), $\mathfrak{a}_T = \omega_1$ also holds in these models.

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[1] For definitions of cardinal characteristics, we refer the reader to [6].

[2] An interesting discussion of the problem can be found in the Appendix of this paper.

On the other hand, given a partition $\mathcal{C}$ of ω^ω into compact sets, Miller introduced a proper forcing notion $\mathbb{Q}(\mathcal{C})$ which has the Laver property and destroys $\mathcal{C}$, that is $\mathcal{C}$ no longer covers ω^ω after forcing with $\mathbb{Q}(\mathcal{C})$. The forcing notion is known as Miller partition forcing and plays an important role in the current article (see Definition 2.1). Spinas showed that $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding, which together with Miller's result establishes the Sacks property of $\mathbb{Q}(\mathcal{C})$, see [49]. Thus, every partition of ω^ω into compact sets can be destroyed with a proper forcing that has the Sacks property, which implies the consistency of $cof(\mathcal{N}) < \mathfrak{a}_T$ and in particular the consistency of $\mathfrak{d} < \mathfrak{a}_T$. In [52, Proposition 4.1.31], Zapletal proved that $\mathbb{Q}(\mathcal{C})$ is forcing equivalent to the quotient of the Borel sets of ω^ω modulo a σ-ideal generated by the closed sets. In this way, the forcing $\mathbb{Q}(\mathcal{C})$ falls into the scope of the theory developed in [52] and [51].

In this article, we study the effect of Miller partition forcing on the independence number $\mathfrak{i}$ and obtain the consistency of $\mathfrak{i} < \mathfrak{a}_T$. The key argument is the fact that Miller partition forcing preserves selective independent families, fact for which we provide two proofs: one building on Laflamme's filter games (see [34] and Definition 3.9) and one building on the notion of fusion with witnesses (see Definition 2.9). Both, the fusion with witnesses, as well as the use of Laflamme's filter game in the context of Miller's partition forcing are highly innovative and do not occur in earlier work on $\mathbb{Q}(\mathcal{C})$.

The notion of *selective independent family* was introduced by S. Shelah in his work on the consistency of $\mathfrak{i} < \mathfrak{u}$ (see [46]). Selective independent families are families with very strong combinatorial properties, which resemble the combinatorial features of Ramsey ultrafilters. Studying the similarities and differences between selective independent families and Ramsey ultrafilters remains a very interesting line of research. For more recent work on maximal independent families see [14, 19, 12, 20, 45].

Employing our notion of fusion with witnesses, we show also that $\mathbb{Q}(\mathcal{C})$ preserves P-points. Together with the fact that Miller partition forcing and its iterations preserve tight mad families, see [25], we obtain the consistency of the following constellation, which implies that in a natural sense $\mathbb{Q}(\mathcal{C})$ is optimal for $\mathfrak{a}_T$.

Theorem. It is relatively consistent that $\mathfrak{i} = \mathfrak{a} = \mathfrak{u} = \omega_1 < \mathfrak{a}_T$.

The question if one can increase simultaneously $\mathfrak{u}$ and $\mathfrak{a}_T$, while preserving small witnesses to $\mathfrak{a}$ and $\mathfrak{i}$ becomes of interest. Further, we show that Shelah's poset $\mathbb{Q}_\mathcal{I}$ for destroying the maximality of a given maximal ideal from [46] strongly preserves tight MAD families. Our results imply that as far as the classical cardinal characteristics of the continuum are concerned, Shelah's $\mathbb{Q}_\mathcal{I}$ is optimal for $\mathfrak{u}$, as it increases the ultrafilter number $\mathfrak{u}$, while it keeps all other (classical) characteristics small. The following result appears as Corollary 4.13 in the article:

Theorem. It is relatively consistent that $\mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{u}$.

Finally, combining Miller partition forcing, Shelah's $\mathbb{Q}_\mathcal{I}$, our preservation results, as well as the preservation results of [25] and [46], in Corollary 4.14 we obtain

Theorem. It is relatively consistent that $\mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{u} = \mathfrak{a}_T = \omega_2$.

2. Miller Partition Forcing

2.1. **Continuous reading of names.** Recall that Sacks forcing $\mathbb{S}$ consists of all perfect trees in $2^{<\omega}$ ordered by inclusion. That is, $p \in \mathbb{S}$ if and only if

(1) $p \subseteq 2^{<\omega}$,
(2) $\forall \sigma \in p\, \forall \tau \in 2^{<\omega}\big(\tau \subseteq \sigma \to \tau \in p\big)$,
(3) $\forall \sigma \in p\, \exists \tau, \tau' \in p\big(\sigma \subseteq \tau \wedge \sigma \subseteq \tau' \wedge \tau \not\subseteq \tau' \wedge \tau' \not\subseteq \tau\big)$.

We will use standard notation: If $p \in \mathbb{S}$ and $\sigma \in p$ we let $p(\sigma) = \{\tau \in p \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$ and call σ a splitting node if $\sigma^\frown i \in p$ for each $i \in 2$. Let $split(p) = \{\sigma \in p \mid \sigma \text{ is a splitting node }\}$. For each $n \in \omega$ let $split_n(p) = \{\sigma \in split(p) \mid |\{\tau \in split(p) \mid \tau \subsetneq \sigma\}| = n\}$ and $stem(p)$ the unique element in $split_0(p)$. Finally, for $p \in \mathbb{S}$ let $[p] = \{f \in 2^\omega \mid \forall n \in \omega(\ f|_n \in p\ )\}$. More about Sacks forcing can be found in [5, 4, 30, 23, 40, 15, 53].

Definition 2.1. (Miller partition forcing) Let $\mathcal{C} \subseteq P(2^\omega)$ be an uncountable partition of 2^ω into closed sets and let

$$\mathbb{Q}(\mathcal{C}) = \{p \in \mathbb{S} \mid \text{for every } K \in \mathcal{C},\ K \cap [p] \text{ is nowhere dense in } [p]\}$$

ordered by reversed inclusion.

This forcing destroys the partition $\mathcal{C}$ in the following way. If G is a $\mathbb{P}(\mathcal{C})$-generic filter, then

$$r_{gen} = \bigcup \bigcap G$$

is an element of 2^ω which does not belong to the interpretation in $V[G]$ of any element of $\mathcal{C}$. So in $V[G]$, $\mathcal{C}$ is no longer a partition of 2^ω. Thus, if we start with a model of CH and define $\mathbb{PM}$ as the resulting model after forcing with a countable support iteration of length ω_2 of all forcing notions of the form $\mathbb{Q}(\mathcal{C})$ with $\mathcal{C}$ ranging over all uncountable partitions in closed sets of 2^ω in all intermediate models, then $\mathbb{PM}$ will not have any uncountable partition in closed sets of 2^ω of size less than ω_2. Note that, Miller defined and used $\mathbb{PM}$ in [42] to show that $cov(\mathcal{M}) = \omega_1$ does not imply that $\mathfrak{a}_T = \omega_1$.

Notice that if $\mathcal{C}$ is the partition of 2^ω into singletons, then $\mathbb{Q}(\mathcal{C}) = \mathbb{S}$. Actually, it can be seen that if $\mathcal{C}$ is an analytic subset of $K(2^\omega)$, where $K(2^\omega)$ is the space of non-empty closed subsets of 2^ω equipped with the Vietoris topology, then $\mathbb{Q}(\mathcal{C})$ is forcing equivalent to the Sacks forcing $\mathbb{S}$.

Theorem 2.2. *Let $\mathcal{C} \subseteq K(2^\omega)$ be an uncountable analytic partition of 2^ω. Then $\mathbb{Q}(\mathcal{C})$ is forcing equivalent to the Sacks forcing $\mathbb{S}$.*

Proof. Let $p \in \mathbb{Q}(\mathcal{C})$. It is enough to find $q \in \mathbb{Q}(\mathcal{C})$ such that $q \leq p$ and $\{r \in \mathbb{Q}(\mathcal{C}) \mid r \leq q\} = \{r \in \mathbb{S} \mid r \subseteq q\}$. To do this consider $f : K(2^\omega) \longrightarrow 2^\omega$ given by $f(A) = minA$ and let $X = \{K \cap [p] \mid K \in \mathcal{C}\} \setminus \{\emptyset\}$. Notice that X is an uncountable analytic subset of $K(2^\omega)$, $f|_X$ is injective and $im(f|_X) \subseteq [p]$. This implies that there is $q \in \mathbb{S}$ such that $[q] \subseteq im(f_X)$. It is easy to see that $q \subseteq p$ and $|[q] \cap K| \leq 1$ for every $K \in \mathcal{C}$. Checking that q is as desired is straightforward. $\square$

A main difficulty in adapting Sacks fusion sequences to $\mathbb{Q}(\mathcal{C})$ is guaranteeing that the fusion is indeed an element of $\mathbb{Q}(\mathcal{C})$. The following proposition can be found in [25].

Proposition 2.3. Let $p \in \mathbb{S}$. Then $p \in \mathbb{Q}(\mathcal{C})$ if and only if there is a dense $D \subseteq [p]$ such that every two different elements of D belong to different elements of $\mathcal{C}$.

The following lemmas play an important role in the proof of Theorem 3.12, which is our main tool in showing that $\mathbb{Q}(\mathcal{C})$ preserves small witnesses to $\mathfrak{i}$.

Lemma 2.4. Let $p \in \mathbb{Q}(\mathcal{C})$ and let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name such that $p \Vdash \dot{f} \in 2^\omega$. Then there exists $q \leq p$, $g \in [q]$ and $h \in 2^\omega$ such that for every $m, n \in \omega$, if $g|_m \in split_n(q)$ then

$$q(g|_m) \Vdash h(n) = \dot{f}(n).$$

Proof. Recursively construct a sequence $\{q_n\}_{n\in\omega} \subseteq \mathbb{Q}(\mathcal{C})$ such that:

(1) $q_0 \leq p$
(2) $\forall n \in \omega\big(q_{n+1} \leq q_n\big)$
(3) $\forall n \in \omega\big(stem(q_n) \subsetneq stem(q_{n+1})\big)$
(4) $\forall n \in \omega \exists i_n \in 2\big(q_n \Vdash \dot{f}(n) = i_n\big)$

Now, let $g = \bigcup\limits_{n\in\omega} stem(q_n)$ and for every $n \in \omega$ let $s_n = |stem(q_n)|$. Define

$$q = \bigcup_{n\in\omega} q_n\big(stem(q_n)^\frown(1 - g(s_n))\big),$$

and let $h \in {}^\omega 2$ such that $h(n) = i_n$ for every $n \in \omega$. Using Proposition 2.3 it is easy to see that $q \in \mathbb{Q}(\mathcal{C})$. Moreover $q \leq p$, g is a branch through q, for every $n \in \omega$ $g|_{s_n} \in split_n(q)$ and $q(g|_{s_n}) \leq q_n$. Thus, in particular, for every $n \in \omega$, we have $q(g|_{s_n}) \Vdash h(n) = \dot{f}(n)$, which completes the proof. □

The following lemma can be deduced from [52, Proposition 4.1.31 and 4.1.2]. For the convenience of the reader, we provide a direct proof.

Lemma 2.5. Let $p \in \mathbb{Q}(\mathcal{C})$ and let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name such that $p \Vdash \dot{f} \in 2^\omega$. Then there exists $q \leq p$ and a continuous $H : [q] \longrightarrow 2^\omega$ such that $q \Vdash H(\dot{r}_{gen}) = \dot{f}$.

Proof. For $q \in \mathbb{Q}(\mathcal{C})$, $g \in [q]$ and $h \in 2^\omega$, we say that the triple (q, g, h) is *good* if it satisfies the conclusion of Lemma 2.4. That is, for every $m, n \in \omega$, if $g|_m \in split_n(q)$ then $q(g|_m) \Vdash h(n) = \dot{f}(n)$. Notice that if (q, g, h) is good and $m \in \omega$ then the triple $(q(g|_m), g, h)$ is also good.

Next, recursively construct a sequence $\{(q_\sigma, g_\sigma, h_\sigma)\}_{\sigma\in 2^{<\omega}}$ of good triples and a sequence $\{T_\sigma\}_{\sigma\in 2^{<\omega}}$ of elements of $\mathcal{K}$ such that:

(a) $q_\emptyset \leq p$,
(b) $\forall \sigma, \tau \in 2^{<\omega}\big(\sigma \subseteq \tau \to q_\tau \leq q_\sigma\big)$.
(c) $\forall \sigma, \tau \in 2^n\big(\sigma \neq \tau \to [q_\sigma] \cap [q_\tau] = \emptyset\big)$.
(d) $\forall \sigma, \tau \in 2^n\big(\sigma \neq \tau \to [q_\sigma] \cap T_\tau = \emptyset\big)$.
(e) $\forall \sigma \in 2^{<\omega} \exists m \in \omega\big((q_{\sigma^\frown 0}, g_{\sigma^\frown 0}, h_{\sigma^\frown 0}) = (q_\sigma(g_\sigma|_m), g_\sigma, h_\sigma)\big)$
(f) $\forall \sigma \in 2^{<\omega}\big(g_\sigma \in T_\sigma\big)$

(g) $\forall\sigma \in 2^n \big(q_\sigma \Vdash f(n) = h_\sigma(n)\big)$

The above can be easily achieved by applying repeatedly Lemma 2.4 and using the fact that each element of $\mathcal{C}$ is nowhere dense in every condition. Once the desired sequences are constructed, define $q = \bigcap_{n\in\omega} \bigcup_{\sigma\in 2^n} q_\sigma$. To see that $q \in \mathbb{Q}(\mathcal{C})$ notice that conditions (b) and (c) imply $q \in \mathbb{S}$, conditions (e) and (f) assure that $g_\sigma \in [q]$ for each $\sigma \in 2^{<\omega}$ and condition (d) together with the fact that necessarily $\{g_\sigma\}_{\sigma\in 2^{<\omega}}$ is dense in q guarantee by Proposition 2.3 that q is in fact in $\mathbb{Q}(\mathcal{C})$.

