Towers, mad families, and unboundedness

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Abstract

We show that Hechler's forcings for adding a tower and for adding a mad family can be represented as finite support iterations of Mathias forcings with respect to filters (so Hechler's forcings are typically σ-centered), and that these filters are $\mathcal{B}$-Canjar for any unbounded family $\mathcal{B}$ of the ground model. In particular, Hechler's forcings preserve the unboundedness of any unbounded scale of the ground model. Moreover, we show that $\mathfrak{b} = \omega_1$ in every extension by Hechler's forcings.

Contents

1 Introduction

In this paper, we analyze Hechler's forcings from [Hec72] for adding a tower (see Section 3) and for adding a mad family (see Section 4), after giving some preliminaries on $\mathcal{B}$-Canjar filters in Section 2.

The forcings consist of finite conditions approximating the generic tower and the generic mad family, respectively. We first show that the forcing for adding a tower can be represented as a finite support iteration, where each iterand adds a single real to the tower (which diagonalizes the initial part of the tower). In fact, each such iterand is equivalent to Mathias forcing with respect to the filter generated by this initial part of the tower (see Lemma 3.3). For the forcing adding a mad family, the situation is analogous (see Lemma 4.5): in this case, the filter is generated by the complements of the elements of the initial part of the mad family. It follows from these representations that Hechler's forcings are typically σ-centered (see Corollary 3.4 and Corollary 4.6).

The main results of this paper show that Hechler's forcings preserve the unboundedness of any unbounded family (with a certain closure property) of the ground model (see Theorem 3.5 and Theorem 4.7); in particular, any unbounded ground model scale is preserved. We actually prove that, for a given unbounded family $\mathcal{B}$ of the ground model, all the filters which are involved in the representation of Hechler's forcings are $\mathcal{B}$-Canjar, i.e., the corresponding Mathias forcings preserve the unboundedness of $\mathcal{B}$. To verify $\mathcal{B}$-Canjarness, we use a combinatorial characterization from [GHMC14] (see Theroem 2.3), together with a genericity argument. In Section 5, we finally conclude that $\mathfrak{b} = \omega_1$ holds true in every extension by one of Hechler's forcings, using that they can be decomposed into a forcing which adds an unbounded family of size ω_1 and a forcing which preserves the unboundedness of this family (see Corollary 5.1 and Corollary 5.2).

In [FKW], the authors of this paper define a forcing which adds a distributivity matrix of regular height λ, i.e., a refining system of mad families of height λ without common refinement. There is always a distributivity matrix of height $\mathfrak{h}$ (which is the minimal possible height), where $\mathfrak{h}$ is the well-known distributivity number. In order to get a model with a distributivity matrix of regular height $\lambda > \mathfrak{h}$, it is shown that the forcing to add the distributivity matrix keeps the bounding number $\mathfrak{b}$ (and hence $\mathfrak{h}$) small: to this end, the forcing is represented as an iteration of Mathias forcings with respect to filters, which are shown to be $\mathcal{B}$-Canjar, where

$\mathcal{B}$ is the family of ground model reals; this ensures that $\mathcal{B}$ is unbounded in the final model, witnessing that $\mathfrak{b}$ is small.

Since a distributivity matrix of height λ consists of mad families as well as (along the branches of its corresponding tree) towers of length λ, the forcing used in [FKW] is an elaborate combination of Hechler's forcings to add a mad family and a tower, respectively. The proof that the forcing from [FKW] preserves the unboundedness of the ground model reals is a more complicated version of the proofs given in this paper.

2 $\mathcal{B}$-Canjar filters

In this section, we will give the neccessary preliminaries about $\mathcal{B}$-Canjar filters and the preservation of unboundedness.

Definition 2.1. Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a filter containing the Frechét filter. *Mathias forcing with respect to* $\mathcal{F}$ (denoted by $\mathbb{M}(\mathcal{F})$) is the set of pairs (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$, where the order is defined as follows: $(t, B) \leq (s, A)$ if

1. $t \trianglerighteq s$, i.e., t extends s
2. $B \subseteq A$
3. for each $n \geq |s|$, if $t(n) = 1$, then $n \in A$.

Note that $\mathbb{M}(\mathcal{F})$ is σ-centered: for $s \in 2^{<\omega}$, the set $\{(s, A) \mid A \in \mathcal{F}\}$ is clearly centered (i.e., finitely many conditions have a common lower bound). Also note that Mathias forcing with respect to a countably generated filter has a countable dense subset, and therefore is forcing equivalent to Cohen forcing $\mathbb{C}$.

