THE SPECTRUM OF INDEPENDENCE

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ABSTRACT. We study the set of possible size of maximal independent families to which we refer as spectrum of independence and denote Spec(mif). Here *mif* abbreviates *maximal independent family*. We show that:

(1) whenever $\kappa_1 < \cdots < \kappa_n$ are finitely many regular uncountable cardinals, it is consistent that $\{\kappa_i\}_{i=1}^n \subseteq \text{Spec}(mif)$;
(2) whenever κ has uncountable cofinality, it is consistent that $\text{Spec}(mif) = \{\aleph_1, \kappa = \mathfrak{c}\}$.

Aassuming large cardinals, in addition to (1) above, we can provide that

$$(\kappa_i, \kappa_{i+1}) \cap \text{Spec}(mif) = \emptyset$$

for each i, $1 \leq i < n$.

1. INTRODUCTION

We study the set of possible sizes of maximal indepndent families. Let $\mathcal{A}$ be a family of infinite subsets of ω. Following [5] we denote by $\text{FF}(\mathcal{A})$ the set of all partial functions $h : \mathcal{A} \to 2$ with finite domain, denoted $\text{dom}(h)$. For $h \in \text{FF}(\mathcal{A})$ let $\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \text{dom}(h)\}$, where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \omega \backslash A$ if $h(A) = 1$. A family $\mathcal{A} \subseteq [\omega]^\omega$ is said to be *independent* if for every $h \in \text{FF}(\mathcal{A})$, the set $\mathcal{A}^h$ is infinite. It is *maximal independent* if in addition, it is not properly included in another maximal independent family. The minimal size of a maximal independent family is denoted $\mathfrak{i}$ and is referred to as the *independence number*.

Compared to the other classical cardinal characteristics of the continuum, the independence number seems to be one of the less studied (for an excellent exposition of the subject of cardinal characteristics, we refer the reader to [1]). In this article we study the set of possible sizes of maximal independent families, to which we refer as *spectrum of independence* and denote Spec(mif). It seems surprisingly difficult to control those possible sizes, one of the *major difficulty* being that a generic *Cohen real* destroys the maximality of all ground model maximal independent families. It is known that both $\mathfrak{d}$ and $\mathfrak{r}$ are below $\mathfrak{i}$, however apart from $\mathfrak{c}$, there are no other known upper bounds. In [5] the second author of the current article shows that consistently $\mathfrak{i} < \mathfrak{u} = \mathfrak{c} = \aleph_2$, construction which will later be observed to provide the existence of a Sacks indestructible maximal independent families. For a detailed proof of the existence of such families

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see [3]. Alternatively the consistency of $\mathfrak{i} < \mathfrak{c}$ can be obtained via a finite support iterations of ccc posets (see [4, Proposition 18.11]), result due to Brendle.

Our article is stuctured as follows: In section 2, to a given independent family $\mathcal{A}$ we associate a family of special filters $\mathcal{U}$, to which we refer as an $\mathcal{A}$-*diagonalization filters*, such that the relativized Mathias poset $\mathbb{M}(\mathcal{U})$ adjoins a generic real $\sigma_\mathcal{A}$ with the following diagonalization property: $\mathcal{A} \cup \{\sigma_\mathcal{A}\}$ is independent and furthermore for each $x \in V \cap ([\omega]^\omega \backslash \mathcal{A})$ such that $\mathcal{A} \cup \{x\}$ is independent, the family $\mathcal{A} \cup \{x, \sigma_\mathcal{A}\}$ is not maximal. This property allows us in an appropriate finite support iteration to guarantee that any finite set of regular cardinals does appear as a subset of $\text{Spec}(mif)$ (see Theorem 5). In Section 3, we study Sacks extensions (extensions obtained via long countable support products of Sacks forcing) of models of CH and show that in those models there are no maximal independent families of intermediate size, i.e. of cardinalities λ where $\aleph_1 < \lambda < \mathfrak{c}$. Finally, in section 4, at the price of assuming large cardinals, we show that the spectrum of independence is not necessarily convex. In fact, the spectrum can exclude finitely many intervals of the form $(\kappa_i, \kappa_{i+1}) = \{\lambda : \kappa_i < \lambda < \kappa_{i+1}\}$. We conclude with some well known open questions, which motivated this work. More is in a paper under preparation.