It remains to observe that by (c) and (f), the function $H : [q] \longrightarrow 2^\omega$ given by

$$H(g)(n) = h_\sigma(n) \text{ if and only if } \sigma \in 2^n \wedge g \in q_\sigma$$

is well defined and continuous. Moreover $q \Vdash H(\dot{r}_{gen}) = \dot{f}$, which completes the proof. □

Remark 2.6. By slight modifications of the proof given above, one can show that $\mathbb{Q}(\mathcal{C})$ has minimal real degree of constructibility, the Sacks property, and is proper.

2.2. **Fusion with witnesses.** We begin with some auxiliary notions.

Definition 2.7. Let $\mathcal{C} = \{C_\alpha\}_{\alpha\in\omega_1}$ be an uncountable partition of 2^ω into closed sets.

(1) We say that $x, y \in {}^\omega 2$ are $\mathcal{C}$-different if x, y belong to different elements of $\mathcal{C}$.
(2) A tree $p \subseteq 2^{<\omega}$ is said to be $\mathcal{C}$-branching if for any $s \in p$ there are $\mathcal{C}$-different branches in $[p]$ extending s.

Note that, a $\mathcal{C}$-branching tree is perfect. We will use the following notation: whenever $\mathcal{C}$ as above is given, for each $x \in 2^\omega$ we denote by α_x the unique ordinal such that $x \in C_{\alpha_x}$.

Lemma 2.8. Let $p \subseteq 2^{<\omega}$ be a tree. The following are equivalent:

(a) $p \in \mathbb{Q}(\mathcal{C})$.
(b) p is $\mathcal{C}$-branching.
(c) p is perfect and $[p]$ contains a countable dense subset with $\mathcal{C}$-different branches.

Proof. ((a) ⇒ (c)) Let $p \in \mathbb{Q}(\mathcal{C})$. p is a perfect tree by the definition. Thus arrange split(p) and assign by induction, to each splitting node s, a real x from $[p]$ extending s which was either already considered or belongs to different set from $\mathcal{C}$ than all previously selected reals. This is possible since any $s \in \text{split}(p)$ may be extended to $t \in \text{split}(p)$ with $[p(t)]$ being disjoint with finitely many sets from $\mathcal{C}$ containing all previously selected reals. The set of all assigned branches is the required dense set.

((c) ⇒ (b)) Trivial.

((b) ⇒ (a)) Let $\beta < \omega_1$ and $s \in p$. There are $x, y \in [p]$ such that $s \subseteq x, y$ and $\alpha_x \neq \alpha_y$. We take $z \in \{x, y\}$ such that $\alpha_z \neq \beta$. Since $z \in [p] \setminus C_\beta$ and C_β is closed, there is $s \subseteq t \subseteq z$ such that $[p_t] \cap C_\beta = \emptyset$. □

The particular enumeration constructed in Lemma 2.8 will be applied several times. Therefore we state explicitly that we may assume the dense set in Lemma 2.8 is enumerated as $\{x_t \colon t \in p\}$ such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$.

Definition 2.9. [Fusion sequence with witnesses]

(1) Let p be a condition in $\mathbb{Q}(\mathcal{C})$. We say that a set $X \subseteq {}^\omega 2$ is a p-witness for the n-th level if $X \subseteq [p]$, for each $s \in \operatorname{split}_n(p)$ there is $x \in X$ extending s, and X has $\mathcal{C}$-different elements. Note that if X is a p-witness for the $(n+1)$-st level then each node from n-th splitting level of p is contained in $\mathcal{C}$-different branches.

(2) Let $(p, X), (q, Y)$ be couples with p, q being conditions in $\mathbb{Q}(\mathcal{C})$, and sets X, Y being p-witness for the $(n+1)$-st level, q-witness for the n-th level, respectively. Then
$$(p, X) \leq^n (q, Y) \text{ if and only if } p \leq q \text{ and } X \supseteq Y.$$
Note that if $(p, X) \leq^n (q, Y)$ then $\operatorname{split}_{<n}(p) = \operatorname{split}_{<n}(q)$.

(3) A sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses if $(p_{n+1}, X_{n+1}) \leq^n (p_n, X_n)$ for each n.

Lemma 2.10. If a sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses then the fusion $\bigcap\{p_n : n \in \omega\}$ is a condition in $\mathbb{Q}(\mathcal{C})$.

Proof. We denote $p = \bigcap\{p_n \colon n \in \omega\}$, $X = \bigcup\{X_n : n \in \omega\}$, and we assume that we have $s \in p$. We take $n \in \omega$ and $t \in \operatorname{split}_n(p)$ such that t extends s. Since $\operatorname{split}_n(p) = \operatorname{split}_n(p_{n+1})$, the set X_{n+1} contains $\mathcal{C}$-different branches extending t. Hence, X is dense in $[p]$. One can easily see that X is contained in $[p]$. Finally, by Lemma 2.8 we conclude that $p \in \mathbb{Q}(\mathcal{C})$. □

A. Miller [42] and O. Spinas [49] applied separate fusion arguments in their proofs, while A. Miller [42] introduced the notion of a fusion even formally. The partial order $\mathbb{Q}(\mathcal{C})$ was recently used in [25], where the notion of a nice sequence was isolated from O. Spinas's fusion arguments. Our definition of fusion sequence covers both approaches. The sequence $\{X_n\}_{n\in\omega}$ in our definition may be obtained as sets of leftmost branches in Miller's fusion argument, and as certain terms of nice sequence in Spinas's approach. In fact, nice sequence may be obtained reenumerating our dense set $\{x_t \colon t \in p\}$ in Lemma 2.8.

In addition to fusion sequences, we shall use two basic schemas to amalgamate conditions. Let us have a condition $p \in \mathbb{Q}(\mathcal{C})$, and for each $s \in \operatorname{split}_n(p)$, $i \in \{0, 1\}$, a condition $q(s, i)$ extending $p(s^\frown i)$. Using Lemma 2.8, one can easily see that the tree
$$q = \bigcup\{q(s, i) \colon s \in \operatorname{split}_n(p), i \in \{0, 1\}\}$$
is a condition in $\mathbb{Q}(\mathcal{C})$ as well. In the second amalgamation technique, we are given a decreasing sequence $\{q_i\}_{i\in\omega}$ of extensions of p with strictly increasing stems $s_n = \operatorname{stem} q_n$. We set $x = \bigcup_{i\in\omega} s_i$ and take $q = \bigcup_{i\in\omega} q_i(s_i^\frown\langle 1 - x(|s_i|)\rangle)$. Again, using Lemma 2.8, one can easily see that q is a condition in $\mathbb{Q}(\mathcal{C})$.

The proof of the fact that $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding is underlying many of the fusion arguments to follow. For convenience of the reader, we repeat it here. We will make use of the following two Lemmas.

Lemma 2.11. Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$ and let h be a function in ${}^\omega\omega \cap V$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$: There is a real $x \in [q]$ and a sequence $\{f_s\}_{s\in x\restriction \operatorname{split}(q)}$ of functions in ${}^{<\omega}\omega$ such that for any $s = x \restriction \operatorname{split}_n(q)$ we have $q(s) \Vdash \dot{f} \restriction h(n) = f_s$.

Proof. Let $p \in \mathbb{Q}(\mathcal{C})$. One can construct a decreasing sequence $\{q_i\}_{i\in\omega}$ of extensions of p with strictly increasing stems such that $q_n \Vdash \dot{f} \restriction h(n) = f_n$ for some $f_n \in {}^{h(n)}\omega$. We denote $s_n = \operatorname{stem} q_n$ and we set $x = \bigcup_{i\in\omega} s_i$. Finally, we take the amalgamation $q = \bigcup_{i\in\omega} q_i(s_i^\frown\langle 1 - x(|s_i|)\rangle)$. ☐

Lemma 2.12. Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$. The set of all conditions q satisfying the following property is dense in $\mathbb{Q}(\mathcal{C})$: For all $m \in \omega$, for all $t \in \operatorname{split}_m(q)$ there is $f_t \in {}^{m+1}\omega$ such that

$$q(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t.$$

Proof. Let $p \in \mathbb{Q}(\mathcal{C})$. We build a fusion sequence $\{(q_n, X_n)\}_{n\in\omega}$ with $q_0 \le q$ such that its fusion q has the required property. Let the condition q_0, branch x, and sequence $\{f_s\}_{s\in x\restriction \operatorname{split}(q_0)}$ be obtained from Lemma 2.11 for p and $h(n) = n + 1$. We set $X_0 = \{x\}$.

Let $0 \le n < \omega$. Suppose we have defined $q_n \in \mathbb{Q}(\mathcal{C})$ and finite $X_n \subseteq [q_n]$. Let $s \in \operatorname{split}_n(q_n)$. Take the unique branch $x \in X_n$ extending s, node $r = x \restriction \operatorname{split}_{n+1}(q_n)$, and number $i = x(|s|)$ in $\{0,1\}$. We set $q(s,i) = q_n(r)$. Let $t \supseteq s^\frown\langle 1-i\rangle$ be such that $[q_n(t)] \cap C_{\alpha_x} = \emptyset$ for all already considered branches x (i.e., all branches in X_n and those assigned to previous nodes in some order of $\operatorname{split}_n(q_n)$). Use Lemma 2.11 for $q_n(t)$ and $h(j) = n + j + 2$ to obtain $q(s, 1-i) \le q_n(t)$, branch x and sequence $\{f_s\}_{s\in x\restriction \operatorname{split}(q_n)}$.

Finally, let X_{n+1} be the set of all considered branches in this step, and

$$q_{n+1} = \bigcup \{q(s,i) \colon s \in \operatorname{split}_n(q_n), i \in \{0,1\}\}.$$

One can verify that the sequence $\{(q_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses. ☐

As an application, we obtain the following straightforward proof of the fact that $\mathbb{P}(\mathcal{K})$ is ${}^\omega\omega$-bounding.

Lemma 2.13 (O. Spinas [49])**.** The poset $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding.

Proof. Let $\dot{f}$ be a $\mathbb{Q}(\mathcal{C})$-name for a function in ${}^\omega\omega$ and let $p \in \mathbb{Q}(\mathcal{C})$. We will show that there is $q \le p$ and $g \in V \cap {}^\omega\omega$ such that $q \Vdash \dot{f} \le^* \check{g}$.

By Lemma 2.12 we can assume that for all $m \in \omega$, for all $t \in \operatorname{split}_m(p)$ there is $f_t \in {}^{m+1}\omega$ such that $p(t) \Vdash \dot{f} \restriction (m+1) = \check{f}_t$. Define $g \in {}^\omega\omega$ as follows:

$$g(n) = \max\{f_s(n) + 1 \colon s \in \operatorname{split}_n(q)\}.$$

Then $q \Vdash \forall n(\dot{f}(n) < g(n))$. ☐

2.3. **P-points preservations.** Next, we show that Miller partition forcing preserves P-points. We will make use of the following notation: Given $\mathcal{G} \subseteq \mathcal{P}(\omega)$, let $\langle\mathcal{G}\rangle_{\text{up}} = \{X \in \mathcal{P}(\omega) \colon \exists G \in \mathcal{G}(G \subseteq X)\}$ and $\langle\mathcal{G}\rangle_{\text{dn}} = \{X \in \mathcal{P}(\omega) \colon \exists G \in \mathcal{G}(X \subseteq G)\}$.

Theorem 2.14. *The forcing notion $\mathbb{Q}(\mathcal{C})$ preserves* P*-points and Ramsey ultrafilters.*

Proof. We prove just first part. The second claim follows from the first one and the fact that the forcing notion $\mathbb{Q}(\mathcal{C})$ is ${}^\omega\omega$-bounding, see [26, Lemma 21.12]. Note that a family $\mathcal{G}$ generates an ultrafilter on ω if and only if $\mathcal{P}(\omega) = \langle\mathcal{G}\rangle_{\text{up}} \cup \langle\mathcal{G}^*\rangle_{\text{dn}}$.

Let $\mathcal{U}$ be an ultrafilter in V. We shall prove that the family $\mathcal{U}$ generates an ultrafilter in $V^{\mathbb{Q}(\mathcal{C})}$, i.e., $V^{\mathbb{Q}(\mathcal{C})} \models \mathcal{P}(\omega) = \langle\mathcal{U}\rangle_{\mathrm{up}} \cup \langle\mathcal{U}^*\rangle_{\mathrm{dn}}$. In $V^{\mathbb{Q}(\mathcal{C})}$, take any set in $\mathcal{P}(\omega)$. We fix $p \in \mathbb{Q}(\mathcal{C})$ and a $\mathbb{Q}(\mathcal{C})$-name $\dot{Y}$ such that $p \Vdash \dot{Y} \subseteq \omega$. By Lemma 2.12 we can assume that for all $m \in \omega$, for all $t \in \mathrm{split}_m(p)$ there is $u_t \in {}^{m+1}2$ such that

$$p(t) \Vdash \dot{Y} \restriction (m+1) = \check{u}_t.$$

Note that the latter property remains true for any stronger condition q, since t in the m-th level of q is an extension of some s in the m-th level of p. Let $\{x_t \colon t \in p\} \subseteq [p]$ be a dense set in $[p]$ containing $\mathcal{C}$-different elements (enumerated such that $s \subseteq x_s$, and if $s \subseteq t \subseteq x_s$ then $x_t = x_s$). We set $Y_t = \bigcup\{u_s \colon s \subseteq x_t\}$.

