DEFINABLE TOWERS AND COHERENT SYSTEMS

VERA FISCHER AND JONATHAN SCHILHAN

ABSTRACT. We show that there are no analytic maximal towers, while in the constructible universe L there are co-analytic maximal towers. The existence of a co-analytic maximal tower is consistent not only with $\mathfrak{c}$ being arbitrarily large, but also $\mathfrak{b}$. We introduce the notion of strong diagonalization of a tower and develop a theory of preservation of maximal towers along matrix iterations. This allows us to show that the existence of a co-analytic maximal tower is consistent with $\mathfrak{s} = \aleph_2$, and also that consistently there are no $\mathbf{\Sigma}^1_2$ definable maximal towers, $\mathfrak{c}$ is arbitrarily large and $\mathfrak{t} = \aleph_1$. In the latter case, we can additionally require that $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$, or $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \mathfrak{c}$.

1. Introduction

A maximal tower is a wellordered subset of $[\omega]^\omega$ with respect to the relation of almost inclusion which has no pseudointersection (see Definition 2.1). In the following, we look at two aspects of the existence of maximal towers. On one side, we study their definability properties and show for example that there are no analytic maximal towers and that in the Constructible Universe, there are co-analytic maximal towers. On the other, we look at the minimal size of a maximal tower, the tower number $\mathfrak{t}$, and its relation to some of the well-known cardinal characteristics of the continuum. Developing the technique of matrix iterations in the context of towers, we show for example that whenever $\kappa < \lambda$ are arbitrary regular uncountable cardinals then consistently $\mathfrak{t} = \kappa < \mathfrak{s} = \mathfrak{b} = \lambda$. Moreover, the methods we develop allow for an interplay, which in particular gives the existence of a co-analytic maximal tower in a model of $\mathfrak{s} = \mathfrak{c} = \aleph_2$ (see Theorem 6.5), as well as the fact that consistently there are no $\mathbf{\Sigma}^1_2$ maximal towers, while the continuum is arbitrarily large and $\mathfrak{t} = \aleph_1$ (see Theorem 6.6). In the latter case, we can also require that $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$, or that $\mathfrak{b} = \aleph_1 < \mathfrak{s} = \mathfrak{c}$ (see Corollary 6.6).

The paper is organized as follows. In section 2, we study the definability properties of towers in $[\omega]^\omega$ and show that there are no analytic maximal towers (see Theorem 2.2), and that every $\mathbf{\Sigma}^1_2$ maximal tower is of size ω_1. In addition, we relax the condition of well-foundedness of the order relation of the members of a maximal tower and consider subsets of $[\omega]^\omega$ which are linearly ordered with respect to almost inclusion and have no pseudointersections. We refer to such families as inextendible linearly ordered towers, abbreviated ilt. We show that there are no analytic ilts and that every $\mathbf{\Sigma}^1_2$ definably ilt contains a cofinal subset of size ω_1 (see Theorems 2.4 and 2.6).

The existence of various nicely definable combinatorial sets of reals, like maximal almost disjoint families (see [15]), maximal cofinitary groups (see [7]), maximal families of orthogonal measures (see [8]), has been of increased interests in the last few decades. Elaborating on Miller's method

Key words and phrases. towers, mad families, matrix iterations, preservation of towers, ultrafilters, definability,cardinal characteristics, forcing.

*Acknowledgments:*The authors would like to thank the Austrian Science Fund, FWF, for generous support through START-Project Y1012-N35.

from [15] (see also [8, 7]), in Section 3, we obtain one of our first positive results, by showing that in the Constructible Universe there is a co-analytic maximal tower (see Theorem 3.2).

In Section 4, we turn our attention towards the preservation of towers along 2D-coherent systems, i.e. matrix iterations (for a general presentation of coherent systems see [6]). Recall that a system of finite support iterations $\langle \mathbb{P}_{\alpha,\beta} : \beta \leq \lambda \rangle$ where $\alpha \leq \kappa$ is called a matrix iteration (introduced by Blass and Shelah in [4], and later studied in [5, 13, 6]) if whenever $\alpha_1 \leq \beta_1$, $\alpha_2 \leq \beta_2$ then $\mathbb{P}_{\alpha_1,\beta_1} \lessdot \mathbb{P}_{\alpha_2,\beta_2}$. The preservation properties for maximal towers which we establish, even though quite different, have been at least partially motivated by existing techniques for preservation of maximal almost disjoint families along matrix iterations. Since mad families are well-known, we give below a brief account of this analogy. A standard method for introducing a maximal almost disjoint family along a finite support iteration is the almost disjoint poset, which given an almost disjoint family $\mathcal{A}$ generically (with finite approximations) adjoins a real a such that for each $b \in \mathcal{A}$, the intersection $b \cap a$ is finite, while for every ground model x which is not in the ideal generated by $\mathcal{A}$, denoted $\mathcal{I}(\mathcal{A})$, the intersection $x \cap a$ is infinite. This property of the almost disjoint poset allows along a finite support iterations whose length is of uncountable cofinality, using just a cofinal family of stages of the iteration, to adjoin a maximal almost disjoint family. Hechler's poset for adjoining a mad family (see [9, 5]) is characterized by an analogous property: If $\mathbb{H}_\kappa$ is the Hechler poset adjoining the mad family $\{a_\gamma\}_{\gamma\in\kappa}$, then for each $\gamma < \kappa$, $\mathbb{H}_\gamma \lessdot \mathbb{H}_\kappa$, $\mathbb{H}_\gamma$ adjoins $\mathcal{A}_\gamma = \{a_\mu\}_{\mu<\gamma}$ and a_γ not only has finite intersection with every member of $\mathcal{A}_\gamma$, but also meets every ground model infinite subset x of ω, $x \notin \mathcal{I}(\mathcal{A}_\gamma)$, in an infinite set. The authors of [5] strengthen the above weak diagonalization property to a property which allows a Hechler mad family adjoined along the first column of a matrix iteration to be preserved along the entire matrix iteration. This preservation result is central to the relative consistency of $\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$, which is one of the main results of see [5].

In order to preserve a Hechler tower (see Definition 4.1) along a matrix iteration, as mentioned above we formulate a notion of strong diagonalization for towers (see Definitions 4.2, 4.3 and Lemma 5.11) and establish its properties under various iterations (see Lemma 4.7 and Lemma 5.11). In particular, we show that Hechler forcing preserves strong diagonalization (see Theorem 5.7) and also, that whenever $M \subseteq N$ are models, $x \in N \cap [\omega]^\omega$ is a real strongly diagonalizing M and $\mathcal{U}$ is an ultrafilter in M, then there is an ultrafilter $\mathcal{V}$ in N extending $\mathcal{U}$ such that for the relativized Mathias posets $\mathbb{M}(\mathcal{U})$ and $\mathbb{M}(\mathcal{V})$ we have $\mathbb{M}(\mathcal{U}) \lessdot_M \mathbb{M}(\mathcal{V})$ and furthermore, if G is $\mathbb{M}(\mathcal{V})$ generic over N, then x strongly diagonalizes $M[G \cap \mathbb{M}(\mathcal{U})]$ (see Theorem 5.1). Our observations on strong diagonalization and the existence of ultrafilters preserving strong diagonalizaiton for towers, allow us, given an arbitrary maximal tower and assuming CH, to inductively construct an ultrafilter, such that the relativized Mathias poset preserves the maximality of the given maximal tower (see Theorem 5.4).

In Section 6, we establish our main results and applications of the above theory. In theorem 6.1 we show that consistently $\mathfrak{t} = \kappa < \mathfrak{s} = \mathfrak{b} = \lambda$, where $\kappa < \lambda$ are arbitrary regular uncountable cardinals. We show that the existence of a Π^1_1 definable tower is consistent not only with $\mathfrak{c}$ being arbitrarily large, but also $\mathfrak{b}$ being arbitrarily large. Furthermore, in Theorem 6.4, we show that the existence of a Π^1_1 maximal tower, is consistent with $\mathfrak{c} = \mathfrak{s} = \aleph_2$. Turning back to our analogy with the definability properties of mad families in models of large continuum, it is worth pointing out that there is still very little known about the cardinal characteristics of the continuum, other than

$\mathfrak{b}$ in models with nicely definable mad families, and that our techniques significantly differ from the techniques used to analyze such families. We conclude the paper, by showing that consistently there are no Σ^1_2 definable inextendible linearly ordered towers, and so no maximal towers, while the continuum is arbitrarily large and $\mathfrak{t} = \omega_1$. Additionally we can control the values of $\mathfrak{b}$ and $\mathfrak{s}$ (see Theorem 6.5 and Corollary 6.6).

2. Towers and Definability

We work with the following definition of a tower. Note that we do not assume the standard requirement that a tower is maximal.

Definition 2.1. A tower is a set $X \subseteq [\omega]^\omega$ which is well ordered with respect to the relation defined by $x \leq y$ iff $y \subseteq^* x$. It is called maximal if it cannot be further extended, i.e. it has no pseudointersection.

