TIGHT MADNESS AND SELECTIVE INDEPENDENCE

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Abstract. Building on earlier results regarding the preservation properties of tight MAD families obtained in [5], as well as the preservation properties of selective independent families obtained in [4, 10], we establish the consistency of $\mathfrak{i} = \mathfrak{a} = \aleph_1 < \mathfrak{u} = \mathfrak{a}_T = \aleph_2$.

1. Introduction

In [10], S. Shelah shows that consistently $\mathfrak{i} < \mathfrak{u}$. To every (proper) maximal ideal $\mathcal{I}$ on ω, he associates a proper, ${}^\omega\omega$-bounding forcing notion $\mathbb{Q}_\mathcal{I}$, which has the Sacks property, destroys the maximality of $\mathcal{I}$ and preservers the maximality of a carefully designed maximal independent family. A family, which remains not only maximal in generic extensions by $\mathbb{Q}_\mathcal{I}$, but also in generic extensions obtained as the countable support iterations of such posets. In [3], the authors set to carefully study the combinatorial properties of Shelah's maximal independent family including dense maximality for independent families (notion which can be traced back to the work of Goldstern and Shelah on the consistency of $\mathfrak{r} < \mathfrak{u}$), two naturally associated ideals to independent families (diagonalization and density ideals), and show that for the class of densely maximal independent families, the two notions coincide. Note that the density independence ideal and more precisely its dual filter, plays an important role in subsequent studies and in particular, the notion of selective independence. Relying on this analysis, in [2] it is shown that Shelah's independent family can be chosen co-analytic in the constructible universe. In [4] we introduce the notion of selective independent family, as a densely maximal independent family whose density filter is Ramsey. Moreover, in the same paper we show that selective independent families are preserved by a class of forcing notions, known as partition forcings and their iterations which leads to the consistency of $\mathrm{cof}(\mathcal{N}) = \mathfrak{i} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2$.

The notion of tight mad families is introduced in [9], while in [5] it is also shown that the partition forcing and its iterations preserve tight mad families in a strong sense. Thus, the question if one can increase simultaneously $\mathfrak{u}$ and $\mathfrak{a}_T$, while preserving small witnesses to $\mathfrak{a}$ and $\mathfrak{i}$ becomes of interest. In this paper we show that:

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Theorem. Shelah's partial order $\mathbb{Q}_{\mathcal{I}}$ associated to a maximal ideal $\mathcal{I}$ strongly preserves tight mad families.

Thus an appropriate iteration leads to the following result.

Theorem. It is relatively consistent that $\mathfrak{a} = \mathfrak{i} = \aleph_1 < \mathfrak{u} = \mathfrak{a}_T = \aleph_2$.

Outline of the paper: In section 2, we introduce the partial order $\mathbb{Q}_{\mathcal{I}}$ and some of its crucial properties. In section 3, we recall the notion of a tight mad families, relevant preservation theorems and obtain the main result of the paper, the fact that $\mathbb{Q}_{\mathcal{I}}$ strongly preserves tight mad families (see Theorem 11). In section 4, we establish the consistency of $\mathrm{cof}(\mathcal{N}) = \mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2$.

2. The poset $\mathbb{Q}_{\mathcal{I}}$

For a maximal ideal $\mathcal{I}$ on ω, the forcing notion $\mathbb{Q}_{\mathcal{I}}$ is the one used by S. Shelah [10]. He has shown that $\mathbb{Q}_{\mathcal{I}}$ is proper [10, Claim 1.13], ${}^{\omega}\omega$-bounding [10, Claim 1.12] and even has the Sacks property [10, Claim 1.12]. In the $\mathbb{Q}_{\mathcal{I}}$-generic extension, $\mathcal{I}$ is no longer a maximal ideal [10, Claim 1.5]. Let us recall some terminology from [10].

Definition 1. Let $\mathcal{I}$ be an ideal on ω.