Claim. We can assume that $\mathcal{Y}_0 = \{Y_s \colon s \in p\}$ is in $\mathcal{U}$ or $\mathcal{Y}_1 = \{\omega \setminus Y_s \colon s \in p\}$ is in $\mathcal{U}$.

Proof. We set $U_0 = \{s \in p \colon Y_s \in \mathcal{U}\}$ and $U_1 = \{s \in p \colon (\omega \setminus Y_s) \in \mathcal{U}\}$. The sets U_0, U_1 are disjoint and their union is p. We may distinguish two cases:

(i) There is $s \in p$ such that $p(s) \subseteq U_0$. In this case, just take $p(s)$.

(ii) For each $s \in p$ there is $t \in p(s)$ such that $t \in U_1$. We build a fusion sequence $\{(p_n, X_n)\}_{n\in\omega}$ such that the fusion has the required properties. Taking $s \in \mathrm{split}_0(p)$ there is $t \in p(s)$ such that $t \in U_1$. We set $p_0 = p(t)$ and $X_0 = \{x_t\}$.

Let $0 \leq n < \omega$. Suppose we have defined $p_n \in \mathbb{Q}_\alpha$ and finite $X_n \subseteq [p_n]$. Let $s \in \mathrm{split}_n(p_n)$. Take node $r = x_s \restriction \mathrm{split}_{n+1}(p_n)$, and number $i = x_s(|s|)$ in $\{0,1\}$. We set $p(s,i) = p_n(r)$. Let $t \supseteq s^\frown\langle 1-i\rangle$ be splitting such that $t \in U_1$. We set $p(s, 1-i) = p(t)$.

Finally, let

$$p_{n+1} = \bigcup\{p(s,i) \colon s \in \mathrm{split}_n(p_n), i \in \{0,1\}\}.$$

and let X_{n+1} be the set of all x_t's for $t \in \mathrm{split}_{n+1}(p_{n+1})$. One can verify that the sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses.

□

We assume that $\mathcal{Y}_0 \in \mathcal{U}$, the other case may be handled analogously. We take a pseudointersection Z of $\mathcal{Y}_0$ in $\mathcal{U}$, with $Z \subseteq Y_\emptyset$. We shall simultaneously build two fusion sequences with witnesses, namely $\{(p^0_n, X^0_n)\}_{n\in\omega}$, $\{(p^1_n, X^1_n)\}_{n\in\omega}$, and a partition of Z into two sets Z_0, Z_1 such that for their respective fusions $q_0, q_1 \leq p$ we obtain $q_0 \Vdash \check{Z}_0 \subseteq \dot{Y}$ and $q_1 \Vdash \check{Z}_1 \subseteq \dot{Y}$.

Let $p^0 = p^1 = p$, $X^0_0 = X^1_0 = \{Y_\emptyset\}$, and $k_0 = 0$, $k_1 = 2$. We assume that p^0_n, p^1_n, k_{2n}, and k_{2n+1} are constructed. Let $t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p^0_n)$, and set $w^*(t) = x_t \restriction \mathrm{split}_{k_{2n+1}}(p)$. For each $i \in \{0,1\}$, we take $w^*(t,i) \in \mathrm{split}_{k_{2n+1}+1}(p)$ extending $w^*(t)^\frown i$. There is $k_{2n+2} > k_{2n+1}+1$ such that

$$Z \setminus k_{2n+2} \subseteq \bigcap\{Y_{w^*(t,i)} \colon t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p^0_n), i \in \{0,1\}\}.$$

We set $w(t,i) = x_{w^*(t,i)} \restriction \mathrm{split}_{k_{2n+2}}(p)$. Take $p_{n+1} = \bigcup\{p(w(t,i)) \colon t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p^0_n), i \in \{0,1\}\}$ and $X_{n+1} = \{x_{w(t,i)} \colon t \in \mathrm{split}_{k_{2n}}(p) \cap \mathrm{split}(p^0_n), i \in \{0,1\}\}$. One can see that $p^0_n \Vdash \check{Z} \cap [k_{2n}, k_{2n+1}) \subseteq \dot{Y}$. The construction of condition p^1_n and the choice of number k_{2n+3} are done similarly, and leads to $p^1_n \Vdash \check{Z} \cap [k_{2n+1}, k_{2n+2}) \subseteq \dot{Y}$. Finally, we define

$$Z_0 = Z \cap \bigcup\{[k_{2n}, k_{2n+1})\colon n \in \omega\} \text{ and } Z_1 = Z \cap \bigcup\{[k_{2n+1}, k_{2n+2})\colon n \in \omega\}.$$

□

3. Selective independence

3.1. **Dense maximality.** Recall the definition:

Definition 3.1. A family $\mathcal{B} \subseteq P(\omega)$ is an *independent family* if for every distinct $A_0, \dots, A_n \in \mathcal{A}$ and $h : \{A_0, \dots, A_n\} \longrightarrow 2$, the set $\bigcap_{i\leq n} A_i^{h(A_i)}$ is infinite where $A_i^0 = A^i$ and $A_i^1 = \omega\backslash A^i$. It is *maximal independent*, if it is independent and maximal under inclusion.

We will be exclusively interested in infinite independent families. For an independent family $\mathcal{A}$ let $\mathrm{FF}(\mathcal{A})$ be the set of all finite be set of all finite partial functions from $\mathcal{A}$ to 2 and order it by inclusion. For $h \in \mathrm{FF}(\mathcal{A})$, we let $\mathcal{A}^h = \bigcap\{A^{h(A)} \mid A \in dom(h)\}$ where $A^0 = \omega\backslash A$ and $A^1 = A$ for $A \subseteq \omega$. The *density ideal* of $\mathcal{A}$, denoted $\mathrm{id}(\mathcal{A})$ is the set of all $X \subseteq \omega$ such that for all $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ in $\mathrm{FF}(\mathcal{A})$ such that $\mathcal{A}^{h'} \cap X$ is finite (or equivalently empty). Dual, to the density ideal of $\mathcal{A}$ is the density filter of $\mathcal{A}$ denoted $\mathrm{fil}(\mathcal{A})$ and consisting of all $X \subseteq \omega$ such that for all $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ in $\mathrm{FF}(\mathcal{A})$ such that $\mathcal{A}^h\backslash X$ is finite (or equivalently empty).[3]

Lemma 3.2. Let $\mathcal{A}$ be an infinite independent family. The following are equivalent:

(1) For all $X \in \mathcal{P}(\omega)$ and all $h \in \mathrm{FF}(\mathcal{A})$ there is $h' \supseteq h$ such that $\mathcal{A}^h \cap X$ or $\mathcal{A}^h\backslash X$ is finite.
(2) For all $h \in \mathrm{FF}(\mathcal{A})$ and all $X \subseteq \mathcal{A}^h$ either $\mathcal{A}^h\backslash X \in \mathrm{id}(\mathcal{A})$ or there is $h' \in \mathrm{FF}(\mathcal{A})$ such that $h' \supseteq h$ and $\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash X$.
(3) For each $X \in \mathcal{P}(\omega)\backslash \mathrm{fil}(\mathcal{A})$ there is $h \in \mathrm{FF}(\mathcal{A})$ such that $X \subseteq \omega\backslash\mathcal{A}^h$.

Proof. First we show that (1) implies (2). Let $h \in \mathrm{FF}(\mathcal{A})$, let $X \subseteq \mathcal{A}^h$ and suppose $\mathcal{A}^h\backslash X \notin \mathrm{id}(\mathcal{A})$. Thus, there is $h' \in \mathrm{FF}(\mathcal{A})$ such that for all $h'' \supseteq h'$ the set $\mathcal{A}^{h''} \cap (\mathcal{A}^h\backslash X)$ is non-empty. Note that if h and h' are incompatible, then $\mathcal{A}^{h'} \cap (\mathcal{A}^h\backslash X) = \emptyset$, which is a contradiction. Therefore h and h' are compatible and without loss of generality, we can assume that $h' \supseteq h$. Thus, we have that for all $h'' \supseteq h'$, the set $\mathcal{A}^{h''}\backslash X \neq \emptyset$. Now, since (1) holds, there is $h'' \supseteq h'$ such that $\mathcal{A}^{h''} \cap X = \emptyset$. That is, $\mathcal{A}^{h''} \subseteq \mathcal{A}^h\backslash X$.

Next, we show that (2) implies (3). Thus, consider any $X \in \mathcal{P}(\omega)\backslash \mathrm{fil}(\mathcal{A})$. Then, in particular $\omega\backslash X \notin \mathrm{id}(\mathcal{A})$ and so there is $h \in \mathrm{FF}(\mathcal{A})$ such that for all $h' \supseteq h$, $|\mathcal{A}^{h'} \cap (\omega\backslash X)| = |\mathcal{A}^{h'}\backslash X| = \omega$. Let $Y = \mathcal{A}^h\backslash X$. Thus, $Y \subseteq \mathcal{A}^h$. By part (2) either $\mathcal{A}^h\backslash Y \in \mathrm{id}(\mathcal{A})$ or there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash Y$. Suppose $\mathcal{A}^h\backslash Y \in \mathrm{id}(\mathcal{A})$. Then, there is $h' \supseteq h$ such that $\mathcal{A}^{h'}\cap(\mathcal{A}^h\backslash Y) = \mathcal{A}^{h'}\backslash Y = \emptyset$. However $\mathcal{A}^{h'}\backslash Y = \mathcal{A}^{h'} \cap X = \emptyset$ and so $X \subseteq \omega\backslash\mathcal{A}^{h'}$ and we are done. If there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h\backslash Y = \mathcal{A}^h \cap X$, then $\mathcal{A}^{h'} \cap (\omega\backslash X) = \mathcal{A}^{h'}\backslash X = \emptyset$, contradicting the choice of h.

To see that (3) implies (2), consider any $h \in \mathrm{FF}(\mathcal{A})$ and $X \subseteq \mathcal{A}^h$. Let $Y = \mathcal{A}^h\backslash X$. If $\omega\backslash Y \in \mathrm{fil}(\mathcal{A})$, then $Y = \mathcal{A}^h\backslash X \in \mathrm{id}(\mathcal{A})$. Otherwise, there is $h*$ such that $\omega\backslash Y \subseteq \omega\backslash\mathcal{A}^{h*}$, which implies that $\mathcal{A}^{h^*} \subseteq Y = \mathcal{A}^h\backslash X$. Note that if $h\perp h^*$, then for some $C \in \mathcal{A}$ we have (without loss

[3] In the notation of [14], $\mathrm{fil}(\mathcal{A}) = \mathcal{F}_{\mathcal{A}}$ and $\mathbf{C}_{\mathcal{A}} = \mathrm{FF}(\mathcal{A})$. The density ideal and filter have been also studied in [19].

of generality) that $\mathcal{A}^{h*} \subseteq C$ and $\mathcal{A}^h \subseteq \omega \backslash C$, which contradicts $\mathcal{A}^{h^*} \subseteq \mathcal{A}^h$. Thus h^* and h are compatible, and so $\mathcal{A}^{h^* \cup h} \subseteq \mathcal{A}^{h^*} \subseteq \mathcal{A}^h \backslash X$.

To see that (2) implies (1) consider any $X \in [\omega]^\omega \backslash \mathcal{A}$ and let $h \in \mathrm{FF}(\mathcal{A})$. We want to show that there is $h' \supseteq h$ such that either $\mathcal{A}^{h'} \cap X = \emptyset$, or $\mathcal{A}^{h'} \backslash X = \emptyset$. Let $Y = X \cap \mathcal{A}^h$. Thus, $Y \subseteq \mathcal{A}^h$. If $\mathcal{A}^h \backslash Y \in \mathrm{id}(\mathcal{A})$, then $\mathcal{A}^h \backslash X \in \mathrm{id}(\mathcal{I})$ and so there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \cap (\mathcal{A}^h \backslash X) = \mathcal{A}^{h'} \backslash X = \emptyset$. Otherwise, there is $h' \supseteq h$ such that $\mathcal{A}^{h'} \subseteq \mathcal{A}^h \backslash Y$ and so $\mathcal{A}^{h'} \cap Y = \emptyset$. However, $\mathcal{A}^{h'} \cap Y = \mathcal{A}^{h'} \cap (X \cap \mathcal{A}^h) = \mathcal{A}^{h'} \cap X = \emptyset$. □

An independent family $\mathcal{A}$ is said to be *densely maximal* if any one of the above three definitions holds. The notion of dense maximality of independent families appears (to the best knowledge of the authors) for the first time in [24]. In particular, we obtain:

Corollary 3.3. Let $\mathcal{A}$ be an infinite independent family. Then, $\mathcal{A}$ is densely maximal iff

$$\mathcal{P}(\omega) = \mathrm{fil}(\mathcal{A}) \cup \langle \omega \backslash \mathcal{A}^h : h \in \mathrm{FF}(\mathcal{A}) \rangle_{dn}.$$[4]

The fact that partial orders with the Sacks property are Cohen preserving will play an important role in our results:

Lemma 3.4 ([14]). Let W be a $\mathbb{P}$-generic extension of V, where $\mathbb{P}$ has the Sacks property. If $\mathcal{A} \in V$ is an independent family, then in W, $\mathrm{fil}(\mathcal{A})$ is generated by $\mathrm{fil}(\mathcal{A})^V$.

