REFINING SYSTEMS OF MAD FAMILIES

VERA FISCHER, MARLENE KOELBING, AND WOLFGANG WOHOFSKY

Abstract. We construct a model in which there exists a refining matrix of regular height λ larger than $\mathfrak{h}$; both $\lambda = \mathfrak{c}$ and $\lambda < \mathfrak{c}$ are possible. A refining matrix is a refining system of mad families without common refinement. Of particular interest in our proof is the preservation of $\mathcal{B}$-Canjarness.

1. Introduction

The Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ has attracted a lot of attention in the last decades. The distributivity of $\mathcal{P}(\omega)/\text{fin}$, the well-known *distributivity number* $\mathfrak{h}$, was introduced in [2], where also the famous base matrix theorem is proved. It is defined as the least number of mad families such that there is no single mad family refining all of them and as we will see is tightly connected to many other structural properties of $\mathcal{P}(\omega)/\text{fin}$. Equivalently, $\mathfrak{h}$ can be defined as the least cardinal on which $\mathcal{P}(\omega)/\text{fin}$ adds a new function into the ordinals and clearly, a system of $\mathfrak{h}$ many mad families can be always chosen to be refining. A review of basic definitions will be given in Section 2.

In this paper, we consider refining matrices of arbitrary height:

Definition 1.1. We say that $\mathcal{A} = \{A_\xi \mid \xi < \lambda\}$ is a *refining matrix of height* λ if

(1) A_ξ is a mad family, for each $\xi < \lambda$,
(2) A_η refines A_ξ whenever $\eta \geq \xi$, and
(3) there is no common refinement, i.e., there is no mad family B which refines every A_ξ.

Note that dropping (2) in the above definition would not yield an interesting notion, because such objects trivially exist whenever $\lambda \geq \mathfrak{h}$. It is straightforward to check that the existence of refining matrices is only a matter of cofinality: if δ is a singular cardinal with $\text{cf}(\delta) = \lambda$, then there exists a refining matrix of height δ if and only if there exists one of height λ. While $\mathfrak{h}$ is the minimal height of a refining matrix, it is easy to check that there can never be a refining matrix of regular height larger than $\mathfrak{c}$. Refining matrices and similar objects have been extensively studied, for example in [2], [11], [13], [23], [10], [1], and [30]. However, to the best of our knowledge, all refining matrices considered in the above works, are of height $\mathfrak{h}$. Thus, the very natural question arises, if a refining matrix is necessarily of height $\mathfrak{h}$, which leads to the main result of the current paper:

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Main Theorem 1.2. *Let V_0 be a model of ZFC which satisfies GCH. In V_0, let $\omega_1 < \lambda \leq \mu$ be cardinals such that λ is regular and $\mathrm{cf}(\mu) > \omega$. Then there is a c.c.c. (and hence cofinality preserving) extension W of V_0 in which there exists a refining matrix of height λ, and $\omega_1 = \mathfrak{h} = \mathfrak{b} < \mathfrak{c} = \mu$.*

Thus, in particular, the existence of refining matrices of two different regular heights is consistent. We construct the model W as follows. We start with V_0 and pass to the Cohen extension V in which $\mathfrak{c} = \mu$. In V, we define a forcing iteration (see Section 3.1) which adds a refining matrix of height λ. Building on ideas from [21], we use c.c.c. iterands which approximate the refining matrix by finite conditions. However we have to use an iteration, because after a single step of the forcing new reals are added, which prevents the generically added almost disjoint families from being maximal. We show that the generic object is actually a refining matrix: in particular, the branches are towers (see Section 4.3) and the levels are mad families (see Section 4.4); for that, we use complete subforcings (see Section 3.4) to capture new subsets of ω (see Section 4.2).

To establish $\omega_1 = \mathfrak{h} = \mathfrak{b}$ in the final model, we show that $\mathfrak{b} = \omega_1$ and use the fact that $\mathfrak{h} \leq \mathfrak{b}$ holds in ZFC. In fact, we show that the ground model reals $\mathcal{B} = \omega^\omega \cap V_0$ remain unbounded. For that, we represent our iteration as a finer iteration of Mathias forcings with respect to carefully selected filters (see Section 6.1) and use a characterization from [20] to show that these filters are $\mathcal{B}$-Canjar (see Section 5 and Section 6.2), i.e., that the corresponding Mathias forcings preserve the unboundedness of $\mathcal{B}$. [1] One can use a genericity argument to show that the chosen filters are $\mathcal{B}$-Canjar at the stage where they appear, however the $\mathcal{B}$-Canjarness of the filters is needed in later stages of the iteration. Since the notion of $\mathcal{B}$-Canjarness of a filter is not absolute (see[2] Example 5.4), we develop a new method allowing us to guarantee that the $\mathcal{B}$-Canjarness of a filter is preserved by Mathias forcings with respect to certain other filters. One basic ingredient is the notion of a "sum" $\mathcal{F}_0 \oplus \mathcal{F}_1$ of two, or finitely many, filters $\mathcal{F}_0$ and $\mathcal{F}_1$ for which the following holds true (see Lemma 5.8.(1)):

Proposition 1.3. *If $\mathcal{B} \subseteq \omega^\omega$ is unbounded and $\mathcal{F}_0 \oplus \mathcal{F}_1$ is $\mathcal{B}$-Canjar, then Mathias forcing with respect to $\mathcal{F}_1$ forces "$\mathcal{F}_0$ is $\mathcal{B}$-Canjar".*

We conclude the paper with further discussion and some open questions. In Section 7.2, we consider the nature of maximal branches through refining matrices. There are two possibilities for such branches: either the branch is cofinal or not. Consistently, there are refining matrices of height $\mathfrak{h}$ without cofinal branches (this was shown in [11] and [13]). In contrast, in the model of Main Theorem 1.2 all maximal branches of the generic refining matrix of height $\lambda > \mathfrak{h}$ are cofinal. In the Cohen model, however, there are no refining matrices of this type of height larger than $\mathfrak{h}$. In Section 7.3, we conclude the paper with a discussion of the notion of a spectrum of refining systems of mad families which our main result gives rise to.

2. Preliminaries

In this section, we recall basic definitions and facts. The reader should feel free to skip this section and only come back if necessary.

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[1]In [15], the same is done for Hechler's original forcings [21] to add a tower or to add a mad family.

[2]We thank Osvaldo Guzmán [19] for providing an example of non-absoluteness.

Let $[\omega]^\omega$ denote the collection of infinite subsets of ω and let $\subseteq^*$ denote the pre-order of almost-inclusion: $b \subseteq^* a$ if $b \setminus a$ is finite. We write $a =^* b$ if $a \subseteq^* b$ and $b \subseteq^* a$. We say that a and b are *almost disjoint* if $a \cap b$ is finite. Moreover, we say that $A \subseteq [\omega]^\omega$ is an *almost disjoint family* if a and a' are almost disjoint whenever $a, a' \in A$ with $a \neq a'$. An almost disjoint family A is *maximal* (called *mad family*) if for each $b \in [\omega]^\omega$ there exists $a \in A$ such that $|b \cap a| = \aleph_0$ (i.e., if A is a maximal antichain in $([\omega]^\omega, \subseteq^*)$). For two almost disjoint families A and B, we say that B *refines* A if for each $b \in B$ there exists an $a \in A$ with $b \subseteq^* a$. Let $spec(\mathfrak{a}) := \{\mu \mid \mu$ is an infinite cardinal and there is a mad family of size $\mu\}$ be the *mad spectrum on* ω, and let $\mathfrak{a} := \min(spec(\mathfrak{a}))$ be the *almost disjointness number*. It is well-known and easy to see that there are always mad families of size $\mathfrak{c}$. For a sequence $\langle a_\xi \mid \xi < \delta \rangle \subseteq [\omega]^\omega$, we say that $b \in [\omega]^\omega$ is a *pseudo-intersection* of $\langle a_\xi \mid \xi < \delta \rangle$ if $b \subseteq^* a_\xi$ for each $\xi < \delta$. We say that $\langle a_\xi \mid \xi < \delta \rangle$ is a *tower of length* δ if $a_\eta \subseteq^* a_\xi$ for any $\eta > \xi$, and it does not have an infinite pseudo-intersection. Let $spec(\mathfrak{t}) := \{\delta \mid \delta$ is regular and there is a tower of length $\delta\}$ be the *tower spectrum*, and let $\mathfrak{t} := \min(spec(\mathfrak{t}))$ be the *tower number*.

Recall from Definition 1.1 that a refining matrix $\mathfrak{A} = \{A_\xi \mid \xi < \lambda\}$ is a refining system of mad families without common refinement. Such a system can be viewed as a tree, which we think of growing downwards: for each $\xi < \lambda$, the elements of the mad family A_ξ form the level ξ of the tree, and for $b \in A_\eta$ and $a \in A_\xi$ with $\eta > \xi$, the element b is below the element a in the tree if and only if $b \subseteq^* a$. Due to the refining structure of the refining matrix, each element of A_η is below exactly one element of A_ξ. Note that this tree is necessarily splitting[3] at some limit levels, because there always appear $\subseteq^*$-decreasing sequences of limit length which have no weakest lower bound. We say that $\langle a_\xi \mid \xi < \delta \rangle$ is a *branch through* $\mathcal{A}$ if $a_\xi \in A_\xi$ for each $\xi < \delta$, and $a_\eta \subseteq^* a_\xi$ for each $\xi \leq \eta < \delta$. We say that the branch is *maximal* if there is no branch through $\mathcal{A}$ strictly extending it. As a matter of fact, a maximal branch through a refining matrix can be cofinal or not; for a discussion of different types of refining matrices (in particular ones without cofinal branches), see Section 7.2. We say that $b \in [\omega]^\omega$ *intersects* $\mathcal{A}$ if for each $\xi < \lambda$ there is an $a \in A_\xi$ with $b \subseteq^* a$. Definition 1.1(3) is in fact equivalent to

(3') $\{b \in [\omega]^\omega \mid b$ intersects $\mathcal{A}\}$ is not dense in $([\omega]^\omega, \subseteq^*)$.

If there is *no* b intersecting $\mathcal{A}$, we call $\mathcal{A}$ *normal*. Note that a refining matrix can always be turned into a normal refining matrix of the same height: let $\bar{a}$ be a witness for (3'), and, for each $\xi < \lambda$, let $\bar{A}_\xi := \{a \cap \bar{a} \mid a \in A_\xi \wedge |a \cap \bar{a}| = \omega\}$; it is easy to check that $\{\bar{A}_\xi \mid \xi < \lambda\}$ is a refining matrix of height λ below $\bar{a}$. We say that $\mathcal{A}$ is a *base matrix* if $\bigcup_{\xi<\lambda} A_\xi$ is dense in $([\omega]^\omega, \subseteq^*)$. This notion goes back to [2] where the existence of base matrices of height $\mathfrak{h}$ has been shown. It is straightforward to check that a base matrix is always normal.

For $f, g \in \omega^\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We say that $\mathcal{B} \subseteq \omega^\omega$ is an *unbounded family*, if there exists no $g \in \omega^\omega$ with $f \leq^* g$ for all $f \in \mathcal{B}$. The *(un)bounding number* $\mathfrak{b}$ is the smallest size of an unbounded family in ω^ω. The following inequalities between the cardinal characteristics are well-known and not too hard to prove (see, e.g., [4] for more details):

$$\omega_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}. \tag{1}$$

[3]See also the discussion in Section 3.1 about the generic refining matrix of Main Theorem 1.2, whose underlying tree is splitting everywhere.

3. Forcing a refining matrix

In this section, we start with the proof of Main Theorem 1.2. In Section 3.1 we define a key forcing notion, study its basic properties, as well as properties of the generic object (see Sections 3.2 and 3.3) and establish a crucial lemma about complete subforcings (see Section 3.4). In Section 4, we will complete the proof that the generic object is indeed a refining matrix. Section 5 and Section 6 are devoted to the remaining part of our main result, i.e. to showing that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ in the final model.

3.1. **Definition of the forcing iteration.** We will define a forcing for adding a refining matrix. The definition has been motivated by the posets for adding towers and mad families from Hechler's paper [21]; see [15] for a representation of these forcings in a form analogous to our definition of $\mathbb{Q}_\alpha$ below.

We proceed as follows. In V_0, let $\mathbb{C}_\mu$ be the usual forcing for adding μ many Cohen reals and let V be the extension by $\mathbb{C}_\mu$. In V, we perform a finite support iteration of length λ to add a refining matrix of height λ. The iterands of this iteration have the countable chain condition (see Lemma 3.3) and are of size continuum. In particular, the size of the continuum stays the same during the whole iteration and $\mathfrak{c} = \mu$ in the final model (see Lemma 3.4.(1)). Our generic refining matrix $\{A_{\xi+1} \mid \xi < \lambda\}$ will be based on $\lambda^{<\lambda}$: each node $\sigma \in \lambda^{<\lambda}$ of successor length will carry an infinite set $a_\sigma \subseteq \omega$ such that for each $\xi < \lambda$,

$$A_{\xi+1} = \{a_\sigma \mid \sigma \in \lambda^{\xi+1}\}$$

is a mad family, and $a_\sigma \subseteq^* a_\tau$ if σ extends τ. All maximal branches of the generic matrix will be cofinal. We write $\tau \trianglelefteq \sigma$ if $\tau \subseteq \sigma$ (i.e., if σ extends τ); we write $\tau \triangleleft \sigma$ if $\tau \trianglelefteq \sigma$ and $\tau \neq \sigma$. The length of σ is denoted by $|\sigma|$. We say that σ is *below* τ if $\tau \trianglelefteq \sigma$; moreover, we say that $\rho^\frown j$ is to the *left* of $\rho^\frown i$ whenever $j < i$, and we call a set of nodes a *block* if it is of the form $\{\rho^\frown i \mid i < \lambda\}$ for some $\rho \in \lambda^{<\lambda}$. Note that our mad families $A_{\xi+1}$ are indexed by successor ordinals only, because there are $\subseteq^*$-decreasing sequences of limit length which do not have weakest lower bounds, and therefore it is necessary that the underlying tree "splits" at such limit levels. Before giving the precise definition of our forcing iteration, let us explain why an iteration is needed. We generically add a set $a_\sigma \subseteq \omega$ for every $\sigma \in \lambda^{<\lambda}$ in V of successor length in such a way that $a_\tau \supseteq^* a_\sigma$ if $\tau \trianglelefteq \sigma$, and $a_\sigma \cap a_\tau =^* \emptyset$ if $|\sigma| = |\tau|$, resulting in a refining system of almost disjoint families. But since new reals and hence new branches through $\lambda^{<\omega}$ are added, the almost disjoint family $A_{\omega+1}$ is not maximal, witnessed by any pseudo-intersection of such a new branch.

As usual, we abuse notation and identify $a_\sigma \subseteq \omega$ with its characteristic function in 2^ω. Throughout succ denotes sequences τ of ordinals, where τ has successor length.

Definition 3.1. We define a finite support iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\}$ of length λ. Let $\mathbb{P}_0 = \{\emptyset\}$. Let $\alpha < \lambda$ and assume $\mathbb{P}_\alpha$ has been defined. For every $\beta \leq \alpha$, let G_β be generic for $\mathbb{P}_\beta$, let $T'_\alpha = \bigcup_{\beta<\alpha}(\lambda^{<\lambda} \cap \mathrm{succ})^{V[G_\beta]}$ and let $T_\alpha = (\lambda^{<\lambda} \cap \mathrm{succ})^{V[G_\alpha]} \setminus T'_\alpha$. In $V[G_\alpha]$, $\mathbb{Q}_\alpha$ consists of all p, where p is a function with finite domain, $\mathrm{dom}(p) \subseteq T_\alpha$, such that for each $\sigma \in \mathrm{dom}(p)$, $p(\sigma) = (s^p_\sigma, f^p_\sigma, h^p_\sigma) = (s_\sigma, f_\sigma, h_\sigma)$ where[4]

(1) $s_\sigma \in 2^{<\omega}$,
(2) if $\tau \triangleleft \sigma$, then $|s_\tau| \geq |s_\sigma|$,
(3) $\mathrm{dom}(f_\sigma) \subseteq (\mathrm{dom}(p) \cup T'_\alpha) \cap \{\tau \in T_\alpha \cup T'_\alpha \mid \tau \triangleleft \sigma\}$ is finite,
(4) $f_\sigma\colon \mathrm{dom}(f_\sigma) \to \omega$,

[4]The paragraph after the definition gives a short intuitive explanation of the roles of s_σ, f_σ, and h_σ.

(5) whenever $\tau \in \mathrm{dom}(f_\sigma) \cap T_\alpha$, $n \in \mathrm{dom}(s_\sigma)$ are such that $n \geq f_\sigma(\tau)$ (hence $n \in \mathrm{dom}(s_\tau)$ by (2) and (3)), we have

$$s_\tau(n) = 0 \to s_\sigma(n) = 0,$$

and whenever $\tau \in \mathrm{dom}(f_\sigma) \cap T'_\alpha$, $n \in \mathrm{dom}(s_\sigma)$ are such that $n \geq f_\sigma(\tau)$, we have

$$a_\tau(n) = 0 \to s_\sigma(n) = 0,$$

(6) $\mathrm{dom}(h_\sigma) \subseteq \mathrm{dom}(p) \cap \{\rho^\frown j \mid j < i\}$, where $\rho \in \lambda^{<\lambda}$ and $i \in \lambda$ are such that $\sigma = \rho^\frown i$,
(7) $h_\sigma \colon \mathrm{dom}(h_\sigma) \to \omega$,
(8) whenever $\tau \in \mathrm{dom}(h_\sigma)$, $n \in \mathrm{dom}(s_\tau) \cap \mathrm{dom}(s_\sigma)$ are such that $n \geq h_\sigma(\tau)$, we have

$$s_\tau(n) = 0 \vee s_\sigma(n) = 0.$$

The order on $\mathbb{Q}_\alpha$ is defined as follows: $q \leq p$ ("q is stronger than p") if

(i) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
(ii) and for each $\sigma \in \mathrm{dom}(p)$, we have
  (a) $s^p_\sigma \trianglelefteq s^q_\sigma$,
  (b) $\mathrm{dom}(f^p_\sigma) \subseteq \mathrm{dom}(f^q_\sigma)$ and $f^p_\sigma(\tau) \geq f^q_\sigma(\tau)$ for each $\tau \in \mathrm{dom}(f^p_\sigma)$,
  (c) $\mathrm{dom}(h^p_\sigma) \subseteq \mathrm{dom}(h^q_\sigma)$ and $h^p_\sigma(\tau) \geq h^q_\sigma(\tau)$ for each $\tau \in \mathrm{dom}(h^p_\sigma)$.

Given a generic filter G for $\mathbb{Q}_\alpha$, we define for each $\sigma \in T_\alpha$,

$$a_\sigma = \bigcup \{s^p_\sigma \mid p \in G \wedge \sigma \in \mathrm{dom}(p)\}.$$

This completes the definition of the forcing.

In the above definition s_σ is a finite approximation of the set a_σ assigned to σ, whereas the functions f_σ and h_σ are promises for guaranteeing that the branches through the generic matrix are $\subseteq^*$-decreasing and the levels are almost disjoint families, respectively. More precisely, f_σ promises that $a_\sigma \setminus f_\sigma(\tau) \subseteq a_\tau$ for each $\tau \in \mathrm{dom}(f_\sigma)$ and h_σ promises that $a_\tau \cap a_\sigma \subseteq h_\sigma(\tau)$ for each $\tau \in \mathrm{dom}(h_\sigma)$ (see Lemma 3.5.(4)).