(1) An equivalence relation E on a subset of ω is an $\mathcal{I}$-equivalence relation if $\mathrm{dom}\, E \in \mathcal{I}^*$ and each E-equivalence class is in $\mathcal{I}$.
(2) For $\mathcal{I}$-equivalence relations E_1, E_2, we denote $E_1 \leq_{\mathcal{I}} E_2$ if $\mathrm{dom}\, E_1 \subseteq \mathrm{dom}\, E_2$, and E_1-equivalence classes are unions of E_2-equivalence classes.
(3) Let $A \subseteq \omega$. A function g is A-n-determined if $g\colon {}^{A}\{0,1\} \to \{0,1\}$ and there is $w \subseteq A \cap (n+1)$ such that for any $\eta, \nu \in {}^{A}\{0,1\}$ with $\eta \restriction w = \nu \restriction w$ we have $g(\eta) = g(\nu)$.

For $i \in A$, by g_i we denote a function from ${}^{A}\{0,1\}$ to $\{0,1\}$ which maps $\eta \in {}^{A}\{0,1\}$ to $\eta(i)$.

Claim 1. Each A-n-determined function is equal to a function $\varphi(g_0, \ldots, g_n)$ which is obtained as a maximum, minimum, and complement (i.e., $1 - g_i$) of $g_0, \ldots, g_n$, $0, 1$.

For an $\mathcal{I}$-equivalence relation E we denote $A = A(E) = \{x\colon x \in \mathrm{dom}\, E, x = \min[x]_E\}$.

Definition 2 (Set of conditions in $\mathbb{Q}_{\mathcal{I}}$)**.** Let $\mathcal{I}$ be an ideal on ω. We define a forcing notion $\mathbb{Q}_{\mathcal{I}}$:

$$p \in \mathbb{Q}_{\mathcal{I}} \text{ iff } p = (H, E) = (H^p, E^p) \text{ where}$$

(1) E is an $\mathcal{I}$-equivalence relation,
(2) H is a function with $\mathrm{dom}\, H = \omega$,
(3) a value $H(n)$ is an $A(E)$-n-determined function,
(4) if $n \in A(E)$ then $H(n) = g_n$,
(5) if $n \in \mathrm{dom}\, E \setminus A(E)$ and nEi for $i \in A(E)$ then $H(n)$ is g_i or $1 - g_i$.

For a condition $q \in \mathbb{Q}_{\mathcal{I}}$, let A^q be $A(E^q)$ in the following.

Definition 3. If $p, q \in \mathbb{Q}_{\mathcal{I}}$ with $A^p \subseteq A^q$ then we write $H^p(n) =^{**} H^q(n)$ if for each $\eta \in {}^{A^p}\{0,1\}$ we have $H^p(n)(\eta) = H^q(n)(\eta')$ where

$$\eta'(j) = \begin{cases} \eta(j) & j \in A^p, \\ H^p(j)(\eta) & j \in A^q \setminus A^p. \end{cases}$$

Definition 4 (The order of $\mathbb{Q}_{\mathcal{I}}$)**.** If $p, q \in \mathbb{Q}_{\mathcal{I}}$ then $p \leq q$ if

(1) $E^p \leq_{\mathcal{I}} E^q$,
(2) If $H^q(n) = g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = H^p(i)$,
(3) If $H^q(n) = 1 - g_i$ for $n \in \operatorname{dom} E^q$ then $H^p(n) = 1 - H^p(i)$,
(4) If $n \in \omega \setminus \operatorname{dom} E^q$ then $H^p(n) =^{**} H^q(n)$.

Finally, $p \leq_n q$ if $p \leq q$ and A^p contains the first n elements of A^q.

The following has been proven in [10]. Items (1) and (2) correspond to [10, Claim 1.7, (2)], item (3) is a straightforward modification of [10, Claim 1.8].