3.2. **Selectivity.** Recall the following definitions. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$. Then $\mathcal{F}$ is centered if for ever finite subfamily $\mathcal{H}$, $\bigcap \mathcal{H} \in \mathcal{F}$; $\mathcal{F}$ is a P-set, if every countably subfamily has a pseudo-intersection in $\mathcal{F}$; $\mathcal{F}$ is a Q-set, if for every partition $\mathcal{E}$ of ω into bounded sets, there is a $X \in \mathcal{F}$ meeting each element of the partition on at most one point, i.e. $|X \cap E| \leq 1$ for each $E \in \mathcal{E}$.

Definition 3.5. Let $\mathcal{F}$ be a filter over ω. We say that $\mathcal{F}$ is a *selective filter* if and only if for every partition $\{X_i\}_{i\in\omega}$ of ω into elements of $\mathcal{F}^*$, where $\mathcal{F}^*$ is the dual ideal of $\mathcal{F}$, there exists $Y \in \mathcal{F}$ such that $|Y \cap X_i| \leq 1$ for each $i \in \omega$.

Note, that a filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is selective (also called Ramsey) if and only if $\mathcal{F}$ extends the Frechét filter and is both a P-set and a Q-set.

Definition 3.6. An independent family is said to be *selective* if it densely maximal and $\mathrm{fil}(\mathcal{A})$ is a selective filter.

Selective independent families exist under CH, result which is due to Shelah (see [46]). Further studies of selective independent families can be found in [14] and [19]. The following preservation theorem, will be central to the proof that iterations of Miller partition forcing, as well as other partial orders which are of interest for this article, preserve selective independent families.

Lemma 3.7 ([46], Lemma 3.2). Let $\mathcal{F}$ be a selective filter and let $\mathcal{H} \subseteq P(\omega) \backslash \mathcal{F}$ be cofinal in $P(\omega) \backslash \mathcal{F}$ with respect to $\subseteq^*$. If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ is a countable support iteration of ω^ω-bounding proper forcing notions such that for all $\alpha < \delta$, we have $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash$ "$\mathcal{H}$ is cofinal in $P(\omega) \backslash \langle \mathcal{F} \rangle$", then the same holds for δ.

[4]Thus, in the notation of [14], $\mathcal{A}$ is densely maximal iff $\mathcal{P}(\omega) = \mathcal{F}_\mathcal{A} \cup \langle \mathcal{C}_\mathcal{A} \rangle_{dn}$.

The forcing iterations, that we will be interested in, all have the Sacks property. Thus, in the corresponding generic extensions $\mathrm{fil}(\mathcal{A})$, is generated by $\mathrm{fil}(\mathcal{A}) \cap V$, where V denotes the ground model (see Lemma 3.4). Thus, if $\mathrm{fil}(A)$ is selective in the ground model, then it will remain selective in the desired generic extensions. Thus, the above preservation theorem implies that in order to guarantee that a given selective independent family remains selective (in our desired generic extensions), it is sufficient to guarantee that each iterand preserves the dense maximality of the family. Note that, the fact that the density filter is selective will play an crucial role in this preservation arguments. That is, our techniques do not imply that densly maximal independnet families are preserved, but only - selective ones. Before giving detailed proofs of these crucial preservation properties of Miller partition forcing (see Corollary 3.14 and Theorem 3.17) which are also the most technical arguments in the paper, we state our main result:

Theorem 3.8. *Assume* CH. *There is a cardinals preserving generic extension in which*

$$\mathrm{cof}(\mathcal{N}) = \mathfrak{a} = \mathfrak{u} = \mathfrak{i} = \omega_1 < \mathfrak{a}_T = \omega_2.$$

Proof. Let V denote the ground model. We assume that $\mathcal{A}$ is a selective independent family in V, $\mathcal{U}$ is a P-point in V, and $\mathcal{E}$ is a tight `MAD` family in V (according to [25]). Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets such that for each α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}(\mathcal{C})$ for some uncountable partition of 2^ω into compact sets and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \omega_2$. $\mathbb{P}_{\omega_2}$ is ${}^\omega\omega$-bounding, and therefore $\mathrm{cof}(\mathcal{N}) = \omega_1$. By Shelah's preservation theorem 3.7 and Corollary 3.14, or alternatively by Theorem 3.17, the family $\mathcal{A}$ remains maximal independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. Similarly, $\mathcal{U}$ generates a P-point in $V^{\mathbb{P}_{\omega_2}}$, so $\mathfrak{u} = \omega_1$ as well. And finally, $\mathfrak{a} = \omega_1$ since $\mathcal{E}$ is a tight `MAD` family (see [25]). □

We proceed with taking care of the successor stages of our forcing construction, i.e. the fact that Miller partition forcing preserves selective independent families. In Subsection 3.3 we give a proof of this fact using Laflamme's filter games and in Subsection 3.4 using the technique we introduced earlier, fusion with witnesses.

3.3. **Laflamme's filter game.** Selective filters have proven useful in preservation results regarding the independence number $\mathfrak{i}$, see [46, 19, 14]. The following game was introduced by C. Laflamme.

Definition 3.9 (Laflamme, [34])**.** Let $\mathcal{F}$ be a filter over ω. The game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$ is defined as follows. On the n turn, Player I plays some $U_n \in \mathcal{F}$ and Player II responds with some $a_n \in U_n$. After ω turns, Player II wins if the sequence $\{a_n\}_{n\in\omega}$ belongs to $\mathcal{F}$. Otherwise, Player I wins.

It is not hard to prove that the Player II never has a winning strategy in this game. On the other hand, we have the following theorem of Laflamme, see [34] (as well as [35]).

Theorem 3.10 (Laflamme)**.** *A filter $\mathcal{F}$ is not selective if and only if Player I does have a winning strategy for the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$.*

Lemma 3.11. Let $\mathcal{F}$ be a selective filter, $p \in \mathbb{S}$ and $H : [p] \longrightarrow P(\omega)$ be a continuous function such that for every $s \in p$,

$$\bigcup H\big[[p(s)]\big] \in \mathcal{F}.$$

Then there are $q \in \mathbb{S}$, $Y \in \mathcal{F}$ such that $q \subseteq p$ and for every $f \in [q]$, $Y \subseteq H(f)$.

Proof. For every $q \in \mathbb{S}$ such that $q \subseteq p$, let $L(q) = \bigcup H\big[[q]\big]$. Now consider the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$. Player I will play the following strategy, while constructing a sequence $\{t_\sigma\}_{\sigma \in 2^{<\omega}} \subseteq p$ such that:
(a) $\forall \sigma \in 2^{<\omega} \forall i \in 2 \big(t_\sigma \subsetneq t_{\sigma^\frown i}\big)$.
(b) $\forall \sigma, \tau \in 2^n \big(\sigma \neq \tau \to (t_\sigma$ and t_τ are incomparable $)\big)$.
On the first turn Player I defines $t_\emptyset = \emptyset$ and plays $U_0 = L(p(t_\emptyset))$. As the rules dictate, Player II responds with some $a_0 \in U_0$. Since $a_0 \in \bigcup H\big[[p(t_\emptyset)]\big]$, there is $f \in [p(t_\emptyset)]$ such that $a_0 \in H(f)$. As H is continuous there is $k \in \omega$ such that for every $g \in [p(f|_k)]$ we have $a_0 \in H(g)$. Now Player I extends $f|_k$ to incomparable $t_0, t_1 \in p$ such that $t_\emptyset \subsetneq t_0, t_1$ and plays $U_1 = L(p(t_0)) \cap L(p(t_1))$. As the rules dictate Player II responds with some $a_1 \in U_1$.

In general, suppose that it is the $n+1$ turn and that Player I has constructed $\{t_\sigma\}_{\sigma \in 2^{\leq n}}$ and for every $m \leq n$ played $U_m = \bigcap\limits_{\sigma \in 2^{\leq m}} L(p(t_\sigma))$. As $a_n \in \bigcap\limits_{\sigma \in 2^n} \big(\bigcup H\big[[p(t_\sigma)]\big]\big)$, for every $\sigma \in 2^n$ there is $f_\sigma \in [p(t_0)]$ such that $a_n \in H(f_\sigma)$. Since H is continuous, there is $k \in \omega$ such that for every $\sigma \in 2^n$ and every $g \in [p(f_\sigma|_k)]$, $a_n \in H(g)$. Now Player I extends each $f_\sigma|_k$ to incomparable $t_{\sigma^\frown 0}, t_{\sigma^\frown 1} \in p$ such that $t_\sigma \subsetneq t_{\sigma^\frown 0}, t_{\sigma^\frown 1}$ and plays $U_{n+1} = \bigcap\limits_{\sigma \in 2^{\leq n+1}} L(p(t_\sigma))$. As the rules dictate, Player II responds with some $a_{n+1} \in U_{n+1}$.

Since $\mathcal{F}$ is a selective filter, this is not a winning strategy for Player I. Therefore there is a match where Player I plays by the above strategy, but Player II wins. Let $\{a_n\}_{n\in\omega}$ and $\{t_\sigma\}_{\sigma \in 2^{<\omega}}$ be the sequences associated to one of these matches and let $q = \{\tau \in p \mid \exists \sigma \in 2^\omega \big(\tau \subseteq t_\sigma\big)\}$. It is straightforward that q and $Y = \{a_n\}_{n\in\omega}$ are the objects we are looking for. □

Theorem 3.12. *Let $\mathcal{F}$ be a selective filter and let G be a $\mathbb{Q}(\mathcal{C})$-generic filter. In $V[G]$, for every $X \in P(\omega)$ one of the following statements occurs:*
(a) *There is $Y \in \mathcal{F} \cap V$ such that $Y \subseteq X$.*
(b) *There is $Z \in V$, such that $Z \notin \mathcal{F}$ and $X \subseteq Z$.*

Proof. Let $\dot{X}$ be a name for a subset of ω, $p \in \mathbb{Q}(\mathcal{C})$ and suppose no condition below p forces (b). By Lemma 2.5 there is a continuous $H : [p] \longrightarrow P(\omega)$ such that $p \Vdash H(\dot{r}_{gen}) = \dot{X}$. For every $q \in \mathbb{S}$ such that $q \subseteq p$, let $L(q) = \bigcup H\big[[q]\big]$ and note that if $q \in \mathbb{Q}(\mathcal{C})$ then $L(q) \in \mathcal{F}$. This holds, because $q \Vdash X \subseteq L(q)$ and q does not force (b). We say that a conditions $q \in \mathbb{S}$, which is not necessarily in $\mathbb{Q}(\mathcal{C})$, is *special* if $q \subseteq p$ and for every $s \in q$ we have that $L(q(s)) \in \mathcal{F}$.

We will make use of the following notion: Given $s \in p$ we say that the pair (q, T) is *s-special* if $q \in \mathbb{S}$ is special, $T \in \mathcal{C}$, $[q] \subseteq [p(s)] \cap T$. We divide the proof in cases.

<u>Case 1</u> For every $s \in p$, there is an s-special pair (q, T).

In this case, consider the game $\mathfrak{G}(\mathcal{F}, \omega, \mathcal{F})$. Player I will play by the following strategy, while recursively constructing sequences $\{q_\sigma\}_{\sigma \in 2^{<\omega}} \subseteq \mathbb{S}$, $\{s_\sigma\}_{\sigma \in 2^{<\omega}} \subseteq p$, and $\{T_\sigma\}_{\sigma \in 2^{<\omega}} \subseteq \mathcal{K}$ such that:

(a) $\forall\sigma \in 2^{<\omega}$ the pair (q_σ, T_σ) is s_σ-special;
(b) $\forall\sigma \in 2^{<\omega}\big(q_{\sigma^\frown 0} \subseteq q_\sigma\big)$;
(c) $\forall\sigma \in 2^{<\omega}\big(T_{\sigma^\frown 0} = T_\sigma\big)$;
(d) $\forall\sigma, \tau \in 2^n\big(\sigma \neq \tau \to T_\sigma \neq T_\tau\big)$;
(e) $\forall\sigma \in 2^{<\omega}\forall i \in 2\big(s_\sigma \subsetneq s_{\sigma^\frown i}\big)$;
(f) $\forall\sigma \in 2^{<\omega}\big(s_{\sigma^\frown 0} \in q_\sigma \wedge [p(s_{\sigma^\frown 1})] \cap T_\sigma = \emptyset\big)$.

On the first turn, Player I defines $s_\emptyset = \emptyset$, an $s_\emptyset$-special pair $(q_\emptyset, T_\emptyset)$ and plays $U_0 = L(q_\emptyset)$. As the rules dictate, Player II responds with some $a_0 \in U_0$. Since $a_0 \in \bigcup H\big[[q_\emptyset]\big]$, there is some $f \in [q_\emptyset]$ such that $a_0 \in H(f)$ and since H is continuous there is $k \in \omega$ such that for every $g \in [p(f|_k)]$, $a_0 \in H(g)$. Notice that since $(q_\emptyset, T_\emptyset)$ is $s_\emptyset$-special, we have that $f|_k$ is compatible with s_σ and moreover we can extend $f|_k$ to incomparable $s_0, s_1 \in p$ such that $s_\emptyset \subsetneq s_0, s_1$, $s_0 \in q_\emptyset$ and $[p(s_1)] \cap T_\emptyset = \emptyset$. Now Player I defines $q_0 = q_\emptyset(s_0)$, $T_0 = T_\emptyset$, an s_1-special pair (q_1, T_1) and plays $U_1 = L(q_0) \cap L(q_1)$. As the rules dictate, Player II responds with some $a_1 \in U_1$.