Remark 3.2. Note that $\mathbb{Q}_\alpha$ is not separative. As an example, we can take p and q as follows: $\mathrm{dom}(p) = \mathrm{dom}(q) = \{\sigma, \tau\}$ where σ is to the left of τ; $p(\tau) = q(\tau) = (\langle 1 \rangle, \emptyset, h)$ where $h(\sigma) = 0$; $p(\sigma) = (\langle\rangle, \emptyset, \emptyset)$ and $q(\sigma) = (\langle 0 \rangle, \emptyset, \emptyset)$. It is easy to see that $p \nleq q$, but any condition stronger than p is compatible with q. Therefore, we later need to provide certain iteration lemmas for the general case of non-separative forcings (see Lemma 4.1.(3)).

3.2. **Countable chain condition and some implications.** We are now going to show that our iterands $\mathbb{Q}_\alpha$ have the c.c.c. and so their finite support iteration $\mathbb{P}_\lambda$ does not change cofinalities or cardinalities.

Lemma 3.3. *$\mathbb{Q}_\alpha$ is precaliber ω_1 (hence in particular c.c.c.) for every $\alpha < \lambda$.*

In fact, $\mathbb{Q}_\alpha$ is even σ-centered: in Section 6, we are going to show that each $\mathbb{Q}_\alpha$ can be represented as a finite support iteration of length strictly less than $\mathfrak{c}^+$ of Mathias forcings with respect to certain filters; since filtered Mathias forcings are σ-centered and σ-centeredness is preserved under finite support iterations of length strictly less than $\mathfrak{c}^+$, it follows that $\mathbb{Q}_\alpha$ is σ-centered (see also Corollary 6.2.(1)).

Proof of Lemma 3.3. Let $\{p_i \mid i < \omega_1\} \subseteq \mathbb{Q}_\alpha$. First note that it is possible to extend[5] all s^p_σ (with $\sigma \in \mathrm{dom}(p)$) of a condition $p \in \mathbb{Q}_\alpha$ to the same length $N_p \in \omega$, by just adding 0's at the end. Therefore we can assume that there exists N such that $|s^{p_i}_\sigma| = N$ for each $i \in \omega_1$ and each $\sigma \in \mathrm{dom}(p_i)$. Apply the Δ-system lemma to $\{\mathrm{dom}(p_i) \mid i \in \omega_1\}$ to find $X \subseteq \omega_1$ of size ω_1 such that $\{\mathrm{dom}(p_i) \mid i \in X\}$ is a Δ-system with root $R \subseteq T_\alpha$. Then we repeatedly apply the Δ-system lemma to obtain $Y \subseteq X$ of size ω_1 such that $\{\mathrm{dom}(f^{p_i}_\sigma) \cap T'_\alpha \mid i \in Y\}$ is a Δ-system with root A_σ for each $\sigma \in R$. Moreover, we can assume that for each $\sigma \in R$, there are s^*_σ, f^*_σ, and h^*_σ such that for all $i \in Y$, we have $s^{p_i}_\sigma = s^*_\sigma$, $f^{p_i}_\sigma \restriction (R \cup A_\sigma) = f^*_\sigma$, and $h^{p_i}_\sigma \restriction R = h^*_\sigma$. Now it is straightforward to check that any two conditions from $\{p_i \mid i \in Y\}$ are compatible; in fact, any finitely many of them have a common lower bound. $\square$

Using standard arguments, one can easily show:

Lemma 3.4.

(1) Let $\alpha \leq \lambda$. Then, in $V[\mathbb{P}_\alpha]$, we have $\mathfrak{c} = \mu$.

(2) Every node $\sigma \in \lambda^{<\lambda}$ from the final model $V[\mathbb{P}_\lambda]$ already appears in some $V[\mathbb{P}_\alpha]$ with $\alpha < \lambda$.

3.3. **The generic refining matrix.** Let G be a generic filter for the iteration $\mathbb{P}_\lambda$. In the final model $V[G]$, we derive our intended generic object as follows. For each $\sigma \in \lambda^{<\lambda} \cap \mathrm{succ}$ fix the minimal $\alpha < \lambda$ such that $\sigma \in V[G_\alpha]$ (see Lemma 3.4.(2)). Then in $V[G_\alpha]$, the node σ belongs to T_α, and letting $G(\alpha)$ be the $\mathbb{Q}_\alpha$ generic filter over $V^{\mathbb{P}_\alpha}$ define $a_\sigma = \bigcup\{s^p_\sigma \mid p \in G(\alpha) \wedge \sigma \in \mathrm{dom}(p)\}$. In $V[G]$, we let for each $\xi < \lambda$, $A_{\xi+1} = \{a_\sigma \mid |\sigma| = \xi + 1\}$. We are now going to show that $\{A_{\xi+1} \mid \xi < \lambda\}$ is a refining system of almost disjoint families.

Lemma 3.5. *Let $\alpha < \lambda$.*

(1) Let $\sigma \in T_\alpha$, $n \in \omega$. Then $D_{\sigma,n} = \{q \in \mathbb{Q}_\alpha \mid \sigma \in \mathrm{dom}(q)$ and $|s^q_\sigma| \geq n\}$ is dense in $\mathbb{Q}_\alpha$.

(2) The set D of conditions q in $\mathbb{Q}_\alpha$ such that for each $\sigma \in \mathrm{dom}(q)$

(a) if $\tau \in \mathrm{dom}(q)$ and $\tau \lhd \sigma$, then $\tau \in \mathrm{dom}(f^q_\sigma)$,

(b) if $\sigma = \rho^\frown i$ then for each $j < i$ such that $\rho^\frown j \in \mathrm{dom}(q)$, we have that $\rho^\frown j \in \mathrm{dom}(h^q_\sigma)$

is dense in $\mathbb{Q}_\alpha$. Moreover, for each $p \in \mathbb{Q}_\alpha$ there is $q \in D$, $q \leq p$ such that $\mathrm{dom}(p) = \mathrm{dom}(q)$.

(3) If $\alpha > 0$, $\tau' \in T'_\alpha$, $p \in \mathbb{Q}_\alpha$ and $\tau' \lhd \sigma$ for some $\sigma \in \mathrm{dom}(p)$, then there is $q \in D$ (here D is from item (2) above) such that $q \leq p$, $\mathrm{dom}(p) = \mathrm{dom}(q)$ and $\tau' \in \mathrm{dom}(f^q_\sigma)$.

(4) Let $p \in \mathbb{Q}_\alpha$ and $\sigma \in \mathrm{dom}(p)$.

(a) If $\tau \in \mathrm{dom}(f^p_\sigma)$, then $p \Vdash a_\sigma \setminus f^p_\sigma(\tau) \subseteq a_\tau$. In particular, $p \Vdash a_\sigma \subseteq^* a_\tau$.

(b) If $\tau \in \mathrm{dom}(h^p_\sigma)$, then $p \Vdash a_\tau \cap a_\sigma \subseteq h^p_\sigma(\tau)$ In particular, $p \Vdash a_\tau \cap a_\sigma =^* \emptyset$.

Proof. (1) Let $p \in \mathbb{Q}_\alpha$. Clearly, we can assume that $\sigma \in \mathrm{dom}(p)$. It is easy to see, using (2), (5), and (8) in Definition 3.1, that, by adjoining 0's, it is possible to extend all s^p_τ in such a way that the resulting q is a condition and $|s^q_\sigma| \geq n$.

(2) Let $p \in \mathbb{Q}_\alpha$, $\sigma \in \mathrm{dom}(p)$. For every $\tau \in \mathrm{dom}(p) \setminus \mathrm{dom}(f^p_\sigma)$ with $\tau \lhd \sigma$, let $f^q_\sigma(\tau) = |s^p_\sigma|$. For every $\rho^\frown j \in \mathrm{dom}(p) \setminus \mathrm{dom}(h^p_\sigma)$ with $j < i$, let $h^q_\sigma(\rho^\frown j) = |s^p_\sigma|$.

(3) Take $f^q_\sigma(\tau') = |s^p_\sigma|$.

(4) This is easy to see, using (5) and (8) in Definition 3.1.

[5] In Lemma 3.9, we will show a stronger fact.

□

Now we can show that in the final model $V[\mathbb{P}_\lambda]$, the sets along branches of $\lambda^{<\lambda}$ are $\subseteq^*$-decreasing and the sets on any level of $\lambda^{<\lambda} \cap \mathrm{succ}$ are pairwise almost disjoint.

Corollary 3.6. *In $V[\mathbb{P}_\lambda]$, the following hold:*

(1) If $\tau, \sigma \in \lambda^{<\lambda} \cap \mathrm{succ}$ such that $\tau \lhd \sigma$, then $a_\sigma \subseteq^* a_\tau$.
(2) If $\rho \in \lambda^{<\lambda}$ and $j < i < \lambda$, then $a_{\rho^\frown j} \cap a_{\rho^\frown i} =^* \emptyset$. Moreover for each $\xi < \lambda$, $\sigma, \sigma' \in \lambda^{\xi+1}$, $\sigma \neq \sigma'$, we have $a_\sigma \cap a_{\sigma'} =^* \emptyset$ and so $A_{\xi+1} = \{a_\sigma \mid \sigma \in \lambda^{\xi+1}\}$ is an almost disjoint family.

Proof. To show (1), let $\eta < \lambda$ be minimal such that $\sigma \in (\lambda^{<\lambda})^{V[\mathbb{P}_\eta]}$. Lemma 3.5.(1), Lemma 3.5.(2), and Lemma 3.5.(4a) imply that the set $\{q \in \mathbb{Q}_\eta \mid q \Vdash a_\sigma \subseteq^* a_\tau\}$ is dense. Hence already $V[\mathbb{P}_{\eta+1}] \models a_\sigma \subseteq^* a_\tau$, which remains true in $V[\mathbb{P}_\lambda]$. To show (2), take $\eta < \lambda$ minimal such that $\rho \in (\lambda^{<\lambda})^{V[\mathbb{P}_\eta]}$. Lemma 3.5.(1), Lemma 3.5.(2) and Lemma 3.5.(4b) imply that the set $\{q \in \mathbb{Q}_\eta \mid q \Vdash a_{\rho^\frown j} \cap a_{\rho^\frown i} =^* \emptyset\}$ is dense. Given $\sigma, \sigma' \in \lambda^{\xi+1}$ such that $\sigma \neq \sigma'$, find $\rho \in \lambda^{<\lambda}$ with $\rho \lhd \sigma, \sigma'$ and $i, j < \lambda$, $j \neq i$ such that $\rho^\frown j \unlhd \sigma$ and $\rho^\frown i \unlhd \sigma'$. Apply the preceding argument and then (1). □

Finally, we show that each a_σ is infinite. By "$s(m) = 1$", we actually mean "$m \in \mathrm{dom}(s)$ and $s(m) = 1$".

Lemma 3.7. *Let $\alpha < \lambda$, $\sigma \in T_\alpha$, $n \in \omega$. Then $D_{\sigma,n} = \{q \in \mathbb{Q}_\alpha \mid \sigma \in \mathrm{dom}(q)$ and $\exists m \geq n(s^q_\sigma(m) = 1)\}$ is dense in $\mathbb{Q}_\alpha$.*

Proof. The proof proceeds by induction on $\alpha < \lambda$. Let $\sigma \in T_\alpha$, $n \in \omega$ and $p \in \mathbb{Q}_\alpha$. By Lemma 3.5.(1), we can assume that $\sigma \in \mathrm{dom}(p)$. Let $N_0 \in \omega$ be bigger than the maximal length of all the s^p_τ with $\tau \in \mathrm{dom}(p)$ and $\tau \unlhd \sigma$, and bigger than n. Let $A = \bigcup\{\mathrm{dom}(f^p_\tau) \cap T'_\alpha \mid \tau \in \mathrm{dom}(p) \wedge \tau \unlhd \sigma\}$.

If A is empty, let $m \in \omega$ be arbitrary with $m \geq N_0$. Otherwise, let $N_1 \geq N_0$ be large enough such that $a_{\psi'} \setminus N_1 \subseteq a_\psi$ for all $\psi, \psi' \in A$ with $\psi \unlhd \psi'$ (see Corollary 3.6). Moreover, let ψ^* be the longest element of the finite set A. By induction, a_{ψ^*} is infinite, so we can fix $m \geq N_1$ such that $a_{\psi^*}(m) = 1$. Therefore $a_\psi(m) = 1$ for each $\psi \in A$. Now, for every $\tau \in \mathrm{dom}(p)$ with $\tau \unlhd \sigma$, extend s^p_τ with 0's to a node $\bar{s}_\tau$ of length m and take $s^q_\tau = \bar{s}_\tau^\frown 1$. It is easy to check that q is a condition, $q \leq p$ and $s^q_\sigma(m) = 1$ as desired. □

Altogether, we proved that $\{A_{\xi+1} \mid \xi < \lambda\}$ is a refining system of almost disjoint families. To show that it forms a refining matrix requires much more, the proof of which will be completed in Section 4.

3.4. **Upwards closed sets and complete subforcings.** The goal of this section is to show that our forcing $\mathbb{Q}_\alpha$ has complete subforcings which use only part of T_α (see Lemma 3.11). In Section 4.2, this will be extended to the whole iteration (see Lemma 4.8), which will be an important ingredient of the proof that the generic object is a refining matrix (see Section 4.3 and Section 4.4). Moreover, Lemma 3.11 will play a crucial role in showing that $\mathfrak{h} = \mathfrak{b} = \omega_1$ in the final extension. We now work in $V[\mathbb{P}_\alpha]$.

Definition 3.8. A condition $p \in \mathbb{Q}_\alpha$ is called *full* if there exists an $N \in \omega$ such that for all $\sigma \in \mathrm{dom}(p)$

(1) $|s^p_\sigma| = N$,
(2) $N > \max(\mathrm{rng}(f^p_\sigma))$ and $N > \max(\mathrm{rng}(h^p_\sigma))$,
(3) $\tau \in \mathrm{dom}(f^p_\sigma)$ for each $\tau \in \mathrm{dom}(p)$ with $\tau \lhd \sigma$, and
(4) if $\sigma = \rho^\frown i$ then for each $j < i$ with $\rho^\frown j \in \mathrm{dom}(p)$, we have $\rho^\frown j \in \mathrm{dom}(h^p_\sigma)$.

A condition $p \in \mathbb{P}_\lambda$ is said to be full, if $p(0)$ is full.

Later, we will use quotients $\mathbb{P}_\lambda / \mathbb{P}_\eta$ and a modification where 0 is replaced by η. Lemma 3.5 gives:

Lemma 3.9. *For every condition $p \in \mathbb{Q}_\alpha$ there exists a full condition q with $q \leq p$ and* $\mathrm{dom}(q) = \mathrm{dom}(p)$. *Hence the set of full conditions in $\mathbb{P}_\lambda$ is dense in $\mathbb{P}_\lambda$.*

Important complete suborders of our forcing are captured by the following notion:

Definition 3.10. Let $C \subseteq \lambda^{<\lambda}$, $\alpha < \lambda$.

(1) $\mathbb{Q}_\alpha^C = \{p \in \mathbb{Q}_\alpha \mid \mathrm{dom}(p) \subseteq C\}$.
(2) C is said to be *α-upwards closed* if for each $\sigma \in C$ and each $\tau \lhd \sigma$ with $\tau \in T_\alpha$, we have $\tau \in C$.
(3) For $p \in \mathbb{Q}_\alpha$, let $p \upharpoonright\upharpoonright C$ be the function p' with $\mathrm{dom}(p') = \mathrm{dom}(p) \cap C$, $s_\sigma^{p'} = s_\sigma^p$, $f_\sigma^{p'} = f_\sigma^p \upharpoonright (C \cup T'_\alpha)$ and $h_\sigma^{p'} = h_\sigma^p \upharpoonright C$ for each $\sigma \in \mathrm{dom}(p')$. Clearly, p' is a condition in $\mathbb{Q}_\alpha^C$ and if C is α-upwards closed, then $f_\sigma^{p'} = f_\sigma^p$.

Recall that $\mathbb{P}'$ is a complete subforcing of $\mathbb{P}$, denoted $\mathbb{P}' \lessdot \mathbb{P}$, if for each q, q' in $\mathbb{P}'$, $q \perp_{\mathbb{P}'} q' \rightarrow q \perp_{\mathbb{P}} q'$, and each $p \in \mathbb{P}$ has a reduction $q \in \mathbb{P}'$ (i.e. a condition q such that if $r \in \mathbb{P}'$ and $r \leq q$ then $r \not\perp_{\mathbb{P}} p$).

Lemma 3.11. *Let $C \subseteq \lambda^{<\lambda}$ be α-upwards closed. Then $\mathbb{Q}_\alpha^C$ is a complete subforcing of $\mathbb{Q}_\alpha$. Moreover, if $p \in \mathbb{Q}_\alpha$ is a full condition, then $p \upharpoonright\upharpoonright C$ is a reduction of p to $\mathbb{Q}_\alpha^C$.*

The sets $\lambda^1 = \{\sigma \in \lambda^{<\lambda} \mid |\sigma| = 1\}$, $1^{<\lambda} = \{\sigma \in \lambda^{<\lambda} \mid \sigma(\xi) = 0$ for every $\xi\}$ are 0-upwards closed. Thus, $\mathbb{Q}_0^{(\lambda^1)}$ and $\mathbb{Q}_0^{(1^{<\lambda})}$ are complete subforcings of $\mathbb{Q}_0$. Note that they are isomorphic to the posets introduced by Hechler [21] to add a mad family or a tower, respectively (for a further study see [15]).

Proof of Lemma 3.11. We give the proof only for the case $\alpha = 0$ and leave the only slightly different general case to the reader. To show that incompatible conditions in $\mathbb{Q}_0^C$ are incompatible in $\mathbb{Q}_0$, let $p_0, p_1 \in \mathbb{Q}_0^C$ and $q \in \mathbb{Q}_0$ with $q \leq p_0, p_1$; then $q' = q \upharpoonright\upharpoonright C$ is in $\mathbb{Q}_0^C$ and $q' \leq p_0, p_1$, as desired. Now, consider any $p \in \mathbb{Q}_0$ and let $p' \leq p$ be a full condition with $\mathrm{dom}(p') = \mathrm{dom}(p)$. We will show that $p' \upharpoonright\upharpoonright C$ is a reduction of p to $\mathbb{Q}_0^C$. Let $q \leq p' \upharpoonright\upharpoonright C$ with $q \in \mathbb{Q}_0^C$.

Let $q' \leq q$ be such that $|s_\sigma^{q'}| = |s_\tau^{q'}|$ for all $\sigma, \tau \in \mathrm{dom}(q')$. Let $\mathrm{dom}(r) = \mathrm{dom}(q') \cup \mathrm{dom}(p')$. For $\sigma \in \mathrm{dom}(p') \setminus \mathrm{dom}(q')$ let $s_\sigma^r = s_\sigma^{p'}$, $f_\sigma^r = f_\sigma^{p'}$, $h_\sigma^r = h_\sigma^{p'}$. Similarly for $\sigma \in \mathrm{dom}(q') \setminus \mathrm{dom}(p')$ let $s_\sigma^r = s_\sigma^{q'}$, $f_\sigma^r = f_\sigma^{q'}$, $h_\sigma^r = h_\sigma^{q'}$. For $\sigma \in \mathrm{dom}(q') \cap \mathrm{dom}(p)$ let $s_\sigma^r = s_\sigma^{q'}$, $f_\sigma^r = f_\sigma^{q'}$, $\mathrm{dom}(h_\sigma^r) = \mathrm{dom}(h_\sigma^{q'}) \cup \mathrm{dom}(h_\sigma^{p'})$, and for $\sigma' \in \mathrm{dom}(h_\sigma^{q'})$ let $h_\sigma^r(\sigma') = h_\sigma^{q'}(\sigma')$, for $\sigma' \in \mathrm{dom}(h_\sigma^{p'}) \setminus \mathrm{dom}(h_\sigma^{q'})$ let $h_\sigma^r(\sigma') = h_\sigma^{p'}(\sigma')$.