Claim 2. Let $p \in \mathbb{Q}_{\mathcal{I}}$. For an initial segment u of A^p, and $h \colon u \to \{0,1\}$, let $p^{[h]}$ be the pair $q = (H^q, E^q)$ defined by (i) and (ii) below:

(i) $E^q = E^p \restriction \bigcup\{[i]_{E^p} \colon i \in A^p \setminus u\}$.
(ii) If $H^p(n)$ is $\varphi(g_0, \dots, g_n)$ then $H^q(n)$ is $\varphi(g_0, \dots, g_i/h(i), \dots, g_n)$, where the substitution is done just for $i \in u$.

Then we have:

(1) $p^{[h]}$ is a condition in $\mathbb{Q}_{\mathcal{I}}$ stronger than p.
(2) The set $\{p^{[h]} \colon h \in {}^u\{0,1\}\}$ is predense below p.
(3) If u is the set of first n elements of A^p, D a dense subset of $\mathbb{Q}_{\mathcal{I}}$ then there is $q \in \mathbb{Q}_{\mathcal{I}}$ such that $q \leq_n p$ and $q^{[h]} \in D$ for any $h \in {}^u\{0,1\}$.

Definition 5 (The game $\mathrm{GM}_{\mathcal{I}}(E)$)**.** $\mathrm{GM}_{\mathcal{I}}(E)$ is the following game. In the n-th move, the first player chooses an $\mathcal{I}$-equivalence relation $E^1_n \leq_{\mathcal{I}} E^2_{n-1}$ ($E^1_0 = E$), and the second player chooses an $\mathcal{I}$-equivalence relation $E^2_n \leq_{\mathcal{I}} E^1_n$. In the end, the second player wins if

$$\bigcup_{n>0} (\operatorname{dom} E^1_n \setminus \operatorname{dom} E^2_n) \in \mathcal{I}.$$

Otherwise, the first player wins.

Remark 6. If the second player wins in the game $\mathrm{GM}_{\mathcal{I}}(E)$, then the game is invariant to taking subsets. That is, the game is invariant to taking $\leq_{\mathcal{I}}$-extensions $\{E^{2,*}_n\}_{n\in\omega}$ with $\operatorname{dom}(E^{2,*}_n) \subseteq \operatorname{dom} E^2_n$.

The next lemma corresponds to [10, Claim 1.10, (1)]

Lemma 7. The game $\mathrm{GM}_{\mathcal{I}}(E)$ is not determined for a maximal ideal $\mathcal{I}$.

3. Tight MAD families

Tight MAD families were investigated in [9, 8, 7]. An AD family $\mathcal{A}$ is called tight if for every $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.

Preservation theorem for tight MAD family under countable support iteration of proper forcing notions was developed by O. Guzmán, M. Hrušák and O. Téllez [5].

Definition 8. Let $\mathcal{A}$ be a tight MAD family. A proper forcing $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$ if for every $p \in \mathbb{P}$, M a countable elementary submodel of $H(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathbb{P}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there is $q \leq p$ an $(M, \mathbb{P})$-generic condition such that

$$q \Vdash \text{“}(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega\text{”},$$

where $\dot{G}$ denotes the name of generic filter.

We restate Corollary 32 by O. Guzmán, M. Hrušák and O. Téllez [5] which is crucial for preserving MAD families in the forthcoming model.

Theorem 9 (O. Guzmán, M. Hrušák, O. Téllez)**.** *Let $\mathcal{A}$ be a tight MAD family. If the sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ is a countable support iteration of proper posets such that*

$$\mathbb{P}_\alpha \Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha \textit{ strongly preserves the tightness of } \mathcal{A}\text{”},$$

then $\mathbb{P}_{\omega_2} \Vdash_\alpha$ “$\mathcal{A}$ is a tight MAD family”.

We need the following fact about the outer hulls observed in [5].

Lemma 10. Let $\mathcal{A}$ be an AD family, $\mathbb{P}$ a partial order, $\dot{B}$ a $\mathbb{P}$-name for a subset of ω and $p \in \mathbb{P}$ such that $p \Vdash$ “$\dot{B} \in \mathcal{I}(\mathcal{A})^+$”. Then the set $\{n : (\exists q \leq p)\ q \Vdash \text{“}n \in \dot{B}\text{”}\}$ is in $\mathcal{I}(\mathcal{A})^+$.