For $f, g \in \omega^\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We say that $\mathcal{B} \subseteq \omega^\omega$ is an *unbounded family*, if there exists no $g \in \omega^\omega$ with $f \leq^* g$ for all $f \in \mathcal{B}$. The *bounding number* $\mathfrak{b}$ is the smallest size of an unbounded family in ω^ω.

A filter $\mathcal{F}$ is Canjar if $\mathbb{M}(\mathcal{F})$ does not add a dominating real over the ground model (i.e., the ground model reals remain unbounded). We are interested in the following generalization of Canjarness:

Definition 2.2. Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. A filter $\mathcal{F}$ on ω is $\mathcal{B}$-*Canjar* if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$ (i.e., $\mathcal{B}$ is still unbounded in the extension by $\mathbb{M}(\mathcal{F})$).

2.1 A combinatorial characterization of $\mathcal{B}$-Canjarness

Later, we will prove that certain filters are $\mathcal{B}$-Canjar; for that, we use the following combinatorial characterization of $\mathcal{B}$-Canjarness by Guzmán-Hrušák-Martínez [GHMC14]. This characterization generalizes a characterization of Canjarness by Hrušák-Minami [HM14].

Let $\mathcal{F}$ be a filter on ω; recall that a set $X \subseteq [\omega]^{<\omega}$ is in $(\mathcal{F}^{<\omega})^+$ if and only if for each $A \in \mathcal{F}$ there is an $s \in X$ with $s \subseteq A$. Note that if $\mathcal{G} \subseteq \mathcal{F}$ are filters and $X \in (\mathcal{F}^{<\omega})^+$, then $X \in (\mathcal{G}^{<\omega})^+$.

Given $\bar{X} = \langle X_n \mid n \in \omega \rangle$ (with $X_n \subseteq [\omega]^{<\omega}$ for each $n \in \omega$), and $f \in \omega^\omega$, let

$$\bar{X}_f = \bigcup_{n\in\omega} (X_n \cap \mathcal{P}(f(n))).$$

Theorem 2.3. *Let $\mathcal{B} \subseteq \omega^\omega$ be an unbounded family. A filter $\mathcal{F}$ on ω is $\mathcal{B}$-Canjar if and only if the following holds: for each sequence $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$, there exists an $f \in \mathcal{B}$ such that $\bar{X}_f \in (\mathcal{F}^{<\omega})^+$.*

Proof. See [GHMC14, Proposition 1]. □

It is well-known that Cohen forcing $\mathbb{C}$ preserves[1] the unboundedness of every unbounded family. As mentioned above, Mathias forcing with respect to a countably generated filter is forcing equivalent to $\mathbb{C}$, and hence any countably generated filter is $\mathcal{B}$-Canjar for every unbounded family $\mathcal{B}$. To illustrate the characterization of $\mathcal{B}$-Canjarness from Theorem 2.3, we also want to provide the following easy combinatorial proof of this fact:

Lemma 2.4. *Let $\mathcal{B}$ be an unbounded family. Then every countably generated filter is $\mathcal{B}$-Canjar.*

Proof. Let $\mathcal{F}$ be a filter generated by $\{a_n \mid n < \omega\}$, i.e., $A \in \mathcal{F}$ if and only if $a_n \subseteq A$ for some $n \in \omega$. Let $\bar{X} = \langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$. For every $n \in \omega$, let $s_n \in X_n$ with $s_n \subseteq \bigcap_{k<n} a_k$ (such s_n exist since $X_n \in (\mathcal{F}^{<\omega})^+$). Let $g \in \omega^\omega$ be such that $g(n) = \max(s_n)$ for every $n \in \omega$. Since $\mathcal{B}$ is unbounded, we can pick $f \in \mathcal{B}$ such that $f(n) > g(n)$ for infinitely many n. It is easy to check that $s_n \in \bar{X}_f$ for infinitely many n, and this implies that $\bar{X}_f \in (\mathcal{F}^{<\omega})^+$, as desired. □

Later, we will actually use the following lemma (which is again based on the characterization from the above Theorem 2.3) to show that a filter is $\mathcal{B}$-Canjar.

[1] In fact, $\mathbb{C}$ is almost bounding.