2. Diagonalizing an independent family

Definition 1. Let $\mathcal{A}$ be an independent family and let $\text{bhull}(\mathcal{A})$ be the set of all boolean combinations of $\mathcal{A}$. That is $\text{bhull}(\mathcal{A}) = \{\mathcal{A}^h : h \in \text{FF}(\mathcal{A})\}$. Note that the Frechét filter, denoted $\mathcal{F}_0$, has the following two properties:

(1) $\forall F \in \mathcal{F}_0 \forall B \in \text{bhull}(\mathcal{A})$, $F \cap B$ is infinite, and
(2) $\mathcal{F}_0 \cap \text{bhull}(\mathcal{A}) = \emptyset$.

A filter $\mathcal{U}$ is said to be an $\mathcal{A}$-*diagonalization filter*, if $\mathcal{U}$ extends $\mathcal{F}_0$ and $\mathcal{U}$ is maximal with respect to the above two properties.

Whenever $\mathcal{U}$ is a filter, denote by $\mathbb{M}(\mathcal{U})$ the Mathias poset relativized to $\mathcal{U}$. The conditions of $\mathbb{M}(\mathcal{U})$ are all pairs of the form $(s, A) \in [\omega]^{<\omega} \times [\omega]^\omega$ where $\max s < \min A$. A condition (s_2, A_2) extends (s_1, A_1), denoted $(s_2, A_2) \leq (s_1, A_1)$, if s_2 end-extends s_1, $s_2 \backslash s_1 \subseteq A_1$ and $A_2 \subseteq A_1$.

Lemma 2. Let $\mathcal{A}$ be an independent family, $\mathcal{U}$ an $\mathcal{A}$-diagonalization filter and let G be $\mathbb{M}(\mathcal{U})$-generic filter. Let $x_G = \bigcup\{s : \exists F(s, F) \in G\}$. Then:

(1) $\mathcal{A} \cup \{x_G\}$ is independent;
(2) If $y \in ([\omega]^\omega \backslash \mathcal{A}) \cap V$ is such that $\mathcal{A} \cup \{y\}$ is independent, then $\mathcal{A} \cup \{x_G, y\}$ is not independent.

Proof. (1) Let $h \in \text{FF}(\mathcal{A})$, $n \in \omega$. Consider the set

$$D_{h,n} := \{(s, F) \in \mathbb{M}(\mathcal{U}) : |s \cap \mathcal{A}^h| > n\}.$$

Pick any $(s, F) \in \mathbb{M}(\mathcal{U})$. Then $F \cap \mathcal{A}^h$ is infinite and so we can find $t \subseteq F \cap \mathcal{A}^h$ such that $\max s < \min t$ and $|t| > n$. Then $(s \cup t, F \backslash (\max t + 1)$ is an extension of (s, F) from $D_{h,n}$ and so $D_{h,n}$ is dense. Since h, n were arbitrary, we obtain that $\mathcal{A}^h \cap x_G$ is infinite for each h.

Again, fix h, n as above and consider the set

$$E_{h,n} := \{(s, F) : |(\min F \backslash \max s) \cap \mathcal{A}^h| > n\}.$$

Consider an arbitrary $(s, F) \in \mathbb{M}(\mathcal{U})$. Find an initial segment t of $\mathcal{A}^h \backslash (\max s + 1)$ such that $|t| > n$. Then $(s \cup t, F \backslash (\max t + 1))$ is an extension of (s, F) from $E_{h,n}$ and so $E_{h,n}$ is dense. Again, since h, n were arbitrary we obtain that $\mathcal{A}^h \backslash x_G$ is infinite, for each h.

(2) Let $y \in ([\omega]^\omega \backslash \mathcal{A}) \cap V$ be such that $\mathcal{A} \cup \{y\}$ is independent. If $y \in \mathcal{U}$ then $x_G \subseteq^* y$ and so $x_G \cap (\omega \backslash y)$ is finite. If $y \notin \mathcal{U}$, the reason must be that either there is $F \in \mathcal{U}$ such that $F \cap y$ is finite, and so $x_G \cap y$ is finite, or there are $F \in \mathcal{U}$ and $h \in \mathrm{FF}(\mathcal{A})$ such that $F \cap y \subseteq \mathcal{A}^h$. Let $C \in \mathrm{dom}(h)$ and wlg assume $h(C) = 1$. Then $F \cap y \subseteq^* \mathcal{A}^h \subseteq C$, which implies that $x_G \cap y \cap (\omega \backslash C)$ is finite. In each of the above cases, $\mathcal{A} \cup \{x_G, y\}$ is not independent. □