Claim. *r is a condition.*

Proof. It is very easy to check that s_σ^r, f_σ^r and h_σ^r are well-defined with the right domains and ranges for all $\sigma \in \mathrm{dom}(r)$. Next, we show that if $\sigma \trianglelefteq \tau$, then $|s_\sigma^r| \geq |s_\tau^r|$: If $\tau \in \mathrm{dom}(q')$, it follows by the α-upwards closure that $\sigma \in \mathrm{dom}(q')$, hence by definition $s_\sigma^r = s_\sigma^{q'}$, $s_\tau^r = s_\tau^{q'}$ and $|s_\sigma^r| \geq |s_\tau^r|$ since q' is a condition. If $\tau \notin \mathrm{dom}(q')$, then $|s_\tau^r| = |s_\tau^{p'}| = |s_\sigma^{p'}|$ and regardless if $\sigma \in \mathrm{dom}(q)$ or not, $|s_\sigma^r| \geq |s_\sigma^{p'}|$.

Let $\sigma, \tau \in \mathrm{dom}(r)$ with $\tau \in \mathrm{dom}(f_\sigma^r)$ and $m \geq f_\sigma^r(\tau)$ and $s_\sigma^r(m) = 1$. We have to show that $s_\tau^r(m) = 1$ (note that, in case $\alpha > 0$, one also has to deal with the case $\tau \in \mathrm{dom}(f_\sigma^r) \setminus \mathrm{dom}(r)$ which works similarly with $a_\tau(m)$ in place of $s_\tau^r(m)$). *Case 1:* σ and τ are both in $\mathrm{dom}(q')$. In this case $s_\sigma^r = s_\sigma^{q'}$, $s_\tau^r = s_\tau^{q'}$,

$f^r_\sigma(\tau) = f^{q'}_\sigma(\tau)$ and $s^r_\tau(m) = 1$ holds since q' is a condition. *Case 2:* $\sigma \in \mathrm{dom}(q')$ and $\tau \notin \mathrm{dom}(q')$. This contradicts the α-upwards closure of C. *Case 3:* $\sigma \notin \mathrm{dom}(q')$ and $\tau \in \mathrm{dom}(q')$. So $f^r_\sigma = f^{p'}_\sigma$ and $s^r_\sigma = s^{p'}_\sigma$. Thus $m < |s^{p'}_\sigma|$ and $\tau \in \mathrm{dom}(p')$ because $\mathrm{dom}(f^r_\sigma) \subseteq \mathrm{dom}(p')$. Since p' is a condition, it follows that $s^{p'}_\tau(m) = 1$ and clearly $s^r_\tau(m) = s^{p'}_\tau(m)$. *Case 4:* $\sigma \notin \mathrm{dom}(q')$ and $\tau \notin \mathrm{dom}(q')$. In this case $s^r_\sigma = s^{p'}_\sigma$, $s^r_\tau = s^{p'}_\tau$, $f^r_\sigma = f^{p'}_\sigma$ and $s^r_\tau(m) = 1$ follows since p' is a condition.

Assume $\rho, \rho' \in \mathrm{dom}(r)$, $\rho' \in \mathrm{dom}(h^r_\rho)$, $m \geq h^r_\rho(\rho')$ and $s^r_\rho(m) = 1$; we have to show that $s^r_{\rho'}(m) = 0$, if it is defined. *Case 1:* $\rho, \rho' \in \mathrm{dom}(q')$. In this case $s^r_\rho = s^{q'}_\rho$, $s^r_{\rho'} = s^{q'}_{\rho'}$ and $h^r_\rho(\rho') = h^{q'}_\rho(\rho')$ and $s^r_{\rho'}(m) = 0$ follows since q' is a condition. *Case 2:* $\rho \in \mathrm{dom}(q')$ and $\rho' \notin \mathrm{dom}(q')$. Since $\rho' \in \mathrm{dom}(h^r_\rho) \setminus \mathrm{dom}(q')$ it follows that $\rho \in \mathrm{dom}(p')$ and $\rho' \in \mathrm{dom}(h^{p'}_\rho)$. If $m \geq |s^{p'}_\rho|$, then $|s^r_{\rho'}(m)|$ is undefined. If $m < |s^{p'}_\rho|$ the fact that p' is a condition implies that $s^{p'}_{\rho'}(m) = 0$ which is the same as $s^r_{\rho'}(m)$. *Case 3:* $\rho \notin \mathrm{dom}(q')$ and $\rho' \in \mathrm{dom}(q')$. It follows that $\rho' \in \mathrm{dom}(h^r_\rho) = \mathrm{dom}(h^{p'}_\rho) \subseteq \mathrm{dom}(p')$ and $m < |s^r_\rho| = |s^{p'}_\rho|$. Since p' is a condition $s^{p'}_{\rho'}(m) = 0$ which is the same as $s^r_{\rho'}(m)$. *Case 4:* $\rho, \rho' \notin \mathrm{dom}(q')$. In this case $s^r_\rho = s^{p'}_\rho$, $s^r_{\rho'} = s^{p'}_{\rho'}$ and $h^r_\rho = h^{p'}_\rho$ and $s^r_{\rho'}(m) = 0$ follows since p' is a condition. □

It is straightforward to check that r extends both q' and p' (and therefore q and p). □

4. No refinement, and madness of levels

This section is dedicated to the central part of the proof that the generic object added by our forcing iteration is a refining matrix of height λ. In Sections 4.4 and 4.3 we show that the levels are mad families with no further refinements. We start with some preliminary lemmas and concepts.

4.1. **On forcing iterations and correct systems.** Next, we gather some key properties of forcing iterations and completeness, which will play a crucial role for our construction later. For further reading, see [18]. Throughout the subsection $\mathbb{P}$, $\mathbb{Q}$, etc. denote arbitrary forcing posets. Recall the following:

Lemma 4.1.

(1) *Suppose that* $\mathbb{P}_0 \lessdot \mathbb{P}$ *and*[6] $\mathbb{P}_1 \lessdot \mathbb{P}$ *satisfying* $\mathbb{P}_0 \subseteq \mathbb{P}_1$. *Then* $\mathbb{P}_0 \lessdot \mathbb{P}_1$. *Moreover, if* $q \in \mathbb{P}_0$ *is a reduction of* $p \in \mathbb{P}_1$ *from* $\mathbb{P}$ *to* $\mathbb{P}_0$, *then* q *is also a reduction of* p *from* $\mathbb{P}_1$ *to* $\mathbb{P}_0$.

(2) *Suppose that* $\mathbb{P}' \lessdot \mathbb{P}$. *Let* φ *be some formula, let* $\dot{x}$, $\dot{y}$, *etc. be* $\mathbb{P}'$*-names, and let* $p \in \mathbb{P}$ *such that* $p \Vdash_{\mathbb{P}} \varphi(\dot{x}, \dot{y}, \ldots)$. *Then for each* $p' \in \mathbb{P}'$ *which is a reduction of* p, *we have* $p' \Vdash_{\mathbb{P}'} \varphi(\dot{x}, \dot{y}, \ldots)$.

(3) *Let* $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\}$ *and* $\{\mathbb{P}'_\alpha, \dot{\mathbb{Q}}'_\alpha \mid \alpha < \delta\}$ *be finite support iterations such that for each* $\alpha < \delta$, $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}'_\alpha \lessdot \dot{\mathbb{Q}}_\alpha$. *Then* $\mathbb{P}'_\delta$ *is a complete subforcing of* $\mathbb{P}_\delta$.

Moreover, if $\mathsf{RED}\colon \mathbb{Q}_0 \to \mathbb{Q}'_0$ *is a map such that* $\mathsf{RED}(q)$ *is a reduction of* q *for each* $q \in \mathbb{Q}_0$, *then for each* $p \in \mathbb{P}_\delta$, *there is a* $p' \in \mathbb{P}'_\delta$ *such that* p' *is a reduction of* p, *and* $p'(0) = \mathsf{RED}(p(0))$, *and, if* $\alpha \geq 1$ *and* $p(\alpha)$ *is a* $\mathbb{P}'_\alpha$*-name with* $p \restriction \alpha \Vdash p(\alpha) \in \dot{\mathbb{Q}}'_\alpha$, *then* $p'(\alpha) = p(\alpha)$.

The first part of item (3) can be found in [9]), however we will need the technical strengthening given by the mapping RED. The iterands in (3) need not be separative, which is essential, as we will apply the Lemma to the $\mathbb{Q}_\alpha$'s from Definition 3.1. The concept below has been introduced by Brendle (see [7, 8]):

[6] In fact, it is sufficient for the proof to go through that incompatible conditions in $\mathbb{P}_1$ are incompatible in $\mathbb{P}$.

Definition 4.2. A system of forcings $\mathbb{R}_0, \mathbb{R}_1 \lessdot \mathbb{R}$ with $\mathbb{R}_0 \cap \mathbb{R}_1 \lessdot \mathbb{R}_0, \mathbb{R}_1$ is *correct* if any two conditions $p_0 \in \mathbb{R}_0$ and $p_1 \in \mathbb{R}_1$ which have a common reduction in $\mathbb{R}_0 \cap \mathbb{R}_1$ are compatible in $\mathbb{R}$.

Under the assumptions of the lemma below, $\mathbb{R} = \mathbb{P} * \dot{\mathbb{Q}}$, $\mathbb{R}_0 = \mathbb{P}$ and $\mathbb{R}_1 = \mathbb{P}' * \dot{\mathbb{Q}}'$ is a correct system. We do not know however, whether the conclusion of the lemma holds for every correct system.

Lemma 4.3. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ and $\mathbb{P}' * \dot{\mathbb{Q}}'$ be two-step iterations satisfying $\mathbb{P}' \lessdot \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}' \lessdot \dot{\mathbb{Q}}$. Then*

$$V[\mathbb{P}' * \dot{\mathbb{Q}}'] \cap V[\mathbb{P}] = V[\mathbb{P}'].$$

Proof. We will only show the special case which we will need later (it is straightforward to extend the proof to the general case): for any $\delta, \varepsilon \in \mathrm{Ord}$,

$$\delta^{\varepsilon} \cap V[\mathbb{P}' * \dot{\mathbb{Q}}'] \cap V[\mathbb{P}] \subseteq V[\mathbb{P}'].$$

Let G be a generic filter for $\mathbb{P}'$, let $\dot{f}_0$ be a $\mathbb{P}$-name, and let $\dot{f}_1$ be a $\mathbb{P}' * \dot{\mathbb{Q}}'$-name. Work in $V[G]$. Assume towards a contradiction that there is a condition $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ with $p \in \mathbb{P}/G$ such that

$$(p, \dot{q}) \Vdash \dot{f}_0 = \dot{f}_1 \in \mathrm{Ord}^{<\mathrm{Ord}} \ \wedge \ \dot{f}_0 \notin V[G]. \tag{2}$$

Let $p' \in G$ be a reduction of p to $\mathbb{P}'$. By standard arguments, we can fix a $\mathbb{P}'$-name $\dot{q}'$ such that $p \Vdash$ "$\dot{q}'$ is a reduction of $\dot{q}$" and $(p', \dot{q}') \in \mathbb{P}' * \dot{\mathbb{Q}}'$. Since p is reduction of $(p, \dot{q})$ to $\mathbb{P}$, it follows from (2) and Lemma 4.1.(2) that $p \Vdash \dot{f}_0 \notin V[G]$. Therefore, we can fix $\gamma \in \varepsilon$ such that p does not decide $\dot{f}_0(\gamma)$ in $\mathbb{P}/G$. Let $(p_1, \dot{q}_1) \leq (p', \dot{q}')$ and $\xi_1 \in \delta$ such that $p_1 \in G$ and $(p_1, \dot{q}_1) \Vdash \dot{f}_1(\gamma) = \xi_1$. Since p does not decide $\dot{f}_0$ at γ, we can fix $p_0 \in \mathbb{P}/G$ with $p_0 \leq p$ and $\xi_0 \in \delta$ with $\xi_0 \neq \xi_1$ such that $p_0 \Vdash \dot{f}_0(\gamma) = \xi_0$. Now we want to find a condition $(p^*, \dot{q}^*)$ which is stronger than $(p, \dot{q}), (p_1, \dot{q}_1)$ and $(p_0, \mathbb{1})$. First note that p_0 and p_1 are compatible, because $p_0 \in \mathbb{P}/G$ and $p_1 \in G$, and fix $p^* \leq p_0, p_1$. Since $p^* \leq p, p_1$ it follows that $p^* \Vdash$ "$\dot{q}'$ is a reduction of $\dot{q}$ and $\dot{q}_1 \leq \dot{q}'$" hence $p^* \Vdash \dot{q}_1 \not\perp \dot{q}$. Let $\dot{q}^*$ be a $\mathbb{P}$-name such that $p^* \Vdash \dot{q}^* \leq \dot{q}_1, \dot{q}$. It is easy to check that $(p^*, \dot{q}^*) \leq (p, \dot{q}), (p_1, \dot{q}_1), (p_0, \mathbb{1})$. Now $(p^*, \dot{q}^*) \Vdash \dot{f}_0 = \dot{f}_1 \wedge \dot{f}_0(\gamma) = \xi_0 \wedge \dot{f}_1(\gamma) = \xi_1$, but $\xi_0 \neq \xi_1$, a contradiction. □

We conclude with an easy observation we will need later on:

Lemma 4.4. *Suppose $\mathbb{P}' \lessdot \mathbb{P}$, $\dot{b}$ is a $\mathbb{P}'$-name and $p \in \mathbb{P}$ is such that $p \Vdash \dot{b} \in [\omega]^{\omega}$. Then for each $N \in \omega$ there are $r \in \mathbb{P}'$, $m > N$ such that $r \Vdash m \in \dot{b}$ and r is compatible with p.*

4.2. **Complete subforcings: hereditarily below γ.** In this section, we give some technical definitions and lemmas as a preparation for the main proofs in Sections 4.3 and 4.4. More precisely, we define, for each $\gamma < \lambda$, the subforcings of "hereditarily below γ" conditions of our iteration, show that they form complete subforcings (see Lemma 4.8) and that each condition is hereditarily below γ for some $\gamma < \lambda$ (see Lemma 4.10). For the rest of the section fix $\eta < \lambda$ and a $\mathbb{P}_\eta$-generic filter G_η.

Definition 4.5. Let $\gamma < \lambda$. In $V[G_\eta]$ define by recursion on $\eta \leq \alpha \leq \lambda$ for a condition $p \in \mathbb{P}_\alpha/G_\eta$ to be *hereditarily below* γ and the poset ${}^{<\gamma}(\mathbb{P}_\alpha/G_\eta)$:

1. $p \in \mathbb{Q}_\eta$ is hereditarily below γ, if $\mathrm{dom}(p) \subseteq \gamma^{<\gamma}$.
2. Let ${}^{<\gamma}(\mathbb{P}_\alpha/G_\eta) = \{p \in \mathbb{P}_\alpha/G_\eta \mid p \text{ hereditarily below } \gamma\}$.
3. Let $\alpha > \eta$, $p \in \mathbb{P}_{\alpha+1}/G_\eta$ is hereditarily below γ, if $p \restriction \alpha \in {}^{<\gamma}(\mathbb{P}_\alpha/G_\eta)$, $p(\alpha)$ is a ${}^{<\gamma}(\mathbb{P}_\alpha/G_\eta)$-name and $p \restriction \alpha \Vdash \mathrm{dom}(p(\alpha)) \subseteq \gamma^{<\gamma}$.

(4) For α limit, $p \in \mathbb{P}_\alpha / G_\eta$ is hereditarily below γ, if $p \restriction \beta \in {}^{<\gamma}(\mathbb{P}_\beta / G_\eta)$ for every $\beta < \alpha$.

For $\alpha \leq \lambda$, a $\mathbb{P}_\alpha / G_\eta$-name $\dot{b}$ is *hereditarily below* γ, if for all $(\dot{x}, p) \in \dot{b}$, $p \in {}^{<\gamma}(\mathbb{P}_\alpha / G_\eta)$ and $\dot{x}$ is hereditarily below γ (this is by recursion).

Clearly, if $p \in \mathbb{P}_\alpha / G_\eta$ is hereditarily below γ and $\gamma' > \gamma$, then p is also hereditarily below γ'. The same holds for a $\mathbb{P}_\alpha / G_\eta$-name $\dot{b}$.

Definition 4.6. Let $\gamma < \lambda$, $\tau \in \lambda^{<\lambda}$. In $V[G_\eta]$ define by recursion on $\eta \leq \alpha \leq \lambda$ for a condition $p \in \mathbb{P}_\alpha / G_\eta$ to be *almost hereditarily below* γ *except for* τ and the poset ${}^{<\gamma+\tau}(\mathbb{P}_\alpha / G_\eta)$:

(1) $p \in \mathbb{Q}_\eta$ is almost hereditarily below γ except for τ, if $\mathrm{dom}(p) \subseteq \gamma^{<\gamma} \cup \{\tau\}$.
(2) Let ${}^{<\gamma+\tau}(\mathbb{P}_\alpha / G_\eta) = \{p \in \mathbb{P}_\alpha / G_\eta \mid p$ almost hereditarily below γ except for $\tau\}$.
(3) Let $\alpha > \eta$, $p \in \mathbb{P}_{\alpha+1} / G_\eta$ is almost hereditarily below γ except for τ, if $p \restriction \alpha$ is almost hereditarily below γ except for τ, $p(\alpha)$ is a[7] ${}^{<\gamma}(\mathbb{P}_\alpha / G_\eta)$-name and $p \restriction \alpha \Vdash \mathrm{dom}(p(\alpha)) \subseteq \gamma^{<\gamma}$.
(4) For α limit, $p \in \mathbb{P}_\alpha / G_\eta$ is almost hereditarily below γ except for τ, if $p \restriction \beta$ is almost hereditarily below γ except for τ, for every $\beta < \alpha$.

For $\alpha \leq \lambda$, a $\mathbb{P}_\alpha / G_\eta$-name $\dot{b}$ is *almost hereditarily below* γ *except for* τ, if for all $(\dot{x}, p) \in \dot{b}$, both p and $\dot{x}$ are almost hereditarily below γ except for τ. We will write *almost hereditarily below* γ and omit the τ if it is clear from the context which τ is meant.