Theorem 2.2. *There is no $\mathbf{\Sigma}^1_1$ definable maximal tower.*

Proof. Assume $X \subseteq [\omega]^\omega$ is a $\mathbf{\Sigma}^1_1$ maximal tower. Then X is uncountable and thus contains an uncountable perfect set $P \subseteq X$ of reals by the Perfect Set Theorem for Analytic sets (see [11, Theorem 29.1]). The set $R = \{(x,y) : x, y \in P \wedge x \subseteq^* y\}$ is Borel. P is an uncountable Polish space and R is Borel as a subset of $P \times P$. But R is a well order of P, which contradicts R having the Baire property by [11, Theorem 8.48]. $\square$

Theorem 2.3. *Every $\mathbf{\Sigma}^1_2$ maximal tower is of size ω_1.*

Proof. If X is $\mathbf{\Sigma}^1_2$ and has size $> \omega_1$, then it contains an uncountable Borel set. Now derive the same contradiction as in the proof of Theorem 2.2. $\square$

Both of the proofs rely mostly on the fact that towers exhibit a well ordered structure and the maximality is inessential. Thus it is natural to ask for a more general version of a tower which is not trivially ruled out by an analytic definition. We call a set $X \subseteq [\omega]^\omega$ an inextendible linearly ordered tower (abreviated as ilt) if it is linearly ordered with respect to $\subseteq^*$ and has no pseudointersection. We call $Y \subseteq X$ cofinal in X if $\forall x \in X \exists y \in Y (y \subseteq^* x)$.

Theorem 2.4. *There is no $\mathbf{\Sigma}^1_1$ definable inextendible linearly ordered tower.*

Proof. Assume $X = p[T]$ is an ilt where T is a tree on $2 \times \omega$.

Claim 2.5. *There is $T' \subseteq T$ so that for every $(s,t) \in T'$, $p[T'_{(s,t)}]$ is cofinal in X.*

Proof. Let $T' = \{(s,t) : p[T_{(s,t)}]$ is cofinal in $X\}$. For every $(u,v) \in T \setminus T'$, we let $x_{u,v}$ be such that $\forall y \in p[T_{(u,v)}](x_{u,v} \subseteq^* y)$. The collection $\{x_{u,v} : (u,v) \in T \setminus T'\}$ is countable and therefore there is $x \in X$ so that $x \subsetneq^* x_{u,v}$ for every $(u,v) \in T \setminus T'$. Now let $(s,t) \in T'$ be arbitrary and $x' \in X$ such that $x' \subseteq^* x$. As $p[T_{(s,t)}]$ is cofinal in X, there is $y \in p[T_{(s,t)}]$ so that $y \subseteq^* x'$. Say $(y,z) \in [T_{(s,t)}]$. For every $n \in \omega$, $(y \restriction n, z \restriction n) \in T'$ because else we get a contradiction to $y \subseteq^* x$. Thus $y \in p[T'_{(s,t)}]$. $\square$

By the claim we can wlog assume that for every $(s,t) \in T$, $p[T_{(s,t)}]$ is cofinal in X. Now consider T as a forcing notion (which is equivalent to Cohen forcing). The generic real will be a new element of $p[T]$ together with a witness. Let $\dot{c}$ be a name for the generic real. Notice that the statement that $p[T]$ is linearly ordered by $\subseteq^*$ is absolute. Thus for every $y \in X$ there is a contition $(s,t) \in T$ and $n \in \omega$ so that either

$$(s,t) \Vdash \dot{c} \subseteq y \setminus n$$

or

$$(s,t) \Vdash y \subseteq \dot{c} \setminus n.$$

The second option is impossible, because $p[T_{(s,t)}]$ is cofinal in X. We can thus find (s,t), $n \in \omega$ and $Y \subseteq X$ cofinal in X, so that for every $y \in Y$, $(s,t) \Vdash \dot{c} \subseteq y \setminus n$. Let $(x,z) \in [T_{(s,t)}]$ be arbitrary. As Y is cofinal in X, there is $y \in Y$ so that $y \subsetneq^* x$. But this clearly contradicts $(s,t) \Vdash \dot{c} \subseteq y \setminus n$. □

Theorem 2.6. *Every $\mathbf{\Sigma}^1_2$ inextendible linearly ordered tower has a cofinal subset of size ω_1.*

Proof. Assume X is $\mathbf{\Sigma}^1_2$. Then it is the union of ω_1 many Borel sets. By Theorem 2.4 each of these Borel sets has a lower bound in X. □

The following result will appear a very useful tool in showing that consistently there are no $\mathbf{\Sigma}^1_2$ definable maximal towers, see Theorem 6.6. For this we need to define Mathias forcing relative to a filter.

Definition 2.7. Let $\mathcal{F}$ be a filter on ω containing all cofinite sets. Then Mathias forcing relative to $\mathcal{F}$ is the poset $\mathbb{M}_{\mathcal{F}}$ consisting of pairs $(s,X) \in [\omega]^{<\omega} \times \mathcal{F}$ such that $\max s < \min X$. The extension relation is defined by $(s,X) \leq (t,Y)$ iff $t \subseteq s$, $X \subseteq Y$ and $t \setminus s \subseteq Y$.

Throughout the paper we denote with $\mathbf{V}$ a fixed ground model.

Theorem 2.8. *Assume that X is a $\mathbf{\Sigma}^1_2$ definable subset of $[\omega]^\omega$, linearly ordered with respect to $\subseteq^*$. Then there is a ccc forcing notion $\mathbb{Q}$ so that for any $\mathbf{V}' \supseteq \mathbf{V}^{\mathbb{Q}}$, X (its evaluation) is not an ilt in $\mathbf{V}'$.*

Proof. As X is $\mathbf{\Sigma}^1_2$ we can write X as the union of ω_1 many Borel sets $\langle B_\alpha : \alpha < \omega_1 \rangle$ so that in any extension $\mathbf{V}'$ of $\mathbf{V}$, $(X)^{\mathbf{V}'} = \bigcup_{\alpha<\omega_1} (B_\alpha)^{\mathbf{V}'}$, where $(X)^{\mathbf{V}'}, (B_\alpha)^{\mathbf{V}'}$ is the evaluation of X, B_α in $\mathbf{V}'$ (see 13.4,13.7 in [10]).

If some B_α is cofinal in X, then this will hold true in any extension by absoluteness (B_α is cofinal in X iff $\forall x \in [\omega]^\omega (x \notin X \vee \exists y \in B_\alpha [y \subseteq^* x])$, which is $\mathbf{\Pi}^1_2$). Thus by Theorem 2.4 we could choose the trivial poset for $\mathbb{Q}$.

If no B_α is cofinal in X, then for any $\alpha < \omega_1$ there is $x_\alpha \in X$ so that $\forall y \in B_\alpha (x_\alpha \subseteq^* y)$. Moreover this will hold true in any extension of $\mathbf{V}$ by absoluteness. As X is linearly ordered wrt $\subseteq^*$, $\{x_\alpha : \alpha < \omega_1\}$ generates a non-principal filter $\mathcal{F}$. Let $\mathbb{Q} = \mathbb{M}(\mathcal{F})$. Then in $\mathbf{V}^{\mathbb{Q}}$ there is a real x so that $x \subseteq^* x_\alpha$ for every $\alpha < \omega_1$. In particular we have that $x \subseteq^* y$ for every $y \in X$ and this will hold true in any further extension. □

Note that the above results can be applied similarly to inextendible linearly ordered subsets of $(\omega^\omega, \leq^*)$.

3. A $\mathbf{\Pi}^1_1$ definable maximal tower in $\mathbf{L}$

In this section we will show how to construct in $\mathbf{L}$ a maximal tower with a $\mathbf{\Pi}^1_1$ definition. For this we apply the coding technique that has been developed in [15] in order to show the existence of various nicely definable combinatorial objects in $\mathbf{L}$.

Let O be the set of odd and E the set of even natural numbers.

Lemma 3.1. *Suppose $z \in 2^\omega$, $y \in [\omega]^\omega$ and $\langle x_\alpha : \alpha < \gamma \rangle$ is a tower, where $\gamma < \omega_1$, so that $\forall \alpha < \gamma (|x_\alpha \cap O| = \omega \wedge |x_\alpha \cap E| = \omega)$. Then there is $x \in [\omega]^\omega$ so that $\forall \alpha < \gamma (x \subseteq^* x_\alpha)$, $|x \cap O| = \omega$, $|x \cap E| = \omega$, $z \leq_T x$ and $|y \cap \omega \setminus x| = \omega$.*

Proof. It is a standard diagonalization to find x so that $\forall \alpha < \gamma (x \subseteq^* x_\alpha)$, $|x \cap O| = \omega$, $|x \cap E| = \omega$ and $|y \cap \omega \setminus x| = \omega$. We assume that z is not eventually constant, else there is nothing to do. Now given x find $\langle n_i \rangle_{i \in \omega}$ increasing in x so that $n_i \in O$ iff $z(i) = 0$. Let $x' = \{n_i : i < \omega\}$. Then x' works. □

Theorem 3.2. *Assume $\mathbf{V} = \mathbf{L}$. Then there is a $\mathbf{\Pi}^1_1$ definable maximal tower with an absolute definition (i.e. it evaluates to the same set in any transitive model of ZFC).*

In what follows, $<_L$ will stand for the canonical global $\mathbf{L}$ well-order. Whenever $r \in 2^\omega$, we write $E_r \subseteq \omega^2$ for the relation defined by

$$mE_r n \text{ iff } r(2^m 3^n) = 0.$$

If E_r is a well-founded and extensional relation then we denote with M_r the unique transitive $\in$-model isomorphic to (ω, E_r). Notice that $\{r \in 2^\omega : E_r \text{ is well-founded and extensional}\}$ is co-analytic.