It is straightforward to verify that the poset $\mathbb{M}(\mathcal{U})$ adjoins an unbounded real. The above Lemma gives rise to the following:

Definition 3. We say that y *diagonalizes* $\mathcal{A}$ *over* V_0 *(in* V_1*)* iff

1. V_1 extends V_0, $(\mathcal{A}$ is independent$)^{V_0}$,
2. $y \in ([\omega]^{\aleph_0})^{V_1} \backslash V_0$, $\mathcal{A} \cup \{y\}$ is independent and
3. whenever $x \in ([\omega]^\omega)^{V_0} \backslash \mathcal{A}$ is such that $V_0 \vDash \mathcal{A} \cup \{x\}$ is independent, then

$$V_1 \vDash \mathcal{A} \cup \{x, y\} \text{ is not independent.}$$

Corollary 4. Let $\mathcal{A}$ be an independent family, $\mathcal{U}$ an $\mathcal{A}$-diagonalization filter and let G be a $\mathbb{M}(\mathcal{U})$-generic filer. Then the Mathias generic real

$$\sigma_G = \bigcup \{s : \exists A (s, A) \in G\}$$

diagonalizes $\mathcal{A}$ over the ground model.

Theorem 5. *(GCH) Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ be finitely many regular uncountable cardinals. Then, it is consitent that $\{\kappa_i\}_{i=1}^n \subseteq$ Spec(mif).*

Proof. Let γ^* be the ordinal product $\kappa_n \cdot \kappa_{n-1} \cdots \kappa_1$. For each $j = 1, \cdots, n$ let $I_j \subseteq \gamma^*$ be such that I_j is unbounded in γ^*, $|I_j| = \kappa_j$ and $\{I_j\}_{j=1}^{j=n}$ are pairwise disjoint. Along I_j inductively we can construct (by forcing) a maximal independent family of cardinality κ_j. Indeed. Define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma^*, \beta < \gamma^* \rangle$ as follows. Fix $\alpha < \gamma$ and suppose for each $k \in \{1, \cdots, n\}$, we have defined sequences of reals $\langle r^k_\gamma : \gamma \in I_k, \gamma < \alpha \rangle$ such that $\mathcal{J}^k_\alpha = \bigcup \{r^k_\gamma : \gamma \in I_\kappa \cap \alpha\}$ is an independent family and for each $\gamma \in I_\kappa$, r^k_γ diagonalizes $\mathcal{J}^k_\gamma = \bigcup \{r^k_\delta : \delta \in I_\kappa \cap \gamma\}$ over $V^{\mathbb{P}_\gamma}$. Proceed as follows. If $\alpha \in I_j$ for some $j \in \{1, \cdots, n\}$, then pick an $\mathcal{J}^j_\alpha$-diagonalizing filter $\mathcal{U}_\alpha$ in $V^{\mathbb{P}_\alpha}$, take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the relativized Mathias poset $\mathbb{M}(\mathcal{U}_\alpha)$ and r^j_α to be the associated Mathias generic real. If $\alpha \notin \bigcup_{k=1}^n I_k$, then take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the Cohen poset. □

The above argument clearly generalizes. Let λ be the intended size of the continuum, where $\mathrm{cof}(\lambda) > \aleph_0$. Partition λ into θ-many disjoint sets $\langle I_j : j \in \theta \rangle$, such that $|I_j| = \sigma_j$ and I_j is cofinal in λ. Using an appropriate bookkeeping function we can do a finite support iteration, such that the iterands corresponding to I_j adjoin a maximal independent family of size σ_j. Then in the final generic extension we will have $\{\sigma_j : j \in \theta\} \subseteq \mathrm{Spec}(mif)$.

3. The Spectrum is not necessarily convex

In the following, we will show that the spectrum is not necessarily convex. In fact, it can be rather small, consisting only of $\aleph_1$ and $\mathfrak{c}$. In [5], in a model of CH, the second author constructed a maximal independent family which remains a natural witness to $\mathfrak{i} = \aleph_1$ in a generic extension with $\mathfrak{u} = \mathfrak{c} = \aleph_2$. The construction gives rise to the existence of a Sacks indestructible maximal independent family. That is a maximal independent family, which remains maximal after the countable support iteration of Sacks forcing. A detailed proof of this fact can be found in [3].