Clearly, if $p \in \mathbb{P}_\alpha / G_\eta$ is almost hereditarily below γ and $\gamma' > \gamma$, then p is also almost hereditarily below γ', and if $p \in \mathbb{P}_\alpha / G_\eta$ is hereditarily below γ, then it is almost hereditarily below γ except for τ for every τ. The same holds for a $\mathbb{P}_\alpha / G_\eta$-name $\dot{b}$. We will make use of the following easy fact:

Lemma 4.7. *Assume $\mathbb{P}'$ is a complete subforcing of $\mathbb{P}$ and G is $\mathbb{P}$-generic. Then in $V[G]$, the set $(\gamma^{<\gamma})^{V[G \cap \mathbb{P}']}$ is α-upwards closed for any $\alpha < \lambda$.*

Proof. Clearly, $(\gamma^{<\gamma})^{V[G \cap \mathbb{P}']}$ is α-upwards closed in $V[G \cap \mathbb{P}']$. Since $V[G]$ and $V[G \cap \mathbb{P}']$ have the same ordinals, the same holds true in $V[G]$. □

We can show now that the suborder of (almost) hereditarily below γ conditions is a complete suborder:

Lemma 4.8. *Let $\gamma < \lambda$. Then ${}^{<\gamma}(\mathbb{P}_\lambda / G_\eta) \lessdot \mathbb{P}_\lambda / G_\eta$. Also, if $\tau \in T_\eta$ is such that $\gamma^{<\gamma} \cup \{\tau\}$ is η-upwards closed, then ${}^{<\gamma+\tau}(\mathbb{P}_\lambda / G_\eta) \lessdot \mathbb{P}_\lambda / G_\eta$. Moreover, if $p \in \mathbb{P}_\lambda / G_\eta$ is full and almost hereditarily below γ except for τ, then $(p(\eta) \restriction \gamma^{<\gamma}, p(\eta+1), p(\eta+2), \dots)$ is a reduction of p to ${}^{<\gamma}(\mathbb{P}_\lambda / G_\eta)$.*

Proof. Inductively on α we show that ${}^{<\gamma}(\mathbb{P}_\alpha / G_\eta)$ and ${}^{<\gamma+\tau}(\mathbb{P}_\alpha / G_\eta))$ are complete subforcings of $\mathbb{P}_\alpha / G_\eta$, whenever $\eta \leq \alpha \leq \lambda$. To simplify notation, we write $\mathbb{P}_\alpha$, ${}^{<\gamma}\mathbb{P}_\alpha$, ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ instead of $\mathbb{P}_\alpha / G_\eta$, ${}^{<\gamma}(\mathbb{P}_\alpha / G_\eta)$, ${}^{<\gamma+\tau}(\mathbb{P}_\alpha / G_\eta)$, respectively. We will define ${}^{<\gamma}\mathbb{P}_\alpha$-names $\dot{\mathbb{Q}}'_\alpha$ such that ${}^{<\gamma}\mathbb{P}_\lambda$ (or ${}^{<\gamma+\tau}\mathbb{P}_\lambda$ respectively) is the finite support iteration of the $\dot{\mathbb{Q}}'_\alpha$'s. The only difference of the two iterations will be the first iterand $\mathbb{Q}'_\eta$.

Initial step $\alpha = \eta + 1$ By Lemma 3.11, ${}^{<\gamma}\mathbb{P}_{\eta+1} = \mathbb{Q}_\eta^{\gamma^{<\gamma}} \lessdot \mathbb{P}_{\eta+1} = \mathbb{Q}_\eta$. Similarly, ${}^{<\gamma+\tau}\mathbb{P}_{\eta+1} = \mathbb{Q}_\eta^{\gamma^{<\gamma} \cup \{\tau\}} \lessdot \mathbb{Q}_\eta$. Take $\mathbb{Q}'_\eta = \mathbb{Q}_\eta^{\gamma^{<\gamma}}$ in the iteration representing ${}^{<\gamma}\mathbb{P}_\lambda$ and $\mathbb{Q}'_\eta = \mathbb{Q}_\eta^{\gamma^{<\gamma} \cup \{\tau\}}$ in the iteration representing ${}^{<\gamma+\tau}\mathbb{P}_\lambda$.

[7] This is not a typo: we really require $p(\alpha)$ to be a ${}^{<\gamma}(\mathbb{P}_\alpha / G_\eta)$-name, not just a ${}^{<\gamma+\tau}(\mathbb{P}_\alpha / G_\eta)$-name.

Successor step $\alpha+1$ Assume ${}^{<\gamma}\mathbb{P}_\alpha$, ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ are complete subforcings of $\mathbb{P}_\alpha$. We will show that ${}^{<\gamma}\mathbb{P}_{\alpha+1}$ and ${}^{<\gamma+\tau}\mathbb{P}_{\alpha+1}$ are complete subforcings of $\mathbb{P}_{\alpha+1}$. In $V[G]$ for a $\mathbb{P}_\alpha$-generic G, let[8] $E = (\gamma^{<\gamma})^{V[G\cap{}^{<\gamma}\mathbb{P}_\alpha]}$. By Lemma 4.7 the set E is α-upwards closed and so by Lemma 3.11, in $V[G]$ we have $\mathbb{Q}^E_\alpha \lessdot \mathbb{Q}_\alpha$. Note that:

Claim 4.9. *$\mathbb{Q}^E_\alpha$ is an element of $V[G\cap{}^{<\gamma}\mathbb{P}_\alpha]$.*

Using the claim, fix a ${}^{<\gamma}\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}'_\alpha$ for $\mathbb{Q}^E_\alpha$. Since ${}^{<\gamma}\mathbb{P}_\alpha \subseteq {}^{<\gamma+\tau}\mathbb{P}_\alpha$ and both are complete suborders of $\mathbb{P}_\alpha$, Lemma 4.1.(1) implies that ${}^{<\gamma}\mathbb{P}_\alpha \lessdot {}^{<\gamma+\tau}\mathbb{P}_\alpha$. Thus, the ${}^{<\gamma}\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}'_\alpha$ is also a ${}^{<\gamma+\tau}\mathbb{P}_\alpha$-name. Now, apply Lemma 4.1.(3) to obtain that ${}^{<\gamma}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ and ${}^{<\gamma+\tau}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ are complete subforcings of $\mathbb{P}_{\alpha+1}$. Since by definition, ${}^{<\gamma}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ is equivalent to ${}^{<\gamma}\mathbb{P}_{\alpha+1}$ and ${}^{<\gamma+\tau}\mathbb{P}_\alpha * \dot{\mathbb{Q}}'_\alpha$ is equivalent to ${}^{<\gamma+\tau}\mathbb{P}_{\alpha+1}$ the successor step is complete.

Limit step α Lemma 4.1.(3) implies that the limit of the finite support iteration of the $\dot{\mathbb{Q}}'_{\alpha'}$ with $\alpha' < \alpha$ is a complete subforcing of $\mathbb{P}_\alpha$ and by definition ${}^{<\gamma}\mathbb{P}_\alpha$ (or ${}^{<\gamma+\tau}\mathbb{P}_\alpha$ respectively) is equivalent to the limit of this finite support iteration.

Now, let p be a full condition in ${}^{<\gamma+\tau}\mathbb{P}_\lambda$. By Lemma 3.11, $p(\eta) \restriction \gamma^{<\gamma} = p(\eta) \restriction\restriction \gamma^{<\gamma}$ is a reduction of $p(\eta)$ to ${}^{<\gamma}\mathbb{P}_{\eta+1}$ (which is $\mathbb{Q}'_\eta$ in the iteration representing ${}^{<\gamma}\mathbb{P}_\lambda$). Since ${}^{<\gamma+\tau}\mathbb{P}_\lambda$ is the iteration of the $\dot{\mathbb{Q}}'_\alpha$ (note that for $\alpha \geq \eta+1$ the iterands of the two iterations coincide), $p \restriction \alpha \Vdash p(\alpha) \in \mathbb{Q}'_\alpha$ for $\alpha \geq \eta+1$ and so Lemma 4.1.(3) completes the proof. □

Proof of Claim 4.9. We work in $V[G]$. Let $G_\beta = G \cap \mathbb{P}_\beta$. It is straightforward to check that $\mathbb{Q}^E_\alpha$ can be defined in $V[G\cap{}^{<\gamma}\mathbb{P}_\alpha]$ provided that $E \cap T'_\alpha$, and hence also $E \cap T_\alpha$, belongs to $V[G\cap{}^{<\gamma}\mathbb{P}_\alpha]$. Note that $E \cap T'_\alpha = \bigcup_{\beta<\alpha}(\gamma^{<\gamma} \cap \mathrm{succ} \cap V[G\cap{}^{<\gamma}\mathbb{P}_\alpha] \cap V[G_\beta])$.. Apply Lemma 4.3 to $\mathbb{P}_\beta * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is the quotient $\mathbb{P}_\alpha / \mathbb{P}_\beta$, and to ${}^{<\gamma}\mathbb{P}_\beta * \dot{\mathbb{Q}}'$, where $\dot{\mathbb{Q}}'$ is the quotient ${}^{<\gamma}\mathbb{P}_\alpha / {}^{<\gamma}\mathbb{P}_\beta$ to obtain $\gamma^{<\gamma} \cap V[G\cap{}^{<\gamma}\mathbb{P}_\alpha] \cap V[G_\beta] = \gamma^{<\gamma} \cap V[G_\beta \cap {}^{<\gamma}\mathbb{P}_\alpha]$. This is possible, since ${}^{<\gamma}\mathbb{P}_\beta \lessdot \mathbb{P}_\beta$ by induction hypothesis and $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}' \lessdot \dot{\mathbb{Q}}$ by Lemma 4.1.(3) for the tail iterations. Therefore, $E \cap T'_\alpha = \bigcup_{\beta<\alpha}(\gamma^{<\gamma} \cap \mathrm{succ})^{V[G_\beta \cap {}^{<\gamma}\mathbb{P}_\alpha]}$, which clearly belongs to $V[G\cap{}^{<\gamma}\mathbb{P}_\alpha]$, as desired. □

The next lemma shows that every condition is (essentially) hereditarily below γ for some $\gamma < \lambda$.

Lemma 4.10. *For every $p \in \mathbb{P}_\lambda / G_\eta$, there is $\gamma < \lambda$ and $p' \in {}^{<\gamma}(\mathbb{P}_\lambda / G_\eta)$ which is forcing equivalent to p.*

Proof. Proceed by induction on α. We only sketch the successor step. Let $(p, \dot{q}) \in (\mathbb{P}_\alpha / G_\eta) * \dot{\mathbb{Q}}_\alpha$. By inductive hypothesis there is $\gamma_p < \lambda$ such that $p \in {}^{<\gamma_p}(\mathbb{P}_\alpha / G_\eta)$. By the c.c.c. of $\mathbb{P}_\alpha$ each element of $\lambda^{<\lambda}$ in $V[\mathbb{P}_\alpha]$ has a nice $\mathbb{P}_\alpha / G_\eta$-name $\dot{\sigma}$ consisting of less than λ many countable antichains of conditions r such that $r \in {}^{<\gamma_r}(\mathbb{P}_\alpha / G_\eta)$ for some γ_r. Let $\gamma_{\dot{\sigma}}$ be the supremum of all these less than λ many γ_r. Then $\dot{\sigma}$ is a ${}^{<\gamma_{\dot{\sigma}}}(\mathbb{P}_\alpha / G_\eta)$-name. Now we can assume that $\dot{q}$ is a nice name which is hereditarily countable except for the nice names $\dot{\sigma}$ for the elements of $\mathrm{dom}(\dot{q})$; then, by the above, one can easily find $\gamma_{\dot{q}}$ such that $\dot{q}$ is in fact a ${}^{<\gamma_{\dot{q}}}(\mathbb{P}_\alpha / G_\eta)$-name. Finally, again by the c.c.c., there exists $\delta < \lambda$ such that $p \Vdash \mathrm{dom}(\dot{q}) \subseteq \delta^{<\delta}$. Let $\gamma = \max(\gamma_p, \gamma_{\dot{q}}, \delta) < \lambda$. Then $(p, \dot{q})$ belongs to ${}^{<\gamma}(\mathbb{P}_{\alpha+1} / G_\eta)$, which finishes the argument. □

Lemma 4.11. *Let G be $\mathbb{P}_\lambda / G_\eta$-generic and $V[G] \models b \subseteq \omega$. Then there is $\gamma < \lambda$ and a $\mathbb{P}_\lambda / G_\eta$-name $\dot{b}$ for b which is hereditarily below γ.*

[8]Note that E is really defined this way for both cases (see also footnote 7).

Proof. Let $\dot{b}$ be a nice name for b. By Lemma 4.10, we can assume that for every condition p in $\dot{b}$, there exists γ_p such that $p \in {}^{<\gamma_p}(\mathbb{P}_\lambda / G_\eta)$. Since b is countable and $\mathbb{P}_\lambda / G_\eta$ has the c.c.c., there exists $\gamma < \lambda$ such that $\dot{b}$ is a ${}^{<\gamma}(\mathbb{P}_\lambda / G_\eta)$-name. □

We conclude with a technical lemma which will be crucial later on:

Lemma 4.12. *Let $\tau \in T_\eta \setminus \gamma^{<\gamma}$. Let $p, r \in \mathbb{P}_\lambda / G_\eta$ be compatible in $\mathbb{P}_\lambda / G_\eta$ such that p is full, almost hereditarily below γ except for τ and r is hereditarily below γ. Then there is $p^* \in {}^{<\gamma+\tau}(\mathbb{P}_\lambda / G_\eta)$ such that $p^* \leq p, r$ and $p^*(\eta)(\tau) = p(\eta)(\tau)$.*

Proof. Without loss of generality we can assume that $\mathrm{dom}(p(\eta)) \supsetneqq \{\tau\}$. Since p is full and almost hereditarily below γ except for τ, by Lemma 4.8 $p^{\mathsf{RED}} = (p(\eta) \restriction \gamma^{<\gamma}, p(\eta+1), p(\eta+2), \dots)$ is a reduction of p to ${}^{<\gamma}(\mathbb{P}_\lambda / G_\eta)$. We show that p^{RED} and r are compatible in ${}^{<\gamma}(\mathbb{P}_\lambda / G_\eta)$. Assume not. Since ${}^{<\gamma}(\mathbb{P}_\lambda / G_\eta) \lessdot \mathbb{P}_\lambda / G_\eta$, it follows that $p^{\mathsf{RED}} \perp_{\mathbb{P}_\lambda / G_\eta} r$. But $p \leq p^{\mathsf{RED}}$, so $p \perp_{\mathbb{P}_\lambda / G_\eta} r$, which is a contradiction to the assumption of the lemma. Let $q^* \in {}^{<\gamma}(\mathbb{P}_\lambda / G_\eta)$ be such that $q^* \leq p^{\mathsf{RED}}, r$. Without loss of generality $q^*(\eta)$ is full. Since $q^*(\eta) \leq p^{\mathsf{RED}}(\eta) = p(\eta) \restriction \gamma^{<\gamma}$ and $p(\eta) \restriction \gamma^{<\gamma} = p(\eta) \restriction\restriction \gamma^{<\gamma}$ is a reduction of $p(\eta)$ by Lemma 3.11, $q^*(\eta)$ is compatible with $p(\eta)$. Let $\bar{q}(\eta)$ be a full witness for that. So $\bar{q}(\eta) \leq p(\eta), r(\eta), q^*(\eta)$.

Now, let $p^*(\eta) = \bar{q}(\eta) \restriction \gamma^{<\gamma} \cup \{(\tau, p(\eta)(\tau))\}$ and for $\alpha > \eta$, let $p^*(\alpha) = q^*(\alpha)$. Then $p^*(\eta)$ is a condition and moreover, $p^*(\eta) \leq q^*(\eta)$, since $\bar{q}(\eta) \leq q^*(\eta)$ and q^* hereditarily below γ except for τ. Thus, p^* is a condition, which is almost hereditarily below γ except for τ and $p^*(\eta)(\tau) = p(\eta)(\tau)$. Since $r(\eta)$ is hereditarily below γ, $p(\eta)$ is almost hereditarily below γ except for τ and $\bar{q}(\eta) \leq r(\eta), p(\eta)$, it is clear that $p^*(\eta)$ extends $r(\eta)$ and $p(\eta)$. Thus $p^* \leq r, p$. □

4.3. **No refinement: branches are towers.** Next we prove that the generic matrix has no refinement. More precisely, we show that the sets along any branch have no pseudo-intersection.

Lemma 4.13. *In $V[\mathbb{P}_\lambda]$, the sequence $\langle a_{\sigma\restriction(\xi+1)} \mid \xi < \lambda\rangle$ is a tower for each $\sigma \in \lambda^\lambda$.*

Proof. Let G_λ be $\mathbb{P}_\lambda$-generic. Work in $V[G_\lambda]$. Fix $\sigma \in \lambda^\lambda$. By Corollary 3.6(1), $\langle a_{\sigma\restriction(\xi+1)} \mid \xi < \lambda\rangle$ is $\subseteq^*$-decreasing. Assume towards a contradiction, that for some infinite $b \subseteq \omega$, $b \subseteq^* a_{\sigma\restriction(\xi+1)}$ for every $\xi < \lambda$. Apply Lemma 4.11 to get $\gamma < \lambda$ and a $\mathbb{P}_\lambda$-name $\dot{b}$ for b which is hereditarily below γ. Let ζ be the least successor ordinal with $\sigma \restriction \zeta \notin \gamma^{<\gamma}$ and $\eta < \lambda$ minimal such that $\sigma \restriction \zeta \in V[G_\eta]$.

Work in $V[G_\eta]$ and consider the tail forcing $\mathbb{P}_\lambda / G_\eta$. The $\mathbb{P}_\lambda$-name $\dot{b}$ can be understood as a $\mathbb{P}_\lambda / G_\eta$-name for b which is hereditarily below γ. Since $b \subseteq^* a_{\sigma\restriction\zeta}$ holds in $V[G_\lambda]$, we can pick $n \in \omega$ and $p \in \mathbb{P}_\lambda / G_\eta$ such that $p \Vdash \dot{b} \setminus n \subseteq a_{\sigma\restriction\zeta}$. From now on, whenever we say "almost hereditarily below γ", we shall mean "almost hereditarily below γ except for $\sigma \restriction \zeta$". Note that (the canonical name for) $a_{\sigma\restriction\zeta}$ is almost hereditarily below γ. Now, by Lemma 4.8 and Lemma 4.1.(2), we can fix p' which is almost hereditarily below γ such that $p' \Vdash \dot{b} \setminus n \subseteq a_{\sigma\restriction\zeta}$. Since η is minimal with $\sigma \restriction \zeta \in V[G_\eta]$, $\mathbb{Q}_\eta$ assigns a set $a_{\sigma\restriction\zeta}$ to $\sigma \restriction \zeta$. Without loss of generality $\sigma \restriction \zeta \in \mathrm{dom}(p'(\eta))$ and p' is a full[9] condition.

By Lemma 4.4 there is $r \in \mathbb{P}_\lambda / G_\eta$ hereditarily below γ and $m > n, |s^{p'(\eta)}_{\sigma\restriction\zeta}|$ such that r is compatible with p', and $r \Vdash m \in \dot{b}$. Apply Lemma 4.12 to obtain $p'' \leq p', r$ such that p'' is almost hereditarily below γ, and moreover

$$p''(\eta)(\sigma \restriction \zeta) = p'(\eta)(\sigma \restriction \zeta).$$

[9]Here we use the modification of Definition 3.8, where 0 is replaced by η, i.e., $p'(\eta)$ is full.

It follows that $p'' \Vdash m \in \dot{b}$. In particular $m > |s^{p''(\eta)}_{\sigma\restriction\zeta}|$, thus we can strengthen p'' to a condition q (only strengthening $p''(\eta)$) by extending $s^{p''(\eta)}_{\sigma\restriction\zeta}$ to length $> m$ with $s^{q(\eta)}_{\sigma\restriction\zeta}(m) = 0$. Then $q \Vdash m \in \dot{b} \wedge m \notin a_{\sigma\restriction\zeta}$, which is a contradiction to the fact that p' forces $\dot{b} \setminus n \subseteq a_{\sigma\restriction\zeta}$. □

4.4. **Levels are mad families.** Finally, we show that the levels of the generic matrix form mad families.

Lemma 4.14. *In $V[\mathbb{P}_\lambda]$, the family $A_{\xi+1} = \{a_\sigma \mid |\sigma| = \xi + 1\}$ is mad for each $\xi < \lambda$.*

Proof. Let G_λ be generic for $\mathbb{P}_\lambda$ and work in $V[G_\lambda]$. The main work lies in the following claim, which guarantees "local madness" below branches. We will prove it after finishing the proof of the lemma.