Proof. Let $\langle y_\xi : \xi < \omega_1 \rangle$ enumerate $[\omega]^\omega$ via the canonical well order of $\mathbf{L}$. We will construct a sequence $\langle \delta(\xi), z_\xi, x_\xi : \xi < \omega_1 \rangle$, where for every $\xi < \omega_1$:

- $\delta(\xi)$ is a countable ordinal
- $z_\xi \in 2^\omega \cap \mathbf{L}_{\delta(\xi)+\omega}$
- $x_\xi \in [\omega]^\omega \cap \mathbf{L}_{\delta(\xi)+\omega}$

The sequence is defined by the following requirements for each $\xi < \omega_1$:

(1) $\delta(\xi)$ is the least ordinal δ greater than $\sup_{\nu<\xi} \delta(\xi)$ so that $y_\xi, \langle \delta(\nu), z_\nu, x_\nu : \nu < \xi \rangle \in \mathbf{L}_\delta$ and $\mathbf{L}_\delta$ projects to ω[1].
(2) z_ξ is the $<_L$ least code for the ordinal $\delta(\xi)$.
(3) $\langle x_\nu : \nu < \xi \rangle$ is a tower and $\forall \nu < \xi (|x_\nu \cap O|) = \omega \wedge |x_\nu \cap E| = \omega)$.
(4) x_ξ is $<_L$ least so that $\forall \nu < \xi (x_\xi \subseteq^* x_\nu)$, $|x_\xi \cap O| = \omega$, $|x_\xi \cap E| = \omega$, $z_\xi \leq_T x$ and $|y_\xi \cap \omega \setminus x| = \omega$.

Notice that z_ξ and x_ξ indeed can be found in $\mathbf{L}_{\delta(\xi)+\omega}$ given that $y_\xi, \langle x_\nu : \nu < \xi \rangle \in \mathbf{L}_{\delta(\xi)}$, and that $\mathbf{L}_{\delta(\xi)}$ projects to ω. It is then straightforward to check that (1)-(4) uniquely determine a sequence $\langle \delta(\xi), z_\xi, x_\xi : \xi < \omega_1 \rangle$ for which $\langle x_\xi : \xi < \omega_1 \rangle$ is a maximal tower.

Claim 3.3. *$\{x_\xi : \xi < \omega_1\}$ is a $\mathbf{\Pi}^1_1$ subset of 2^ω.*

Proof. Let $\Psi(v)$ be the formula expressing that for some $\xi < \omega_1$, $v = \langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle$. More precisely, $\Psi(v)$ says that v is a sequence $\langle \rho_\nu, \zeta_\nu, \tau_\nu : \nu \leq \xi \rangle$ of some length $\xi + 1$, that satisfies the clauses (1)-(4) for every $\nu \leq \xi$.

The formula $\Psi(v)$ is absolute for transitive models of some finite fragment Th of ZFC which holds at limit stages of the $\mathbf{L}$ hierarchy. Namely we need absoluteness of the formula $\varphi_1(\xi, y)$ expressing

[1]This means that over $\mathbf{L}_\delta$ there is a definable surjection to ω. The set of such δ is unbounded in ω_1.

that $y = y_\xi$, $\varphi_2(\delta, M)$ expressing that $M = \mathbf{L}_\delta$ projects to ω and $\varphi_3(z, \delta)$ expressing that z is the $<_L$ least code for δ.

Moreover we have that $\langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle \in \mathbf{L}_{\delta(\xi)+\omega}$ and that

$$\mathbf{L}_{\delta(\xi)+\omega} \models \Psi(\langle \delta(\nu), z_\nu, x_\nu : \nu \leq \xi \rangle)$$

for every $\xi < \omega_1$.

Now let $\Phi(r, x)$ be a formula expressing that E_r is a well founded and extensional relation, $M_r \models \mathrm{Th}$ and for some $v \in M_r$,

$$M_r \models v \text{ is a sequence } \langle \rho_\nu, \zeta_\nu, \tau_\nu : \nu \leq \xi \rangle \wedge \Psi(v) \wedge \tau_\xi = x.$$

We thus have that $x = x_\xi$ for some $\xi < \omega_1$ iff $\exists r \in 2^\omega \Phi(r, x)$. $\Phi(r, x)$ can clearly be taken as a $\mathbf{\Pi}^1_1$ formula.

For any $\xi < \omega_1$, the well order $\delta(\xi)$ is coded by z_ξ and $z_\xi \leq_T x_\xi$. Thus $\delta(\xi) + \omega < \omega_1^{x_\xi}$ and there is $r \in \mathbf{L}_{\omega_1^{x_\xi}}$ so that $M_r = \mathbf{L}_{\delta(\xi)+\omega}$. In particular

$$\exists r \in \mathbf{L}_{\omega_1^{x_\xi}}(\Phi(r, x_\xi)).$$

We get that

$$\exists \xi < \omega_1 (x = x_\xi) \leftrightarrow \exists r \in \mathbf{L}_{\omega_1^x}(\Phi(r, x)).$$

The right hand side can be expressed by a $\mathbf{\Pi}^1_1$ formula. $\square$

$\square$

The fact that the $\mathbf{\Pi}^1_1$ definition above will give rise to the same set in any extension of $\mathbf{L}$ follows from the following general observation:

Remark 3.4. Any $\mathbf{\Sigma}^1_2$ maximal tower has an absolute definition. This follows from the Mansfield-Solovay Theorem (see e.g. [14, Theorem 21.1]) and because a tower cannot contain a perfect set (see the proof of Theorem 2.2).

4. Towers and Diagonalization

Definition 4.1. [Hechler [9]] For an ordinal γ we define the poset $\mathbb{T}_\gamma$ consisting of all finite partial functions $p\colon \gamma \times \omega \to 2$ with $\operatorname{dom} p = F_p \times n_p$ for some $F_p \in [\gamma]^{<\omega}$ and $n_p \in \omega$. $q \leq p$ if $q \supseteq p$ and whenever $\alpha < \beta \in F_p$, $n \in n_q \setminus n_p$ and $q(\beta, n) = 1$ then $p(\alpha, n) = 1$.

For any ordinal γ the poset $\mathbb{T}_\gamma$ is ccc. Moreover, whenever G is $\mathbb{T}_\gamma$ generic, the sets $T_\alpha := \{n \in \omega : \exists p \in G(p(\alpha, n) = 1)\}$ for $\alpha < \gamma$ form a tower $\langle T_\alpha : \alpha < \gamma \rangle$, i.e. a decreasing sequence in $([\omega]^\omega, \subseteq^*)$. When $\operatorname{cof}(\gamma) \geq \omega_1$ then $\langle T_\alpha : \alpha < \gamma \rangle$ is maximal in $V[G]$ (see [9]).

For any $\delta < \gamma$ we have that $\mathbb{T}_\delta \lessdot \mathbb{T}_\gamma$ and we can explicitly find a quotient $\dot{\mathbb{T}}_{[\delta,\gamma)}$, so that $\mathbb{T}_\gamma$ is forcing equivalent to $\mathbb{T}_\delta * \dot{\mathbb{T}}_{[\delta,\gamma)}$. In $V[G]$, where G is $\mathbb{T}_\delta$ generic, $\mathbb{T}_{[\delta,\gamma)}$ will consist of all pairs (p, H) such that p is a finite partial function $p\colon [\delta, \gamma) \times \omega \to 2$, with $\operatorname{dom} p = F_p \times n_p$ and $H \in [\delta]^{<\omega}$. The order is defined by $(q, K) \leq (p, H)$ if q extends p in the sense of $\mathbb{T}_\gamma$, $H \subseteq K$ and for all $\alpha \in F_p$, $\beta \in H$, $n \in n_q \setminus n_p$ if $q(\alpha, n) = 1$ then $n \in T_\beta$.

Definition 4.2. [weak diagonalization] Assume $M \subseteq N$ are transitive models of set theory, $x \in [\omega]^\omega \cap N$. We say that x *weakly diagonalizes* M if for all $y \in [\omega]^\omega \cap M$, $|y \cap (\omega \setminus x)| = \omega$.

Definition 4.3. [strong diagonalization] Assume $M \subseteq N$ are transitive models of set theory, $x \in [\omega]^\omega \cap N$. We say that x *strongly diagonalizes* M if for every $f \in M$, $k \in \omega$, where $f : \omega \to [\omega]^{\leq k}$ is such that $f(n) \subseteq \omega \setminus n$ for every n, there is some $n \in \omega$ such that $f(n) \subseteq \omega \setminus x$.

Notice that strong diagonalization implies weak diagonalization. To see this, given $y \in M$ just let $f(n) = \{\min y \cap (\omega \setminus n)\}$ for every n. There is another way to get weak diagonalization as we will show now:

Definition 4.4. Let $M \subseteq N$ be models of set theory, $c \in N \cap \omega^\omega$. We say that c is unbounded over M if for every $f \in M \cap \omega^\omega$, $c \not<^* f$.