Theorem 6 ([3], Corollary 36; [5])**.** *(CH) There is a maximal independent family, which remains maximal after the countable support iteration of Sacks forcing, as well as after an arbitrarily long countable support product of Sacks forcing.*

Theorem 7. *(CH) Let λ be a cardinal of uncountable cofinality. Let G be $\mathbb{P}$-generic, where $\mathbb{P}$ is the countable support product of Sacks forcing of length λ. Then $V[G] \vDash Spec(mif) = \{\aleph_1, \lambda\}$.*

Proof. Fix κ such that $\aleph_1 < \kappa < \lambda$. We will show that if $\mathcal{A}$ is a maximal independent family of cardinality κ, then $\mathcal{A}$ is not maximal. Towards a contradiction suppose there is $p_\star \in \mathbb{P}$ and a family $\{\tau_\alpha : \alpha < \kappa\}$ of $\mathbb{P}$-names for subsets of ω such that $p_\star \Vdash (\{\tau_\alpha : \alpha < \kappa\}$ is max independent). For $\alpha < \aleph_2$, let $p_\alpha \leq p_\star$ and let $\mathcal{U}_\alpha \in [\lambda]^{\aleph_0}$ be such that the support of p_α, $\mathrm{dom}(p_\alpha) = \mathcal{U}_\alpha$ and below p_α we can read τ_α continuously. Whenever τ is a nice $\mathbb{P}$-name for an infinite subset of ω and $p \in \mathbb{P}$, we denote by $\tau(\leq p)$ the natural restriction of τ below p. Now, we can find $S \in [\omega_2]^{\aleph_2}$ such that

$(\star)$ $\langle \mathcal{U}_\alpha : \alpha \in S\rangle$ is a Δ-system with root $\mathcal{U}_\star$, the sequence $\langle \mathrm{otp}(\mathcal{U}_\alpha) : \alpha \in S\rangle$ is constant, and for $\alpha \neq \beta$ from S, if $\pi_{\alpha,\beta}$ is the order preserving function from $\mathcal{U}_\beta$ onto $\mathcal{U}_\alpha$, then $\pi_{\alpha,\beta} \restriction \mathcal{U}_\star = \mathrm{id}_{\mathcal{U}_\star}$, $\pi_{\alpha,\beta}$ maps $\tau_\beta(\leq p_\beta)$ onto $\tau_\alpha(\leq p_\alpha)$.

Now, each τ_α is wlog the $\mathbb{P}$-name depending only on $\aleph_1$ conditions $\{p_{\alpha,i} : i < \omega_1\}$ (because $\mathbb{P}$ is $\aleph_2$-cc). Let $W_\alpha = \bigcup_i \mathrm{dom}(p_{\alpha,i})$. Let $W = \bigcup_{\alpha<\kappa} W_\alpha$. Then $|W| \leq \kappa \times \aleph_1 < \lambda$. We can find $\mathcal{U}$ such that $\mathcal{U} \subseteq \lambda$, $\mathrm{otp}(\mathcal{U}) = \mathrm{otp}(\mathcal{U}_\alpha)$ for $\alpha \in S$, $\mathcal{U} \cap W = \mathcal{U}_\star$. If $\alpha \in S$ let $\pi_{\alpha,\star}$ be the order preserving function from $\mathcal{U}$ onto $\mathcal{U}_\alpha$. Then consider the condition $p = \pi^{-1}_{\alpha,\star}(p_\alpha)$ and the naturally defined name $\tau = \pi^{-1}(\tau_\alpha \leq p_\alpha)$. Now $p \leq p_\alpha$ and $p \Vdash (\{\tau\} \cup \{\tau_\alpha : \alpha \in \kappa\}$ is independent), which contradicts $p_\star \Vdash (\{\tau_\alpha : \alpha < \kappa\}$ is maximal). □

4. Excluding Values

Let κ be a measurable cardinal and let $\mathcal{D} \subseteq \mathcal{P}(\kappa)$ be a κ-complete ultrafilter. Let $\mathbb{P}$ be a partial order. Then $\mathbb{P}^\kappa/\mathcal{D}$ is defined as the set of all equivalence classes

$$[f] = \{g \in {}^\kappa\mathbb{P} : \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{D}\}$$

and is supplied with the partial order relation $[f] \leq [q]$ iff

$$\{\alpha \in \kappa : f(\alpha) \leq_{\mathbb{P}} g(\alpha)\} \in \mathcal{D}.$$

We can identify each $p \in \mathbb{P}$ with the equivalence class $[p] = [f_p]$, where $f_p(\alpha) = p$ for each $\alpha \in \kappa$ and so we can assume $\mathbb{P} \subseteq \mathbb{P}^\kappa/\mathcal{D}$. The following claims can be found in [2, Lemmas 0.1 and 0.2].