Claim 4.15. *Let $\rho \in \lambda^{<\lambda}$ and let $b \subseteq \omega$ be such that $b \cap a_{\rho\restriction\zeta}$ is infinite for every successor $\zeta \leq |\rho|$. Then there exists an $i < \lambda$ such that $b \cap a_{\rho^\frown i}$ is infinite.*

Fix $\xi < \lambda$. By Corollary 3.6(2), $A_{\xi+1}$ is almost disjoint. Using the claim, we will show that $A_{\xi+1}$ is actually mad. Let $b \subseteq \omega$ be infinite. To find $\sigma \in \lambda^{\xi+1}$ such that $b \cap a_\sigma$ is infinite, we construct by induction on ζ, a branch $\langle \rho_\zeta \mid \zeta \leq \xi + 1 \rangle$ with $|\rho_\zeta| = \zeta$ for each ζ and $\rho_{\zeta'} \trianglelefteq \rho_\zeta$ for $\zeta' \leq \zeta$ such that $b \cap a_{\rho_\zeta}$ is infinite for every successor $\zeta \leq \xi + 1$. Let $\rho_0 = \langle\rangle$. Assume $\langle \rho_{\zeta'} \mid \zeta' < \zeta \rangle$ is constructed. If ζ is a limit, let $\rho_\zeta = \bigcup\{\rho_{\zeta'} \mid \zeta' < \zeta\}$. If $\zeta = \zeta' + 1$ is a successor, $\rho_{\zeta'}$ fulfills the assumptions of the claim by induction. Let $i < \lambda$ be given by the claim and let $\rho_\zeta = \rho_{\zeta'}{}^\frown i$. Then $b \cap a_{\rho_\zeta}$ is infinite. Finally take $\sigma = \rho_{\xi+1}$. □

Proof of Claim 4.15. Assume towards a contradiction that $b \cap a_{\rho\restriction\zeta}$ is infinite for every successor $\zeta \leq |\rho|$, but $b \cap a_{\rho^\frown i}$ is finite for every $i < \lambda$. Let η be minimal with $\rho \in V[G_\eta]$. Thus $a_{\rho^\frown i}$ (for any i) is not defined in $V[G_\eta]$ but it will be defined in $V[G_{\eta+1}]$. From now on, we work in $V[G_\eta]$. Consider the tail forcing $\mathbb{P}_\lambda / G_\eta$ and apply Lemma 4.11 to get a $\mathbb{P}_\lambda/G_\eta$-name $\dot{b}$ for b and $\gamma' < \lambda$ such that $\dot{b}$ is hereditarily below γ'. Let $\gamma < \lambda$ be strictly above $|\rho| + 1$, $\sup(\mathrm{rng}(\rho))$, γ' and pick $n \in \omega$, $p \in \mathbb{P}_\lambda / G_\eta$ such that

(1) $p \Vdash \dot{b} \cap a_{\rho^\frown\gamma} \subseteq n$,
(2) $p \Vdash \dot{b} \cap a_{\rho^\frown i}$ is finite, for each $i < \gamma$, and
(3) $p \Vdash \dot{b} \cap a_{\rho\restriction\zeta}$ is infinite, for each successor $\zeta \leq |\rho|$.

From now on, whenever we say "almost hereditarily below γ", we shall mean "almost hereditarily below γ except for $\rho^\frown\gamma$". Clearly, $\dot{b}$, $a_{\rho^\frown i}$ for $i \leq \gamma$ and $a_{\rho\restriction\zeta}$ for each successor $\zeta \leq |\rho|$ are almost hereditarily below γ. By Lemma 4.8 and Lemma 4.1.(2), we can fix p' which is almost hereditarily below γ such that items (1), (2), and (3) above hold true for p' in place of p. Without loss of generality, p' is full and $\rho^\frown\gamma \in \mathrm{dom}(p'(\eta))$. Define $R = \mathrm{dom}(p'(\eta)) \cap \{\rho^\frown i \mid i < \gamma\}$ and $R' = \mathrm{dom}(f^{p'(\eta)}_{\rho^\frown\gamma})$.

Let $\dot{x}$ be a $\mathbb{P}_\lambda/G_\eta$-name such that $\Vdash \dot{x} = \bigcap_{\tau\in R'} (\dot{b} \cap a_\tau) \setminus \bigcup_{\tau\in R} a_\tau$. Since ${}^{<\gamma}(\mathbb{P}_\lambda / G_\eta) \lessdot \mathbb{P}_\lambda / G_\eta$ by Lemma 4.8 and all names used to define $\dot{x}$ are hereditarily below γ, we can assume that $\dot{x}$ is hereditarily below γ as well. Since R and R' are finite, p' forces $\dot{x}$ to be infinite. By Lemma 4.4 there is $r \in {}^{<\gamma}(\mathbb{P}_\lambda / G_\eta)$ and $m > n, |s^{p'(\eta)}_{\rho^\frown\gamma}|$ such that r is compatible with p' and $r \Vdash m \in \dot{x}$. Apply Lemma 4.12 to obtain $p'' \leq p', r$ such that p'' is almost hereditarily below γ and

$$p''(\eta)(\rho^\frown\gamma) = p'(\eta)(\rho^\frown\gamma).$$

In particular $p'' \Vdash m \in \dot{x}$ and $p'' \Vdash m \in \bigcap_{\tau\in R'} a_\tau \setminus \bigcup_{\tau\in R} a_\tau$. Now extend p'' to a condition q so that the following holds: For $\alpha > \eta$, $q(\alpha) = p''(\alpha)$; for $\tau \in (R \cup R') \cap \mathrm{dom}(p''(\eta))$, $s^{q(\eta)}_\tau$ extends $s^{p''(\eta)}_\tau$ and $|s^{q(\eta)}_\tau| > m$; $s^{q(\eta)}_{\rho^\frown\gamma}$ extends $s^{p''(\eta)}_{\rho^\frown\gamma}$ and $s^{q(\eta)}_{\rho^\frown\gamma}(i) = 0$ for $|s^{p''(\eta)}_{\rho^\frown\gamma}| \leq i < m$, $s^{q(\eta)}_{\rho^\frown\gamma}(m) = 1$.

Note that in particular, $s_\tau^{q(\eta)}(m) = 1$ for $\tau \in R' \cap \mathrm{dom}(p''(\eta))$, $a_\tau(m) = 1$ for $\tau \in R' \setminus \mathrm{dom}(p''(\eta))$, and $s_\tau^{q(\eta)}(m) = 0$ for $\tau \in R$. But then, $q \Vdash m \in \dot{x} \cap a_{\rho^\frown\gamma}$, which is a contradiction to $p' \Vdash \dot{x} \cap a_{\rho^\frown\gamma} \subseteq n$. □

This finishes the proof that the generic matrix is a refining matrix of height λ. It remains to prove that $\mathfrak{b}$ (and hence $\mathfrak{h}$) is small in our final model; this is the subject of Sections 5 and 6.

5. $\mathcal{B}$-Canjar filters

In this section, we give the preliminaries regarding $\mathcal{B}$-Canjar filters and preservation of unboundedness, which are needed in Section 6. Let $\mathfrak{F} \subseteq \mathcal{P}(\omega)$ be a set with the finite intersection property, i.e., the intersection of any finitely many of its elements is infinite. We write $\langle\mathfrak{F}\rangle_{\mathrm{Fr}}$ to denote the filter generated by $\mathfrak{F}$ together with the Fréchet filter, i.e., $B \in \langle\mathfrak{F}\rangle_{\mathrm{Fr}}$ if $B \supseteq \bigcap_{i<n} a_i \setminus m$ for some $n, m \in \omega$ and $\{a_i \mid i < n\} \subseteq \mathfrak{F}$. Recall, that for a filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ containing the Fréchet filter, *Mathias forcing with respect to* $\mathcal{F}$, denoted $\mathbb{M}(\mathcal{F})$, is the poset of pairs (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$, where the extension relation is defined as follows: $(t, B) \leq (s, A)$ if $t \trianglerighteq s$, $B \subseteq A$, and for each $n \geq |s|$, if $t(n) = 1$, then $n \in A$. The generic real for $\mathbb{M}(\mathcal{F})$ is a pseudo-intersection of $\mathcal{F}$, $\mathbb{M}(\mathcal{F})$ is σ-centered and Mathias forcing with respect to the Fréchet filter is forcing equivalent to Cohen forcing $\mathbb{C}$. A filter $\mathcal{F}$ is said to be *Canjar* if $\mathbb{M}(\mathcal{F})$ does not add a dominating real over the ground model. We will need the following generalization of Canjarness: For an unbounded family $\mathcal{B} \subseteq \omega^\omega$, a filter $\mathcal{F}$ is said to be *$\mathcal{B}$-Canjar* if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$.

5.1. **A combinatorial characterization of $\mathcal{B}$-Canjarness.** We will make use of the following combinatorial characterization of $\mathcal{B}$-Canjarness, due to Guzmán-Hrušák-Martínez [20]. The characterization generalizes an earlier result of Hrušák-Minami [22]. For a given filter $\mathcal{F}$ on ω, a set $X \subseteq [\omega]^{<\omega}$ is in $(\mathcal{F}^{<\omega})^+$ if and only if for each $A \in \mathcal{F}$ there is an $s \in X$ with $s \subseteq A$. Given $\bar{X} = \langle X_n \mid n \in \omega\rangle$ with $X_n \subseteq [\omega]^{<\omega}$ for each $n \in \omega$ and $f \in \omega^\omega$, let $\bar{X}_f = \bigcup_{n\in\omega}(X_n \cap \mathcal{P}(f(n)))$.

Theorem 5.1. *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. A filter $\mathcal{F}$ is $\mathcal{B}$-Canjar if and only if for each sequence $\bar{X} = \langle X_n \mid n \in \omega\rangle \subseteq (\mathcal{F}^{<\omega})^+$ there exists an $f \in \mathcal{B}$ such that $\bar{X}_f \in (\mathcal{F}^{<\omega})^+$.*

Proof. See [20, Proposition 1]. □

Lemma 5.2. *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. Then:*

(1) The Fréchet filter is $\mathcal{B}$-Canjar.
(2) If $\mathcal{F}$ is a $\mathcal{B}$-Canjar filter extending the Fréchet filter and $\{a_n \mid n < \omega\}$ is such that $\mathcal{F} \cup \{a_n \mid n < \omega\}$ has the finite intersection property, then $\langle\mathcal{F} \cup \{a_n \mid n < \omega\}\rangle_{\mathrm{Fr}}$ is $\mathcal{B}$-Canjar.
(3) Every countably generated filter is $\mathcal{B}$-Canjar.

Proof. For item (1), note that Cohen forcing $\mathbb{C}$ preserves the unboundedness of every unbounded family. To see item (2), let $\bar{X} = \langle X_n \mid n \in \omega\rangle \subseteq (\langle\mathcal{F} \cup \{a_n \mid n < \omega\}\rangle_{\mathrm{Fr}}{}^{<\omega})^+$. Let $Y_n = \{s \in X_n \mid s \subseteq \cap_{k<n} a_k\}$ and let $\bar{Y} = \langle Y_n \mid n \in \omega\rangle$. It is easy to see that $Y_n \in (\mathcal{F}^{<\omega})^+$ for each n. By the assumption and Theorem 5.1 there exists $f \in \mathcal{B}$ such that $\bar{Y}_f \in (\mathcal{F}^{<\omega})^+$. To show that $\bar{Y}_f \in (\langle\mathcal{F} \cup \{a_n \mid n < \omega\}\rangle_{\mathrm{Fr}}{}^{<\omega})^+$, let $B \in \langle\mathcal{F} \cup \{a_n \mid n < \omega\}\rangle_{\mathrm{Fr}}$. We have to find $s \in \bar{Y}_f$ with $s \subseteq B$. Clearly we can assume that $\emptyset \notin \bar{Y}_f$. Fix $A \in \mathcal{F}$ and $n \in \omega$ with $B \supseteq A \cap \bigcap_{k<n} a_k$. Since $\mathcal{F}$ contains the Fréchet filter and $\bar{Y}_f \in (\mathcal{F}^{<\omega})^+$, there exist infinitely many $s \in \bar{Y}_f$ with $s \subseteq A$. So there exists $m \geq n$ and $s \in Y_m \cap \bar{Y}_f$ with $s \subseteq A$; note that $s \in Y_m$ implies $s \subseteq \bigcap_{k<n} a_k$, so $s \subseteq B$, as desired. Clearly $\bar{Y}_f \subseteq \bar{X}_f$, so $\bar{X}_f \in (\langle\mathcal{F} \cup \{a_n \mid n < \omega\}\rangle_{\mathrm{Fr}}{}^{<\omega})^+$.

Item (3) follows immediately from items (1) and (2) of the Lemma. □

5.2. **Preservation of unboundedness at limits.** We will make use of the following preservation theorem, a more general version of which can be found in [24, Theorem 2.2].

Theorem 5.3. *Suppose* $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\}$ *is a finite support iteration of c.c.c. partial orders of limit length* δ*, and* $\mathcal{B} \subseteq \omega^\omega$ *is unbounded; also suppose that* $\mathcal{B}$ *is countably directed, i.e., it satisfies*

$$\forall \mathcal{A} \subseteq \mathcal{B}\ (|\mathcal{A}| = \aleph_0 \to \exists f \in \mathcal{B}\ \forall g \in \mathcal{A}\ g \leq^* f). \tag{3}$$

If $\forall \alpha < \delta \Vdash_{\mathbb{P}_\alpha}$ *"*$\mathcal{B}$ *is an unbounded family", then* $\Vdash_{\mathbb{P}_\delta}$ *"*$\mathcal{B}$ *is an unbounded family".*

5.3. **Preservation of $\mathcal{B}$-Canjarness and finite sums of filters.** The notion of $\mathcal{B}$-Canjarness of a filter is not absolute in general:

Example 5.4 (from [19])**.** *Let* $\mathcal{B}$ *be the ground model reals and* $\mathcal{U}$ *be a* $\mathcal{B}$*-Canjar ultrafilter. Let* $\mathbb{P}$ *be Grigorieff forcing with respect to* $\mathcal{U}$*, which forces that* $\mathcal{U}$ *cannot be extended to a P-point. It is well-known that* $\mathbb{P}$ *preserves the unboundedness of* $\mathcal{B}$*, and it can be shown that* $\mathcal{U}$ *is not a* P^+*-filter in* $V[\mathbb{P}]$*; since any Canjar filter is a* P^+*-filter, it follows that* $\mathcal{U}$ *is no longer* $\mathcal{B}$*-Canjar.*

Grigorieff forcing is proper, but not c.c.c.; however, Grigorieff forcing can be decomposed into a σ*-closed and a c.c.c. forcing (see [26]). Since a* σ*-closed forcing does not destroy the* $\mathcal{B}$*-Canjarness of a filter, the above example also yields an example of a c.c.c. forcing destroying the* $\mathcal{B}$*-Canjarness of a filter.*

We will now provide a method, which will allow us to guarantee that the $\mathcal{B}$-Canjarness of a filter is not destroyed by Mathias forcings with respect to certain other filters. As a tool, we introduce finite sums of filters and consider Mathias forcings with respect to these sums.

Lemma 5.5. *Let* $\mathcal{F}$ *be a filter,* $\mathcal{B} \subseteq \omega^\omega$ *and* $\mathbb{P}$ *be a forcing notion. Then the following are equivalent:*

(1) $\mathbb{P}$ *forces that*[10] $\mathcal{F}$ *is* $\mathcal{B}$*-Canjar.*
(2) $\mathbb{M}(\mathcal{F}) \times \mathbb{P}$ *forces that* $\mathcal{B}$ *is unbounded.*

Even though we will apply the lemma only in case $\mathcal{B}$ is unbounded and $\mathcal{F}$ is $\mathcal{B}$-Canjar in the ground model, this is not necessary for the proof. If one of these assumptions fails, both (1) and (2) are false.

Proof of Lemma 5.5. Let $\mathbb{Q} = \mathbb{M}(\mathcal{F})$. Note that (1) holds if and only if $\mathbb{P}$ forces $\mathbb{M}(\langle\check{\mathcal{F}}\rangle_{\mathrm{Fr}}) \Vdash$ "$\mathcal{B}$ unbounded". Further $\mathbb{P}$ forces that $\check{\mathbb{Q}}$ is dense in, and hence forcing equivalent to $\mathbb{M}(\langle\check{\mathcal{F}}\rangle_{\mathrm{Fr}})$. So, (1) holds if and only if $\mathbb{P} * \check{\mathbb{Q}}$ forces that $\mathcal{B}$ is unbounded, which is the same as (2) since $\mathbb{P} * \check{\mathbb{Q}}$ is equivalent to $\mathbb{P} \times \mathbb{Q} = \mathbb{Q} \times \mathbb{P}$. □

Definition 5.6. For $A, B \subseteq \omega$, let $A \oplus B = \{2n \mid n \in A\} \cup \{2m+1 \mid m \in B\}$. For two filters $\mathcal{F}_0$ and $\mathcal{F}_1$, let $\mathcal{F}_0 \oplus \mathcal{F}_1 = \{A \oplus B \mid A \in \mathcal{F}_0, B \in \mathcal{F}_1\}$ and inductively, let $\bigoplus_{k<m+1} \mathcal{F}_k = \left(\bigoplus_{k<m} \mathcal{F}_k\right) \oplus \mathcal{F}_m$.

Note that $\mathcal{F}_0 \oplus \mathcal{F}_1$ is a filter if $\mathcal{F}_0$ and $\mathcal{F}_1$ are filters and hence also the finite sum of filters is a filter. Clearly, reordered sums are isomorphic by an isomorphism induced by a permutation of ω. This implies that the $\mathcal{B}$-Canjarness of a finite sum of filters does not depend on the order of the sum.

Lemma 5.7. *Let* $\mathcal{F}_0$ *and* $\mathcal{F}_1$ *be two filters. Then* $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ *is forcing equivalent to* $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$*.*

[10] To be more precise, one should write $\langle\check{\mathcal{F}}\rangle_{\mathrm{Fr}}$ instead of $\mathcal{F}$.

Proof. Let $D_\times \subseteq \mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ be the set of all $((s_0, A_0), (s_1, A_1)) \in \mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ with $|s_0| = |s_1|$, and let $D_\oplus \subseteq \mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ be the set of all $(s, A) \in \mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ with $|s|$ being an even number. Note that $D_\times$ is a dense subforcing of $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$, and $D_\oplus$ is a dense subforcing of $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$.

For $s_0, s_1 \in 2^{<\omega}$ with $L = |s_0| = |s_1|$, let $s_0 \oplus s_1 \in 2^{<\omega}$ be such that $|s_0 \oplus s_1| = 2L$ and satisfies $(s_0 \oplus s_1)(2n) = s_0(n)$ and $(s_0 \oplus s_1)(2n+1) = s_1(n)$. Define $\iota\colon D_\times \to D_\oplus$ as follows: $((s_0, A_0), (s_1, A_1)) \mapsto (s_0 \oplus s_1, A_0 \oplus A_1)$. It is easy to see that ι is an isomorphism between $D_\times$ and $D_\oplus$. Consequently, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ and $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ are forcing equivalent. □

The next Lemma provides the main ingredients for the induction in Lemma 6.3:

Lemma 5.8.

(1) If $\mathcal{F}_0 \oplus \mathcal{F}_1$ is $\mathcal{B}$-Canjar, then $\mathbb{M}(\mathcal{F}_1)$ forces that $\mathcal{F}_0$ is $\mathcal{B}$-Canjar.

