THE SPECTRUM OF INDEPENDENCE, II

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ABSTRACT. We study the set $\mathrm{sp}(\mathfrak{i}) = \{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^\omega$ is a maximal independent family$\}$, referred to as the spectrum of independence. We develop a forcing notion, which allows us to adjoin a maximal independent family of arbitrary cardinality, and so in particular of cardinality $\aleph_\omega$. Moreover, given an arbitrary set Θ of uncountable cardinals, our techniques allow to obtain a cardinal preserving generic extension in which $\Theta \subseteq \mathfrak{sp}(\mathfrak{i})$, thus showing that $\mathfrak{sp}(\mathfrak{i})$ can be arbitrarily large. For finite Θ, as well as certain countably infinite Θ, we can obtain a precise equality, i.e. models of $\mathfrak{sp}(\mathfrak{i}) = \Theta$.

1. Introduction

The study of the spectrum of various extremal, also referred to as combinatorial sets of reals, has already a comparatively long history. Already in [10], it is shown that given an arbitrary set of uncountable cardinals Θ one can obtain a cardinal preserving generic extension in which for each cardinal $\theta \in \Theta$ there is a maximal almost disjoint family of cardinality θ. Thus, if we denote by $\mathfrak{sp}(\mathfrak{a})$ the set of cardinalities of infinite maximal almost disjoint families, the results of [10] show that for Θ as above, consistently $\Theta \subseteq \mathfrak{sp}(\mathfrak{a})$. Obtaining precise equality, i.e. realizing a given set of uncountable values as the spectrum of almost disjointness, has proven to be a more difficult task. Imposing a number of restrictions on Θ, Blass shows in [1] that for certain Θ, Hechler's techniques not only provide a model of $\Theta \subseteq \mathfrak{sp}(\mathfrak{a})$, but also $\Theta = \mathfrak{sp}(\mathfrak{a})$. The task of guaranteeing that a certain undesired cardinal does not appear in $\mathfrak{sp}(\mathfrak{a})$ has been achieved via an isomorphism of names argument. Such arguments, have their precursors, the most simple of which is probably the proof that in the Cohen extension over a model of GCH, every (infinite) maximal almost disjoint family is either of cardinality $\aleph_1$ or of cardinality $\mathfrak{c}$ (see [2]). Improving on the restrictions on Θ, Shelah and Spinas show that in fact $\mathfrak{sp}(\mathfrak{a})$ can be quite arbitrary, with only some exceptional cases remaining open (see [12]). Similar studies, regarding the spectrum of towers and maximal cofinitary groups can be found in [10, 4] respectively.

We focus our attention on independent families and their spectrum. Recall, that an independent family is a family $\mathcal{A} \subseteq [\omega]^\omega$ such that for all pairwise disjoint non-empty subfamilies $\mathcal{A}_0, \mathcal{A}_1$ of $\mathcal{A}$ the set $\bigcap \mathcal{A}_0 \setminus \bigcup \mathcal{A}_1$ is infinite. An independent family is said to be maximal if it is not properly included into another independent family. The existence of maximal independent families follows

2000 *Mathematics Subject Classification.* 03E15, 03E35.

Key words and phrases. consistency; combinatorial cardinal characteristics; independent families; spectrum.

The authors would like to thank for the generous support of: the National Science Foundation of USA through grant DMS 1833363, the Austrian Science Fund (FWF) through grant Y1012-N35 (Fischer) and the Israel Science Foundation (ISF) through grant 1838/19 (Shelah). Paper 1225 in the second author list.

from the Axiom of Choice. Classical examples of maximal independent families of cardinality $\mathfrak{c}$, among others, examples due to Hausdorff, as well as Fichtenholz and Kantorovic, can be found in (see [8]). The consistency of $\mathfrak{i} < \mathfrak{c}$, where $\mathfrak{i}$ denotes the minimal cardinality of a maximal independent family, is due to Brendle (see [9]). The study of the spectrum of independence, i.e. of the set $\mathfrak{sp}(\mathfrak{i})$ of all cardinalities of maximal independent families has been initiated in [6], where we show that the $\mathfrak{sp}(\mathfrak{i})$ can contain any desired *finite* set of uncountable *regular* cardinals. In the current paper, we significantly improve the above results.

Hechler's post for adjoining an uncountable maximal almost disjoint family with finite conditions has played a key role in showing that consistently $\mathfrak{a} = \aleph_\omega$, see [3]. Note, that while using Solovay's almost disjoint coding, can produce generic extensions in which there are maixmal almost disjoint families of cardinality κ, where κ is of uncountable cofinality, the technique does not allow to produce extensions with maximal almost disjoint families of cardinality $\aleph_\omega$. One of the advantages of Hechler's poset is that it allows, arbitrary cardinalities, including $\aleph_\omega$ to be realized as elements of $\mathfrak{sp}(\mathfrak{a})$. The situation with some close relatives of the almost disjointess number is similar. In [13] Zhang developed a forcing notion, which similarly to Solovay's poset, allows to adjoin a single new generator to a given cofinitary group and so showed that consistently there are maximal cofinitary groups of cardinality smaller than $\mathfrak{c}$ (and moreover, that consistently $\mathfrak{a}_g < \mathfrak{c}$). Cardinalities of countable cofinalities remained beyond the reach of Zhang's technique. The problem was addressed in [7], where the authors develop a forcing notion which given an arbitrary uncountable index set I, adjoins a family of cofinitary permutations $\mathcal{G} = \{g_i\}_{i\in I}$, which generates a maximal cofinitary groups. The technique not only allows to obtain a generic extension in which there is a maximal cofinitary group of countable cofinality, but also, similarly to the almost disjointness number case, allows to obtain a model in which $\mathfrak{a}_g = \aleph_\omega$ (see [7]).