Lemma 4.5. *Let $M \subseteq N$ be models of set theory, $x \in [\omega]^\omega \cap N$. Let f_x be defined by $f_x(n) = \min x \setminus n$. If f_x is unbounded over M, then x weakly diagonalizes M.*

Proof. Assume that $y \in M$ is such that $y \subseteq^* x$, say $y \setminus m \subseteq x$. Then clearly for every $n \geq m$, $f_y(n) \geq f_x(n)$, where f_y is defined in the same way as f_x. $\square$

We are now showing that the generics added by $\mathbb{T}_\gamma$ satisfy strong diagonalization and unboundedness in a certain sense.

Lemma 4.6. *Let γ be an ordinal. Let $G_{\gamma+1}$ be $\mathbb{T}_{\gamma+1}$ generic and $G_\gamma := G_{\gamma+1} \cap \mathbb{T}_\gamma$. Furthermore, as before, let $T_\alpha := \{i \in \omega : \exists p \in G(p(\alpha, i) = 1)\}$ for $\alpha < \gamma + 1$. Then:*

(1) T_γ strongly diagonalizes $\mathbf{V}[G_\gamma]$,

(2) f_{T_γ} is unbounded over $\mathbf{V}[G_\gamma]$, where f_{T_γ} is defined as in Lemma 4.5.

Proof. (1) We work in $\mathbf{V}[G_\gamma]$. Let f, k be as in the definition of strong diagonalization and $(p, H) \in T_{[\gamma,\gamma+1)}$ be arbitrary with $\operatorname{dom} p = \{\gamma\} \times n_p$ for $n_p \in \omega$. As f is diverging, there is some n so that $f(n) \cap n_p = \emptyset$. Now extend (p, H) to (q, H) so that $q(\gamma, i) = 0$ for all $i \in f(n)$. Then we have that $(q, H) \Vdash f(n) \subseteq \omega \setminus T_\gamma$.

(2) is analogous to (1). $\square$

Lemma 4.7. *Assume $\langle x_\alpha : \alpha < \kappa \rangle$ is a maximal tower and M is a countable model of set theory. Then there is $\alpha < \kappa$ so that x_α is strongly diagonalizing over M.*

Proof. It suffices to show that for any $k \in \omega$, $f : \omega \to [\omega]^k$ so that $\forall n(f(n) \subseteq \omega \setminus n)$, there is $\alpha < \kappa$ so that $\exists^\infty n \in \omega(f(n) \cap x_\alpha = \emptyset)$.

We show this by induction on k. For $k = 1$ this is just saying that $\langle x_\alpha : \alpha < \kappa \rangle$ is a maximal tower. Let $f : \omega \to [\omega]^{k+1}$. Consider $g : \omega \to [\omega]^k$ defined by $g(n) = f(n) \setminus \{\max f(n)\}$. Then if we know the statement for k, there is $\alpha < \kappa$ so that $X = \{n \in \omega : g(n) \cap x_\alpha = \emptyset\}$ is infinite. Let $x := \{\max f(n) : n \in X\}$. Then there is $\beta > \alpha$ so that $|x \cap \omega \setminus x_\beta| = \omega$. Clearly now $\{n \in X : \max f(n) \in \omega \setminus x_\beta\} \subseteq^* \{n \in \omega : f(n) \cap x_\beta = \emptyset\}$. $\square$

5. Preservation results

Lemma 5.1. *Assume $\mathbb{P}_0 \lessdot \mathbb{P}_1$, $\dot{x}$ a $\mathbb{P}_1$ name for a real so that $\Vdash_{\mathbb{P}_1}$ "$\dot{x}$ is strongly diagonalizing over $\mathbf{V}^{\mathbb{P}_0}$". Further assume that $\dot{\mathbb{Q}}_l$ is a $\mathbb{P}_l$ name for a poset, $l \in \{0, 1\}$. Then,*

(1) $\mathbb{P}_0 \star \mathbb{1} \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}_1$ and $\Vdash_{\mathbb{P}_1 \star \dot{\mathbb{Q}}_1}$ "$\dot{x}$ is strongly diagonalizing over $\mathbf{V}^{\mathbb{P}_0 \star \mathbb{1}}$",

(2) $\mathbb{P}_0 \star \dot{\mathbb{Q}}_0 \lessdot \mathbb{P}_1 \star \dot{\mathbb{Q}}_0$ and $\Vdash_{\mathbb{P}_1 \star \dot{\mathbb{Q}}_0}$ "$\dot{x}$ is strongly diagonalizing over $\mathbf{V}^{\mathbb{P}_0 \star \dot{\mathbb{Q}}_0}$".

Proof. (1) and the first part of (2) are straightforward. Let us show that $\Vdash_{\mathbb{P}_1 \star \dot{\mathbb{Q}}_0}$ "$\dot{x}$ is strongly diagonalizing over $\mathbf{V}^{\mathbb{P}_0 \star \dot{\mathbb{Q}}_0}$". Let $\dot{f} \in \mathbf{V}^{\mathbb{P}_0}$ be a $\mathbb{Q}_0$ name for a function $f\colon \omega \to [\omega]^{\leq k}$ so that $f(n) \subseteq \omega \setminus n$ for all n. Also let $q \in \mathbb{Q}_0$ be a condition. Then we can find in $\mathbf{V}^{\mathbb{P}_0}$ a decreasing sequence $\langle q_n : n \in \omega \rangle$ of conditions below q and a function g so that $q_n \Vdash g(n) = \dot{f}(n)$ for every n. By strong diagonalization, there is $n \in \omega$ so that $g(n) \cap x = \emptyset$ and thus in $\mathbf{V}^{\mathbb{P}_1}$, $q_n \Vdash \dot{f}(n) \cap x = \emptyset$. We have thus shown that in $\mathbf{V}^{\mathbb{P}_1}$, the set of conditions q forcing that there is some n with $\dot{f}(n) \cap x = \emptyset$ is dense in $\mathbb{Q}_0$. □

5.1. **Mathias forcing.**

Lemma 5.2. *Let $M \subseteq N$ be models of set theory, $\mathcal{V} \in M$ an ultrafilter in M, $x \in N$ strongly diagonalizes M. Then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{V}$ in N, so that:*

(1) every maximal antichain of $\mathbb{M}(\mathcal{V})$ in M is a maximal antichain of $\mathbb{M}(\mathcal{U})$ in N,

(2) whenever G is $\mathbb{M}(\mathcal{U})$-generic over N then x strongly diagonalizes $M[G]$.

Proof. Work in N. Let $C \subseteq \mathbb{M}(\mathcal{V})$ be a maximal antichain in M and let $s \in [\omega]^{<\omega}$. We call $X \in \mathcal{V}^+$ ($\mathcal{V}$ positive sets) *forbidden by* s, C if (s, X) is incompatible with every $p \in C$.

For $k \in \omega$, $\dot{f} \in M$ an $\mathbb{M}(\mathcal{V})$ name for a function $\omega \to [\omega]^{\leq k}$ so that $\dot{f}(n) \subseteq \omega \setminus n$ there are (in M) maximal antichains $D_n^{\dot{f}} \subseteq \mathbb{M}(\mathcal{V})$ and functions $g_n^{\dot{f}} : D_n^{\dot{f}} \to [\omega]^{\leq k}$ so that $p \Vdash \dot{f}(n) = g_n^{\dot{f}}(p)$ for $p \in D_n^{\dot{f}}$. We say $Y \in \mathcal{V}^+$ *is forbidden by* $\dot{f}, k, t$ if for all $n \in \omega$, (t, Y) is incompatible with all conditions $p \in D_n^{\dot{f}}$ so that $g_n^{\dot{f}}(p) \subseteq \omega \setminus x$.

Let $\mathcal{I}$ be the ideal generated by all forbidden sets.

Claim 5.3. *$\mathcal{I} \cap \mathcal{V} = \emptyset$.*

Proof. Assume there are sets

$$X_0, \dots X_{l-1}; Y_0, \dots, Y_{l-1}$$

forbidden by

$$s_0, C_0, \dots, s_{l-1}, C_{l-1}; \dot{f}_0, k_0, t_0, \dots, \dot{f}_{l-1}, k_{l-1}, t_{l-1}$$

so that $\bigcup_{i<l} X_i \cup \bigcup_{i<l} Y_i \supseteq Z \in \mathcal{V}$.

For $t \in [\omega]^{<\omega}$ and $C \subseteq \mathbb{M}(\mathcal{V})$ we say that C permits t if there is $(s, X) \in C$ so that $(t, X) \leq (s, X)$. Further let $k = \sum_{i<l} k_i$.

Sublaim 5.4. *There is a function $h\colon \omega \to \omega$ in M and a function $g\colon \omega \to [\omega]^{\leq k}$ in M so that for any $n \in \omega$, whenever we cover $Z \cap h(n)$ with $2l$ pieces, then at least one piece s of the covering has the following property:*

- *for all $i < l$ there is $t \subseteq s$ such that C_i permits $s_i \cup t$,*
- *for all $i < l$ there is $t \subseteq s$ such that some $p \in D_n^{\dot{f}_i}$ with $g_n^{\dot{f}_i}(p) \subseteq g(n)$ permits $t_i \cup t$.*

Proof. Work in M. Let $G : \omega \to [\omega]^{\leq k}$ be such that $|G^{-1}(a)| = \omega$ for every $a \in [\omega]^{\leq k}$.