Lemma 2.5. *Let $V \subseteq W$ be models of ZFC, and assume that $\mathcal{B} \subseteq \omega^\omega \cap V$ is unbounded in W, and that $\mathcal{F} \in W$ is a filter on ω. Moreover, assume the following: for each sequence $\langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$ there exists a sequence $\langle s_n \mid n \in \omega \rangle$, as well as a model V' with $V \subseteq V' \subseteq W$ such that*

1. *$\langle s_n \mid n \in \omega \rangle \in V'$,*
2. *$s_n \in X_n$ for each $n \in \omega$,*
3. *for each $D \in [\omega]^\omega \cap V'$ and for each $A \in \mathcal{F}$, there exists $n \in D$ such that $s_n \subseteq A$.*

Then $\mathcal{F}$ is $\mathcal{B}$-Canjar (in W).

Proof. We want to show that $\mathcal{F}$ is $\mathcal{B}$-Canjar by proving its characterization given by Theorem 2.3.

So suppose a sequence $\langle X_n \mid n \in \omega \rangle \subseteq (\mathcal{F}^{<\omega})^+$ is given. By the hypothesis of the lemma, we can fix $\{s_n \mid n \in \omega\}$ and V' satisfying (1)–(3).

Due to (1), there is $g \in V'$ such that $s_n \subseteq g(n)$ for each $n \in \omega$. Since $\mathcal{B}$ is unbounded in W, there is an $f \in \mathcal{B}$ such that $g \not\geq^* f$ (i.e., if $g(n) < f(n)$ for infinitely many $n \in \omega$); to finish the proof, we want to show that

$$\bar{X}_f = \bigcup_{n \in \omega} (X_n \cap \mathcal{P}(f(n)))$$

is in $(\mathcal{F}^{<\omega})^+$; so fix $A \in \mathcal{F}$; we will find $s \in \bar{X}_f$ with $s \subseteq A$.

Note that both f (which is actually in V) and g are in V', so there is an infinite set $D \in V'$ such that $g(n) \leq f(n)$ for each $n \in D$. Now use (3) to obtain an $n \in D$ with $s_n \subseteq A$; observe that $s_n \in X_n$ by (2), and $s_n \subseteq g(n) \leq f(n)$, hence $s_n \in \bar{X}_f$, as desired. □

2.2 Preservation of unboundedness at limits

We will also use the following theorem by[2]Judah-Shelah [JS90] about preservation of unboundedness in finite support iterations:

Theorem 2.6. *Suppose $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta\}$ is a finite support iteration of c.c.c. partial orders of limit length δ, and $\mathcal{B} \subseteq \omega^\omega$ is unbounded and satisfies*

$$\forall \mathcal{A} \subseteq \mathcal{B}\ (|\mathcal{A}| = \aleph_0 \rightarrow \exists f \in \mathcal{B}\ \forall g \in \mathcal{A}\ g \leq^* f); \tag{1}$$

[2]In fact, [JS90, Theorem 2.2] is a much more general version than the one presented here.

moreover, suppose that

$$\forall \alpha < \delta \Vdash_{\mathbb{P}_\alpha} \text{“}\mathcal{B} \text{ is an unbounded family”}.$$

Then $\Vdash_{\mathbb{P}_\delta}$ *“$\mathcal{B}$ is an unbounded family”.*

Proof. See [Fis08, Theorem 3.5.2]. □

3 Hechler's tower forcing

In this section, we analyze Hechler's forcing from [Hec72] to add a tower. First, we give some basic definitions.

For $a, b \in [\omega]^\omega$, we say that $b \subseteq^* a$ if $b \setminus a$ is finite, i.e., $\subseteq^*$ denotes almost-inclusion. For a sequence $\langle a_\xi \mid \xi < \delta\rangle \subseteq [\omega]^\omega$, we say that $b \in [\omega]^\omega$ is a *pseudo-intersection* of $\langle a_\xi \mid \xi < \delta\rangle$ if $b \subseteq^* a_\xi$ for each $\xi < \delta$. We say that $\langle a_\xi \mid \xi < \delta\rangle$ is a *tower of length* δ if $a_\eta \subseteq^* a_\xi$ for any $\eta > \xi$, and it does not have an infinite pseudo-intersection.

The definition of the forcing we are giving here is not exactly as in [Hec72], but it is easy to see that it is equivalent. Let λ be a regular uncountable cardinal.