In the current paper, we show that for every uncountable cardinal κ there is a ccc forcing notion, which adjoins a maximal independent family of cardinality κ, and so in particular, we obtain a generic extension in which there are maximal independent families of cardinality $\aleph_\omega$ (see Theorem 2.4). Moreover our techniques allow an arbitrarily large set Θ of uncountable cardinals to be realized as a subset of $\Theta \subseteq \mathfrak{sp}(\mathfrak{i})$ (see Theorem 3.1). For Θ finite, or Θ countably infinite and subject to some additional requirements, we obtain precise equality, i.e. a generic extensions, in which $\Theta = \mathfrak{sp}(\mathfrak{i})$ (see Theorem 4.5). Even though the results are significant improvement of [6], there are interesting remaining open question which we discuss briefly in the end of the paper.

2. Countable Cofinalities

In this section, we show that consistently there are maximal independent families of any desired cardinality (including $\aleph_\omega$). To obtain this we improve on the techniques introduced in [6] and in particular make a heavy use of the notion of a diagonalization filter.

For a given independent family $\mathcal{A}$ we denote by $\mathrm{FF}(\mathcal{A})$ the set of all finite (partial) functions $h : \mathcal{A} \to \{0,1\}$. Thus any $h \in \mathrm{FF}(\mathcal{A})$ denotes in a natural way a Boolean combination associated to the family $\mathcal{A}$, namely the set $\bigcap\{A : A \in h^{-1}(0)\} \backslash \bigcup\{A : A \in h^{-1}(1)\}$, which we denote by $\mathcal{A}^h$. We will use also the following notation: If $A \in \mathcal{A}$, then $A^0 = A$ and $A^1 = \omega\backslash A$. Thus for $h \in \mathrm{FF}(\mathcal{A})$ the boolean combination $\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \mathrm{dom}(h)\}$.

Recall also that for a given filter $\mathcal{U}$ the Mathias partial order relativized to $\mathcal{U}$, denoted $\mathbb{M}(\mathcal{U})$, is the poset of all pairs $(s,A)\in[\omega]^{<\omega}\times\mathcal{U}$ such that $\max s<\min A$ with extension relation defined as follows: $(t,B)\leq(s,A)$ provided that t end-extends s, $t\backslash s\subseteq A$ and $B\subseteq A$. For a condition $p=(s,A)\in\mathbb{M}(\mathcal{U})$ let $p_1=s$ and $p_2=A$.

Definition 2.1. Let $\mathcal{A}$ be an independent family. A filter $\mathcal{F}$ is said to be *an* $\mathcal{A}$-diagonalization filter, if $\mathcal{F}$ extends the Frechét Filter and is maximal with respect to the following property $\forall F\in\mathcal{F}\forall h\in\mathrm{FF}(\mathcal{A})(|F\cap\mathcal{A}^h|=\omega)$.

Diagonalization filters are dual to the so called diagonalization ideals, which have been studied in detail in [5]. In [6] it has been shown that:

Lemma 2.2. Let $\mathcal{A}$ be an independent family, let $\mathcal{F}$ be a $\mathcal{A}$-diagonalization filter and let G be $\mathbb{M}(\mathcal{F})$ generic over V and let $x_G=\bigcup\{s:\exists A(s,A)\in G\}$. Then $\mathcal{A}\cup\{x_G\}$ is independent and $\forall y\in V\cap([\omega]^\omega\backslash\mathcal{A})$ the family $\mathcal{A}\cup\{x_G,y\}$ is not independent.

Thus, in particular (in the above lemma) if y is an infinite subset of ω from the ground model extending $\mathcal{A}$ to a strictly larger independent family then $\mathcal{A}\cup\{x_G,y\}$ is not independent in $V[G]$. We say that x_G is a $\mathcal{A}$-diagonalization real over V. Diagonalization filters can be used to adjoin, along the length of a finite support iteration, maximal independent families of regular uncountable cardinalities.

Here, we obtain the following strengthening of the above Lemma.

Lemma 2.3. Let $\mathcal{A}$ be an independent family and $\mathcal{U}$ a diagonalization filter for $\mathcal{A}$. Let $I\neq\emptyset$ and for each $i\in I$, let $\mathcal{U}_i=\mathcal{U}$. Let $\mathbb{P}$ be the finite support product $\prod_{i\in I}\mathbb{M}(\mathcal{U}_i)$ and let $G=\prod_{i\in I}G_i$ be $\mathbb{P}$-generic. Then in $V[G]$ the family $\mathcal{A}\cup\{x_i\}_{i\in I}$ is independent and for each $i\in I$ and each $y\in V\cap([\omega]^\omega\backslash\mathcal{A})$ the family $\mathcal{A}\cup\{x_i,y\}$ is not independent, where $x_i=x_{G_i}$ for each $i\in I$.