Assume that for some n, we cannot find an appropriate $h(n)$ and $g(n) = G(h(n))$. By a compactness argument we find a covering of Z with $2l$ pieces so that for any initial covering the conclusion of the subclaim fails for $g(n)$ chosen according to G. One of the pieces of the covering has to be in $\mathcal{V}$, say W. Since C_i is a maximal antichain, there is $p \in C$ compatible to (s_i, W). But then there is

$t \subseteq W$ so that p permits $s_i \cup t$. Similarly there are $p_i \in D_n^{\dot f}$ and $t \subseteq W$ so that p_i permits $s_i \cup t$. Let $a = \bigcup_{i<l} g_n^{\dot f}(p_i)$. Then if we choose $h(n)$ large enough we can get $a = G(h(n))$ and $s = W \cap h(n)$ having the required properties for $g(n) = G(h(n))$, which contradicts the assumption. $\square$

Now let h and g be given as in the subclaim. Notice that by strong diagonalization of x over M we have that for some $n \in \omega$, $g(n) \subseteq \omega \setminus x$.

Fix such an $n \in \omega$ and consider the covering $\{X_i \cap h(n), Y_i \cap h(n) : i < l\}$. Since (s_i, X_i) is incompatible with all conditions from C_i, there is no $t \subseteq X_i \cap h(n)$ such that C_i permits $s_i \cup t$. So $X_i \cap h(n)$ is not the piece that the subclaim claims to exist. Thus a piece of the form $Y_i \cap h(n)$ is as claimed by the subclaim. This means that there are $t \subseteq Y_i \cap h(n)$ and $p \in D_n^{\dot f_i}$ with $g_n^{\dot f_i}(p) \subseteq g(n) \subseteq \omega \setminus x$ and p permits $t_i \cup t$. But this contradicts Y_i being forbidden by $\dot f_i, k_i, t_i$.

$\square$

Now that the claim is proven, construct $\mathcal{U} \supseteq \mathcal{V}$ in N such that $\mathcal{U} \cap \mathcal{I} = \emptyset$. It is clear that then (1) and (2) hold true. $\square$

Theorem 5.5. *(CH) For any maximal tower $\langle x_\alpha : \alpha < \omega_1 \rangle$ there is an ultrafilter $\mathcal{U}$ so that $\mathbb{M}(\mathcal{U})$ preserves $\langle x_\alpha : \alpha < \omega_1 \rangle$, i.e. $\Vdash_{\mathbb{M}(\mathcal{U})}$ "$\langle x_\alpha : \alpha < \omega_1 \rangle$ is maximal".*

Proof. The ultrafilter $\mathcal{U}$ is constructed recursively using Lemma 4.7 and 5.2. More precisely, given an enumeration $\langle y_\alpha : \alpha < \omega_1 \rangle$ of 2^ω, we define by induction a sequence $\langle \mathcal{U}_\alpha, M_\alpha, \xi(\alpha) : \alpha < \omega_1 \rangle$ so that the following hold true for all $\alpha < \omega_1$:

(1) M_α is the transitive collapse of a countable elementary submodel of $H(\omega_2)$ and y_α, $\langle \mathcal{U}_\beta, M_\beta : \beta < \alpha \rangle \in M_\alpha$.
(2) $x_{\xi(\alpha)} \in M_{\alpha+1}$ is strongly diagonalizing over M_α.
(3) $\mathcal{U}_\alpha \in M_\alpha$ is an ultrafilter over $M_\alpha \cap \mathcal{P}(\omega)$.
(4) $\mathcal{U}_{\alpha+1} \supseteq \mathcal{U}_\alpha$ is such that every maximal antichain of $\mathbb{M}(\mathcal{U}_\alpha)$ in M_α is still a maximal antichain of $\mathbb{M}(\mathcal{U}_{\alpha+1})$ in $M_{\alpha+1}$.
(5) $\mathcal{U}_{\alpha+1} \supseteq \mathcal{U}_\alpha$ is such that whenever G is $\mathbb{M}(\mathcal{U}_{\alpha+1})$ generic over $M_{\alpha+1}$, then $x_{\xi(\alpha)}$ is strongly diagonalizing over $M_\alpha[G]$.
(6) When α is limit, then $\mathcal{U}_\alpha$ is such that every maximal antichain of $\mathbb{M}(\mathcal{U}_\beta)$ in M_β is still a maximal antichain of $\mathbb{M}(\mathcal{U}_\alpha)$ in M_α for any $\beta < \alpha$.

Using Lemma 4.7 we can get (2) and using Lemma 5.2 we get (3),(4) and (5). (6) can be attained by a straightforward strengthening of Lemma 5.2, whose proof is almost the same (see also [4, p. 266]).

Given this sequence, let $\mathcal{U} = \bigcup_{\alpha<\omega_1} \mathcal{U}_\alpha$. First notice that for any $\alpha < \omega_1$, any maximal antichain $A \in M_\alpha$ of $\mathbb{M}(\mathcal{U}_\alpha)$ is still maximal in $\mathbb{M}(\mathcal{U})$. If $\dot x$ is any $\mathbb{M}(\mathcal{U})$-name for a real consisting of countably many pairs of the form $\langle \check n, (s, X) \rangle$, then $\dot x \in M_\alpha$ for some α. Whenever G is $\mathbb{M}(\mathcal{U})$ generic then G is also $\mathbb{M}(\mathcal{U}_\alpha)$ generic over M_α, and by (5), $x_{\xi(\alpha)}$ is strongly diagonalizing over $M_\alpha[G]$. In particular $\dot x[G] \not\subseteq^* x_{\xi(\alpha)}$.

This proves that $\mathbb{M}(\mathcal{U})$ preserves the tower $\langle x_\alpha : \alpha < \omega_1 \rangle$ maximal. $\square$

5.2. **Hechler forcing.** We use the following version of Hechler forcing.

Definition 5.6. Let IS denote the subset of $\omega^{<\omega}$ consisting of increasing sequences. Hechler forcing for adding a dominating real is the poset $\mathbb{D}$ consisting of pairs $(s,f) \in IS \times \omega^\omega$. The extension relation is defined by $(s,f) \leq (t,g)$ iff $t \subseteq s$, $g \leq f$, $\forall n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)(s(n) \geq f(n))$.

A very useful tool for the study of combinatorial properties of Hechler forcing is the notion of a rank function.

Definition 5.7. Assume D is a dense open subset of $\mathbb{D}$. Then we define a corresponding rank function $\mathrm{rk}\colon IS \to \omega_1$ as follows:

- $\mathrm{rk}(s) = 0$ iff there is $g \in \omega^\omega$ so that $(s,g) \in D$,
- $\mathrm{rk}(s) \leq \alpha$ iff there is $l \in \omega$ and a sequence $\langle t_i : i \in \omega \rangle$ of extensions of s so that for every $i \in \omega$, $|t_i| = l$, $t_i(n) > i$ for every $n \in l \setminus \mathrm{dom}\, s$, and $\mathrm{rk}\, t_i < \alpha$.

Lemma 5.8. *If D is open dense then the corresponding rank is defined for every $s \in IS$.*

Proof. Assume $\mathrm{rk}(s)$ is undefined. For any $l \in \omega$ we let T_l be the set of all minimal extensions t of s with $\mathrm{rk}(t)$ being defined and $|t| = l$.

We claim that $\{t(l-1) : t \in T_l\}$ is bounded for every $l \in \omega$. Assume $l \in \omega$ is such that $\{t(l-1) : t \in T_l\}$ is unbounded, then we find a sequence $\langle t_i : i \in \omega \rangle$ in T_l so that $t_i(l-1) > i$ for every $i \in \omega$. Using the minimality of the t_i we find that $t_i(l-2)$ is also unbounded so we can pass to a subsequence $\langle t_{i_n} : n \in \omega \rangle$ so that $t_{i_n}(l-1), t_{i_n}(l-2) > n$ for $n \in \omega$. Continuing like this we find a sequence $\langle s_i : i \in \omega \rangle$ with $|s_i| = l$, $\mathrm{rk}(s_i)$ is defined, and $s_i(n) > i$ for every $n \in l \setminus \mathrm{dom}(s)$ and every $i \in \omega$. This poses a contradiction to $\mathrm{rk}(s)$ being undefined.

Now let $f(n) = \max\{t(n) : t \in T_{n+1}\} + 1$. Then (s,f) can be extended to a condition $(t,g) \in D$. $\mathrm{rk}(t)$ is defined and there is a minimal $t' \subseteq t$, $s \subseteq t'$, for which $\mathrm{rk}(t')$ is defined, say $|t'| = l$. But then $t'(l-1) \geq f(l-1)$ which is a contradiction. $\square$

Theorem 5.9. *Assume $M \subseteq N$ are models of set theory and $x \in N \cap [\omega]^\omega$ strongly diagonalizes M. Then, whenever G is $\mathbb{D}$ generic over N, then x strongly diagonalizes $M[G]$.*

Proof. Assume the statement of the theorem is false. This means that there is a $\mathbb{D}$ name $\dot{f} \in M$ so that $\Vdash \forall n \in \omega[\dot{f}(n) \subseteq \omega \setminus n \wedge |\dot{f}(n)| = k]$ for some $k \in \omega$, and there is some condition $(s,g) \in \mathbb{D} \cap N$ so that $(s,g) \Vdash \forall n(\dot{f}(n) \cap x \neq \emptyset)$.