Definition 3.1. $\mathbb{TOW}_\lambda$ is defined as follows: $p \in \mathbb{TOW}_\lambda$ if p is a function with finite domain, $\mathrm{dom}(p) \subseteq \lambda$, and for each $\alpha \in \mathrm{dom}(p)$, we have

$$p(\alpha) = (s^p_\alpha, f^p_\alpha) = (s_\alpha, f_\alpha),$$

where

1. $s_\alpha \in 2^{<\omega}$,
2. for each $\beta \in \mathrm{dom}(p)$ with $\beta < \alpha$, $|s_\beta| \geq |s_\alpha|$,
3. $\mathrm{dom}(f_\alpha) \subseteq \mathrm{dom}(p) \cap \alpha$,
4. $f_\alpha : \mathrm{dom}(f_\alpha) \to \omega$,
5. whenever $\beta \in \mathrm{dom}(f_\alpha)$, and $n \in \omega$ with $n \in \mathrm{dom}(s_\beta) \cap \mathrm{dom}(s_\alpha)$ and $n \geq f_\alpha(\beta)$, we have
$$s_\beta(n) = 0 \to s_\alpha(n) = 0.$$

The order on $\mathbb{TOW}_\lambda$ is defined as follows: $q \leq p$ (“q is stronger than p”) if

1. $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
2. and for each $\alpha \in \mathrm{dom}(p)$, we have
   (a) $s^p_\alpha \trianglelefteq s^q_\alpha$,
   (b) $\mathrm{dom}(f^p_\alpha) \subseteq \mathrm{dom}(f^q_\alpha)$ and $f^p_\alpha(\beta) \geq f^q_\alpha(\beta)$ for each $\beta \in \mathrm{dom}(f^p_\alpha)$,

Given a generic filter G for $\mathbb{TOW}_\lambda$, we define, for each $\alpha < \lambda$,

$$a_\alpha := \bigcup \{s^p_\alpha \mid p \in G \wedge \alpha \in \mathrm{dom}(p)\}.$$

This completes the definition of the forcing.

The generic object $\langle a_\alpha \mid \alpha < \lambda \rangle$ added by $\mathbb{TOW}_\lambda$ is a tower of length λ.

3.1 Complete subforcings

For any $\alpha \leq \lambda$, let

$$\mathbb{TOW}_\alpha = \{p \in \mathbb{TOW}_\lambda \mid \mathrm{dom}(p) \subseteq \alpha\}.$$

It is quite easy to verify that these subforcings are complete in each other:

Lemma 3.2. *Let $\beta < \alpha \leq \lambda$. Then $\mathbb{TOW}_\beta$ is a complete subforcing of $\mathbb{TOW}_\alpha$.*

Proof. We first show that $\mathbb{TOW}_\beta \subseteq_{ic} \mathbb{TOW}_\alpha$: Let $p_0, p_1 \in \mathbb{TOW}_\beta$ and $q \in \mathbb{TOW}_\alpha$ with $q \leq p_0, p_1$. We have to show that there exists a condition $q' \in \mathbb{TOW}_\beta$ with $q' \leq p_0, p_1$. Let $q' := q \restriction \beta$. It is very easy to check that q' is as we wanted.

Let $p \in \mathbb{TOW}_\alpha$. We want to define a reduction of p to $\mathbb{TOW}_\beta$. Let $\mathsf{RED}(p) := p \restriction \beta$. Let $q \in \mathbb{TOW}_\beta$ with $q \leq \mathsf{RED}(p)$. We have to show that q is compatible with p. To show this, we define a witness r as follows: $\mathrm{dom}(r) := \mathrm{dom}(p) \cup \mathrm{dom}(q)$. For $\gamma \in \mathrm{dom}(q)$ let $r(\gamma) := q(\gamma)$ and for $\gamma \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$, let $r(\gamma) := p(\gamma)$.

First we check that r is a condition:

It is very easy to check that s^r_γ and f^r_γ are well-defined with the right domains and ranges for all $\gamma \in \mathrm{dom}(r)$.