Proof. The fact that $\mathcal{A}\cup\{x_i,y\}$ is not independent holds, since each x_i is a $\mathcal{A}$-diagonalization real over V. It remains to show that $\mathcal{A}\cup\{x_i\}_{i\in I}$ is independent. Thus, it is sufficient to show that for each $h\in\mathrm{FF}(\mathcal{A})$, $j:I\to\{0,1\}$ with finite domain and $n\in\omega$ the set of conditions $\bar{q}\in\mathbb{P}$ such that

$$\bar{q}\Vdash\exists i^*>n(i^*\in\bigcap_{i\in\mathrm{dom}(j)}\dot{x}_i^{j(i)}\cap\bigcap\mathcal{A}^h)$$

is dense. Fix h,j,n as above and let $\bar{p}\in\mathbb{P}$ be an arbitrary condition. Without loss of generality $\mathrm{dom}(\bar{p})=\mathrm{dom}(j)$. Let $I_0=j^{-1}(0)$, $I_1=j^{-1}(1)$ and $\bar{p}=\prod_{i\in\mathrm{dom}(j)}(s_i,F_i)$. Since $\mathcal{U}$ is a diagonalization filter for $\mathcal{A}$, the set $\mathcal{A}^h\cap\bigcap_{i\in\mathrm{dom}(j)}F_i$ is unbounded and it contains i^* such that $i^*>\max\{n,\max_{i\in I_0}s_i\}$. Then for each $i\in I_0$

$$q_i=(s_i\cup\{i^+\},F_i\backslash(i^*+1))\leq(s_i,F_i)\text{ and }q_i\Vdash_{\mathbb{M}(\mathcal{U}_i)}i^+\in\dot{x}_i\cap\mathcal{A}^h,$$

while for each $i\in I_1$

$$q_i=(s_i,F_i\backslash(i^*+1))\leq(s_i,F_i)\text{ and }q_i\Vdash_{\mathbb{M}(\mathcal{U}_i)}i^*\in(\omega\backslash\dot{x}_i)\cap\mathcal{A}^h.$$

Thus we can find a condition $\bar{q}$ extending $\bar{p}$ as desired. □

The above partial order can be used to adjoin via forcing a maximal independent family of arbitrary desired size. We will use the following standard terminology on trees, which can be also found in [11]. For σ and θ given cardinals, ${}^{<\sigma}\theta$ is the set of all $\tau : \alpha \to \theta$ where $\alpha < \sigma$. Then ${}^{<\sigma}\theta$ has a tree structure under end-extensions, i.e. under the relation $\trianglelefteq$, where $\eta \trianglelefteq \tau$ iff $\mathrm{dom}(\eta) \leq \mathrm{dom}(\tau)$ and $\eta = \tau \restriction \mathrm{dom}(\eta)$. If $S \subseteq {}^{<\sigma}\theta$ is closed under initial segments, then S is a tree under $\trianglelefteq$, the set of predecessors of $\tau \in S$, denoted $\tau \downarrow_S$ (sometimes we just write $\tau\downarrow$ when S is clear from context) is the set of all proper initial segments of τ and the set of successors of τ in S, is the set of all $\mu \in S$ such that $\tau \trianglelefteq \mu$. Recall that the height of a node τ of the tree, denoted $\mathrm{height}_S(\tau)$, is the order type of the set of its predecessors. Moreover the α-th level of S, denoted $\mathcal{L}_\alpha(S)$ or S_α, is the collection of all nodes of height α in S and the height of S is the least ordinal α such that $\mathcal{L}_\alpha(S) = \emptyset$. For $\eta \in S$, we say that $\mu \in S$ is an immediate successor of η in S if there is $\varepsilon \in \theta$ such that $\mu = \tau^\frown\langle\varepsilon\rangle$. We denote by $\mathrm{succ}_S(\tau)$ the set of all successor nodes of τ in S. We say that a tree S is θ-splitting, if for every $\tau \in S$, $|\mathrm{succ}_S(\tau)| = \theta$. A path through a tree S is a chain $P \subseteq T$ such that $P \cap \mathcal{L}_\alpha(S) \neq \emptyset$ for all $\alpha < \mathrm{height}(T)$.

Theorem 2.4. *Let θ be an uncountable cardinal. Then there is a ccc generic extension in which there is a maximal independent family of cardinality θ.*

Proof. Let $S = {}^{<\omega_1}\theta$. Thus, in particular, S is a θ-splitting tree of height ω_1, each branch of which is of length ω_1. For each $\alpha < \omega_1$, let S_α denote the α-th level of the tree (and so $\mathcal{L}_\alpha(S) = S_\alpha = S \cap {}^{\alpha+1}\theta$).