We define the dense open sets $D_n = \{t \in IS : \exists h \in \omega^\omega[(t,h) \Vdash \dot{f}(n) = a]$ for some $a \in [\omega]^{\leq k}\}$ and the respective rank functions $\mathrm{rk}_n\colon IS \to \omega_1$.

For each $n \in \omega$ and for $t \in IS$ we define the set

$$Z_n(t) = \{a \in [\omega]^k : \forall f \in \omega^\omega \exists (t',f') \leq (t,f)[(t',f') \Vdash \dot{f}(n) = a]\}$$

Claim 5.10. $\forall m \in \omega \exists n \geq m(Z_n(s) \neq \emptyset)$.

Proof. Fix $m \in \omega$ and assume that for every $n \geq m$, $Z_n(s) = \emptyset$. For any $n \geq m$ we define a partial function $v_n\colon IS \to [\omega]^{\leq k} \setminus \{\emptyset\}$ in M so that the following will hold true for every $t \in \mathrm{dom}\, v_n$:

(1) $\forall h \in \omega^\omega \exists (t',h') \leq (t,h)[(t',h') \Vdash v_n(t) \subseteq \dot{f}(n)]$,
(2) if $(t,g) \leq (s,g)$ then $v_n(t) \cap x \neq \emptyset$.

The definition is done recursively in M step by step as follows:

- Initial step: For $t \in D_n$ let $v_n(t)$ be such that $\exists h \in \omega^\omega[(t,h) \Vdash \dot{f}(n) = v_n(t)]$. Then clearly (1) holds true and (2) holds true because we assumed that $(s,g) \Vdash \dot{f}(n) \cap x \neq \emptyset$.

- Assume we defined v_n already on a set $D \subseteq IS$. Assume $t \in IS \setminus D$ is such that there is $l \in \omega$ and a sequence $\langle t_i : i \in \omega \rangle$ of extensions of t with $|t_i| = l$, $t_i \in D$ for every $i \in \omega$ and $t_i(j) > i$ for every $j \in \operatorname{dom} t_i \setminus t$.

  Further assume that the following hold true of the sequence $\langle t_i : i \in \omega \rangle$:
  - $v_n(t_i)$ has the same size $J \leq k$ for every $i \in \omega$.
  - There is $a \neq \emptyset$, so that $a \subseteq v_n(t_i)$ for every $i \in \omega$ and either $|a| = J$ (i.e. $a = v_n(t_i)$ for every i) or $\langle \min v_n(t_i) \setminus a : i \in \omega \rangle$ is strictly increasing.

  Then we set $v_n(t) = a$.

  We have to show that (1) and (2) hold true.

  (1): Let $h \in \omega^\omega$ be given. If we let $i = \max_{j<l} h(j)$, then $t_i > h$ so that $(t_i, h) \leq (t, h)$. We know that (1) was true of t_i, so there is $(t', h') \leq (t_i, h)$ with $(t', h') \Vdash v_n(t_i) \subseteq \dot{f}(n)$. In particular $(t', h') \Vdash v_n(t) = a \subseteq v_n(t_i) \subseteq \dot{f}(n)$.

  (2): Assume $(t, g) \leq (s, g)$. If $|a| = J$ then definitely $v_n(t) \cap x \neq \emptyset$ has to hold true because $v_n(t) = v_n(t_i)$ for every i and in particular for some large enough i where $(t_i, g) \leq (s, g)$. Now assume $|a| < J$, and thus $\langle \min v_n(t_i) \setminus a : i \in \omega \rangle$ is strictly increasing. As x is strongly diagonalizing over M there must be $i \in \omega$ large enough so that $v_n(t_i) \setminus a \subseteq \omega \setminus x$ and $(t_i, g) \leq (s, g)$. $v_n(t_i) \cap x = ((v_n(t_i) \setminus a) \cup a) \cap x \neq \emptyset$ by assumption and thus $a \cap x \neq \emptyset$.

  We might have different possible choices for $v_n(t)$ because a lot of sequences $\langle t_i : i \in \omega \rangle$ as above might exist, so we pick it arbitrarily if it exists.

Sublaim 5.11. *$v_n(s)$ is defined.*

Proof. Assume $v_n(s)$ was not defined. We know that $\operatorname{rk}_n(s)$ is defined. Let $\langle t_i : i \in \omega \rangle$ be such that for $i \in \omega$, $|t_i| = l$, $\operatorname{rk}_n(t_i) < \operatorname{rk}_n(s)$ and $t_i(j) > i$ for every $j \in \operatorname{dom} t_i \setminus s$.

Then $v_n(t_i)$ must be undefined for almost every $i \in \omega$. Because assume $v_n(t_i)$ is defined for infinitely many i. Then we could assume wlog, by passing to a subsequence, that $v_n(t_i)$ is defined for all i, $|v_n(t_i)|$ has constant value $J \leq k$ and either

- $\langle \min v_n(t_i) : i \in \omega \rangle$ is strictly increasing,
- or there is $a \neq \emptyset$ so that $\forall i \in \omega (a \subseteq v_n(t_i))$ and in case $|a| < J$, $\langle \min v_n(t_i) \setminus a : i \in \omega \rangle$ is strictly increasing.

In the first case we would get $v_n(t_i) \subseteq \omega \setminus x$ for some large enough i where $(t_i, g) \leq (s, g)$ contradicting (2) above. In the second case $v_n(s)$ would be defined, contrary to our assumption.

Thus consider $i \in \omega$ large enough so that $v_n(t_i)$ is undefined and $(t_i, g) \leq (s, g)$ and let $t^0 = t_i$.

By arguing in the same way for t^0, we get a sequence $t^0, t^1, \dots$ in N so that $\operatorname{rk}_n(t^{i+1}) < \operatorname{rk}_n(t^i)$, $(t^{i+1}, g) \leq (t^i, g)$ and $v_n(t^i)$ is undefined. This sequence has to end eventually with $\operatorname{rk}(t^i) = 0$. But then $v_n(t^i)$ is defined as in the initial step. We get a contradiction. $\square$

Finally unfix n. We get a sequence $\langle v_n(s) : n \geq m \rangle$ defined in M, so that $v_n(s) \subseteq \omega \setminus n$, $v_n(s) \in [\omega]^{\leq k}$ and $v_n(s) \cap x \neq \emptyset$. But by strong diagonalization we know that there is some $n \geq m$ so that $v_n(s) \subseteq \omega \setminus x$. Again we have a contradiction and the claim is proven.

$\square$

Now that we proved the claim, we are finished. Namely, pick in M for every $n \in \omega$ where $Z_n(s) \neq \emptyset$, $a_n \in Z_n(s)$. Then by strong diagonalization of x, there is some n so that $a_n \subseteq \omega \setminus x$. But then we can extend (s, g) to force that $\dot{f}(n) = a_n \subseteq \omega \setminus x$, which is a contradiction.

□

The following Theorem was proven by Baumgartner and Dordal in [1]. We give a proof of it based on Theorem 5.9 and Lemma 4.7.

Theorem 5.12. *Suppose $\langle x_\alpha : \alpha < \kappa\rangle$ is a maximal tower. Then $\mathbb{D}$ preserves $\langle x_\alpha : \alpha < \kappa\rangle$ maximal.*

Proof. Suppose $\dot{x}$ is $\mathbb{D}$ name for an element of $[\omega]^\omega$. Let M be a countable elementary submodel of $H(\theta)$ containing $\dot{x}$, where θ is a large enough regular cardinal. Then by Lemma 4.7 there is some $\alpha < \kappa$ so that x_α is strongly diagonalizing over M. By Theorem 5.9 x_α stays strongly diagonalizing over $M[G]$, where G is a Hechler generic. In particular $\dot{x}[G] \not\subseteq^* x_\alpha$. □

5.3. **Limits of fsi.**

Lemma 5.13. *Let $\langle \mathbb{P}_{l,i}, \dot{\mathbb{Q}}_{l,i} : i < \delta\rangle$, $l \in \{0,1\}$ be finite support iterations so that for every $i < \delta$, $\mathbb{P}_{0,i} \lessdot \mathbb{P}_{1,i}$. Further assume that $\dot{x}$ is a $\mathbb{P}_{1,0}$ name for a real so that for every $i < \delta$, $\Vdash_{\mathbb{P}_{1,i}}$ "$\dot{x}$ is strongly diagonalizing over $\mathbf{V}^{\mathbb{P}_{0,i}}$". Then,*

(1) $\mathbb{P}_{0,\delta} \lessdot \mathbb{P}_{1,\delta}$,
(2) $\Vdash_{\mathbb{P}_{1,\delta}}$ "$\dot{x}$ is strongly diagonalizing over $\mathbf{V}^{\mathbb{P}_{0,\delta}}$".