Claim 8.

(1) The poset $\mathbb{P}$ is a complete suborder of $\mathbb{P}^\kappa/D$ if and only if $\mathbb{P}$ is κ-cc. Thus, if $\mathbb{P}$ is ccc, then $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathcal{D}$.
(2) If $\mathbb{P}$ has the countable chain condition, then so does $\mathbb{P}^\kappa/\mathcal{D}$.

Taking ultrapowers destroys the maximality of small independent families.

Lemma 9. Let $\lambda \geq \kappa$ and let $\mathcal{A}$ be a $\mathbb{P}$-name for an independent family. Then

$$\Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \mathcal{A} \text{ is not maximal.}$$

Proof. Use averages. □

We denote by *Even* the class of all ordinals α such that $\alpha = \beta + 2k$ for some limit β and $k \in \omega$, and by *Odd* the class of ordinals α which can be written in the form $\alpha = \beta + 2k + 1$ where β is a limit and $k \in \omega$.

Theorem 10. *Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ be measurable cardinals witnessed by κ_i-complete ultrafilters $\mathcal{D}_i \subseteq \mathcal{P}(\kappa_i)$. Then there is a ccc generic extnesion in which*

$$\{\kappa_i\}_{i=1}^n \subseteq \mathit{Spec}(mif) \text{ and } (\kappa_i, \kappa_{i+1}) \cap \mathit{Spec}(mif) = \emptyset$$

for each $1 \leq i < n$.

Proof. We will modify the proof of Theorem 5 as follows. Thus, fix γ^* and $I_j \subseteq \gamma$ for each $j = 1, \cdots, n$ as in the proof of 5, but assume in addition that I_j consists of successor cardinals and $\gamma^* = \sup\{\gamma \in I_j : \gamma \in \mathit{Even}\} = \sup\{\gamma \in I_j : \gamma \in \mathit{Odd}\}$. Proceed with the recursive definition of a ccc finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma^*, \beta < \gamma^* \rangle$. Fix $\alpha < \gamma$ and suppose for each $k \in \{1, \cdots, n\}$, we have defined sequences of reals $\langle r^k_\gamma : \gamma \in I_k \cap \text{Even}, \gamma < \alpha \rangle$ such that $\mathcal{J}^k_\alpha = \bigcup\{r^k_\gamma : \gamma \in I_\kappa \cap \text{Even} \cap \alpha\}$ is an independent family and for each $\gamma \in I_\kappa \cap \mathit{Even}$, r^k_γ diagonalizes $\mathcal{J}^k_\gamma$. Proceed as follows: If $\alpha \in I_k \cap \mathit{Even}$ for some $k \in \{1, \cdots, n\}$, then pick an $\mathcal{J}_\alpha$-diagonalizing filter $\mathcal{U}_\alpha$ in $V^{\mathbb{P}_\alpha}$, take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the relativized Mathias poset $\mathbb{M}(\mathcal{U}_\alpha)$ and r^k_α to be the associated Mathias generic real. If $\alpha \in I_k \cap \mathit{Odd}$ for some $k \in \{1, \cdots, n\}$ then $\alpha = \beta + 1$ for some β and we can take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\beta$-name for the quotient poset $\mathbb{R}_\alpha$, where $\mathbb{P}^{\kappa_k}/\mathcal{D}_k = \mathbb{P}_\beta * \mathbb{R}$. If $\alpha \notin \bigcup_{k=1}^n I_k$ for each k, then take $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for the Cohen poset. □

5. Concluding Remarks

Even though, we just gave an initial analysis of the spectrum of independence our results can be viewed as a very preliminary attempt to address the following two questions:

1. Is it consistent that $\mathfrak{i} < \mathfrak{a}$? Note that the consistency of $\mathfrak{a} < \mathfrak{i}$ holds in the random model.

2. Is it consistent that $\mathfrak{i}$ is of countable cofinality?

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