Assume $\gamma' \in \mathrm{dom}(f^r_\gamma)$, $m \geq f^r_\gamma(\gamma')$ and $s^r_{\gamma'}(m) = 0$. We have to show that $s^r_\gamma(m) = 0$, if it is defined. We have three cases, depending on where γ and γ' are. *Case 1:* $\gamma, \gamma' \in \mathrm{dom}(q)$. So the requirement follows because q is a condition. *Case 2:* $\gamma, \gamma' \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$. In this case the requirement holds, because p

is a condition. *Case 3:* $\gamma \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ and $\gamma' \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. So $\gamma' \in \mathrm{dom}(f^p_\gamma) \setminus \mathrm{dom}(f^q_\gamma)$. Hence for $m < |s^p_\gamma|$ this holds because that depends just on p. For $m \geq |s^p_\gamma| = |s^r_\gamma|$, $s^r_\gamma(m)$ is not defined, and we have nothing to show. Other combinations are not possible, because $\gamma' \in \mathrm{dom}(f^r_\gamma)$.

If $\gamma \in \mathrm{dom}(q) \cap \mathrm{dom}(p)$ then $s^p_\gamma = s^{p'}_\gamma \subseteq s^q_\gamma = s^r_\gamma$, $\mathrm{dom}(f^p_\gamma) \subseteq \mathrm{dom}(f^q_\gamma) = \mathrm{dom}(f^r_\gamma)$ and for $\gamma' \in \mathrm{dom}(f^p_\gamma) \cap \mathrm{dom}(f^q_\gamma)$, $f^r_\gamma(\gamma') = f^q_\gamma(\gamma') \leq f^p_\gamma(\gamma')$, so r is a condition which extends both q and p. □

3.2 Iteration via filtered Mathias forcings

For $\alpha < \lambda$, $\mathbb{TOW}_\alpha$ is a complete subforcing of $\mathbb{TOW}_{\alpha+1}$ by Lemma 3.2, so we can form the quotient $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$. Moreover, because conditions in $\mathbb{TOW}_\lambda$ have finite domain,

$$\mathbb{TOW}_\alpha = \bigcup_{\delta<\alpha} \mathbb{TOW}_\delta$$

for each limit ordinal $\alpha \leq \lambda$; in other words, $\mathbb{TOW}_\alpha$ is the direct limit of the forcings $\mathbb{TOW}_\delta$ for $\delta < \alpha$. So $\mathbb{TOW}_\lambda$ is forcing equivalent to the finite support iteration of the quotients $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$ for $\alpha < \lambda$.

Recall that $\mathbb{M}(\mathcal{F})$ denotes Mathias forcing with respect to the filter $\mathcal{F}$ (see Definition 2.1). We are now going to show that $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha)$ for a filter $\mathcal{F}_\alpha$. Work in an extension by $\mathbb{TOW}_\alpha$, and note that, for each $\beta < \alpha$, a set a_β has been added by $\mathbb{TOW}_\alpha$. Let

$$\mathcal{F}_\alpha := \langle \{a_\beta \mid \beta < \alpha\} \rangle_{\text{Frechét}},$$

i.e., $\mathcal{F}_\alpha$ is the filter generated by (the Frechét filter and) the $\subseteq^*$-decreasing sequence $\{a_\beta \mid \beta < \alpha\}$ added by $\mathbb{TOW}_\alpha$.

The quotient $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$ adds the set a_α. The following lemma will provide a dense embedding from $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$ to $\mathbb{M}(\mathcal{F}_\alpha)$ which preserves (the finite appoximations of) the generic real a_α. Therefore, a_α is also the generic real for $\mathbb{M}(\mathcal{F}_\alpha)$. Recall that the generic real for $\mathbb{M}(\mathcal{F})$ is a pseudo-intersection of $\mathcal{F}$, and the definition of $\mathcal{F}_\alpha$ ensures that a pseudo-intersection of it is almost contained in a_β for each $\beta < \alpha$, as it is the case for the real a_α.

Lemma 3.3. $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$ *is densely embeddable into* $\mathbb{M}(\mathcal{F}_\alpha)$.

Proof. We work in a fixed extension by $\mathbb{TOW}_\alpha$. The embedding ι is defined as follows: $\iota(p) := (s^p_{\alpha+1}, A)$ where

$$A := \bigcap_{\beta\in\mathrm{dom}(f_{\alpha+1})} (a_\beta \cup f_{\alpha+1}(\beta)) \setminus |s^p_{\alpha+1}|.$$

To see that it is a dense embedding, we have to check the following conditions:

1. (Density) For every condition $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha)$ exists a condition p such that $\iota(p) \leq (s, A)$.

2. (Incompatibility preserving) If p and p' are incompatible, then so are $\iota(p)$ and $\iota(p')$.

3. (Order preserving) If $p' \leq p$ then $\iota(p') \leq \iota(p)$.