In general, suppose that is the $n+1$ turn and Player I has constructed q_σ, s_σ and T_σ for every $\sigma \in 2^{\leq n}$. Moreover, suppose that for every $m \leq n$ Player I has played $U_m = \bigcap_{\sigma\in 2^m} L(q_\sigma)$. Since $a_n \in \bigcap_{\sigma \in 2^n} \big(\bigcup H\big[[q_\sigma]\big]\big)$, for every $\sigma \in 2^n$ there is $f_\sigma \in [q_\sigma]$ such that $a_n \in H(f_\sigma)$. As H is continuous, there is $k \in \omega$ such that for every $\sigma \in 2^n$ and every $g \in [p(f_\sigma|_k)]$, $a_n \in H(g)$. As each (q_σ, T_σ) is s_σ-special, we have that $f_\sigma|_k$ is compatible with s_σ and that $\bigcup_{\sigma\in 2^n} T_\sigma \cap [p]$ is nowhere dense in $[p]$. Then we can extend each $f_\sigma|_k$ to incomparable $s_{\sigma^\frown 0}, s_{\sigma^\frown 1} \in p$ such that $s_\sigma \subsetneq s_{\sigma^\frown 0}$, $s_{\sigma^\frown 1}$, $s_{\sigma^\frown 0} \in q_\sigma$ and $p(s_{\sigma^\frown 1})] \cap T_\sigma = \emptyset$. Now Player I defines $q_{\sigma^\frown 0} = q_\sigma(s_{\sigma^\frown 0})$, $T_{\sigma^\frown 0} = T_\sigma$, an $s_{\sigma^\frown 1}$-special pair $(q_{\sigma^\frown 1}, T_{\sigma^\frown 1})$ and plays $U_{n+1} = \bigcap_{\sigma\in 2^{n+1}} L(q_\sigma)$. As the rules dictate, Player II responds with some $a_{n+1} \in U_{n+1}$.

Since $\mathcal{F}$ is a selective filter, the above is not a winning strategy for Player I and so, there is a match where Player I follows the strategy, but Player II wins. Let $\{a_n\}_{n\in\omega}$, $\{q_\sigma\}_{\sigma\in 2^\omega}$, $\{s_\sigma\}_{\sigma\in 2^\omega}$ and $\{T_\sigma\}_{\sigma\in 2^\omega}$ be the sequences associated to one of these matches. To finish Case 1, define $q = \{\tau \in p \mid \exists\sigma \in 2^\omega\big(\tau \subseteq s_\sigma\big)\}$. Moreover, if c_0 is the constant 0 function in 2^ω and $\sigma \in 2^{<\omega}$ then $g_\sigma = \bigcup\{s_\tau \mid \tau \subseteq \sigma^\frown c_0\} \in T_\sigma$ and the set $Q = \{g_\sigma \mid \sigma \in 2^{<\omega}\}$ is dense in $[q]$. But then, by Proposition 2.3, $q \in \mathbb{Q}(\mathcal{C})$. Since for every $n \in \omega$, every $\sigma \in 2^{n+1}$ and every $g \in [p(s_\sigma)]$, we have that $a_n \in H(g)$ and every $g \in [q]$ satisfies this condition for some $\sigma \in 2^{n+1}$, we obtain that for every $g \in [q]$, $\{a_n\}_{n\in\omega} \subseteq H(g)$. In particular we have that $q \Vdash \{a_n\}_{n\in\omega} \subseteq \dot{X}$. Since $\{a_n\}_{n\in\omega} \in \mathcal{F}$, we are done.

<u>*Case 2*</u> There is $s_0 \in p$ for which there is no s_0-special pair (q, T). That is, every ordered pair (q, T) does not satisfy one of the following conditions: $q \in \mathbb{S}$ is special, $T \in \mathcal{K}$, or $[q] \subseteq [p(s_0)] \cap T$.

In this case, we use Lemma 3.11 to find $q \in \mathbb{S}$ and $Y \in \mathcal{F}$ such that $q \subseteq p(s_0)$ and for every $f \in [q]$, $Y \subseteq H(f)$. Notice that q is special. Suppose towards a contradiction that $q \notin \mathbb{Q}(\mathcal{C})$. Since every element of $\mathcal{C}$ is closed, this means that there is some $T \in \mathcal{C}$ such that $T \cap [q]$ has non-empty interior in $[q]$ and so we can find $\tau \in q$ such that $[q(\tau)] \subseteq T$. Then $(q(\tau), T)$ is s_0-special, which is a contradiction. Therefore $q \in \mathbb{Q}(\mathcal{C})$. To finish this case, just note that as before $q \Vdash Y \subseteq \dot{X}$. $\square$

Suppose that $\mathcal{C}$ is the partition of 2^ω in singletons. Then $\mathbb{Q}(\mathcal{C}) = \mathbb{S}$ and so *Case 1* of Theorem 3.12 never occurs. Therefore Lemma 3.11 actually yields a complete proof of Theorem 3.12 for Sacks forcing. Additionally, we obtain once again:

Corollary 3.13. The poset $\mathbb{Q}(\mathcal{C})$ preserves selective ultrafilters.

Corollary 3.14. Let $\mathcal{A}$ be a selective independent family and let G be a $\mathbb{Q}(\mathcal{C})$-generic filter. Then, in $V[G]$, $\mathcal{A}$ is still selective independent.

Proof. Since $\mathbb{Q}(\mathcal{C})$ is proper and has the Sacks property, $\langle \text{fil}(\mathcal{A})^V \rangle$ is a selective filter in $V[G]$, but by Lemma 3.4.(1) we know that $\text{fil}(\mathcal{A})^{V[G]} = \langle \text{fil}(\mathcal{A})^V \rangle$. To show that $\mathcal{A}$ remains dense in $V[G]$, note that by Theorem 3.12, the family $P(\omega)^V \setminus \text{fil}(\mathcal{A})^V$ is cofinal in $P(\omega)^{V[G]} \setminus \langle \text{fil}(\mathcal{A})^V \rangle$. However, by hypothesis $\{\omega \setminus \mathcal{A}^h : h \in \text{FF}(\mathcal{A})\}$ is cofinal in $P(\omega)^V \setminus \text{fil}(\mathcal{A})^V$ and so by Lemma 3.4 we are done. ☐

3.4. **Fusion sequences and selectivity.** We use a combinatorial characterization of Q-filters, a similar one to a characterization of happy families, see Proposition 0.7 by A. Mathias [37] or Proposition 11.6 in [26].

Lemma 3.15. Let $\mathcal{F}$ be a filter. The following are equivalent:

(a) $\mathcal{F}$ is a Q-filter.
(b) For any increasing function $f \in {}^\omega\omega$ there is $\{k(n) \colon n \in \omega\} \in \mathcal{F}$ such that $f(k(n)) < k(n+1)$.

Proof. ((a) $\Rightarrow$ (b)) Inductively, choose a sequence $\{n(l)\}_{l \in \omega}$ such that $n(0) = 0$ and

$$n(l+1) = \min\{n \colon n_l < n \text{ and } \forall m \leq n_l (f(m) \leq n)\}.$$

We consider the partition $\mathcal{E}_0 = \{[n_{3l}, n_{3l+3})\}_{l \in \omega}$. There is $C_1 \in \mathcal{F}$ such that C_1 is a semi-selector for $\mathcal{E}_0$. Now, define an equivalence relation $\mathcal{E}_1$ on C_1 as follows:

$$m \sim_{\mathcal{E}_1} k \text{ iff } m = k \vee m < k \leq f(m) \vee k < m \leq f(k).$$

Each $\mathcal{E}_1$ equivalence relation has at most two members. Indeed, if there were three numbers $m_1 < m_2 < m_3$ in one equivalence class of $\mathcal{E}_1$ then $m_1 < m_2 < m_3 \leq f(m_1)$. There are $l_1 < l_2 < l_3$ such that $m_i \in [n_{3l_i}, n_{3l_i+3})$. Then $m_1 < n_{3l_2} \leq m_2 < n_{3l_3} \leq m_3 \leq f(m_1)$. However, on the other hand by the definition of sequence $\{n(l)\}_{l \in \omega}$ we have $f(m_1) \leq n_{3l_2+1} < n_{3l_3}$, a contradiction.

Extend $\mathcal{E}_1$ to an equivalence relation $\mathcal{E}_2$ on ω by defining

$$m \sim_{\mathcal{E}_2} k \text{ iff } m = k \vee m \sim_{\mathcal{E}_1} k.$$

There is C_2 in $\mathcal{F}$ such that C_2 is a semi-selector for $\mathcal{E}_2$. Without loss of generality $C_2 \subseteq C_1$ and $0 \in C_2$. Let $\{k(n)\}_{n \in \omega}$ enumerate in increasing order C_2. Thus for all n, n' we have that $k(n) \not\sim_{\mathcal{E}_2} k(n')$. Thus, if $n < n'$ then $k(n') \not\leq f(k(n))$ and so for all $n \in \omega$, $f(k(n)) < k(n+1)$.

((b) $\Rightarrow$ (a)) Let $\mathcal{E}$ be a bounded partition of ω. We set

$$f(n) = \max \bigcup \{E \in \mathcal{E} \colon (\exists i \leq n)\ i \in E\}.$$

There is $\{k(n)\colon n\in\omega\}\in\mathcal{F}$ such that $f(k(n))<k(n+1)$ for each $n\in\omega$. The set $\{k(n)\colon n\in\omega\}$ is a semi-selector for $\mathcal{E}$. Indeed, $k(n)\leq f(k(n))<k(n+1)$ and therefore $k(n+1)$ is from different set of partition $\mathcal{E}$ than all $k(i)$ for $i\leq n$. $\square$

In particular, we get the following, which we state for completeness.

Lemma 3.16. An ${}^{\omega}\omega$-bounding forcing notion preserves Q-filters.

Proof. If $\mathbb{P}$ is an ${}^{\omega}\omega$-bounding forcing notion, $\mathcal{F}$ a Q-filter in V, then we use part (2) of Lemma 3.15 for $f\in V\cap{}^{\omega}\omega$ dominating function $g\in V^{\mathbb{P}}\cap{}^{\omega}\omega$. $\square$

Theorem 3.17. (CH) *Let $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\beta\colon\alpha\leq\omega_2,\beta<\omega_2\rangle$ be a countable support iteration such that for each α, $\mathbb{Q}_\alpha=\mathbb{Q}(\mathcal{C}_\alpha)$ for some partition $\mathcal{C}_\alpha$ of 2^ω. If $\mathcal{A}$ is a selective independent family then ($\mathcal{A}$ is a selective independent family)$^{V^{\mathbb{P}_{\omega_2}}}$.*

Proof. We begin with a proof that $\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}$ has a property similar to being a happy family by A. Mathias [37], see [26] as well. Note that A. Mathias [37, Proposition 0.10] has shown that an ultrafilter $\mathcal{G}$ is Ramsey if and only if $\mathcal{G}$ is happy (see Proposition 11.7 in [26] as well).

Claim 3.18. In $V^{\mathbb{P}_\alpha}$, let $\{\mathcal{G}_n\}_{n\in\omega}$ be a sequence of finite subsets of $\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}$. There is $\{k(n)\colon n\in\omega\}\in\mathrm{fil}(\mathcal{A})\cap V$ such that

$$k(n+1)\in\bigcap\mathcal{G}_{k(n)}.$$

Proof. $\mathcal{G}$ is a p-set and therefore there is $C_0\in\mathcal{G}$ such that $C_0\subseteq^* G$ for each $G\in\bigcup\{\mathcal{G}_n\colon n\in\omega\}$. Thus, for some function $f\in{}^{\omega}\omega$

$$(\forall n\in\omega)\ C_0\setminus f(n)\subseteq\bigcap\mathcal{G}_n.$$

Since $\mathbb{P}_\alpha$ is ${}^{\omega}\omega$-bounding, without loss of generality $f\in V\cap{}^{\omega}\omega$, f is strictly increasing, and $n+2<f(n)$. Let us take $\{k(n)\colon n\in\omega\}\in\mathrm{fil}(\mathcal{A})\cap V$ from Lemma 3.15 such that $C=\{k(n+1)\colon n\in\omega\}\subseteq C_0$. Hence, we have $k(n+1)\in C_0\setminus f(k(n))$, and so $k(n+1)\in\bigcap\mathcal{G}_{k(n)}$. $\square$

We will prove by induction on $\alpha\leq\omega_2$ that the family $\mathcal{A}$ remains densely maximal in $V^{\mathbb{P}_\alpha}$. Suppose first that α is a limit. Note that for each $\beta\leq\alpha$, $(\mathrm{fil}(\mathcal{A}))^{V^{\mathbb{P}_\beta}}=\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}$. Now, suppose for each $\beta<\alpha$, $V^{\mathbb{P}_\beta}\vDash\mathcal{A}$ is densely maximal. That is

$$V^{\mathbb{P}_\beta}\vDash\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}\cup\langle\{\omega\backslash\mathcal{A}^h\colon h\in\mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}}=\mathcal{P}(\omega).$$

However, by Shelah's preservation theorem, $V^{\mathbb{P}_\alpha}\vDash\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}\cup\langle\{\omega\backslash\mathcal{A}^h\colon h\in\mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}}=\mathcal{P}(\omega)$. Thus $\mathcal{A}$ remains densely maximal in $V^{\mathbb{P}_\alpha}$.