Recursively define a finite support iteration $\mathbb{P}_S = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_1, \beta < \omega_1\rangle$ as follows. Let $\mathbb{P}_0 = \{\emptyset\}$, $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for the trivial poset. Let $\mathcal{A}_0$ be the empty independent family and let $\mathcal{U}_0$ be a $\mathcal{A}_0$-diagonalization filter, i.e. an arbitrary ultrafilter extending the Frechét filter. For each $\eta \in S_1$, let $\mathcal{U}_\eta = \mathcal{U}_0$ and let $\mathbb{Q}_1 = \prod_{\eta\in S_1} \mathbb{M}(\mathcal{U}_\eta)$ with finite supports. In $V^{\mathbb{P}_1 * \dot{\mathbb{Q}}_1}$ for each $\eta \in S_1$ let a_η be the $\mathbb{M}(\mathcal{U}_\eta)$-generic real. Now, suppose $\alpha \geq 2$ and for each $\eta \in S_\alpha$, let $\mathcal{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}_S(\eta \restriction \xi), \xi < \alpha\}$ be an independent family in $V^{\mathbb{P}_\alpha}$. For each $\eta \in S_\alpha$ let $\mathbb{Q}_\alpha$ be the finite support product $\prod_{\eta\in S_\alpha} \mathbb{M}(\mathcal{U}_\eta)$. In $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha}$ for each $\eta \in S_\alpha$ and each $\nu \in \mathrm{succ}_{S_\alpha}(\eta)$, let a_ν be the $\mathbb{M}(\mathcal{U}_\eta)$-generic real.

In $V^{\mathbb{P}_S}$ for each path g of S let $\mathcal{A}_g = \{a_\nu : \nu \in \mathrm{succ}(g \restriction \xi), \xi < \omega_1\}$. Then $\mathcal{A}_g$ is a maximal independent family of cardinality θ. Maximality follows from the diagonalization properties and the fact that the length of the iteration is of uncountable cofinality. □

3. The spectrum can be large

In [6], it is shown that in the Sacks model, or in a model obtained by a large product of Sacks forcing, every maximal independent family is either of cardinality $\aleph_1$, or of cardinality $\mathfrak{c}$. Thus, in such extensions $\mathfrak{sp}(\mathfrak{i}) = \{\aleph_1, \mathfrak{c}\}$ is naturally small. Below, we show that to the opposite, $\mathfrak{sp}(\mathfrak{i})$ can be arbitrarily large.

Theorem 3.1. *Let Θ be a set of uncountable cardinals. Then there is a ccc generic extension in which $\Theta \subseteq \mathfrak{sp}(\mathfrak{i})$.*

Proof. Let $\sigma = \aleph_1$. Let $\bar{S} = \langle S_\theta : \theta \in \Theta \rangle$ be a family of pairwise disjoint trees such that for each $\theta \in \Theta$, S_θ is a θ-splitting tree of height ω_1 such that each branch is of length ω_1. For example, take $S_\theta = {}^{<\omega_1}(\theta \times \{\theta\})$. For each $\alpha < \omega_1$ let $S_{\theta,\alpha}$ denote the α-th level of S_θ and let $\bar{S}_\alpha = \bigcup_{\alpha<\theta} S_{\theta,\alpha}$. Define a finite support iteration $\mathbb{P}(\Theta) = \mathbb{P}_{\bar{S}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \le \sigma, \beta < \sigma \rangle$ recursively as follows:

(1) Let $\mathbb{P}_0 = \{\emptyset\}$ and let $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for the trivial poset.
(2) Let $\mathcal{A}_0$ be the empty independent family and let $\mathcal{U}_0$ be a $\mathcal{A}_0$-diagonalization filter, i.e. an arbitrary ultrafilter extending the Frechét filter. For each $\eta \in \bar{S}_1$ let $\mathcal{U}_\eta = \mathcal{U}_0$ and let $\mathbb{Q}_1 = \prod_{\eta \in S_1} \mathbb{M}(\mathcal{U}_\eta)$ with finite supports. In $V^{\mathbb{P}_1 * \dot{\mathbb{Q}}_1}$ let a_η be a $\mathbb{M}(\mathcal{U}_\eta)$-generic real.
(3) Suppose $\alpha \ge 2$. For each $\theta \in \Theta$ and each $\eta \in S_{\theta,\alpha}$ let
$$\mathcal{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}_{S_\theta}(\eta \restriction \xi), \xi < \alpha\}$$
be an independent family in $V^{\mathbb{P}_\alpha}$ and let $\mathcal{U}_\eta$ be a diagonalization filter for $\mathcal{A}_\eta$ also in $V^{\mathbb{P}_\alpha}$. Now, let $\mathbb{Q}_\alpha$ be the finite support product of $\prod_{\eta \in \bar{S}_\alpha} \mathbb{M}(\mathcal{U}_\eta)$ and in $V^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha}$ for each $\eta \in \bar{S}_\alpha$ let a_η be the $(\mathcal{U}_\eta)$-generic real.

With this the definition of $\mathbb{P}$ is complete. In $V^{\mathbb{P}}$ for each $\theta \in \Theta$ and each path g in S_θ, the family
$$\mathcal{A}_g = \{a_\nu : \nu \in \mathrm{succ}_{S_{\theta,\alpha}}(g \restriction \alpha), \alpha < \sigma\}$$
is a maximal independent family of cardinality θ. The maximality follows from the diagonalization properties of the Mathias generics and the fact that the length of the iteration is of uncountable cofinality. □

The above theorem leads us to the following definition:

Definition 3.2. Let Θ be a set of uncountable cardinals. If

(1) $\min \Theta = \sigma$ is regular,
(2) $\sup \Theta = \max \Theta = \lambda$ is of uncountable cofinality,
(3) if Θ is infinite, then Θ is closed with respect to singular limits of cofinality $\aleph_0$,

then we say that Θ is a *pre-independence-spectrum* or (σ, λ)*-pre-independence spectrum*.