(2) Let $\mathcal{B}$ be a countably directed family, α a limit and $\{\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha\}$ a finite support iteration. If $\mathbb{P}_\beta$ forces that $\mathcal{F}$ is $\mathcal{B}$-Canjar for every $\beta < \alpha$, then $\mathbb{P}_\alpha$ forces that $\mathcal{F}$ is $\mathcal{B}$-Canjar.

(3) Let $\mathcal{F}_0$ be $\mathcal{B}$-Canjar and $\mathcal{F}_1$ be countably generated. Then $\mathcal{F}_0 \oplus \mathcal{F}_1$ is $\mathcal{B}$-Canjar.

Proof. To see (1), note that by assumption and Lemma 5.7, $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ forces that $\mathcal{B}$ is unbounded; apply Lemma 5.5 to finish the proof. To see (2) observe that by assumption and Lemma 5.5, $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta$ forces that $\mathcal{B}$ is unbounded for every $\beta < \alpha$. However $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$ is the direct limit of $\langle \mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta \mid \beta < \alpha \rangle$ (and $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\beta \lessdot \mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$) and so can be written as the limit of a finite support iteration. Then by Theorem 5.3, also $\mathbb{M}(\mathcal{F}) \times \mathbb{P}_\alpha$ forces that $\mathcal{B}$ is unbounded. The conclusion follows by Lemma 5.5.

For (3), note that by Lemma 5.7, $\mathbb{M}(\mathcal{F}_0 \oplus \mathcal{F}_1)$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$. Now, in the extension by $\mathbb{M}(\mathcal{F}_0)$, $\mathcal{B}$ is unbounded since $\mathcal{F}_0$ is $\mathcal{B}$-Canjar. Moreover the filter generated by $\mathcal{F}_1$ in the same extension is countably generated and so by Lemma 5.2.(3) it is $\mathcal{B}$-Canjar. It remains to observe that by Lemma 5.5. $\mathbb{M}(\mathcal{F}_0) \times \mathbb{M}(\mathcal{F}_1)$ forces that $\mathcal{B}$ is unbounded. □

Using the fact that sums can be reordered (see the remark after Definition 5.6), we obtain the following stronger statement: Let $\mathcal{F}_0, \ldots, \mathcal{F}_{m-1}$ be filters such that the sum of the filters which are not countably generated is $\mathcal{B}$-Canjar; then $\bigoplus_{k<m} \mathcal{F}_k$ is $\mathcal{B}$-Canjar.

6. Preserving unboundedness: $\mathfrak{h} = \mathfrak{b} = \omega_1$

In this section, we complete the proof of Main Theorem 1.2 by showing that $\mathfrak{b} = \omega_1$ in the final model W. Hence $\mathfrak{h} = \omega_1$ and therefore in W there are refining matrices of heights $\mathfrak{h}$ and λ. More specifically, in Section 6.1, we will show that our iteration $\mathbb{P}_\lambda$ can be represented as a finer iteration whose iterands are Mathias forcings with respect to filters. In Section 6.2, we show that these filters are $\mathcal{B}$-Canjar, where $\mathcal{B}$ is the set of reals of V_0. For a similar, but less involved argument showing that Hechler's original forcings [21] to add a tower or to add a mad family can be represented as an iteration of Mathias forcings with respect to $\mathcal{B}$-Canjar filters see [15].

6.1. Finer iteration via filtered Mathias forcings.

The notation in this section refers to our construction in Section 3.1. Fix $\alpha < \lambda$. As a preparation, we introduce a "nice" enumeration of T_α. We go through the nodes in T_α level by level and blockwise (items (1) and (2) below, respectively). More precisely, let $\{\sigma^\nu_\alpha \mid \nu < \Lambda_\alpha\}$ enumerate T_α (note that $|T_\alpha| = \mathfrak{c}$ and hence Λ_α is an ordinal with $\mathfrak{c} < \Lambda_\alpha < \mathfrak{c}^+$) so that

(1) $|\sigma_\alpha^{\bar\nu}| < |\sigma_\alpha^\nu| \to \bar\nu < \nu$,
(2) if $\rho \in \lambda^{<\lambda}$ and $\{\rho^\frown i \mid i < \lambda\} \subseteq T_\alpha$, then there is $\nu < \Lambda_\alpha$ such that $\rho^\frown i = \sigma_\alpha^{\nu+i}$ for each $i < \lambda$.

For $\beta \leq \Lambda_\alpha$ let $\mathbb{Q}_\alpha^{<\beta} = \mathbb{Q}_\alpha^{\{\sigma_\alpha^\nu \mid \nu<\beta\}}$ and for $\beta < \Lambda_\alpha$ let $\mathbb{Q}_\alpha^{\leq\beta} = \mathbb{Q}_\alpha^{\{\sigma_\alpha^\nu \mid \nu\leq\beta\}}$. Note that $\mathbb{Q}_\alpha^{<\Lambda_\alpha} = \mathbb{Q}_\alpha$ and that $\{\sigma_\alpha^\nu \mid \nu < \beta\}$ is α-upwards closed for each $\beta \leq \Lambda_\alpha$. Therefore, by Lemma 3.11, $\mathbb{Q}_\alpha^{<\beta} \lessdot \mathbb{Q}_\alpha$. By Lemma 4.1.(1), $\mathbb{Q}_\alpha^{<\beta} \lessdot \mathbb{Q}_\alpha^{\leq\beta}$ and so we can form the quotient $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$. Moreover, because conditions in $\mathbb{Q}_\alpha$ have finite domain, $\mathbb{Q}_\alpha^{<\beta} = \bigcup_{\delta<\beta} \mathbb{Q}_\alpha^{<\delta}$ for each limit ordinal $\beta \leq \Lambda_\alpha$. In other words, $\mathbb{Q}_\alpha^{<\beta}$ is the direct limit of the forcings $\mathbb{Q}_\alpha^{<\delta}$ for $\delta < \beta$. So $\mathbb{Q}_\alpha$ is forcing equivalent to the finite support iteration of the quotients $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$ for $\beta < \Lambda_\alpha$.

We will show that $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$ is in fact forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha^\beta)$ for some filter $\mathcal{F}_\alpha^\beta$. Work in an extension by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}$. Note that for each $\tau \in T_\eta$ with $\eta < \alpha$, a_τ is added by $\mathbb{P}_\alpha$ and for $\nu < \beta$, $a_{\sigma_\alpha^\nu}$ is added by $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta}$. These sets[11] define $\mathcal{F}_\alpha^\beta$ as follows: Let $\rho \in \lambda^{<\lambda}$, $i < \lambda$ be such that $\sigma_\alpha^\beta = \rho^\frown i$ and let

$$\mathfrak{F}_\alpha^\beta = \{a_{\rho\restriction(\xi+1)} \mid \xi + 1 \leq |\rho|\} \cup \{\omega \setminus a_{\rho^\frown j} \mid j < i\}.$$

That is, $\mathfrak{F}_\alpha^\beta$ is the collection of all sets assigned to the nodes above σ_α^β and the complements of the sets assigned to the nodes to the left of σ_α^β within the same block. Now $\mathfrak{F}_\alpha^\beta$ has the finite intersection property, becuse any finite intersection of elements of $\mathfrak{F}_\alpha^\beta$ almost contains $a_{\rho^\frown j}$ for some $j < \lambda$. Let $\mathcal{F}_\alpha^\beta$ be the filter generated by $\mathfrak{F}_\alpha^\beta$ together with the Fréchet filter, i.e. $\mathcal{F}_\alpha^\beta = \langle \mathfrak{F}_\alpha^\beta \rangle_{\mathrm{Fr}}$. Note that the quotient $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$ adds a_σ where $\sigma = \sigma_\alpha^\beta$. The lemma below provides a dense embedding from $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$ to $\mathbb{M}(\mathcal{F}_\alpha^\beta)$ with the property that a_σ is the generic real for $\mathbb{M}(\mathcal{F}_\alpha^\beta)$.

Lemma 6.1. *$\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$ is densely embeddable into $\mathbb{M}(\mathcal{F}_\alpha^\beta)$.*

Proof. For simplicity of notation, let $\sigma = \sigma_\alpha^\beta$. Let G be a $\mathbb{Q}_\alpha^{<\beta}$-generic filter. We work in the extension by G, so $\mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta} = \{p \in \mathbb{Q}_\alpha^{\leq\beta} \mid \forall q \in G(p \text{ is compatible with } q)\}$. Define $\iota\colon \mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta} \to \mathbb{M}(\mathcal{F}_\alpha^\beta)$ as follows: for $p \in \mathbb{Q}_\alpha^{\leq\beta}/\mathbb{Q}_\alpha^{<\beta}$, let $p(\sigma) = (s_\sigma, f_\sigma, h_\sigma)$ and let $\iota(p) = (s_\sigma, A)$, where

$$A = \bigcap_{\tau\in\mathrm{dom}(f_\sigma)} (a_\tau \cup f_\sigma(\tau)) \cap \bigcap_{\rho\in\mathrm{dom}(h_\sigma)} ((\omega \setminus a_\rho) \cup h_\sigma(\rho)) \setminus |s_\sigma|.$$

To see that ι is a dense embedding, we have to check the following:

(1) (Density) For every condition $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha^\beta)$, there exists a condition p such that $\iota(p) \leq (s, A)$.
(2) (Incompatibility preserving) If p and p' are incompatible, then so are $\iota(p)$ and $\iota(p')$.
(3) (Order preserving) If $p' \leq p$, then $\iota(p') \leq \iota(p)$.

To show (1), let $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha^\beta)$. Since $A \in \mathcal{F}_\alpha^\beta$, there exist finite sets $\{\rho_i \mid i < m\}$, $\{\tau_j \mid j < l\}$ and $N \in \omega$ such that $\bigcap_{j<l} a_{\tau_j} \cap \bigcap_{i<m}(\omega \setminus a_{\rho_i}) \setminus N \subseteq A$. Extend s with 0's to s_σ such that $|s_\sigma| = \max(|s|, N)$ and let $\mathrm{dom}(h_\sigma) = \{\rho_i \mid i < m\}$, $h_\sigma(\rho_i) = |s_\sigma|$ for every i, $\mathrm{dom}(f_\sigma) = \{\tau_j \mid j < l\}$ and $f_\sigma(\tau_j) = |s_\sigma|$ for every j. Now, let $p = \{(\sigma, (s_\sigma, f_\sigma, h_\sigma))\} \cup \{(\tau, (\langle\rangle, \emptyset, \emptyset)) \mid \tau \in (\mathrm{dom}(f_\sigma) \cap T_\alpha) \cup \mathrm{dom}(h_\sigma)\}$. To see that p is in the quotient, consider an arbitrary $q \in G$. It is easy to check that $q \cup \{(\tau, (s_\tau, f_\tau, h_\tau)) \mid \tau \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)\} \leq p, q$. By definition, $\iota(p) = (s_\sigma, A')$, where $A' = \bigcap_{\tau\in\mathrm{dom}(f_\sigma)} (a_\tau \cup f_\sigma(\tau)) \cap \bigcap_{\rho\in\mathrm{dom}(h_\sigma)} ((\omega \setminus a_\rho) \cup h_\sigma(\rho)) \setminus |s_\sigma|$. It follows

[11] It is possible (see the base step $\beta^* = 0$ of the proof of Lemma 6.3(3)) that only sets a_τ with $\tau \in T_\eta$ for some $\eta < \alpha$ are used. This is the case if ρ is pre-T_α-minimal and $i = 0$.

that $A' \overset{(*)}{=} \bigcap_{\tau\in\mathrm{dom}(f_\sigma)} a_\tau \cap \bigcap_{\rho\in\mathrm{dom}(h_\sigma)} (\omega \setminus a_\rho) \setminus |s_\sigma| \subseteq \bigcap_{j<l} a_{\tau_j} \cap \bigcap_{i<m} (\omega \setminus a_{\rho_i}) \setminus N \subseteq A$, where (*) holds because $|s_\sigma| \geq f_\sigma(\tau), h_\sigma(\rho)$ for every τ, ρ in the respective domains. Therefore $s_\sigma \trianglerighteq s$, $A' \subseteq A$ and $s_\sigma(n) = 0$ for all $n \geq |s|$. So $\iota(p) = (s_\sigma, A') \leq (s, A)$.

We prove (2) by showing the contrapositive. Assume $\iota(p)$ and $\iota(p')$ are compatible. Define q as follows. Let $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \mathrm{dom}(p')$. For every $\tau \in \mathrm{dom}(q)$, let $s^q_\tau = s^p_\tau \cup s^{p'}_\tau$, $\mathrm{dom}(f^q_\tau) = \mathrm{dom}(f^p_\tau) \cup \mathrm{dom}(f^{p'}_\tau)$ and for $\rho \in \mathrm{dom}(f^q_\tau)$ let $f^q_\tau(\rho) = \min(f^p_\tau(\rho), f^{p'}_\tau(\rho))$, and the same for h: $\mathrm{dom}(h^q_\tau) = \mathrm{dom}(h^p_\tau) \cup \mathrm{dom}(h^{p'}_\tau)$ and for $\rho \in \mathrm{dom}(h^q_\tau)$ let $h^q_\tau(\rho) = \min(h^p_\tau(\rho), h^{p'}_\tau(\rho))$. It is easy to check that q is in the quotient, $q \leq p, p'$.

To show (3), let $p' \leq p$. By definition, $s^{p'}_\sigma \trianglerighteq s^p_\sigma$, $\mathrm{dom}(h^{p'}_\sigma) \supseteq \mathrm{dom}(h^p_\sigma)$, $\mathrm{dom}(f^{p'}_\sigma) \supseteq \mathrm{dom}(f^p_\sigma)$, and $f^{p'}_\sigma(\tau) \leq f^p_\sigma(\tau)$ for $\tau \in \mathrm{dom}(f^p_\sigma)$, $h^{p'}_\sigma(\rho) \leq h^p_\sigma(\rho)$ for $\rho \in \mathrm{dom}(h^p_\sigma)$. Then

$$A' = \bigcap_{\tau\in\mathrm{dom}(f^{p'}_\sigma)} (a_\tau \cup f^{p'}_\sigma(\tau)) \cap \bigcap_{\rho\in\mathrm{dom}(h^{p'}_\sigma)} ((\omega \setminus a_\rho) \cup h^{p'}_\sigma(\rho)) \setminus |s^{p'}_\sigma|$$

is a subset of

$$A := \bigcap_{\tau\in\mathrm{dom}(f^{p}_\sigma)} (a_\tau \cup f^{p}_\sigma(\tau)) \cap \bigcap_{\rho\in\mathrm{dom}(h^{p}_\sigma)} ((\omega \setminus a_\rho) \cup h^{p}_\sigma(\rho)) \setminus |s^{p}_\sigma|.$$

By definition, $\iota(p) = (s^p_\sigma, A)$ and $\iota(p') = (s^{p'}_\sigma, A')$. To show that $(s^{p'}_\sigma, A') \leq (s^p_\sigma, A)$, it remains to show that for $n \geq |s^p_\sigma|$ with $s^{p'}_\sigma(n) = 1$, we have $n \in A$. First fix $\rho \in \mathrm{dom}(h^p_\sigma)$ and show that $n \in (\omega \setminus a_\rho) \cup h^p_\sigma(\rho)$. If $n < h^p_\sigma(\rho)$, this is clear. If $n \geq h^p_\sigma(\rho)$, we know that $s^{p'}_\sigma$ respects h^p_σ, and so $n \in \omega \setminus a_\rho$. So in both cases, $n \in (\omega \setminus a_\rho) \cup h^p_\sigma(\rho)$. To show that for $\tau \in \mathrm{dom}(f^p_\sigma)$, $n \in a_\tau \cup f^p_\sigma(\tau)$ argue in the same way as for h. If $n < f^p_\sigma(\tau)$, this is clear. If $n \geq f^p_\sigma(\tau)$, we know that $s^{p'}_\sigma$ respects f^p_σ, and so $n \in a_\tau$. Thus in both cases, $n \in a_\tau \cup f^p_\sigma(\tau)$ finishing the proof. □

The following facts will be used in the proof of Claim 6.4. Item (2) below generalizes a well-known fact about trees (see [25, Lemma 3.8]); for a proof, see [14].

Lemma 6.2.

(1) $\mathbb{P}_\alpha$ *is* σ*-centered for each* $\alpha \leq \lambda$ *and more generally, the same holds for* $\mathbb{P}_\alpha / \mathbb{P}_\eta$ *for* $\eta < \alpha$.

(2) If $\mathbb{P} \times \mathbb{P}$ *has the c.c.c. and* $\mathrm{cf}(\delta) > \omega$*, then, in* $V[\mathbb{P}]$*, every new function from* δ *to the ordinals has an initial segment which is new.*

Proof. To see item (1), note that since Mathias forcing with respect to a filter is σ-centered and $\mathbb{Q}^{\leq\beta}_\alpha/\mathbb{Q}^{<\beta}_\alpha$ is densely embeddable into such a forcing by the above lemma, also $\mathbb{Q}^{\leq\beta}_\alpha/\mathbb{Q}^{<\beta}_\alpha$ is σ-centered. Since $\Lambda_\eta < \mathfrak{c}^+$ for every $\eta < \alpha$, and $\alpha \leq \lambda \leq \mathfrak{c}$, $\mathbb{P}_\alpha$ is a finite support iteration of σ-centered forcings of length strictly less than $\mathfrak{c}^+$. It is well-known that such iterations are σ-centered (see [31, proof of Lemma 2] or [5]). □

6.2. **The filters are $\mathcal{B}$-Canjar.** To finish the proof of Main Theorem 1.2, we still have to show that $\mathfrak{b} = \omega_1$ holds true in the final extension. Recall that the ground model V_0 satisfies CH. Let $\mathcal{B} = \omega^\omega \cap V_0$. We add μ many Cohen reals to obtain V. Thus in V, $\mathcal{B}$ is still unbounded. In Section 6.1, we have defined filters $\mathcal{F}^\beta_\alpha$ for $\alpha < \lambda$ and $\beta < \Lambda_\alpha$ and have shown that $\mathbb{Q}_\alpha$ is equivalent to the finite support iteration of the Mathias forcings $\mathbb{M}(\mathcal{F}^\beta_\alpha)$. In particular, $\mathbb{P}_\alpha * \mathbb{Q}^{<\beta}_\alpha * \mathbb{M}(\mathcal{F}^\beta_\alpha) = \mathbb{P}_\alpha * \mathbb{Q}^{\leq\beta}_\alpha$, and $\mathbb{P}_\alpha * \mathbb{Q}^{<\Lambda_\alpha}_\alpha = \mathbb{P}_{\alpha+1}$. As clearly $\mathcal{B}$ is countably directed, by Theorem 5.3 it suffices to show that for each $\alpha < \lambda$ and $\beta < \Lambda_\alpha$ the unboundedness of $\mathcal{B}$ is preserved by $\mathbb{M}(\mathcal{F}^\beta_\alpha)$. More precisely, we will make use of the following Lemma.

Lemma 6.3. *For all $\alpha < \lambda$, $\beta^* < \Lambda_\alpha$ the following holds:*

(1) $\mathcal{B}$ is unbounded in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$,

(2) if $m \in \omega$, $\beta_0, \dots, \beta_{m-1} < \Lambda_\alpha$ and $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ for every $k < m$, then $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. In particular $\mathcal{F}_\alpha^{\beta^*}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$.

Proof. We prove (1) and (2) by simultaneous induction on (α, β^*). Suppose the Lemma holds for each $(\alpha', \beta') <_{lex} (\alpha, \beta^*)$.