Suppose $V^{\mathbb{P}_\alpha}\vDash\mathcal{A}$ is densely maximal. We will show that $V^{\mathbb{P}_{\alpha+1}}\vDash\mathcal{A}$ is densely maximal. In $V^{\mathbb{P}_{\alpha+1}}$, take any $Y\in\mathcal{P}(\omega)\backslash\langle\mathrm{fil}(\mathcal{A})\cap V\rangle_{\mathrm{up}}$. Suppose $Y\notin\langle\{\omega\backslash\mathcal{A}^h:h\in\mathrm{FF}(\mathcal{A})\}\rangle_{\mathrm{dn}}$. Thus, for all $h\in\mathrm{FF}(\mathcal{A})$, $Y\not\subseteq\omega\backslash\mathcal{A}^h$ and so for all $h\in\mathrm{FF}(\mathcal{A})$, $|Y\cap\mathcal{A}^h|=\omega$. Therefore in $V^{\mathbb{P}_\alpha}$ we can fix $p\in\mathbb{Q}_\alpha$ and a $\mathbb{Q}_\alpha$-name $\dot{Y}$ for Y such that for all $h\in\mathrm{FF}(\mathcal{A})$, $p\Vdash|\dot{Y}\cap\mathcal{A}^h|=\infty$.

By Lemma 2.12 we can assume that for all $m\in\omega$, for all $t\in\mathrm{split}_m(p)$ there is $u_t\in{}^{m+1}2$ such that $p(t)\Vdash\dot{Y}\restriction(m+1)=\check{u}_t$. Now, in $V^{\mathbb{P}_\alpha}$ for each $t\in p$, let

$$Y_t=\{m\in\omega:p(t)\nVdash\check{m}\notin\dot{Y}\}.$$

Claim 3.19.

(i) $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$.
(ii) If $s \subseteq t$ then $Y_t \subseteq Y_s$.
(iii) $Y_t \in \mathrm{fil}(\mathcal{A}) \cap V^{\mathbb{P}_\alpha}$.
(iv) If $m \in Y_s$ for $s \in \mathrm{split}_n(p)$, and $n < m$ then there is $t \in \mathrm{split}_m(p)$ extending s such that $p(t) \Vdash \check{m} \in \dot{Y}$.

Proof. (i) Let $m \in \dot{Y}[G]$ for a generic G containing $p(t)$. If $p(t) \Vdash \check{m} \notin \dot{Y}$ then $m \notin \dot{Y}[G]$, a contradiction.

(ii) Since $p(t) \subseteq p(s)$, from $p(t) \nVdash \check{m} \notin \dot{Y}$ we obtain $p(s) \nVdash \check{m} \notin \dot{Y}$.

(iii) If $Y_t \notin \mathrm{fil}(\mathcal{A}) \cap V^{\mathbb{P}_\alpha}$ then there is $h \in \mathrm{FF}(\mathcal{A})$ such that $Y_t \subseteq \omega \backslash \mathcal{A}^h$, i.e. $Y_t \cap \mathcal{A}^h = \emptyset$. Since $p(t) \Vdash \dot{Y} \subseteq \check{Y}_t$, then $p(t) \Vdash \mathcal{A}^h \cap \dot{Y} = \emptyset$. However, $p(t) \Vdash |\dot{Y} \cap \mathcal{A}^h| = \infty$, which is a contradiction.

(iv) Since $p(s) \nVdash \check{m} \notin \dot{Y}$ there is a condition $q \leq p(s)$ such that $q \Vdash \check{m} \in \dot{Y}$. However, by our assumption on p due to Lemma 2.12, for any $t \in \mathrm{split}_m(p)$ we have either $p(t) \Vdash \check{m} \in \dot{Y}$ or $p(t) \Vdash \check{m} \notin \dot{Y}$. Since $\{p(t) \colon t \in \mathrm{split}_m(p), t \supseteq s\}$ is pre-dense in $p(s)$, there is $t \in \mathrm{split}_m(p)$ extending s such that $p(t) \Vdash \check{m} \in \dot{Y}$. □

Claim 3.20. We can assume that a dense set $X \subseteq [p]$ with $\mathcal{C}$-different elements has the associated family $\{y_x \colon x \in X\}$ of sets in $\mathrm{fil}(\mathcal{A})$ such that if $t = x \restriction \mathrm{split}_n(p)$ then $p(t) \Vdash y_x(n) \in \dot{Y}$.

Proof. Hence, $Y_t \in \langle \mathrm{fil}(\mathcal{A}) \cap V \rangle_{\mathrm{up}}$ for each $t \in \mathrm{split}(p)$. By Claim 3.18 for $\mathcal{G}_n$ being the family of all Y_t's with $t \in \mathrm{split}_{\leq n+2}(p)$, we obtain $\{k(n) \colon n \in \omega\} \in \mathrm{fil}(\mathcal{A})$ such that

$$k(n+1) \in \bigcap \{Y_t \colon t \in \mathrm{split}_{\leq k(n)+2}(p)\}.$$

Moreover, by part (4) of Claim 3.19, for any $s \in \mathrm{split}_{k(n)+1}(p)$ there is $t \in \mathrm{split}_{k(n+1)}(p)$ extending s such that $p(t) \Vdash \check{k}(n+1) \in \dot{Y}$. For each branch $x \in [p]$ we consider set

$$i(x) = \{i \colon p(t) \Vdash \check{k}(i+1) \in \dot{Y} \text{ for } t = x \restriction \mathrm{split}_{k(i+1)}(p)\}.$$

We say that $x \in [p]$ is acceptable branch if $i(x)$ is cofinite. The smallest n with $i(x) \supseteq [n, \infty)$ is called a degree of acceptability of x. Due to part (4) of Claim 3.19 there are acceptable branches extending each $s \in p$. Note that for each acceptable branch x, $y_x = \{k(i+1) \colon i \in i(x)\} \in \mathrm{fil}(\mathcal{A})$. We continue using a fusion argument. We build a fusion sequence $\{(p_n, X_n)\}_{n \in \omega}$.

To define p_0, take some acceptable branch x extending some node in $\mathrm{split}_{k(0)+1}(p)$ with degree of acceptability at most 1, and a node $s = x \restriction \mathrm{split}_{k(1)}(p)$. We set $p_0 = p(s)$ and $X_0 = \{x\}$.

Let us assume that p_n and X_n are defined, and consider $s \in \mathrm{split}_{k(n)}(p) \cap \mathrm{split}(p_n)$. Take the unique acceptable branch $x \in X_n$ extending s. We set $q(s,i) = q_n(r)$. Define $i = x(|s|) \in \{0,1\}$ and $s_i = x \restriction \mathrm{split}_{k(n+1)}(p)$. Then we set s_{1-i} to be an extension of $s^\frown\langle 1-i \rangle$ such that:

(i) $[p(s_{1-i})] \cap C_{\alpha_x} = \emptyset$ for all already considered acceptable branches x (i.e., all branches in X_n and those assigned to previous nodes in some order of $\mathrm{split}_{k(n)}(p) \cap \mathrm{split}(p_n)$).
(ii) $s_{1-i} \in \mathrm{split}_{k(m+1)}(p)$ with $m \in i(x)$ for some acceptable branch $x \in [p]$ with degree of acceptability at most m.

Finally, let X_{n+1} be the set of all considered acceptable branches in this step, and

$$p_{n+1} = \bigcup\{p(s_i)\colon s \in \mathrm{split}_{k(n)}(p) \cap \mathrm{split}(p_n), i \in \{0,1\}\}.$$

One can see that the sequence $\{(p_n, X_n)\}_{n\in\omega}$ is a fusion sequence with witnesses. Moreover, the fusion $q = \bigcap\{p_n\colon n \in \omega\}$ satisfies the requirements. We set $X = \bigcup\{X_n\colon n \in \omega\}$.

We shall show that the family $\{y_x\colon x \in X\}$ possesses the desired properties. Indeed, let $x \in X$. For each $n \in \omega$ we have $y_x(n) = k(i(x)(n) + 1)$. Due to construction of q we have $x \restriction \mathrm{split}_n(q) = x \restriction \mathrm{split}_{k(j_n+1)}(p)$ for some increasing sequence $\{j_i\}_{i\in\omega}$, and if $t = x \restriction \mathrm{split}_n(q)$ then $p(t) \Vdash \check{k}(j_n + 1) \in \dot{Y}$. Thus $k(j_n + 1) \in y_x$ and consequently $j_n \geq i(x)(n)$ for each n. Let us now fix n and consider $t = x \restriction \mathrm{split}_n(q)$. The definition of y_x guaranties that $p(s) \Vdash \check{k}(i(x)(n) + 1) \in \dot{Y}$ for $s = x \restriction \mathrm{split}_{k(i(x)(n)+1)}(p)$. Thus we have $q(s) \Vdash \check{y}_x(n) \in \dot{Y}$. On the other hand, $s = x \restriction \mathrm{split}_{k(i(x)(n)+1)}(p) \subseteq x \restriction \mathrm{split}_{k(j_n+1)}(p) = x \restriction \mathrm{split}_n(q) = t$. □

Our last part of the proof resembles the proof of previous claim. Let x_s for $s \in \mathrm{split}(p)$ be the branch in X extending s such that if $s \subseteq t \subseteq x_s$ then $x_t = x_s$. The corresponding y_{x_s} is denoted y_s. The set y_s belongs to $\langle \mathrm{fil}(\mathcal{A}) \cap V\rangle_{\mathrm{up}}$. By Claim 3.18 for $\mathcal{G}_n$ being the family of all y_t's with $t \in \mathrm{split}_{\leq n+2}(p)$, we obtain $\{l(n)\colon n \in \omega\} \in \mathrm{fil}(\mathcal{A})$ such that

$$l(n + 1) \in \bigcap\{y_t\colon t \in \mathrm{split}_{\leq l(n)+2}(p)\}.$$

Let us denote $C = \{l(n+1)\colon n \in \omega\}$. We shall construct a condition $q \leq p$ such that $q \Vdash \check{C} \subseteq \dot{Y}$. Then $q \Vdash \dot{X} \in \mathrm{fil}(\mathcal{A})$ which is a contradiction.

We build a fusion sequence $\{(p_n, X_n)\}_{n\in\omega}$. Let $p_0 = p$, $X_0 = \{x_t\}$ for $t \in \mathrm{split}_0(p)$, and suppose we have defined p_n. For each $t \in \mathrm{split}_n(p_n)$ and each $i \in \{0,1\}$ take $w^*(t,i) \in \mathrm{split}_{l(n)+1}(p)$ such that $w^*(t,i)$ end-extends $t^\frown i$. Then

$$l(n + 1) \in \bigcap\{y_{w^*(t,i)}\colon t \in \mathrm{split}_n(p_n), i \in \{0,1\}\}$$

and so for each t,i we take $w(t,i) = x_{w^*(t,i)} \restriction \mathrm{split}_{l(n+1)}(p)$. Note that by Claim 3.20 and the fact that $l(n+1) \geq j$ for $l(n+1) = y_{w^*(t,i)}(j)$ we obtain

$$p(w(t,i)) \Vdash \check{l}(n + 1) \in \dot{Y}.$$

Take $p_{n+1} = \bigcup\{p(w(t,i))\colon t \in \mathrm{split}_n(p_n), i \in \{0,1\}\}$, $X_{n+1} = \{x_{w(t,i)}\colon t \in \mathrm{split}_n(p_n), i \in \{0,1\}\}$. □

The successor case of the above proof gives once again:

Corollary 3.21. Let $\mathcal{A}$ be a selective independent family and let G be a $\mathbb{Q}(\mathcal{C})$-generic filter. Then, in $V[G]$, $\mathcal{A}$ is still selective independent.

4. No small ultrafilter bases and tightness

4.1. **The poset $\mathbb{Q}_{\mathcal{I}}$.** For a maximal ideal $\mathcal{I}$ on ω, below $\mathbb{Q}_{\mathcal{I}}$ denotes the forcing notion introduced by S. Shelah in [46] for obtaining the consistency of $\mathfrak{i} < \mathfrak{u}$. In [46] it is shown that $\mathbb{Q}_{\mathcal{I}}$ is proper [46, Claim 1.13], ${}^\omega\omega$-bounding [46, Claim 1.12] and even has the Sacks property [46, Claim 1.12]. In the $\mathbb{Q}_{\mathcal{I}}$-generic extension, $\mathcal{I}$ is no longer a maximal ideal [46, Claim 1.5]. For completeness of the presentation we repeat below the definition and some of the key properties of $\mathbb{Q}_{\mathcal{I}}$.

Definition 1. *Let $\mathcal{I}$ be an ideal on ω.*

(1) An equivalence relation E on a subset of ω is an $\mathcal{I}$-equivalence relation if $\operatorname{dom} E \in \mathcal{I}^*$ and each E-equivalence class is in $\mathcal{I}$.
(2) For $\mathcal{I}$-equivalence relations E_1, E_2, we denote $E_1 \leq_{\mathcal{I}} E_2$ if $\operatorname{dom} E_1 \subseteq \operatorname{dom} E_2$, and E_1-equivalence classes are unions of E_2-equivalence classes.
(3) Let $A \subseteq \omega$. A function g is A-n-determined if $g\colon {}^A\{0,1\} \to \{0,1\}$ and there is $w \subseteq A \cap (n+1)$ such that for any $\eta, \nu \in {}^A\{0,1\}$ with $\eta \restriction w = \nu \restriction w$ we have $g(\eta) = g(\nu)$.