Proof. (1) is well known (see e.g. [5, Lemma 10]). For (2) assume that $\dot{f}$ is a $\mathbb{P}_{0,\delta}$ name for a function $\omega \to [\omega]^k$ so that $\dot{f}(n) \subseteq \omega \setminus n$ for every $n \in \omega$. Further assume that $p \in \mathbb{P}_{1,\delta}$ forces that $\forall n \in \omega(\dot{f}(n) \cap \dot{x} \neq \emptyset)$, say $p \in \mathbb{P}_{1,i}$ for $i < \delta$. Let $G_{1,i}$ be a $\mathbb{P}_{1,i}$-generic containing p and let $G_{0,i} = G_{1,i} \cap \mathbb{P}_{0,i}$. In $\mathbf{V}[G_{0,i}]$, let $\mathbb{R}^0_{i,\delta}$ be the quotient poset $\mathbb{P}_{0,\delta}/G_{0,i}$ and $\dot{f}'$ the quotient name $\dot{f}/G_{0,i}$. Find in $\mathbf{V}[G_{0,i}]$ a sequence $\langle q_n\rangle$ of $\mathbb{R}^0_{i,\delta}$ conditions and a sequence $\langle a_n\rangle$ so that $q_n \Vdash \dot{f}'(n) = a_n$. As p forced that $\forall n \in \omega(\dot{f}(n) \cap \dot{x} \neq \emptyset)$ we must have that $a_n \cap \dot{x}[G_{0,i}] \neq \emptyset$ for every n. But this is a contradiction to $\Vdash_{\mathbb{P}_{1,i}}$ "$\dot{x}$ is strongly diagonalizing over $\mathbf{V}^{\mathbb{P}_{0,i}}$". □

The following was implicitly proven in [1] (see Theorem 3.3 and 3.5 there).

Lemma 5.14. *Suppose $\langle x_\alpha : \alpha < \kappa\rangle$ is a maximal tower. Assume that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i \leq \delta\rangle$ is a finite support iteration of ccc posets so that for every $i < \delta$, $\Vdash_{\mathbb{P}_i}$ "$\langle x_\alpha : \alpha < \kappa\rangle$ is maximal", then $\Vdash_{\mathbb{P}_\delta}$ "$\langle x_\alpha : \alpha < \kappa\rangle$ is maximal".*

6. Applications

Theorem 6.1. *(GCH) Let $\kappa < \lambda$ be regular uncountable cardinals. Then there is a ccc forcing extension in which $\mathfrak{t} = \kappa < \mathfrak{b} = \mathfrak{s} = \lambda$ and $\lambda^{<\lambda} = \lambda$.*

Proof. Consider a matrix iteration $\langle \mathbb{P}_{\alpha,\beta}, \dot{\mathbb{Q}}_{\alpha,\beta} : \alpha \leq \kappa, \beta < \lambda\rangle$ where $\mathbb{P}_{0,\kappa}$ is Hechler's poset $\mathbb{T}_\kappa$ (see Definition 4.1) for adjoining a maximal tower $T = \{t_\gamma\}_{\gamma\in\kappa}$. Let $\mathbf{V}$ denote the ground model and for each $\alpha \leq \kappa, \beta \leq \lambda$, let $\mathbf{V}_{\alpha,\beta} = \mathbf{V}^{\mathbb{P}_{\alpha,\beta}}$. Use successor stages β such that $\beta \equiv 1 \bmod 3$ to guarantee that $V^{\mathbb{P}_{\kappa,\lambda}} \vDash \kappa \leq \mathfrak{p} = \mathfrak{t}$. For this it is sufficient to arrange, that whenever $\mathcal{F}$ is a family with the strong finite intersection property in $\mathbf{V}^{\mathbb{P}_{\kappa,\lambda}}$ of caridnality $< \kappa$, then there is $\beta < \lambda$ and $\alpha < \kappa$ such that $\mathcal{F} \in \mathbf{V}^{\mathbb{P}_{\alpha,\beta}}$ and the posets $\{\dot{\mathbb{Q}}_{\zeta,\beta}\}_{\zeta\in\kappa}$ are used to add a pseudointersection to $\mathcal{F}$. This can be achieved by an appropriate bookkeeping function and after fixing α, β as above, defining $\dot{\mathbb{Q}}_{\zeta,\beta}$ to be a $\mathbb{P}_{\zeta,\beta}$ name for the trivial poset whenever $\zeta < \alpha$, and a $\mathbb{P}_{\zeta,\beta}$ name for the Mahtias forcing relativized to $\mathcal{F}$ whenever $\zeta \geq \alpha$. Successor stages β for which $\beta \equiv 2 \bmod 3$ are used to adjoin full Hechler reals, i.e. for each $\zeta \in \kappa$ define $\dot{\mathbb{Q}}_{\zeta,\beta}$ to be a $\mathbb{P}_{\zeta,\beta}$ name for Hechler's forcing for adjoining a dominating real. This clearly guarantees that $\mathbf{V}^{\mathbb{P}_{\kappa,\lambda}} \vDash \mathfrak{b} = \lambda$. Finally, to guarantee that $\mathfrak{s} = \lambda$ in $\mathbf{V}^{\mathbb{P}_{\kappa,\lambda}}$, one can use successor β's such that $\beta \equiv 0 \bmod 3$ and Lemma 5.1. to construct an increasing sequence of ultrafilters $\{\mathcal{U}_{\zeta,\beta}\}_{\zeta\in\kappa}$, where for each $\mathcal{U}_{\zeta,\beta}$ is an ultrafilter in $\mathbf{V}^{\mathbb{P}_{\zeta,\beta}}$ such that $\mathbb{M}(\mathcal{U}_{\zeta,\beta}) \lessdot_{\mathbf{V}_{\zeta,\beta}} \mathbb{M}(\mathcal{U}_{\eta,\beta})$ whenever $\zeta < \eta < \kappa$ and such that if t_ζ strongly diagonalizes $\mathbf{V}_{\zeta,\beta}$, then t_ζ strongly diagonalizes $\mathbf{V}_{\zeta,\beta+1}$ for each $\zeta \in \kappa$. Then clearly $\mathbf{V}^{\mathbb{P}_{\kappa,\lambda}} \vDash \mathfrak{s} = \lambda$.

The preservation properties developed in the previous two sections, and in particular Lemmas 4.7, 5.1, 5.7 and 5.11 imply that T remains a maximal tower in $V^{\mathbb{P}_{\kappa,\lambda}}$ and so a witness to $\mathfrak{t} = \kappa$ in the final generic extension. □

Remark 6.2. The consistency of $\mathfrak{t} = \kappa < \mathfrak{b} = \mathfrak{s} = \lambda$ can be proven in several different ways. The simplest one is to use the fact that for any infinite $\theta < \mathfrak{t}$, $2^\theta = \mathfrak{c}$ (see [2, Theorem 6.14]). Using this we can start with a model where $2^\kappa > \lambda$ and construct an iteration in which we add λ many dominating reals, λ many unsplit reals and destroy small filter bases. But this approach gives strong restrictions to cardinal arithmetic and we don't get that $\lambda^{<\lambda} = \lambda$. The other one is to use a result of Blass ([3, Theorem 2]) that says that whenever $\mathbf{V}$ is the union of an increasing sequence $\langle \mathbf{V}_\alpha \rangle_{\alpha<\kappa}$ of models of ZFC, where $[\omega]^\omega \cap (\mathbf{V}_{\alpha+1} \setminus \mathbf{V}_\alpha) \neq \emptyset$ and $\langle [\omega]^\omega \cap \mathbf{V}_\alpha \rangle_{\alpha<\kappa} \in \mathbf{V}$, then $\mathfrak{t} \leq \kappa$ (even $\mathfrak{g} \leq \kappa$ where $\mathfrak{g}$ is the groupwise density number). Thus in the above matrix iteration we may have simply used Cohen reals in the first column to get the same result. But our approach shows how to additionaly preserve a generically added tower and get an explicit witness for $\mathfrak{t} = \kappa$.

Remark 6.3. For $\kappa < \lambda$ regular uncountable cardinals, assuming GCH one can easily obtain the consistency of $\mathfrak{t} = \mathfrak{b} = \kappa < \mathfrak{s} = \lambda$. Indeed, one can use a matrix iteration $\langle \mathbb{P}_{\alpha,\beta}, \dot{\mathbb{Q}}_{\alpha,\beta} : \alpha \leq \kappa, \beta < \lambda \rangle$, where $\mathbb{P}_{\kappa,0} = \mathbb{C}_\kappa$ is the standard poset for adjoining κ-many Cohen reals. Successors β, such that $\beta \equiv 1 \bmod 2$ and an appropriate bookkeeping (see [5]) can be used to guarantee that in the final extension $V_{\kappa,\lambda}$ there are no unbounded families of cardinality $< \kappa$, i.e. we use these columns to force with appropriate (depending on the bookkeeping function) restricted Hechler posets. Successor stages β, where $\beta \equiv 0 \bmod 2$ can be used to define increasing sequences of ultrafilters $\{\mathcal{U}_{\zeta,\beta}\}_{\zeta<\kappa}$ such that forcing with the relativized Mathias poset over $V_{\zeta,\beta}$ not only adjoins an unsplit real, but also preserves the unboundedness of c_γ, the γ-th Cohen real over $V_{\gamma,0}$. This can be achieved as in [5], and results in $V_{\kappa,\lambda} \vDash \mathfrak{s} = \mathfrak{c} = \lambda$. The preservation properties of the construction, see [5], imply that the family of Cohen reals remains unbounded in $V_{\kappa,\lambda}$ and so a witness to $\mathfrak{b} = \kappa$ in this model.