Remark 3.3. The requirement that σ is regular uncountable reflects only our construction. While $\mathfrak{c} \in \mathfrak{sp}(\mathfrak{i})$ always and so λ representing the continuum in the intended generic extension has to be of uncountable cofinality, the requirement that σ is regular (or more generally of uncountable cofinality) is too restrictive. We also do not know if the third requirement is in general necessary.

Theorem 3.4. *(GCH) Let Θ be a (σ, λ)-pre-independence-spectrum. Then there is a ccc generic extension in which $\Theta \subseteq \mathfrak{sp}(\mathfrak{i})$, $\mathfrak{i} = \min \Theta = \sigma$ and $\mathfrak{c} = \max \Theta$.*

Proof. Choosing a sequence $\bar{S} = \langle S_\theta : \theta \in \Theta \rangle$ of pairwise disjoint trees where for each $\theta \in \Theta$, S_θ is θ-splitting tree of height σ, each branch of length σ (e.g. simply take $S_\theta = {}^{<\sigma}(\theta \times \{\theta\})$) and let $\mathbb{P} = \mathbb{P}(\bar{S})$ be defined as in the previous theorem, but σ is not necessarily $\aleph_1$. Then in $V^{\mathbb{P}}$, $\mathfrak{c} = \lambda$. The Cohen reals adjoined along the length of the iteration imply $\sigma \le \mathfrak{d}$ and since $\mathfrak{d} \le \mathfrak{i} \le \sigma$, we obtain $\mathfrak{i} = \sigma$. □

4. Excluding cardinalities

In order to exclude cardinals from sp(i) we have to provide a more careful analysis of the construction given above. For this, we will introduce some general notation and terminology.

Definition 4.1.

(1) Given a (σ, λ)-pre-independence-spectrum Θ, let $\mathbf{m} = \mathbf{m}(\Theta)$ be the collection of all sequences $S_{\mathbf{m}} = \langle S_\theta : \theta \in \Theta \rangle$ consisting of pairwise disjoint trees such that each S_θ is a θ-splitting tree of height σ, each branch of length σ. For $\alpha < \sigma$, we let $S_{\theta,\alpha}$ denote the α-th level of S_θ and let $S_{\theta,<\alpha} = \bigcup_{\beta<\alpha} S_{\theta,\beta}$ denote the tree S_θ below level α. Moreover, for each $\alpha < \sigma$ we let $S_{\mathbf{m},\alpha} = \bigcup_{\theta\in\Theta} S_{\theta,\alpha}$ and $S_{\mathbf{m},<\alpha} = \bigcup_{\theta\in\Theta} S_{\theta,<\alpha}$. We refer to the elements of $\mathbf{m}$ as $\mathfrak{sp}(\mathfrak{i})$*-parameters for* Θ.

(2) For a $\mathfrak{sp}(i)$-parameter $S_{\mathbf{m}}$, let $\mathbb{Q}_{\mathbf{m}}$ be the collection of all forcing notions $\mathbf{q} = \mathbb{P}(\Theta) = \mathbb{P}(S_{\mathbf{m}})$, defined as in Theorem 3.1 using the given σ. For $p \in \mathbb{P}(\Theta)$ we let $\text{supt}(p) = \{\alpha : \Vdash_{\mathbb{P}_\alpha(\theta)} p(\alpha) \neq 1_{\mathbb{Q}_\alpha}\}$. We can assume that for each $\alpha \in \text{supt}(p)$, $\text{supt}(p(\alpha)) \in [S_{\mathbf{m},\alpha}]^{<\omega}$. Moreover, we can assume that for all $\alpha \in \text{supt}(p)$ and all $\eta \in \text{supt}(p(\alpha))$, the finite part of $p(\alpha)(\eta)$ is an actual finite subset of ω and the infinite part of $p(\alpha)(\eta)$ has the form $\mathbf{B}(\{a_{\eta_i}\}_{i<\mathbf{i}(\alpha,\eta,p)})$ where $\mathbf{i}(\alpha,\eta,p) \leq \omega$, $\mathbf{B}$ is a Borel function and $\{\eta_i\}_{i<\mathbf{i}(\alpha,\eta,p)} \subseteq S_{\mathbf{m},<\alpha}$. We refer to $\{\eta_i\}_{i<\mathbf{i}(\alpha,\eta,p)}$ as the *actual support of* $p(\alpha)(\eta)$ and denote it $\text{asupt}(p(\alpha)(\eta))$. We let
$$\text{asupt}(p(\alpha)) = \bigcup\{\text{asupt}(p(\alpha)(\eta)) : \eta \in \text{supt}(p(\alpha))\}$$
and refer to it as *actual support of* $p(\alpha)$. Finally, we let
$$\text{fsupt}(p) = \bigcup\{\text{asupt}(p(\alpha)) : \alpha \in \text{supt}(p)\}$$
and refer to it as *the full support of* p.

(3) Additionally, let $\text{dom}(p) = \bigcup\{\text{supt}(p(\alpha)) : \alpha \in \text{supt}(p)\}$.

Of particular importance for us is the following.