For $i \in A$, by g_i we denote a function from ${}^A\{0,1\}$ to $\{0,1\}$ which maps $\eta \in {}^A\{0,1\}$ to $\eta(i)$.

Claim 4.1. Each A-n-determined function is equal to a function $\varphi(g_0, \dots, g_n)$ which is obtained as an interpretation of a formula $\varphi(a_0, \dots, a_n)$ of propositional calculus. The symbols $\wedge$, $\vee$, $\neg$ are interpreted as a maximum, minimum, and complement (i.e., $1 - g_i$), respectively. The formula $\varphi(a_0, \dots, a_n)$ may contain constant symbols $0, 1$ which are interpreted as constant functions $0, 1$.

For an $\mathcal{I}$-equivalence relation E we denote $A = A(E) = \{x\colon x \in \operatorname{dom} E, x = \min[x]_E\}$.

Definition 4.2 (Set of conditions in $\mathbb{Q}_{\mathcal{I}}$)**.** Let $\mathcal{I}$ be an ideal on ω. We define a forcing notion $\mathbb{Q}_{\mathcal{I}}$:

$$p \in \mathbb{Q}_{\mathcal{I}} \text{ iff } p = (H, E) = (H^p, E^p) \text{ where}$$

(1) E is an $\mathcal{I}$-equivalence relation,
(2) H is a function with $\operatorname{dom} H = \omega$,
(3) a value $H(n)$ is an $A(E)$-n-determined function,
(4) if $n \in A(E)$ then $H(n) = g_n$,
(5) if $n \in \operatorname{dom} E \setminus A(E)$ and nEi for $i \in A(E)$ then $H(n)$ is g_i or $1 - g_i$.

For a condition $q \in \mathbb{Q}_{\mathcal{I}}$, let A^q be $A(E^q)$ in the following.

Definition 4.3. If $p, q \in \mathbb{Q}_{\mathcal{I}}$ with $A^p \subseteq A^q$ then we write $H^p(n) =^{**} H^q(n)$ if for each $\eta \in {}^{A^p}\{0,1\}$ we have $H^p(n)(\eta) = H^q(n)(\eta')$ where

$$\eta'(j) = \begin{cases} \eta(j) & j \in A^p, \\ H^p(j)(\eta) & j \in A^q \setminus A^p. \end{cases}$$

Definition 4.4 (The order of $\mathbb{Q}_{\mathcal{I}}$)**.** If $p, q \in \mathbb{Q}_{\mathcal{I}}$ then $p \leq q$ if

(1) $E^p \leq_{\mathcal{I}} E^q$,
(2) If $H^q(n) = g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = H^p(i)$,
(3) If $H^q(n) = 1 - g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = 1 - H^p(i)$,
(4) If $n \in \omega \setminus \operatorname{dom} E^q$ then $H^p(n) =^{**} H^q(n)$.

Finally, $p \leq_n q$ if $p \leq q$ and A^p contains the first n elements of A^q.

The following has been proven in [46]. Items (1) and (2) correspond to [46, Claim 1.7, (2)], item (3) is a straightforward modification of [46, Claim 1.8].

Claim 4.5. Let $p \in \mathbb{Q}_{\mathcal{I}}$. For an initial segment u of A^p, and $h\colon u \to \{0,1\}$, let $p^{[h]}$ be the pair $q = (H^q, E^q)$ defined by (i) and (ii) below:

(i) $E^q = E^p \restriction \bigcup\{[i]_{E^p} : i \in A^p \setminus u\}$.
(ii) If $H^p(n)$ is $\varphi(g_0, \dots, g_n)$ then $H^q(n)$ is $\varphi(g_0, \dots, g_i/h(i), \dots, g_n)$, where the substitution is done just for $i \in u$.

Then we have:

(1) $p^{[h]}$ is a condition in $\mathbb{Q}_{\mathcal{I}}$ stronger than p.
(2) The set $\{p^{[h]} : h \in {}^u\{0,1\}\}$ is predense below p.
(3) If u is the set of first n elements of A^p, D a dense subset of $\mathbb{Q}_{\mathcal{I}}$ then there is $q \in \mathbb{Q}_{\mathcal{I}}$ such that $q \leq_n p$ and $q^{[h]} \in D$ for any $h \in {}^u\{0,1\}$.

Definition 4.6 (The game $\mathrm{GM}_{\mathcal{I}}(E)$)**.** $\mathrm{GM}_{\mathcal{I}}(E)$ is the following game. In the n-th move, the first player chooses an $\mathcal{I}$-equivalence relation $E^1_n \leq_{\mathcal{I}} E^2_{n-1}$ ($E^1_0 = E$), and the second player chooses an $\mathcal{I}$-equivalence relation $E^2_n \leq_{\mathcal{I}} E^1_n$. In the end, the second player wins if

$$\bigcup_{n>0} (\operatorname{dom} E^1_n \setminus \operatorname{dom} E^2_n) \in \mathcal{I}.$$

Otherwise, the first player wins.

Remark 4.7. If the second player wins in the game $\mathrm{GM}_{\mathcal{I}}(E)$, then the game is invariant to taking subsets. That is, the game is invariant to taking $\leq_{\mathcal{I}}$-extensions $\{E^{2,*}_n\}_{n\in\omega}$ with $\operatorname{dom}(E^{2,*}_n) \subseteq \operatorname{dom} E^2_n$.

The next lemma corresponds to [46, Claim 1.10, (1)]

Lemma 4.8. The game $\mathrm{GM}_{\mathcal{I}}(E)$ is not determined for a maximal ideal $\mathcal{I}$.

4.2. **Tight `MAD` families.** Tight `MAD` families were investigated in [36, 33, 25]. An `AD` family $\mathcal{A}$ is called tight if for every $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.

Preservation theorem for tight `MAD` family under countable support iteration of proper forcing notions was developed by O. Guzmán, M. Hrušák and O. Téllez [25].

Definition 4.9. Let $\mathcal{A}$ be a tight `MAD` family. A proper forcing $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$ if for every $p \in \mathbb{P}$, M a countable elementary submodel of $H(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathbb{P}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there is $q \leq p$ an $(M, \mathbb{P})$-generic condition such that

$$q \Vdash \text{“}(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega\text{”},$$

where $\dot{G}$ denotes the name of generic filter.

We restate Corollary 32 by O. Guzmán, M. Hrušák and O. Téllez [25] which is crucial for preserving `MAD` families in the forthcoming model.

Theorem 4.10. *[O. Guzmán, M. Hrušák, O. Téllez] Let $\mathcal{A}$ be a tight* MAD *family. If the sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2 \rangle$ is a countable support iteration of proper posets such that*

$$\mathbb{P}_\alpha \Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \text{ strongly preserves the tightness of } \mathcal{A}\text{”},$$

then $\mathbb{P}_{\omega_2} \Vdash_\alpha$ “$\mathcal{A}$ is a tight MAD *family”.*

We need the following fact about the outer hulls observed in [25].

Lemma 4.11. Let $\mathcal{A}$ be an AD family, $\mathbb{P}$ a partial order, $\dot{B}$ a $\mathbb{P}$-name for a subset of ω and $p \in \mathbb{P}$ such that $p \Vdash$ “$\dot{B} \in \mathcal{I}(\mathcal{A})^+$”. Then the set $\{n \colon (\exists q \leq p)\ q \Vdash \text{“}n \in \dot{B}\text{”}\}$ is in $\mathcal{I}(\mathcal{A})^+$.

And now we are ready to show the main result of the paper.

Theorem 4.12. *Let $\mathcal{A}$ be a tight* MAD *family, $\mathcal{I}$ being a maximal proper ideal on ω. The poset $\mathbb{Q}_\mathcal{I}$ strongly preserves the tightness of $\mathcal{A}$.*

Proof. Let $p \in \mathbb{Q}_\mathcal{I}$, M a countable elementary submodel of $H(\kappa)$ such that $\mathcal{I}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. We fix an enumeration $\{D_n \colon n \in \omega\}$ of all open dense subsets of $\mathbb{Q}_\mathcal{I}$ that are in M, and an enumeration $\{\dot{Z}_n \colon n \in \omega\}$ of all $\mathbb{Q}_\mathcal{I}$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M with names repeating infinitely many times.

We define a strategy for the first player in the game $\mathrm{GM}_\mathcal{I}(E)$, which cannot be winning in all rounds.

We set $p_0 = q_0 = p$ and $u_0 = \emptyset$. We assume that the first player has chosen E^1_n, q_n, p_n, u_n, and the second one an E^2_n. We give instructions to choose E^1_{n+1}, q_{n+1}, p_{n+1}, u_{n+1}. We begin with q_{n+1}:

(1) $\operatorname{dom} E^{q_{n+1}} = \operatorname{dom} E^{p_n}$,
(2) $x E^{q_{n+1}} y$ iff one of the following holds:
  (i) $x E^2_n y$.
  (ii) There is $k \in u_n$ with $x, y \in [k]_{E^{p_n}}$ and $x, y \notin \operatorname{dom} E^2_n$.
  (iii) There are $k_0, k_1 \notin \bigcup\{[i]_{E^{p_n}} \colon i \in u_n\}$ with $x \in [k_0]_{E^{p_n}}$, $y \in [k_1]_{E^{p_n}}$ and $k_0, k_1 \notin \operatorname{dom} E^2_n$.
(3) $H^{q_{n+1}}$ is chosen such that:
  (i) If $l \in \omega \setminus \operatorname{dom} E^{p_n}$ then $H^{q_{n+1}}(l) =^{**} H^{p_n}(l)$.
  (ii) If $l \in \operatorname{dom} E^{p_n} \setminus A^{q_{n+1}}$, $H^{p_n}(l) = g_i$ then $H^{q_{n+1}}(l) = H^{q_{n+1}}(i)$.
  (iii) If $l \in \operatorname{dom} E^{p_n} \setminus A^{q_{n+1}}$, $H^{p_n}(l) = 1 - g_i$ then $H^{q_{n+1}}(l) = 1 - H^{q_{n+1}}(i)$.
  (iv) If $l \in A^{p_n} \setminus A^{q_{n+1}}$ then $H^{q_{n+1}}(l) =^{**} H^{p_n}(\min[l]_{E^{q_{n+1}}})$.

Note that for the already defined condition q_{n+1} we have $q_{n+1} \leq_n p_n$. Take $u_{n+1} = u_n \cup \{\min(A^{q_{n+1}} \setminus u_n)\}$. By Lemma 4.11, the set $D'_n = \{r \in \mathbb{Q}_\mathcal{I} \colon r \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n\text{”}\}$ is open dense below p (and also below q_{n+1}). Then $D'_n \cap D_n$ is dense below q_{n+1}. Therefore we can apply Lemma 4.8 to obtain $p_{n+1} \leq_{n+1} q_{n+1}$ such that for each $h \in {}^{u_{n+1}}\{0,1\}$, the condition $p^{[h]}_{n+1} \in D'_n \cap D_n \cap M$. In particular, if $h \in {}^{u_{n+1}}\{0,1\}$ then $p^{[h]}_{n+1} \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n \neq \emptyset\text{”}$ and $p^{[h]}_{n+1} \in D_n \cap M$. By Lemma 4.8 we have $p_{n+1} \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n \neq \emptyset\text{”}$. Finally, we set

$$E^1_{n+1} = E^{p_{n+1}} \restriction (\operatorname{dom} E^{p_{n+1}} \setminus \bigcup\{[i]_{E^{p_{n+1}}} \colon i \in u_{n+1}\}).$$

We define a fusion q of a sequence $\langle p_n : n \in \omega \rangle$. Relation E^q has $\operatorname{dom} E^q = \bigcap\{\operatorname{dom} E^{p_n} : n \in \omega\}$, and xE^qy if for every n large enough, $xE^{p_n}y$. Function H^q is equal to H^{p_n} for large enough n. In order to guarantee $q \in \mathbb{Q}_{\mathcal{I}}$, it is necessary to choose a play with the first player using described strategy, but he looses. Thus the second player wins and by Remark 4.7, we can assume that $\min \operatorname{dom} E^2_n > \max u_{n+1}$. Consequently, $\operatorname{dom} E^{p_n} \setminus \operatorname{dom} E^2_n \subseteq \bigcup\{[k]_{E^{q_{n+1}}} : k \in u_{n+1}\}$, and thus $\operatorname{dom} E^q \in \mathcal{I}^*$. One can check that other properties for $q \in \mathbb{Q}_{\mathcal{I}}$ are satisfied by the definition of q.

Finally, condition q is $(M, \mathbb{Q}_{\mathcal{I}})$-generic, and $q \leq_n p_n$ for each n. Hence, we have $q \Vdash$ "$(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega$". □

As a corollary we obtain that in Shelah's model of $\mathfrak{i} < \mathfrak{u}$ (see [46]), also the almost disjointness number is small. Thus, in a certain sense $\mathbb{Q}_{\mathcal{I}}$ is optimal for the ultrafilter number $\mathfrak{u}$.

Corollary 4.13. It is relatively consistent that $\mathfrak{a} = \mathfrak{i} < \mathfrak{u}$.