Proof of (1): *For* $\alpha = \beta^* = 0$, this is clear since $\mathcal{B}$ is unbounded in $V[\mathbb{P}_0 * \mathbb{Q}_0^{<0}] = V$. *In case* $\beta^* = \beta' + 1$ *is a successor ordinal*, we use the fact that (1) holds for α and β' by induction; By Lemma 6.1, $\mathbb{Q}_\alpha^{\leq\beta'} / \mathbb{Q}_\alpha^{<\beta'}$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha^{\beta'})$. Since by induction (2) holds for β', $\mathbb{M}(\mathcal{F}_\alpha^{\beta'})$ preserves the unboundedness of $\mathcal{B}$, hence the same is true for $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}$, as desired. *In case* (α, β^*) *is a limit point of the lexicographic order*, use the fact that $\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}$ is the limit of a finite support iteration of c.c.c. forcings and that (1) holds for each $(\alpha', \beta') <_{lex} (\alpha, \beta^*)$ to apply Theorem 5.3 and conclude (1) for (α, β^*).

Proof of (2): Fix α. By (1), $\mathcal{B}$ is unbounded in $V[\mathbb{P}_\alpha]$. We say that $\rho \in \lambda^{<\lambda}$ is *pre-T_α-minimal* if it is the predecessor of a minimal node of T_α. It is straightforward to check that this is the case if and only if

- $\rho \in V[\mathbb{P}_\alpha]$,
- $\rho \notin V[\mathbb{P}_\eta]$ for any $\eta < \alpha$, and
- for every $\gamma < |\rho|$, there exists $\eta < \alpha$ with $\rho \restriction \gamma \in V[\mathbb{P}_\eta]$.

Note that for $\alpha = 0$, the only pre-T_α-minimal node is the root $\langle\rangle$ and for $\alpha > 0$ all pre-T_α-minimal nodes have limit length. We proceed by induction on β^*.

Base step: $\beta^* = 0$. Let $\beta_0, \dots, \beta_{m-1}$ be such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<0}] = V[\mathbb{P}_\alpha]$ for each $k < m$. Therefore $\sigma_\alpha^{\beta_k} = \rho_k^\frown 0$ for some pre-T_α-minimal node ρ_k: Indeed, observe that $\mathfrak{F}_\alpha^\beta$ contains elements which are only added by $\mathbb{Q}_\alpha$ (and hence $\mathfrak{F}_\alpha^\beta \notin V[\mathbb{P}_\alpha]$) whenever $\sigma_\alpha^\beta = \rho^\frown i$ with ρ not pre-T_α-minimal or $i > 0$. If $\mathrm{cf}(|\rho_k|)$ is countable for all $k < m$, the filter $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is countably generated and hence by Lemma 5.2.(3) it is $\mathcal{B}$-Canjar. In particular, for $\alpha = 0$, the only pre-T_α-minimal node is $\rho = \langle\rangle$, which completes the proof for $\alpha = \beta^* = 0$. So assume $\alpha > 0$ for the rest of the proof of the base step. If $\mathrm{cf}(\alpha) \leq \omega$ all pre-T_α-minimal nodes ρ have $\mathrm{cf}(|\rho|) = \omega$:

Claim 6.4. *Let ρ be a pre-T_α-minimal node and* $\mathrm{cf}(|\rho|) > \omega$. *Then*

(1) $\mathrm{cf}(\alpha) > \omega$, *and*

(2) there exists no $\eta < \alpha$ such that $\rho \restriction \gamma \in V[\mathbb{P}_\eta]$ for all $\gamma < |\rho|$.

Proof. Let us first show (2). Fix some $\eta < \alpha$. Since ρ is pre-T_α-minimal, $\rho \in V[\mathbb{P}_\alpha] \setminus V[\mathbb{P}_\eta]$. By Lemma 6.2.(1), $\mathbb{P}_\alpha / \mathbb{P}_\eta$ is σ-centered, hence in particular $(\mathbb{P}_\alpha / \mathbb{P}_\eta) \times (\mathbb{P}_\alpha / \mathbb{P}_\eta)$ has the c.c.c.. Therefore, by Lemma 6.2.(2), the new function ρ has an initial segment which is not in $V[\mathbb{P}_\eta]$. Now let us show (1). Assume towards a contradiction that $\mathrm{cf}(\alpha) \leq \omega$ and let $\langle \alpha_n \mid n \in \omega\rangle$ be increasing cofinal in α (in case α is a successor, let α_n be its predecessor for every n). For every $\gamma < |\rho|$, let $n \in \omega$ be such that $\rho \restriction \gamma \in V[\mathbb{P}_{\alpha_n}]$. Since $\mathrm{cf}(|\rho|) > \omega$, there exists $n^* \in \omega$ such that $\rho \restriction \gamma \in V[\mathbb{P}_{\alpha_{n^*}}]$ for cofinally many $\gamma < |\rho|$ (and hence for all $\gamma < |\rho|$), contradicting (2). □

Thus we can assume $\mathrm{cf}(\alpha) > \omega$. We argue that $\mathrm{cf}(|\rho|) > \omega$ for all pre-T_α-minimal nodes ρ. Suppose ρ is a counterexample. Let $\langle \gamma_n \mid n \in \omega\rangle$ be increasing cofinal in $|\rho|$. For every $n < \omega$, let $\alpha_n < \alpha$ be such that

$\rho \restriction \gamma_n \in V[\mathbb{P}_{\alpha_n}]$. Fix $\alpha' < \alpha$ such that $\alpha_n < \alpha'$ for every n. As there are no new countable sequences of elements of $V[\mathbb{P}_{\alpha'}]$ in $V[\mathbb{P}_\alpha]$, we conclude that $\rho \in V[G_{\alpha''}]$ for some $\alpha'' < \alpha$, a contradiction.

Now we will show that $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha]$, using the characterization from Theorem 5.1. Let $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\alpha]$ be positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. We want to show that there exists $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. Clearly there exists $\eta < \alpha$ with $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\eta]$. Moreover, let η be large enough such that for all $j < k < m$ with $\rho_j \neq \rho_k$, there exists a successor $\delta < |\rho_j|, |\rho_k|$ such that $\rho_j \restriction \delta \neq \rho_k \restriction \delta$ and $a_{\rho_j \restriction \delta}, a_{\rho_k \restriction \delta} \in V[\mathbb{P}_\eta]$. For every $k < m$, let $\gamma_k < |\rho_k|$ be such that $a_{\rho_k \restriction \gamma_k} \notin V[\mathbb{P}_\eta]$. Such γ_k exist, because the ρ_k are pre-T_α-minimal, using Claim 6.4.(2). Clearly $a_{\rho_k \restriction \gamma_k} \in V[\mathbb{P}_\alpha]$ for every $k < m$. The filter $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle_{\mathrm{Fr}}$ is countably generated and hence $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha]$. Clearly $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle_{\mathrm{Fr}} \subseteq \bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$, hence $\langle X_n \mid n \in \omega\rangle$ is positive for $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle_{\mathrm{Fr}}$. Therefore we can fix $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \langle a_{\rho_k \restriction \gamma_k}\rangle_{\mathrm{Fr}}$. Note that $\bar{X}_f \in V[\mathbb{P}_\eta]$. We will use a genericity argument to show that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. It is enough to show that for all successors $\delta_k < |\rho_k|$, for all $l_k \in \omega$ there exists $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m}(a_{\rho_k \restriction \delta_k} \setminus l_k)$, because sets of this form are a basis for the filter. If $\delta_k \leq \gamma_k$ for all k, this holds by the choice of f.

We show by induction on $\eta \leq \eta' < \alpha$ that for all successors $\delta_k < |\rho_k|$ and all $l_k < \omega$, if all $a_{\rho_k \restriction \delta_k} \in V[\mathbb{P}_{\eta'}]$ then $V[\mathbb{P}_{\eta'}] \models$ "$\exists s \in \bar{X}_f \ s \subseteq \bigoplus_{k<m}(a_{\rho_k \restriction \delta_k} \setminus l_k)$". Note that this holds for $\eta' = \eta$ by choice of f, and that at limit steps of the induction no new $a_{\rho_k \restriction \delta_k}$ appear, so we only have to show it for successors. Assume that it holds for η' and show it for $\eta' + 1$. For every $k < m$, let $\delta_k < |\rho_k|$ with $a_{\rho_k \restriction \delta_k} \in V[\mathbb{P}_{\eta'+1}]$ and $l_k \in \omega$ be given. Let $p \in \mathbb{Q}_{\eta'}$. We will show that there exists $q \leq p$ and $s \in \bar{X}_f$ such that $q \Vdash s \subseteq \bigoplus_{k<m}(a_{\rho_k \restriction \delta_k} \setminus l_k)$. Without loss of generality $\rho_k \restriction \delta_k \in \mathrm{dom}(p)$ for all $k < m$ with $\rho_k \restriction \delta_k \in T_{\eta'}$ and p is a full condition. For every $k < m$, define Σ_k as follows: If $\rho_k \restriction \delta_k \in T_{\eta'}$, let $\Sigma_k = \bigcup\{\mathrm{dom}(f^p_{\rho_k \restriction \gamma}) \cap T'_{\eta'} \mid \gamma \leq \delta_k \wedge \rho_k \restriction \gamma \in \mathrm{dom}(p)\}$. If $\rho_k \restriction \delta_k \notin T_{\eta'}$, let $\Sigma_k = \{\rho_k \restriction \delta_k\}$. Let $\Sigma = \bigcup_{k<m} \Sigma_k$. For every $k < m$, let σ_k be the longest initial segment of ρ_k which belongs to Σ in case there exists such and let[12] $\sigma_k = \rho_k \restriction 1$ otherwise. Note that $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$ and that $a_{\sigma_k} \in V[\mathbb{P}_{\eta'}]$ for every $k < m$. Now let $N \in \omega$ be large enough such that

- $N \geq l_k$ for every $k < m$,
- $N \geq |s^p_\sigma|$ for every $\sigma \in \mathrm{dom}(p)$,
- $a_{\sigma_k} \setminus N \subseteq a_\tau$ for all $\tau \in \Sigma_k$, for all $k < m$.

By hypothesis, in $V[\mathbb{P}_{\eta'}]$ we can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m}(a_{\sigma_k} \setminus N)$. To get q extend p as follows. For every $k < m$ and $\gamma \leq \delta_k$ with $\rho_k \restriction \gamma \in \mathrm{dom}(p)$, let $s^q_{\rho_k \restriction \gamma} = s^p_{\rho_k \restriction \gamma}{}^\frown(\mathbf{0} \restriction [|s^p_{\rho_k \restriction \gamma}|, N))^\frown(a_{\sigma_k} \restriction [N, \max(s)])$. Observe that by the choice of η and since $\eta' \geq \eta$, if $\rho_k \neq \rho_j$ then there is no $\tau \in \mathrm{dom}(p)$ with $\tau \trianglelefteq \rho_k$ and $\tau \trianglelefteq \rho_j$. Now if $\rho_0 = \rho_j$ then $\sigma_k = \sigma_j$ and so the above is well-defined. Note that η was chosen large enough such that for all $\gamma_k \leq \delta_k$, $\gamma_j \leq \delta_j$, if $\rho_j \restriction \gamma_j \neq \rho_k \restriction \gamma_k$ then they are not in the same block. In particular, $\rho_j \restriction \gamma_j \notin \mathrm{dom}(h^p_{\rho_k \restriction \gamma_k})$ and $\rho_k \restriction \gamma_k \notin \mathrm{dom}(h^p_{\rho_j \restriction \gamma_j})$. Therefore Definition 3.1.(8) holds for q. The rest of Definition 3.1 is easy to verify and so q is a condition. Moreover q forces $s \subseteq \bigoplus_{k<m}(a_{\rho_k \restriction \delta_k} \setminus l_k)$.

Successor step Let us say that a filter $\mathcal{F}_\alpha^\beta$ and the associated $\mathfrak{F}_\alpha^\beta$ is *new in* $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ if $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ and $\mathfrak{F}_\alpha^\beta \notin V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta^*}]$ for all $\delta^* < \beta^*$. Now assume that we have shown (2) for β^*. We will show it for $\beta^* + 1$.

[12] We just have to choose any initial segment of ρ_k which belongs to $T'_{\eta'}$ and make sure that $\sigma_k = \sigma_j$ if $\rho_k = \rho_j$. Alternatively, in such cases, we could replace a_{σ_k} by ω below.

If $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ for every $k < m$, then by induction hypothesis $\left(\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}\right) \oplus \mathcal{F}_\alpha^{\beta^*}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. Hence, by Lemma 5.8.(1), $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*} * \mathbb{M}(\mathcal{F}_\alpha^{\beta^*})] = V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$. It is easy to check that there are exactly two new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$: $\mathcal{F}_\alpha^{\beta}$ where β is such that $\sigma_\alpha^\beta = \sigma_\alpha^{\beta^*}{}^\frown 0$ and $\mathcal{F}_\alpha^{\beta'}$ where $\beta' = \beta^* + 1$ (i.e., $\sigma_\alpha^{\beta'} = \rho^\frown(i+1)$ and $\sigma_\alpha^{\beta^*} = \rho^\frown i$). Both $\mathfrak{F}_\alpha^\beta$ and $\mathfrak{F}_\alpha^{\beta'}$ are extensions of $\mathfrak{F}_\alpha^{\beta^*}$ by one new set. Therefore the filter $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is an extension of $\bigoplus_{k<m} \mathcal{F}_\alpha^{\tilde{\beta}_k}$ by finitely many sets, where $\tilde{\beta}_k = \beta^*$ if $\beta_k = \beta$ or $\beta_k = \beta'$, and $\tilde{\beta}_k = \beta_k$ otherwise. By the above, $\bigoplus_{k<m} \mathcal{F}_\alpha^{\tilde{\beta}_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$. Then by Lemma 5.2.(2), also $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*+1}]$.

Limit step Now assume that β^* is a limit and that we have shown (2) for all $\delta^* < \beta^*$. Let us prove it for β^*. If for each $k < m$ there exists $\delta_k^* < \beta^*$ such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta_k^*}]$ then there exists $\bar{\delta}^* < \beta^*$ such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\bar{\delta}^*}]$ for all $k < m$. By induction hypothesis $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta^*}]$ for every $\bar{\delta}^* \leq \delta^* < \beta^*$ and so by Lemma 5.8.(2) it is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$.

Now we have to consider new filters. There are two cases: either β^* is such that $\sigma_\alpha^{\beta^*}$ has 0 as its last entry, or is such that $\sigma_\alpha^{\beta^*}$ has a limit ordinal i as its last entry.

First case: $\sigma_\alpha^{\beta^*} = \rho^\frown 0$ for some ρ. Let us first argue that there are no new filters unless $|\rho|$ is a limit and $\sigma_\alpha^{\beta^*}$ is the first node of its level in the enumeration (i.e., $|\sigma_\alpha^\delta| < |\rho|$ for each $\delta < \beta^*$). If $\sigma_\alpha^{\beta^*}$ is not the first node of its level, then there are no new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$: If $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$, then there exists $\delta^* < \beta^*$ such that $\mathfrak{F}_\alpha^\beta \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\delta^*}]$, because $\mathfrak{F}_\alpha^\beta$ contains – from the sets of this level – only boundedly many sets within only one block. Similarly, if $|\rho|$ is a successor and $\sigma_\alpha^{\beta^*}$ is the first node of its level then there are no new filters in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ because any $\mathfrak{F}_\alpha^\beta$ contains only boundedly many sets from level $|\rho|$.

So we assume from now on that $|\rho|$ is a limit and $\sigma_\alpha^{\beta^*}$ is the first node of its level. In this case, there are many new filters $\mathcal{F}_\alpha^\beta$ in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. Moreover, it is easy to check that $\mathcal{F}_\alpha^\beta$ is new if and only if the following holds: $\sigma_\alpha^\beta = \bar{\rho}^\frown 0$ for some $\bar{\rho}$ with $|\bar{\rho}| = |\rho|$ and $\bar{\rho}$ not pre-T_α-minimal. Observe that $\mathfrak{F}_\alpha^\beta = \{a_{\bar{\rho}\restriction\gamma} \mid \gamma < |\rho|\}$. Let $\beta_0, \ldots, \beta_{m-1}$ be such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ for each $k < m$. We want to show that $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$.

In case $\mathrm{cf}(|\rho|) = \omega$, we can use Lemma 5.8 and the remark thereafter to finish the proof: $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is a sum of filters, in which the new filters are countably generated, whereas the sum of the filters which are not new is $\mathcal{B}$-Canjar (see the first paragraph of the limit step).

Assume from now on that $\mathrm{cf}(|\rho|) > \omega$. Let $\mathsf{new} \subseteq m$ be the set of $k < m$ such that $\mathcal{F}_\alpha^{\beta_k}$ is a new filter and $\mathsf{old} = m \setminus \mathsf{new}$. For each $k \in \mathsf{new}$, fix ρ_k such that $\sigma_\alpha^{\beta_k} = \rho_k^\frown 0$ (with $|\rho_k| = |\rho|$ and ρ_k not pre-T_α-minimal). Let $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ be positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. Clearly there exists a hereditarily countable name for $\langle X_n \mid n \in \omega\rangle$. Let $\gamma < |\rho|$ be a successor ordinal large enough such that the following hold:

- $\langle X_n \mid n \in \omega\rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$ (this is possible due to $\mathrm{cf}(|\rho|) > \omega$).
- For all $j, k \in \mathsf{new}$, if $\rho_j \neq \rho_k$, then ρ_j and ρ_k split before γ.
- For all $k \in \mathsf{old}$, either $|\sigma_\alpha^{\beta_k}| < \gamma$ or[13] $|\sigma_\alpha^{\beta_k}| > |\rho|$.
- $a_{\rho_k\restriction\gamma} \notin V[\mathbb{P}_\alpha]$ for all $k \in \mathsf{new}$, i.e., $\rho_k \restriction \gamma \in T_\alpha$ (possible since ρ_k is not pre-T_α-minimal).

For $k < m$ with $k \in \mathsf{old}$, $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$ by the choice of γ and we define $\tilde{\mathfrak{F}}_\alpha^{\beta_k} = \mathfrak{F}_\alpha^{\beta_k}$. For $k \in \mathsf{new}$, let $\tilde{\mathfrak{F}}_\alpha^{\beta_k} = \{a_{\rho_k\restriction\gamma}\}$. Now, we can use Lemma 5.8 and the remark thereafter to show that $\bigoplus_{k<m} \langle\tilde{\mathfrak{F}}_\alpha^{\beta_k}\rangle_{\mathrm{Fr}}$ is

[13]Note that $|\sigma_\alpha^{\beta_k}| > |\rho|$ is only possible if $\sigma_\alpha^{\beta_k} = \tilde{\rho}^\frown 0$ for a pre-T_α-minimal node $\tilde{\rho}$.

$\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. Indeed, $\bigoplus_{k\in\mathsf{old}} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle_{\mathrm{Fr}}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ by the first paragraph of the limit step and for each $k \in \mathsf{new}$, $\langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle_{\mathrm{Fr}}$ is countably generated. Moreover, $\langle X_n \mid n \in \omega \rangle$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ and $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle_{\mathrm{Fr}} \subseteq \bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$, hence $\langle X_n \mid n \in \omega \rangle$ is positive for $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle_{\mathrm{Fr}}$. So we can fix $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle_{\mathrm{Fr}}$. Since $\langle X_n \mid n \in \omega \rangle$ and $\bigoplus_{k<m} \langle \tilde{\mathfrak{F}}_\alpha^{\beta_k} \rangle_{\mathrm{Fr}}$ are in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$, and being positive is absolute, this holds in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}]$.