Definition 4.2. Let $S_{\mathbf{m}}$ be an $\mathfrak{sp}(\mathfrak{i})$-parameter. A group of permutations $K = K(S_{\mathbf{m}})$ of $\bigcup S_{\mathbf{m}}$ is said to be an $S_{\mathbf{m}}$-group if for each $\pi \in K$ the following holds:

(1) If $\eta \in S_{\theta,\alpha}$ then $\pi(\eta) \in S_{\theta,\alpha}$.
(2) If $\eta, \nu \in S_\theta$ and η is an initial segment of ν, then $\pi(\eta)$ is an initial segment of $\pi(\eta)$.
(3) Given π_1, π_2 in K, if there is $\alpha < \sigma$ and $\eta \in S_{\mathbf{m},\alpha}$ such that $\pi_1(\eta) = \pi_2(\eta)$ then for each $\nu \in S_{\mathbf{m},<\alpha}$ we have $\pi_1(\nu) = \pi_2(\nu)$.

For each $\alpha \leq \sigma = \sigma_{\mathbf{m}}$, we let $K_\alpha = \{\pi \restriction S_{\mathbf{m},<\alpha} : \pi \in K\}$.

Definition 4.3. Given an $S_{\mathbf{m}}$-group K, we let $\mathbf{Q}_{\mathbf{m},K}$ to be the class of all $\mathbf{q} \in \mathbf{Q}_{\mathbf{m}}$ such that if $\mathbf{q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \sigma, \beta < \sigma \rangle$ then for every $\alpha \leq \sigma$ and $\pi \in K_\alpha$, π induces an automorphism $\hat{\pi}$ of $\mathbb{P}_\alpha$ with the property that for each $\theta \in \Theta$ and each $\eta \in S_{\mathbf{m},\theta}$, $\hat{\pi}$ maps $\mathcal{U}_\eta$ to $\mathcal{U}_{\pi(\eta)}$.

Lemma 4.4. If K is an $S_{\mathbf{m}}$-group, then $\mathbf{Q}_{\mathbf{m},K}$ is non-empty.

Proof. Straightforward. □

Lemma 4.5. Let Θ be a countable (σ,λ)-pre-independence spectrum, let $S_{\mathbf{m}}$ be a $\mathfrak{sp}(\mathfrak{i})$-parameter for Θ, $K = K(S_{\mathbf{m}})$ and let $\theta_\bullet$ be a cardinal such that $\sigma < \theta_\bullet < \lambda$ and $\theta_\bullet \notin \Theta$. Let $\mathbf{q} = \mathbb{P}(S_{\mathbf{m}}) \in \mathbb{Q}_{\mathbf{m},K}$. A sufficient condition for

$$\Vdash_{\mathbb{P}(S_{\mathbf{m}})} \text{“}\theta_\bullet \notin \mathfrak{sp}(\mathfrak{i})\text{”}$$

is the following:

$$\text{there is } \theta_* \in [\sigma,\theta_\bullet) \text{ with } \theta_* \geq \sup(\Theta \cap \theta_\bullet), \theta_*^{<\sigma} < \theta_\bullet.$$

Proof. Assume towards contradiction that there is $p_1 \in \mathbb{P}(S_{\mathbf{m}})$ such that $p_1 \Vdash \theta_\bullet \in \mathfrak{sp}(\mathfrak{i})$. Hence for some $\langle \underset{\sim}{a}_\alpha : \alpha < \theta_\bullet \rangle$

$$p_1 \Vdash \text{“}\langle \underset{\sim}{a}_\alpha : \alpha < \theta_\bullet \rangle \text{ is a maximal independent family”}.$$

Without loss of generality each $\underset{\sim}{a}_\alpha$ is a canonical $\mathbb{P}(S_{\mathbf{m}})$-name, i.e. it is

$$\langle p^\alpha_{n,l}, \mathbf{t}^\alpha_{n,l} : n, l < \omega \rangle$$

where each $\langle p^\alpha_{n,l} : l < \omega \rangle$ is a maximal antichain in $\mathbb{P}(S_{\mathbf{m}})$, each $\mathbf{t}^\alpha_{n,l}$ is a truth value and

$$p^\alpha_{n,l} \Vdash n \in a_\alpha \text{ iff } \mathbf{t}^\alpha_{n,l} \text{ is truth.}$$

Moreover, we can fix $\bar{t}_\alpha = \langle t_{\alpha,k} \rangle_{k\in\omega} \subseteq \bigcup S_{\mathbf{m}}$ with no repetitions and including $\operatorname{fsupt}(\underset{\sim}{a}_\alpha)$, where

$$\operatorname{fsupt}(\underset{\sim}{a}_\alpha) = \bigcup_{n,l\in\omega} \operatorname{fsupt}(p^\alpha_{n,l}).$$

Claim. There is $W_1 \in [W_0]^{\theta_*^+}$ such that for all $\theta \in \Theta \cap \theta_\bullet$ and all $h \in \varepsilon_\bullet$, $\alpha, \beta \in W_1$

$$\text{if } t_{\alpha,h} \in S_\theta \text{ and } t_{\beta,h} \in S_\theta \text{ then } t_{\alpha,h} = t_{\beta,h}.$$

Proof. Counting argument. □

Furthermore, we have the following:

Claim. There is $W_2 \in [\theta_\bullet]^{\theta_*^+}$ such that

(1) for all $\alpha, \beta \in W_2$ and all $l < \varepsilon_\bullet$, $\lg(t_{\alpha,l}) = \lg(t_{\beta,l})$,
(2) for all $\alpha, \beta \in W_2$, $h, l \in \varepsilon_\bullet$

$$(\exists\theta \in \Theta)(t_{\alpha,h} \in S_\theta \wedge t_{\alpha,l} \in S_\theta \wedge t_{\alpha,h} <_{S_\theta} t_{\alpha,l}) \text{ iff } (\exists\theta \in \Theta)(t_{\beta,h} \in S_\theta \wedge t_{\beta,l} \in S_\theta \wedge t_{\beta,h} <_{S_\theta} t_{\beta,l})$$

Proof. Counting argument. □

For each $\varepsilon \in \varepsilon_\bullet$, $\alpha \in W_2$ let $t_{\alpha,\varepsilon} \in S_{\theta_{\alpha,\varepsilon}}$. We can find $W_3 \in [W_2]^{\theta_*^+}$ such that for each $\varepsilon \in \varepsilon_\bullet$ the sequence $\langle \theta_{\alpha,\varepsilon} : \alpha \in W_3 \rangle$ is a constant θ_ε. Indeed, since $|W_2| = \theta_*^+ > |\Theta|$, at least one tree, say θ_ε, appears θ_*^+ many times in $\{\theta_{\alpha,\varepsilon}\}_{\alpha\in W_2}$.

Moreover, subject to further thinning out, we can assume that for all $\alpha, \beta \in W_2$:

(1) for all k, n, l in ω: $t_{\alpha,k} \in \operatorname{fsupt}(p^\alpha_{n,l})$ iff $t_{\beta,k} \in \operatorname{fsupt}(p^\beta_{n,l})$;
(2) $t_{\alpha,k} \in \operatorname{dom}(p^\alpha_{n,l})$ iff $t_{\beta,k} \in \operatorname{dom}(p^\beta_{n,l})$;
(3) if $t_{\alpha,k} \in \operatorname{dom}(p^\alpha_{n,l})$ then $\operatorname{trunk}(p^\alpha_{n,l}(t_{\alpha,k})) = \operatorname{trunk}(p^\beta_{n,l}(t_{\beta,k}))$

(4) $\bar{t}_\alpha$, $\bar{t}_\beta$ realize the same quantifier free type in $S_{\mathbf{m}}$ and so in particular the length $\lg(t_{\alpha,l}) = \lg(t_{\beta,l})$, $t_{\alpha,l} \in S_\theta$ if and only if $t_{\beta,l} \in S_\theta$, etc.

Now, consider the equivalence relation E on $\varepsilon_\bullet$ defined as follows:

$$\varepsilon_1 E \varepsilon_2 \text{ iff } \theta_{\varepsilon_1} = \theta_{\varepsilon_2}.$$

For each $\varepsilon \in \varepsilon_\bullet$, let v_ε be the equivalence class of ε, i.e. $v_\varepsilon = [\varepsilon]_E$. Thus, in particular, $\varepsilon_\bullet = \bigcup_{\varepsilon\in\varepsilon_\bullet} v_\varepsilon$. Let $\{v_l\}_{l<\sigma_\bullet}$ be an enumeration of all equivalence classes, where $\sigma_\bullet \leq \varepsilon_\bullet < \sigma$ and for each $l < \sigma_\bullet$, let θ_l be $\theta_{\alpha,\varepsilon}$ where $\varepsilon \in v_l$. Moreover, we can arranged that the sequence $\langle \bar{t}^+_\alpha \restriction v_l : \alpha \in W_3\rangle$ forms a Δ-system and that for each v_l, the elements of the sequence $\langle \bar{t}^+_\alpha \restriction v_l : \alpha \in W_3\rangle$ realize the same type in S_{θ_l}. Now, for each equivalence class v_l, let

$$A^{\theta_l} = \bigcap\{\mathrm{Range}(\bar{t}^+_\alpha \restriction v_l) : \alpha \in W_3\} = \langle t^l_\varepsilon : \varepsilon \in w_l\rangle,$$

where $w_l = \{\varepsilon \in v_l : t_{\alpha,\varepsilon} \in A^{\theta_l}\}$ and $t^l_\varepsilon = t_{\alpha,\varepsilon}$ for some $\alpha \in W_3$.

Let $p_2 \leq p_1$, let $\bar{t} \in {}^\omega(\bigcup S_{\mathbf{m}})$ with no repetitions, be such that $\mathrm{fsupt}(p_2)$ is contained in $\bar{t}$ and let $\bar{t}^+$ be the downwards closure of $\bar{t}$. Thus, $\bar{t}^+$ is a sequence of length $< \sigma$. There is a sequence $\bar{t}^\odot = \langle t_\varepsilon : \varepsilon < \varepsilon_\bullet\rangle$ such that for each $l < \sigma_\bullet$:

(1) $\bar{t}^\odot \restriction v_l$ is a subtree of S_{θ_l} and realizes in S_{θ_l} the same q. f. type as $\bar{t}^+_\alpha \restriction v_l$ for each $\alpha \in W_3$,
(2) $\bar{t}^\odot \restriction w_l = \langle t^l_\varepsilon : \varepsilon \in w_l\rangle = A^{\theta_l}$;
(3) if $\varepsilon \in v_l \backslash w_l$ then $t_\varepsilon \notin \bigcup\{\mathrm{Range}(\bar{t}^+_\alpha) : \alpha < \theta_\bullet\} \cup \mathrm{Range}(\bar{t}^+)$.