Theorem 6.4. *The existence of a $\mathbf{\Pi}^1_1$ definable maximal tower is consistent with arbitrarily large continuum and with $\mathfrak{b}$ being arbitrarily large.*

Proof. Start in $\mathbf{L}$. For κ uncountable consider $\mathbb{C}_\kappa$ the forcing adding κ many Cohen reals. It is well known that $\mathbb{C}_\kappa$ preserves every maximal tower. Thus the tower given by Theorem 3.2 will stay maximal in any forcing extension of $\mathbf{L}$ via $\mathbb{C}_\kappa$ and have the same definition there.

For $\mathfrak{b}$ consider $\mathbb{D}_\kappa$ the κ length fsi of Hechler forcing. By Theorem 5.12 and Lemma 5.14, any maximal tower is preserved by $\mathbb{D}_\kappa$. □

Theorem 6.5. *The existence of a $\mathbf{\Pi}^1_1$ definable maximal tower is consistent with $\mathfrak{s} = \omega_2$.*

Proof. Work in $\mathbf{V} = \mathbf{L}$ and let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be the $\mathbf{\Pi}^1_1$ maximal tower given by Theorem 3.2. Using Theorem 5.5 and Lemma 5.14, construct a finite support iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i \leq \omega_2 \rangle$ such that for every $i < \omega_2$

(1) $\Vdash_{\mathbb{P}_i}$ "$\langle x_\alpha : \alpha < \omega_1 \rangle$ is a maximal tower",
(2) $\Vdash_{\mathbb{P}_i} \mathfrak{c} = \omega_1$,
(3) $\Vdash_{\mathbb{P}_i} \dot{\mathbb{Q}}_i = \mathbb{M}(\dot{\mathcal{U}}_i)$ where $\dot{\mathcal{U}}_i$ is a $\mathbb{P}_i$ name for an ultrafilter preserving $\langle x_\alpha : \alpha < \omega_1 \rangle$ maximal.

Again by Lemma 5.14, in any extension via $\mathbb{P}_{\omega_2}$, $\langle x_\alpha : \alpha < \omega_1 \rangle$ is maximal and thus there is a $\mathbf{\Pi}^1_1$ maximal tower. Each iterand $\dot{\mathbb{Q}}_i$ adds an unsplit real over $\mathbf{V}^{\mathbb{P}_i}$. Thus $\mathbf{V}^{\mathbb{P}_{\omega_2}} \models \omega_2 \leq \mathfrak{s} \leq \mathfrak{c} \leq \omega_2$. □

Theorem 6.6. *It is consistent with arbitrarily large continuum that there is no $\mathbf{\Sigma}^1_2$ definable inextendible linearly ordered tower (in particular no maximal tower), but $\mathfrak{t} = \omega_1$.*

Proof. Fix an uncountable regular cardinal κ and assume GCH. As before, we will construct a matrix iteration $\langle \mathbb{P}_{\alpha,\beta} : \alpha \leq \omega_1, \beta \leq \kappa \rangle$ in which we explicitly add a generic maximal tower that will be preserved. Formally we also fix a bookkeeping function $F \colon \kappa \to [H(\kappa)]^\omega$ enumerating all countable subsets of $H(\kappa)$, repeating every set unboundedly often.

(1) If $\beta = 0$, then $\mathbb{P}_{\alpha,0}$ is Hechler's poset $\mathbb{T}_\alpha$ for adding a tower of length α.
(2) If $\beta > 0$ is a limit ordinal, then $\mathbb{P}_{\alpha,\beta}$ is the direct limit of the posets $\langle \mathbb{P}_{\alpha,\zeta} : \zeta < \beta \rangle$.
(3) If $\beta = \zeta + 1$ and $F(\zeta) = \dot{c}$ is a $\mathbb{P}_{\omega_1,\zeta}$ name for a real, encoding a $\mathbf{\Sigma}^1_2$ subset X of $[\omega]^\omega$ that is linearly ordered wrt $\subseteq^*$, then $\dot{c}$ is a $\mathbb{P}_{\alpha,\zeta}$ name for some $\alpha < \omega_1$. By Theorem 2.8 there is a $\mathbb{P}_{\alpha,\zeta}$ name $\dot{\mathbb{Q}}$ for a ccc forcing, so that X will not be an ilt in any extension of $\mathbf{V}^{\mathbb{P}_{\alpha,\zeta} \star \dot{\mathbb{Q}}}$. In this case let $\mathbb{P}_{\gamma,\beta} = \mathbb{P}_{\gamma,\zeta} \star \mathbb{1}$ for $\gamma < \alpha$ and $\mathbb{P}_{\gamma,\beta} = \mathbb{P}_{\gamma,\zeta} \star \dot{\mathbb{Q}}$ for $\gamma \geq \alpha$.
(4) If $\beta = \zeta + 1$ and we are not in case (3) let $\mathbb{P}_{\alpha,\beta} = \mathbb{P}_{\alpha,\zeta} \star \mathbb{1}$ for every $\alpha < \omega_1$.

Our preservation theorems imply that $\mathbf{V}^{\mathbb{P}_{\omega_1,\kappa}} \models \mathfrak{t} = \omega_1$, while clearly $\mathbf{V}^{\mathbb{P}_{\omega_1,\kappa}} \models \mathfrak{c} = \kappa$. Also it follows from the construction that there is no $\mathbf{\Sigma}^1_2$ ilt in $\mathbf{V}^{\mathbb{P}_{\omega_1,\kappa}}$. □

Corollary 6.7. Let κ be an arbitrary regular uncountable cardinal. The lack of a $\mathbf{\Sigma}^1_2$ definable ilt is consistent with $\mathfrak{t} = \omega_1$ and each of the following:

(1) $\mathfrak{b} = \mathfrak{s} = \mathfrak{c} = \kappa$,
(2) $\mathfrak{b} = \omega_1 \leq \mathfrak{s} = \kappa$.

Proof. The techniques of Theorem 6.1 and Theorem 6.6 can clearly be combined to obtain the desired result. □

We would like to conclude the paper with the following questions: Is it consistent that there is a co-analytic maximal tower and $\mathfrak{s} > \aleph_2$? Is there an ilt in Solovay's model?

References

[1] J. Baumgartner, P. Dordal *Adjoining dominating functions.* J. Symbolic Logic 50 (1985), no. 1, 94 - 101.
[2] A. Blass *Combinatorial cardinal characteristics of the continuum.* Handbook of set theory. Vols. 1, 2, 3, 395 - 489, Springer, Dordrecht, 2010.
[3] A. Blass *Applications of superperfect forcing and its relatives.* Set theory and its applications (Toronto, ON, 1987), 18 - 40, Lecture Notes in Math., 1401, Springer, Berlin, 1989.
[4] A. Blass, S. Shelah *Ultrafilters with small generating sets.* Israel J. Math. 65 (1989), no. 3, 259 - 271.

[5] J. Brendle, V. Fischer *Mad families, splitting families and large continuum* Journal of Symbolic Logic, 76, 1, 198-208, March 2011.

[6] V. Fischer, S. Friedman, D. Mejia, D. Montoya *Coherent systems of finite support iterations* accepted at the Journal of Symbolic Logic.

[7] V. Fischer, D. Schrittesser, A. Törnquist *A co-analytic Cohen indestructible maximal cofinitary group* accepted at the Journal of Symbolic Logic.

[8] V. Fischer, A. Törnquist *A co-analytic maximal set of orthogonal measures.* Journal of Symbolic Logic, 75, 4, pp. 1403 - 1414.

[9] S. Hechler *short completed nested sequences in $\beta\mathbb{N}\backslash\mathbb{N}$ and small maximal almost-disjoint families.* General Topology and its Applications 2 (1972), 139 - 149, North-Holland Publishing Company.

[10] A. Kanamori *The higher infinite. Large cardinals in set theory from their beginnings.* Second edition. Paperback reprint of the 2003 edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2009.

[11] A. Kechris *Classical descriptive set theory.* Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995.

[12] A. R. D. Mathias *Happy families.* Ann. Math. Logic 12 (1977), no. 1, 59 - 111.

[13] D. Mejia, *Matrix iterations and Cichons diagram.* Arch. Math. Logic, 52(3-4), 261 278, 2013.

[14] A. Miller *Descriptive set theory and forcing. How to prove theorems about Borel sets the hard way.* Lecture Notes in Logic, 4. Springer-Verlag, Berlin, 1995.

[15] A. Miller *Infinite combinatorics and definability* Annals of Pure and Applied Logic, vol. 41 (1989), 179 203.

Kurt Gödel Research Center, University of Vienna, Währinger Strasse 25, 1090 Vienna, Austria
E-mail address: vfischer@logic.univie.ac.at

Kurt Gödel Research Center, University of Vienna, Währinger Strasse 25, 1090 Vienna, Austria
E-mail address: jonathan.schilhan@univie.ac.at