And now we are ready to show the main result of the paper.

Theorem 11. *Let $\mathcal{A}$ be a tight MAD family, $\mathcal{I}$ being a maximal proper ideal on ω. The poset $\mathbb{Q}_\mathcal{I}$ strongly preserves the tightness of $\mathcal{A}$.*

Proof. Let $p \in \mathbb{Q}_\mathcal{I}$, M a countable elementary submodel of $H(\kappa)$ such that $\mathcal{I}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$ for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. We fix an enumeration $\{D_n : n \in \omega\}$ of all open dense subsets of $\mathbb{Q}_\mathcal{I}$ that are in M, and an enumeration $\{\dot{Z}_n : n \in \omega\}$ of all $\mathbb{Q}_\mathcal{I}$-names for elements of $\mathcal{I}(\mathcal{A})^+$ that are in M with names repeating infinitely many times.

We define a strategy for the first player in the game $\mathrm{GM}_\mathcal{I}(E)$, which cannot be winning in all rounds.

We set $p_0 = q_0 = p$ and $u_0 = \emptyset$. We assume that the first player has chosen E^1_n, q_n, p_n, u_n, and the second one an E^2_n. We give instructions to choose E^1_{n+1}, q_{n+1}, p_{n+1}, u_{n+1}. We begin with q_{n+1}:

(1) $\operatorname{dom} E^{q_{n+1}} = \operatorname{dom} E^{p_n}$,
(2) $x E^{q_{n+1}} y$ iff one of the following holds:
  (i) $x E^2_n y$.

(ii) There is $k \in u_n$ with $x, y \in [k]_{E^{p_n}}$ and $x, y \notin \operatorname{dom} E^2_n$.

(iii) There are $k_0, k_1 \notin \bigcup\{[i]_{E^{p_n}} : i \in u_n\}$ with $x \in [k_0]_{E^{p_n}}$, $y \in [k_1]_{E^{p_n}}$ and $k_0, k_1 \notin \operatorname{dom} E^2_n$.

(3) $H^{q_{n+1}}$ is chosen such that:

(i) If $l \in \omega \setminus \operatorname{dom} E^{p_n}$ then $H^{q_{n+1}}(l) =^{**} H^{p_n}(l)$.

(ii) If $l \in \operatorname{dom} E^{p_n} \setminus A^{q_{n+1}}$, $H^{p_n}(l) = g_i$ then $H^{q_{n+1}}(l) = H^{q_{n+1}}(i)$.

(iii) If $l \in \operatorname{dom} E^{p_n} \setminus A^{q_{n+1}}$, $H^{p_n}(l) = 1 - g_i$ then $H^{q_{n+1}}(l) = 1 - H^{q_{n+1}}(i)$.

(iv) If $l \in A^{p_n} \setminus A^{q_{n+1}}$ then $H^{q_{n+1}}(l) =^{**} H^{p_n}(\min[l]_{E^{q_{n+1}}})$.

Note that for the already defined condition q_{n+1} we have $q_{n+1} \leq_n p_n$. Take $u_{n+1} = u_n \cup \{\min(A^{q_{n+1}} \setminus u_n)\}$. By Lemma 10, the set $D'_n = \{r \in \mathbb{Q}_{\mathcal{I}} : r \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n\text{”}\}$ is open dense below p (and also below q_{n+1}). Then $D'_n \cap D_n$ is dense below q_{n+1}. Therefore we can apply Lemma 7 to obtain $p_{n+1} \leq_{n+1} q_{n+1}$ such that for each $h \in {}^{u_{n+1}}\{0,1\}$, the condition $p^{[h]}_{n+1} \in D'_n \cap D_n \cap M$. In particular, if $h \in {}^{u_{n+1}}\{0,1\}$ then $p^{[h]}_{n+1} \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n \neq \emptyset\text{”}$ and $p^{[h]}_{n+1} \in D_n \cap M$. By Lemma 7 we have $p_{n+1} \Vdash \text{“}(\dot{Z}_n \cap B) \setminus n \neq \emptyset\text{”}$. Finally, we set

$$E^1_{n+1} = E^{p_{n+1}} \restriction (\operatorname{dom} E^{p_{n+1}} \setminus \bigcup\{[i]_{E^{p_{n+1}}} : i \in u_{n+1}\}).$$