Moreover, using the preservation results of the current article, together with the preservation results of [25], as well as the fact that $\mathbb{Q}_{\mathcal{I}}$ has the Sacks property, we obtain:

Corollary 4.14. It is relatively consistent that $\operatorname{cof}(\mathcal{N}) = \mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2$.

Proof. Work over a model of CH. Let $\mathcal{A}_0$ be Shelah's selective independent family and let $\mathcal{A}_1$ be a tight mad family. Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ of posets such that for even α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{P}(\mathcal{K})$ for some uncountable partition of 2^ω into compact sets, for odd α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}_{\mathcal{I}}$ for some maximal ideal on ω, and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \mathfrak{u} = \omega_2$. The iteration $\mathbb{P}_{\omega_2}$ has the Sacks property and therefore $\operatorname{cof}(\mathcal{N}) = \omega_1$. By the indestructibility of selective independence the family $\mathcal{A}_0$ remains maximal independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. Moreover, by the preservation properties of tight MAD families, see [25], and the above preservation theorems, $\mathcal{A}_1$ is a witness to $\mathfrak{a} = \omega_1$ in the final model. □

Appendix: The problem of Vaughan

We conclude the paper with an overview of the problem of Vaughan and point out towards the many difficulties surrounding a possible solution of it, in particular the fact that the most common forcing methods do not seem to help with the problem:

(1) *Finite support iteration of ccc forcings of length a regular cardinal over a model of* CH. This approach can not work since in the models obtained in this way, the size of the continuum is equal to *cov* $(\mathcal{M})$ and it is known that *cov* $(\mathcal{M}) \leq \mathfrak{i}$.[5]

(2) *Countable support iteration of definable proper forcings of length* ω_2 *over a model of* CH. It follows by the results of M. Džamonja, M. Hrušák and J. Moore in [43] that in all of these models the equality $\mathfrak{b} = \mathfrak{a}$ will hold, so in particular we will have that $\mathfrak{a} \leq \mathfrak{i}$.

[5]S. Shelah proved that $\mathfrak{d} \leq \mathfrak{i}$ (see the appendix of [50]). This result was improved by B. Balcar, F. Hernandez-Hernández and M. Hrušák in [2] where they proved that *cof* $(\mathcal{M}) \leq \mathfrak{i}$.

(3) *Countable support iteration of non-definable proper forcings of length ω_2 over a model of CH.* This approach could work, however a model of $\mathfrak{i} < \mathfrak{a}$ obtained by this method will also be a model of $\omega_1 = \mathfrak{d} < \mathfrak{a}$, (since $\mathfrak{d} \leq \mathfrak{i}$) thus solving the problem of Roitman, which is considered to be one of the hardest problems on the theory of cardinal invariants.

(4) *Forcing with ultrapowers and iterating along a template.* The method of forcing with ultrapowers and iterating along a template was introduced by S. Shelah in [47] to build models of $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$. This is a very powerful method that has been very useful and has been successfully applied to this day. Unfortunately, it seems that all forcings obtained using this method, tend to increase $\mathfrak{i}$ for the same reason they increase $\mathfrak{a}$. To learn more about this powerful method, see [9, 10, 8, 22, 38, 18, 16].

(5) *Short finite support iterations over models of MA.* Performing a finite support iteration of length ω_1 over a model of MA (for example) is a powerful method to add "small witnesses" of some cardinal invariants while keeping others large. Models obtained in this way are often called "dual models" (see [13] for several interesting results and applications of this methods). In [2] a dual model was constructed to add a small maximal independent family in order to build a model of $\mathfrak{i} < non(\mathcal{N})$. Unfortunately, it is not clear how one could avoid adding a small MAD family with this method. Moreover, it seems likely that the principle $\diamondsuit_{\mathfrak{d}}$ of M. Hrušák will hold in this models[6] (see [27]).

In principle, it could be possible to construct a model of $\mathfrak{i} < \mathfrak{a}$ using matrix iterations (see [7], [11] and [39]) to learn more about this method), but one would need to be very careful in order to avoid problems like in the points 1 and 5 above.

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References

[1] L. Aurichi, L. Zdomskyy *Covering properties of ω-mad families* Arch. Math. Logic, 59 (3-4), 445–452, 2020.
[2] B. Balcar, F. Hernaández-Hernández, and M. Hrušák *Combinatorics of dense subsets of the rationals* Fund. Math., 183(1), 59-80, 2004.
[3] T. Bartoszyński, H. Judah *Set theory: on the structure of the real line.*, ISBN 1-56881-044-X, ix + 546, 1995, Wellesley, MA: A. K. Peters Ltd.
[4] J. Baumgartner *Sacks forcing and the total failure of Martin's axiom* Topology Appl., 19 (3), 211-225, 1985.
[5] J. Baumgartner, R. Laver *Iterated perfect-set forcing* Ann. Math. Logic, 17(3), 271-288, 1979.
[6] A. Blass *Combinatorial cardinal characteristics of the continuum*, in: Foreman M., Kanamori A. (eds) Handbook of Set Theory. Springer, Dordrecht (2010), 24–27.
[7] A. Blass, S. Shelah *Ultrafilters with small generating sets* Israel J. Math., 65(3):259-271, 1989.
[8] J. Brendle *The almost-disjointness number may have countable cofinality*, Trans. Amer. Math. Soc., 355 (7), 2633–2649, 2003.
[9] J. Brendle *Mad families and iteration theory*, Logic and algebra, Contemp. Math., 302, 1–31, Amer. Math. Soc., Providence, RI, 2002.
[10] J. Brendle *Mad families and ultrafilters* Acta Univ. Carolin. Math. Phys., 48(2):19-35, 2007.

[6] $\diamondsuit_{\mathfrak{d}}$ is the following principle: There is a family $\{d_\alpha \mid \alpha \in \omega_1\}$ such that $d_\alpha : \alpha \longrightarrow \omega$ and for every $F : \omega_1 \longrightarrow \omega_1$ there is $\alpha \geq \omega$ such that $F \restriction \alpha \leq^* d_\alpha$. In [27] it was proved that $\diamondsuit_{\mathfrak{d}}$ implies both $\mathfrak{d} = \omega_1$ and $\mathfrak{a} = \omega_1$.

[11] J. Brendle, V. Fischer *Mad families, splitting families and large continuum* J. Symbolic Logic, 76(1):198-2008, 2011.

[12] J. Brendle, V. Fischer, Y. Khomskii *Definable maximal independent families*, Proc. Amer. Math. Soc., 147 (8), 3547–3557, 2019.

[13] J. Brendle, J. Flašková *Generic existence of ultrafilters on the natural numbers* Fund. Math., 236 (3), 201–245, 2017.

[14] D. Chodounský, V. Fischer, J. Grebík *Free sequences in* $\mathcal{P}(\omega)/\text{fin}$ Arch. Math. Logic, 58 (7-8), 1035–1051, 2019.

[15] D. Fernández-Bretón, M. Hrušák *A parametrized diamond principle and union ultrafilters* Colloq. Math., 153(2), 261-271, 2018.

[16] V. Fischer, S. D. Friedman, D. Mejía, D. Montoya *Coherent systems of finite support iterations* J. Symb. Log., 83 (1), 208–236, 2018.

[17] V. Fischer, S. D. Friedman, L. Zdomskyy *Projective wellorders and mad families with large continuum*, Ann. Pure Appl. Logic, 162 (11), 853–862, 2011.

[18] V. Fischer, D. Mejía *Splitting, bounding, and almost disjointness can be quite different*, Canad. J. Math., 69 (3), 502–531, 2017.

[19] V. Fischer, D. Montoya *Ideals of independence* Archive for Mathematical Logic 58(5-6), 767-785, 2019

[20] V. Fischer, S. Shelah *The spectrum of independence*, Arch. Math. Logic, 58 (7-8), 877–884, 2019.

[21] V. Fischer, J. Šupina *Selective independence,* preprint.

[22] V. Fischer, A. Törnquist *Template iterations and maximal cofinitary groups*, Fund. Math., 230 (3), 205–236, 2015.

[23] S. Geschke, S. Quickert *On Sacks forcing and the Sacks property*, Classical and new paradigms of computation and their complexity hierarchies, Trends Log. Stud. Log. Libr., 23, 95–139, Kluwer Acad. Publ., Dordrecht, 2004.

[24] M. Goldstern, S. Shelah *Ramsey ultrafilters and the reaping number Con*($\mathfrak{r} < \mathfrak{u}$) Ann. Pure Appl. Logic 49 (1990), no. 2, 121-142.

[25] O. Guzmán, M. Hrušák, O. Téllez *Restricted* MAD *families,* J. Symbolic Logic 85 (2020), pp. 149–165.

[26] L. Halbeisen, *Combinatorial Set Theory. With a gentle introduction to forcing,* 2nd edition, Springer Monogr. Math., Springer, London, 2017, pp. xvi+594.

[27] M. Hrušák *Another* $\diamondsuit$*-like principle*, Fund. Math, 167, 277–289, 2001.

[28] M. Hrušák *Selectivity of almost disjoint families* Acta Univ. Carolin. Math. Phys., 4182, 13-21, 2000.

[29] M. Hrušák *Almost disjoint families and topology*, Recent progress in general topology. III, 601–638, Atlantis Press, Paris, 2014.

[30] M. Hrušák *Life in the Sacks model* 29th Winter School on Abstract Analysis (Lhota nad Rohanovem/Zahrádky u České Lípy, 2001), Acta Univ. Carolin. Math. Phys., 42, 43–58, 2001.

[31] M. Hrušák, S. García Ferreira *Ordering* MAD *families a la Katětov,* J. Symbolic Logic 68 (2003), pp. 1337–1353.

[32] A. Kechris *Classical descriptive set theory*, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, xviii+402, 1995.

[33] M. Kurilič *Cohen-stable families of subsets of integers,* J. Symbolic Logic 66 (2001), pp. 257–270.

[34] C. Laflamme *Filter games and combinatorial properties of strategies*, Set theory (Boise, ID, 1992–1994), Contemp. Math., 192, 51–67, Amer. Math. Soc., Providence, RI, 1996.

[35] C. Laflamme, C. C. Leary *Filter games on* ω *and the dual ideal*, Fund. Math., 159–173 173(2), 2002.

[36] V.I. Malykhin *Topological properties of Cohen generic extensions,* Trans. Moscow Math. Soc. 52 (1990), pp. 1–32.

[37] A.R.D. Mathias, *Happy families,* Ann. Math. Log. 12 (1977), pp. 59–111.

[38] D. A. Mejía *Template iterations with non-definable ccc forcing notions*, Ann. Pure Appl. Logic, 166(11), 1071–1109, 2015.

[39] D. A. Mejía *Matrix iterations and Cichon's diagram*, Arch. Math. Logic, 52 (3-4), 261–278, 2013.

[40] A. W. Miller *Mapping a set of reals onto the reals*, J. Symbolic Logic, 48, 575–584, 1983.
[41] A. W. Miller *Some properties of measure and category* Transactions of the American Mathematical Society, 266 (1), 93–114, 1981.
[42] A. W. Miller *Covering* 2^ω *with* ω_1 *Disjoint Closed Sets* Jon Barwise and H. Jerome Keisler and Kenneth Kunen, Studies in Logic and the Foundations of Mathematics, Elsevier, 101, 415-421, 1980.
[43] J. T. Moore, M. Hrušák, M. Džamonja *Parametrized* $\diamondsuit$ *principles*, Trans. Amer. Math. Soc., 356 (6), 2281-2306, 2004.
[44] L. Newelski *On partitions of the real line into compact sets*, Journal of Symbolic Logic (52), 353-359, 1987.
[45] M. J. Perron *On the Structure of Independent Families*, Thesis (Ph.D.)–Ohio University, ProQuest LLC, Ann Arbor, MI, 2017.
[46] S. Shelah *Con*$(\mathfrak{u} > \mathfrak{i})$*,* Arch. Math. Logic 31 (6), 1992, pp. 433–443.
[47] S. Shelah *Two cardinal invariants of the continuum* $(\mathfrak{d} < \mathfrak{a})$ *and FS linearly ordered iterated forcing* Acta Math., 192(2), 187–223, 2004.
[48] S. Shelah *Proper and improper forcing,* Perspectives in Mathematical Logic, Vol. 5, 2nd ed., Springer-Verlag, Berlin, 1998.
[49] O. Spinas *Partition numbers*, Ann. Pure Appl. Logic, 90 (1-3), 243–262, 1997.
[50] J. E. Vaughan *Small uncountable cardinals and topology* In Open problems in topology, 195-218. North-Hollan, amsterdam, 1990. With appendix by S. Shelah
[51] J. Zapletal *Descriptive set theory and definable forcing*, Mem. Amer. Math. Soc., 167 (793), viii+141, 2004.
[52] J. Zapletal *Forcing idealized*, Cambridge Tracts in Mathematics, 174, Cambridge University Press, Cambridge, 2008.
[53] Y. Y. Zheng *Selective ultrafilters on FIN*, Proc. Amer. Math. Soc., 145 (12), 5071–5086, 2017.

Centro de Ciencias Matemáticas, UNAM, Mexico
Email address: chapi@matmor.unam.mx

Institute of Mathematics, University of Vienna, Augasse 2-6, 1090 Vienna, Austria
Email address, corresponding author: vera.fischer@univie.ac.at

Centro de Ciencias Matemáticas, UNAM, Mexico
Email address: oguzman@matmor.unam.mx

Institute of Mathematics, P.J. Šafárik University in Košice, Jesenná 5, 040 01 Košice, Slovakia
Email address: jaroslav.supina@upjs.sk