Consider the set

$$\mathrm{bad}_{\bar{t}} = \{\alpha \in W_3 : \exists l < \varepsilon_\bullet \langle t_{\beta,l} : \beta \in W_3\rangle \text{ has no repetitions and } t_{\alpha,l} \in \mathrm{Range}(\bar{t}^+)\}.$$

Since $|\mathrm{Range}(\bar{t}^+)| < \sigma$, for each l there are strictly less than σ many α-s such that $t_{\alpha,l} \in \mathrm{Range}(\bar{t}^+)$ and since $\varepsilon_\bullet < \sigma$, we obtain $|\mathrm{bad}_{\bar{t}}| < \sigma$.

Note that for each $\alpha \in W_3\backslash\mathrm{bad}_{\bar{t}}$, there is an automorphism π_α of $\bigcup S_{\mathbf{m}}$ such that $\pi_\alpha(\bar{t}^+) = \bar{t}^+$ and $\pi_\alpha(\bar{t}^+_\alpha) = t^\odot$. Fix any $\alpha \in W_3\backslash\mathrm{bad}_{\bar{t}}$ and let $\underset{\sim}{a}^\odot = \hat{\pi}(\underset{\sim}{a}_\alpha)$.

Take an arbitrary finite subfamily $\{\underset{\sim}{a}_{\alpha_l} : l \in k\}$. We will show that

$$p_2 \Vdash \text{“}\{\underset{\sim}{a}_{\alpha_l} : l \in k\} \cup \{\underset{\sim}{a}^\odot\} \text{ is independent”},$$

thus reaching a contradiction to the choice of p_1. Note that there are $\{\alpha^l\}_{l\in k} \subseteq W_3$, $\gamma \in W_3\backslash\{\alpha^l\}_{l\in k}$ and $\pi \in K$ such that $\{\underset{\sim}{a}_{\alpha^l}\}_{l\in k}$ are pairwise distinct and

(1) π is the identity on $\bar{t}^+$ and so $\hat{\pi}(p_2) = p_2$,
(2) $\hat{\pi}$ maps $\underset{\sim}{a}_{\alpha^l}$ to $\underset{\sim}{a}_{\alpha_l}$ for each $l \in k$,
(3) $\hat{\pi}$ maps $\underset{\sim}{a}_\gamma$ to $\underset{\sim}{a}^\odot$.

Now, since

$$p_2 \Vdash \text{“}\{\underset{\sim}{a}_{\alpha^l} : l \in k\} \cup \{\underset{\sim}{a}_\gamma\} \text{ is independent”},$$

we have

$$\hat{\pi}(p_2) \Vdash \text{“}\{\hat{\pi}(\underset{\sim}{a}_{\alpha^l}) : l \in k\} \cup \{\hat{\pi}(\underset{\sim}{a}_\gamma)\} \text{ is independent”},$$

and so

$$p_2 \Vdash \text{“}\{\underset{\sim}{a}_{\alpha_l} : l \in k\} \cup \{\underset{\sim}{a}^\odot\} \text{ is independent”},$$

which is a contradiction. □

Theorem 4.6. *(GCH) Let C be a countable set of uncountable cardinals which is closed with respect to singular limits,* $\min C = \sigma$ *is regular uncountable and* $\max C = \sup C = \lambda$ *is of uncountable cofinality. Then there is a cardinal preserving generic extension in which* $\mathfrak{sp}(\mathfrak{i}) = C$.

Proof. Let $S_{\mathbf{m}}$ be a (σ, λ)-independence parameter, $K = K(S_{\mathbf{m}})$ and $\mathbf{q} = \mathbb{P}(S_{\mathbf{m}}) \in \mathbb{Q}_{\mathbf{m},K}$. Then by Theorem 3.1 $V^{\mathbb{P}(S_{\mathbf{m}})} \vDash C \subseteq \mathfrak{sp}(\mathfrak{i})$, while by Lemma 4.5 in fact we have equality, i.e. $V^{\mathbb{P}(S_{\mathbf{m}})} \vDash C = \mathfrak{sp}(\mathfrak{i})$. □

5. Questions

Even though the techniques developed in the above article show that the spectrum of maximal independent families can be quite arbitrary and that consistently there are maximal independent families of arbitrary uncountable cardinalities. Of interest however remains the following question:

Question 1. Is it consistent that $\mathfrak{i}$ is of countable cofinality?

Since $\mathfrak{d} \leq \mathfrak{i}$, a model of $\mathfrak{i} < \mathfrak{a}$ is necessarily a model of $\mathfrak{d} < \mathfrak{a}$. However, in all known models of $\mathfrak{d} < \mathfrak{a}$ (templates, ultrapowers), we have $\mathfrak{a} = \mathfrak{i}$ and the following question, known as Vaughan's problem is still open:

Question 2. Is it consistent that $\mathfrak{i} < \mathfrak{a}$?

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