Now we use a genericity argument in $\mathbb{Q}_\alpha^{<\beta^*} / \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}$ to show that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. We have to show that for all $\langle A_k \mid k < m \rangle$ with $A_k \in \mathcal{F}_\alpha^{\beta_k}$ there is $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} A_k$. For $k \in \mathsf{new}$, we can assume that $A_k = a_{\rho_k \restriction \delta_k} \setminus l_k$ with $\gamma < \delta_k < |\rho|$ and $l_k \in \omega$, because these sets form filter bases. For $k \in \mathsf{old}$ let $B_k = A_k$ and for $k \in \mathsf{new}$ (in this case $|\sigma_\alpha^{\beta_k}| = |\rho| + 1 > \gamma$) let $B_k = a_{\rho_k \restriction \gamma}$. By the choice of f for all $N \in \omega$ there is an $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} (B_k \setminus N)$.

Let $p \in \mathbb{Q}_\alpha^{<\beta^*} / \mathbb{Q}_\alpha^{\lambda^{\leq\gamma}}$. Without loss of generality $\rho_k \restriction \delta_k \in \mathrm{dom}(p)$ if $k \in \mathsf{new}$. For every $k \in \mathsf{new}$, define $\Sigma_k = \bigcup \{\mathrm{dom}(f^p_{\rho_k \restriction \delta}) \cap \lambda^{\leq\gamma} \mid \delta \leq \delta_k \wedge \rho_k \restriction \delta \in \mathrm{dom}(p)\}$. Now let $N \in \omega$ be large enough such that

- $N \geq l_k$ for every $k \in \mathsf{new}$; $N \geq |s^p_\sigma|$ for every $\sigma \in \mathrm{dom}(p)$;
- $a_{\rho_k \restriction \gamma} \setminus N \subseteq a_\tau$ for all $\tau \in \Sigma_k$, for all $k \in \mathsf{new}$.

By the above, we can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m} (B_k \setminus N)$. To get q, extend p as follows. For every $k \in \mathsf{new}$, for every $\delta \leq \delta_k$ with $\rho_k \restriction \delta \in \mathrm{dom}(p)$, let $s^q_{\rho_k \restriction \delta} = s^p_{\rho_k \restriction \delta} {}^\frown (\mathbf{0} \restriction [|s^p_{\rho_k \restriction \delta}|, N)) {}^\frown (a_{\rho_k \restriction \gamma} \restriction [N, \max(s)])$. By the choice of γ, for $j, k \in \mathsf{new}$, for each $\gamma < \delta < |\rho|$ either $\rho_k \restriction \delta = \rho_j \restriction \delta$ or they are not in the same block. In particular $\rho_j \restriction \delta \notin \mathrm{dom}(h^p_{\rho_k \restriction \delta})$ and $\rho_k \restriction \delta \notin \mathrm{dom}(h^p_{\rho_j \restriction \delta})$. Therefore Definition 3.1.(8) holds for q. The rest of Definition 3.1, as well as the fact that q forces $s \subseteq \bigoplus_{k<m} A_k$, are easy to check.

Second case: $\sigma_\alpha^{\beta^*} = \rho^\frown i$ with $i > 0$ limit. In this case, $\mathcal{F}_\alpha^{\beta^*}$ is the only new filter in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. Let $\beta_0, \dots, \beta_{m-1}$ be such that $\mathfrak{F}_\alpha^{\beta_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ for each $k < m$. We want to show that $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. Let $\langle X_n \mid n \in \omega \rangle \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$ be positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. Let $\dot{X}$ be a hereditarily countable name for $\langle X_n \mid n \in \omega \rangle$ and let D be countable containing all the domains of conditions occurring in the name $\dot{X}$. Let $\beta^{**} < \beta^*$ be such that $\rho^\frown 0 = \sigma_\alpha^{\beta^{**}}$ Let $\bar{D} = \{\tau \in D \mid \exists j < i \ (\tau = \rho^\frown j)\}$. and let $C = \{\sigma_\alpha^\nu \mid \nu < \beta^{**}\} \cup \bar{D}$. Clearly, $\{\sigma_\alpha^\nu \mid |\sigma_\alpha^\nu| \leq |\rho|\} \subseteq C \subseteq \{\sigma_\alpha^\nu \mid |\sigma_\alpha^\nu| \leq |\rho| + 1\}$, hence C is α-upwards closed. Lemma 3.11 implies that $\mathbb{Q}_\alpha^C \lessdot \mathbb{Q}_\alpha$ and $\mathbb{Q}_\alpha^C \subseteq \mathbb{Q}_\alpha^{<\beta^*}$. Since $\mathbb{Q}_\alpha^{<\beta^*} \lessdot \mathbb{Q}_\alpha$, by Lemma 4.1.(1), $\mathbb{Q}_\alpha^C \lessdot \mathbb{Q}_\alpha^{<\beta^*}$.

Observe that for all $k < m$ either $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^{**}}] = \mathfrak{F}_\alpha^{\beta_k}$ or $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^{**}}] = \mathfrak{F}_\alpha^{\beta^{**}}$. In particular, for every $k < m$ there exists β'_k such that $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^{**}}] = \mathfrak{F}_\alpha^{\beta'_k}$ and $\mathfrak{F}_\alpha^{\beta'_k} \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^{**}}]$. Hence $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^{**}}] \rangle_{\mathrm{Fr}}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. For every $k < m$, $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$ is the set $\mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^{**}}]$ together with countably many new sets (some of the sets $\omega \setminus a_\tau$ with $\tau \in \bar{D}$), therefore $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle_{\mathrm{Fr}}$ is a filter generated by $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^{**}}] \rangle_{\mathrm{Fr}}$ together with countably many new sets. Hence, by Lemma 5.2.(2), $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle_{\mathrm{Fr}}$ is $\mathcal{B}$-Canjar in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^{<\beta^*}]$. Since the sets $\langle X_n \mid n \in \omega \rangle$ are positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$ and $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle_{\mathrm{Fr}} \subseteq \bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$, the sets $\langle X_n \mid n \in \omega \rangle$ are also positive for $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle_{\mathrm{Fr}}$. So we can fix $f \in \mathcal{B}$ such that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle_{\mathrm{Fr}}$. Since $\langle X_n \mid n \in \omega \rangle$ and $\bigoplus_{k<m} \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle_{\mathrm{Fr}}$ are in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$, and being positive is absolute, this holds in $V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$.

Now use a genericity argument in $\mathbb{Q}_\alpha^{<\beta^*} / \mathbb{Q}_\alpha^C$ to show that $\bar{X}_f$ is positive for $\bigoplus_{k<m} \mathcal{F}_\alpha^{\beta_k}$. Fix $\langle A_k \mid k < m \rangle$ with $A_k \in \mathcal{F}_\alpha^{\beta_k}$. For simplicity of notation assume there is $m' \leq m$ such that $A_k \in V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C]$ if and only if $k < m'$. For $m' \leq k < m$ there exists $B_k \in \langle \mathfrak{F}_\alpha^{\beta_k} \cap V[\mathbb{P}_\alpha * \mathbb{Q}_\alpha^C] \rangle_{\mathrm{Fr}}$, $\ell_k \in \omega$ and $\langle j_r^k \mid r < \ell_k \rangle \subseteq i$ such that $A_k = B_k \cap \bigcap_{r<\ell_k} (\omega \setminus a_{\rho^\frown j_r^k})$. So $\bigoplus_{k<m} A_k = \bigoplus_{k<m'} A_k \oplus \bigoplus_{m' \leq k < m} (B_k \cap \bigcap_{r<\ell_k} (\omega \setminus a_{\rho^\frown j_r^k}))$. Let $p \in \mathbb{Q}_\alpha^{<\beta^*} / \mathbb{Q}_\alpha^C$. Without loss of generality assume that $\rho^\frown j_r^k \in \mathrm{dom}(p)$ if $\rho^\frown j_r^k \notin C$. Let $N > |s_\tau^p|$ for every $\tau \in \mathrm{dom}(p)$. We can fix $s \in \bar{X}_f$ with $s \subseteq \bigoplus_{k<m'} A_k \oplus \bigoplus_{m' \leq k < m} (B_k \setminus N)$. To get q, extend each $s^p_{\rho^\frown j_r^k}$ with 0's to have length $\max(s) + 1$. It is easy to check that q is a condition forcing $s \subseteq \bigoplus_{k<m} A_k$, as desired. $\square$

By Lemma 6.3, $\mathcal{B}$ is unbounded in $V[\mathbb{P}_\alpha]$ for every $\alpha < \lambda$, so by Theorem 5.3, $\mathcal{B}$ is unbounded in $V[\mathbb{P}_\lambda]$. Thus in $V[\mathbb{P}_\lambda]$, $\mathfrak{b} = \omega_1$ and so $\mathfrak{h} = \omega_1$ as well. This concludes the proof of Main Theorem 1.2.

7. Further discussion and questions

In this section, we discuss the structure of refining matrices, as well as a notion of spectrum of refining systems of mad families. For basic definitions and facts, see Section 2.

7.1. **Possible variants of the main theorem.** It is possible to derive a bit more from the proof of Main Theorem 1.2 than what is stated in the theorem. Our forcing construction is based on the tree $\lambda^{<\lambda}$ and therefore results in a specific kind of refining matrix of height λ: first, all its maximal branches are cofinal, and second, the underlying tree has λ-splitting everywhere; more precisely, its underlying tree structure is $\lambda^{<\lambda} \cap \mathrm{succ}$. In particular, it immediately follows that $\lambda \in \mathit{spec}(\mathfrak{a})$ and hence $\mathfrak{a} \leq \lambda$.

We can modify the construction (by changing the underlying tree) to obtain different kinds of refining matrices of height λ. In fact, the following generalization of Main Theorem 1.2 holds true:

Generalized Main Theorem 7.1. *Let V_0 be a model of ZFC satisfying GCH. In V_0, let $\omega_1 \leq \lambda \leq \mathrm{cf}(\theta)$ and $\theta \leq \mu$ be cardinals such that λ is regular and $\mathrm{cf}(\mu) > \omega$. Then there is a c.c.c. extension W of V_0 in which there exists a refining matrix whose underlying tree structure is $\theta^{<\lambda} \cap \mathrm{succ}$ and $\omega_1 = \mathfrak{h} = \mathfrak{b} \leq \mathfrak{c} = \mu$.*

In this model, clearly $\theta \in \mathit{spec}(\mathfrak{a})$. For the proof of the generalization, the forcing construction is based on the tree $\theta^{<\lambda}$ instead of $\lambda^{<\lambda}$, and sets a_σ are generically added to its nodes of successor length. In Definition 4.5 and Definition 4.6, we need a pair of ordinals (ε, δ) in place of γ, where $\varepsilon < \theta$ and $\delta < \lambda$, and $\gamma^{<\gamma}$ has to be replaced by $\varepsilon^{<\delta}$.

The reason why we have to require $\lambda \leq \mathrm{cf}(\theta)$ is Lemma 4.10: a nice name for a node in $\theta^{<\lambda}$ contains less than λ many conditions r, and the corresponding ε_r have to be bounded in θ. It would even be possible to have different splitting at different nodes, provided that all the splitting sizes have cofinality at least λ. This way, we can get more values into $\mathit{spec}(\mathfrak{a})$ (compare with Hechler's paper [21]).

Observe that it is always possible to turn a refining matrix with θ-splitting into a refining matrix with $\mathfrak{c}$-splitting (of the same height), by just taking every ωth level and deleting all other levels. It is not clear whether it is possible to do it the other way round, i.e., to get a refining matrix with θ-splitting (for $\theta \in \mathit{spec}(\mathfrak{a})$) from a refining matrix with $\mathfrak{c}$-splitting.

The Cohen model satisfies $\mathit{spec}(\mathfrak{a}) = \{\omega_1, \mathfrak{c}\}$ (see, e.g., [6, Proposition 3.1]). Thus, if $\omega_1 < \theta < \mathfrak{c}$, there are no mad families of size θ in the Cohen model, and hence no refining matrices with θ-splitting. To obtain a model with such a matrix, one can apply Generalized Main Theorem 7.1 for $\lambda = \omega_1 < \theta < \mu$. On the other hand, the model of Generalized Main Theorem 7.1 with $\lambda = \theta = \omega_1 < \mu$ coincides with the Cohen model with $\mathfrak{c} = \mu$: this can be seen by representing the iteration as an iteration of Mathias forcings with respect to filters, as described in Section 6.1; since $\lambda = \theta = \omega_1$, all the filters are countably generated, therefore the respective Mathias forcings are equivalent to Cohen forcing. Thus we obtain:

Observation 7.2. *Let $\mu > \omega_1$. Then, in the Cohen model with $\mathfrak{c} = \mu$ (i.e., in the extension of a GCH model by $\mathbb{C}_\mu$) there exists a refining matrix whose underlying tree structure is $\omega_1^{<\omega_1} \cap \mathrm{succ}$.*

7.2. **Branches through refining matrices.** We will now discuss the structure of refining matrices with respect to cofinal branches. It is straightforward to check that every maximal branch which is not cofinal is a tower. We call a refining matrix normal if no element of $[\omega]^\omega$ intersects it. Recall from Section 2 that whenever there is a refining matrix of height λ, then there is also a normal refining matrix of height λ.

In case $\mathfrak{t} = \mathfrak{h}$ (so in particular under $\mathfrak{h} = \omega_1$) there are no towers of length strictly less than $\mathfrak{h}$, hence all maximal branches of a refining matrix of height $\mathfrak{h}$ are cofinal.

On the other hand, it is possible to have a refining matrix of height $\mathfrak{h}$ which has no cofinal branches. In fact, it was shown by Dow that this is the case in the Mathias model (see [13, Lemma 2.17]):

Theorem 7.3. *Assume CH. In the extension by the countable support iteration of Mathias forcing of length ω_2, there is a refining matrix of height $\mathfrak{h}$ without cofinal branches*[14] *(and $\omega_1 = \mathfrak{t} < \mathfrak{h} = \mathfrak{c} = \omega_2$).*

We do not know whether there is a normal[15] refining matrix of height $\mathfrak{h}$ *with* cofinal branches in the Mathias model; this would imply that $\mathfrak{h} = \omega_2 \in spec(\mathfrak{t})$. We also do not know whether $\omega_2 \in spec(\mathfrak{t})$ in the Mathias model.

It is actually consistent that *no* normal refining matrix of height $\mathfrak{h}$ has cofinal branches. This was proved by Dordal by constructing a model in which $\mathfrak{h} \notin spec(\mathfrak{t})$ (see [11] or[16] [12, Corollary 2.6]):

Theorem 7.4. *It is consistent with ZFC that $spec(\mathfrak{t}) = \{\omega_1\}$ and $\mathfrak{h} = \omega_2 = \mathfrak{c}$.*

Let us now discuss refining matrices of regular height strictly above $\mathfrak{h}$. Recall that $\omega_1 = \mathfrak{t} = \mathfrak{h}$ holds true in the model of Main Theorem 1.2; in particular, there are refining matrices of height ω_1 (all whose maximal branches are cofinal). All maximal branches through the generic refining matrix of height $\lambda > \omega_1$ are cofinal, because the forcing construction is based on the tree $\lambda^{<\lambda}$. Moreover, as shown in Section 4.3, all these maximal branches are actually towers (i.e., the matrix is normal). In particular, λ belongs to $spec(\mathfrak{t})$.

In the Cohen model, the situation is different. Again, $\omega_1 = \mathfrak{t} = \mathfrak{h}$ holds true, so there are refining matrices of height ω_1 (all whose maximal branches are cofinal). We do not know the following:

Question 7.5. Is there a refining matrix of regular height larger than $\mathfrak{h}$ in the Cohen model?

In any case, there is a crucial difference to the model of our main theorem: in the Cohen model, there is no normal refining matrix of regular height $\lambda > \omega_1$ with cofinal branches, due to the following well-known fact.

[14] In fact, there is even a base matrix of this kind.

[15] Here and in similar cases, it is necessary to demand that the refining matrix is normal, due to the fact that there are always trivial examples of refining matrices with constant cofinal branches.

[16] In fact, [12, Corollary 2.6]) also works for getting $\mathfrak{h} = \mathfrak{c}$ larger than ω_2, and for certain tower spectra which are more complicated than $\{\omega_1\}$.

Proposition 7.6. *Assume CH, and let μ be a cardinal with $\mathrm{cf}(\mu) > \omega$. Then $spec(\mathfrak{t}) = \{\omega_1\}$ holds true*[17] *in the extension by $\mathbb{C}_\mu$ (where $\mathbb{C}_\mu$ is the forcing for adding μ many Cohen reals).*

Finally, let us remark that the generic refining matrix from Main Theorem 1.2 cannot be a base matrix. This can be seen by a slight generalization of the proof of Lemma 3.7 which yields the following. For each infinite ground model set $b \subseteq \omega$, each a_σ has infinitely many 1's (and also infinitely many 0's) within b. If b is infinite and co-infinite, it follows that b splits a_σ. In particular, $a_\sigma \not\subseteq^* b$, so b witnesses that the generic matrix is not a base matrix.

7.3. **The spectrum of refining systems of mad families.** The study of refining matrices of various heights naturally gives rise to the following notion. Let

$$spec(\mathfrak{h}) := \{\lambda \mid \lambda \text{ is regular and there is a refining matrix of height } \lambda\}$$

be the *spectrum of refining systems of mad families*. Recall that the existence of refining matrices is only a matter of cofinality. Clearly, the minimum of $spec(\mathfrak{h})$ is the distributivity number $\mathfrak{h}$.

Spectra have been considered for several cardinal characteristics, but not for $\mathfrak{h}$. For example, spectra for the tower number $\mathfrak{t}$ have been investigated in [21] and [12], spectra for the almost disjointness number $\mathfrak{a}$ in [21], [6], and [29], spectra for the bounding number $\mathfrak{b}$ in [12], spectra for the ultrafilter number $\mathfrak{u}$ in [27], [28], and [17], and spectra for the independence number $\mathfrak{i}$ in [16]. Furthermore, [3] develops a framework for dealing with several spectra.

Let $[\mathfrak{h}, \mathfrak{c}]_{Reg}$ denote the set of regular cardinals δ with $\mathfrak{h} \leq \delta \leq \mathfrak{c}$. As already mentioned, it is easy to check that there can never be a refining matrix of regular height larger than $\mathfrak{c}$, hence $spec(\mathfrak{h}) \subseteq [\mathfrak{h}, \mathfrak{c}]_{Reg}$. Recall that the model of Main Theorem 1.2 satisfies $\{\omega_1, \lambda\} \subseteq spec(\mathfrak{h})$ (where $\lambda > \omega_1$ is the regular cardinal chosen there). In particular, by choosing $\lambda = \mu = \omega_2$, we obtain a model in which $\{\omega_1, \omega_2\} = spec(\mathfrak{h}) = [\mathfrak{h}, \mathfrak{c}]_{Reg}$.

Question 7.7. Is it consistent that $spec(\mathfrak{h})$ contains more than 2 elements?

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[17] In fact, the following stronger statement holds true in the extension: Let $\lambda > \omega_1$ be regular, and let $\langle a_\alpha \mid \alpha < \lambda \rangle$ be a $\subseteq^*$-decreasing sequence; then there exists an $\alpha_0 < \lambda$ such that $a_{\alpha_0} =^* a_\beta$ for every $\beta \geq \alpha_0$.

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Institute of Mathematics, University of Vienna, Kolingasse 14–16, 1090 Wien, Austria
Email address: `vera.fischer@univie.ac.at`

Institute of Mathematics, University of Vienna, Kolingasse 14–16, 1090 Wien, Austria
Email address: `marlenekoelbing@web.de`

Institute of Mathematics, University of Vienna, Kolingasse 14–16, 1090 Wien, Austria
Email address: `wolfgang.wohofsky@gmx.at`