We define a fusion q of a sequence $\langle p_n : n \in \omega\rangle$. Relation E^q has $\operatorname{dom} E^q = \bigcap\{\operatorname{dom} E^{p_n} : n \in \omega\}$, and $xE^q y$ if for every n large enough, $xE^{p_n}y$. Function H^q is equal to H^{p_n} for large enough n. In order to guarantee $q \in \mathbb{Q}_{\mathcal{I}}$, it is necessary to choose a play with the first player using described strategy, but he looses. Thus the second player wins and by Remark 6, we can assume that $\min \operatorname{dom} E^2_n > \max u_{n+1}$. Consequently, $\operatorname{dom} E^{p_n} \setminus \operatorname{dom} E^2_n \subseteq \bigcup\{[k]_{E^{q_{n+1}}} : k \in u_{n+1}\}$, and thus $\operatorname{dom} E^q \in \mathcal{I}^*$. One can check that other properties for $q \in \mathbb{Q}_{\mathcal{I}}$ are satisfied by the definition of q.

Finally, condition q is $(M, \mathbb{Q}_{\mathcal{I}})$-generic, and $q \leq_n p_n$ for each n. Hence, we have $q \Vdash \text{“}(\forall \dot{Z} \in \mathcal{I}(\mathcal{A}) \cap M[\dot{G}])\ |\dot{Z} \cap B| = \omega\text{”}$. □

4. Models

The assumptions of the next theorem is satisfied in the constructible universe. Note that the following is the strengthening of the consistency result in [4] by evaluating the value of almost disjointness number. Even though, we refer the reader to [4] for the properties of the partition forcing, for completeness we recall its definition here: Given a partition $\mathcal{C} = \{C_\alpha\}_{\alpha \in \omega_1}$ of 2^ω into closed sets, the partition forcing associated to $\mathcal{C}$ is the poset $\mathbb{Q}(\mathcal{C})$ consisting of all perfect trees $p \subseteq 2^{<\omega}$ with the property that each C_α is nowhere dense in $[p]$ and extension relation inclusion.

Theorem 12. *Assume $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $\diamondsuit_{(\delta<\omega_2 : \operatorname{cf}(\delta)=\omega_1)}$. There is a cardinals preserving generic extension in which*

$$\operatorname{cof}(\mathcal{N}) = \mathfrak{i} = \mathfrak{a} = \omega_1 < \mathfrak{a}_T = \mathfrak{u} = \omega_2.$$

Proof. Let V denote the ground model. We assume that $\mathcal{A}_0$ is a selective independent family and $\mathcal{A}_1$ is a tight mad family, both in V.

Using an appropriate bookkeeping device define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ of posets such that for even α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}(\mathcal{C})$ for some uncountable partition of 2^ω into compact sets, for odd α, $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha = \mathbb{Q}_{\mathcal{I}}$ for some maximal ideal on ω, and such that $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{a}_T = \mathfrak{u} = \omega_2$. The iteration $\mathbb{P}_{\omega_2}$ has the Sacks property and therefore $\mathrm{cof}(\mathcal{N}) = \omega_1$. By the indestructibility of selective independence, see [4], the family $\mathcal{A}_0$ remains maximal independent in $V^{\mathbb{P}_{\omega_2}}$ and so a witness to $\mathfrak{i} = \omega_1$. Moreover, by the preservation properties of tight MAD families, see [5], and the above preservation theorems, $\mathcal{A}_1$ is a witness to $\mathfrak{a} = \omega_1$ in the final model. $\square$

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