To show (1): Let $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha)$. Since $A \in \mathcal{F}_\alpha$, there exist finitely many $\{\beta_j \mid j < l\}$ and $N \in \omega$ such that $\bigcap_{j<l} a_{\beta_j} \setminus N \subseteq A$. Extend s with 0's to $s_{\alpha+1}$ such that $|s_{\alpha+1}| = \max(|s|, N)$ and let $\operatorname{dom}(f_{\alpha+1}) := \{\beta_j \mid j < l\}$ and $f_{\alpha+1}(\beta_j) := |s_{\alpha+1}|$ for every j. Let $p := \{(\alpha + 1, (s_{\alpha+1}, f_{\alpha+1}))\} \cup \{(\beta, (\langle\rangle, \emptyset, \emptyset)) \mid \beta \in \operatorname{dom}(f_{\alpha+1})\}$.

To see, that p is in the quotient, let $q \in G$ arbitrary, it is easy to check, that $q \cup \{(\beta, (s_\beta, f_\beta)) \mid \beta \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)\} \leq p, q$.

$\iota(p) = (s_{\alpha+1}, A')$ where

$$A' = \bigcap_{\beta \in \operatorname{dom}(f_{\alpha+1})} (a_\beta \cup f_{\alpha+1}(\beta)) \setminus |s_{\alpha+1}|.$$

It follows that

$$A' \overset{(*)}{=} \bigcap_{\beta \in \operatorname{dom}(f_{\alpha+1})} a_\beta \setminus |s_{\alpha+1}| \subseteq \bigcap_{j<l} a_{\beta_j} \setminus N \subseteq A$$

(where (*) holds because $|s_{\alpha+1}| \geq f_{\alpha+1}(\beta)$ for every β in their domain). Therefore $s_{\alpha+1} \trianglerighteq s$, $A' \subseteq A$, and $s_{\alpha+1}(n) = 0$ for all $n \geq |s|$. So $\iota(p) = (s_{\alpha+1}, A') \leq (s, A)$.

We prove (2) by showing the contrapositive: Assume $\iota(p) = (s, A)$ and $\iota(p') = (s', A')$ are compatible. Let (t, B) be a witness. Define q as follows: $\operatorname{dom}(q) := \operatorname{dom}(p) \cup \operatorname{dom}(p')$, for every $\beta \in \operatorname{dom}(q)$ let $s^q_\beta := s^p_\beta \cup s^{p'}_\beta$, $\operatorname{dom}(f^q_\beta) := \operatorname{dom}(f^p_\beta) \cup \operatorname{dom}(f^{p'}_\beta)$ and for $\rho \in \operatorname{dom}(f^q_\beta)$ let $f^q_\beta(\rho) = \min(f^p_\beta(\rho), f^{p'}_\beta(\rho))$. It is easy to check, that q is a condition in the quotient and $q \leq p, p'$.

To show (3): Let $p' \leq p$. By definition that means: $|s^{p'}_{\alpha+1}| \geq |s^p_{\alpha+1}|$ and $\operatorname{dom}(f^{p'}_{\alpha+1}) \supseteq \operatorname{dom}(f^p_{\alpha+1})$, and $f^{p'}_{\alpha+1}(\beta) \leq f^p_{\alpha+1}(\beta)$ for $\beta \in \operatorname{dom}(f^p_{\alpha+1})$; so

$$A' := \bigcap_{\beta \in \operatorname{dom}(f^{p'}_{\alpha+1})} (a_\beta \cup f^{p'}_{\alpha+1}(\beta)) \setminus |s^{p'}_{\alpha+1}| \subseteq \bigcap_{\beta \in \operatorname{dom}(f^p_{\alpha+1})} (a_\beta \cup f^p_{\alpha+1}(\beta)) \setminus |s^p_{\alpha+1}| =: A.$$

Let $n \geq |s^p_{\alpha+1}|$ and $s^{p'}_{\alpha+1}(n) = 1$. We have to show that $n \in A$; fix $\beta \in \operatorname{dom}(f^p_{\alpha+1})$ and show that $n \in a_\beta \cup f^p_{\alpha+1}(\beta)$. If $n < f^p_{\alpha+1}(\beta)$, this is clear. If $n \geq f^p_{\alpha+1}(\beta)$, we know that $s^{p'}_{\alpha+1}$ respects $f^p_{\alpha+1}$, and so $n \in a_\beta$. So in both cases, $n \in a_\beta \cup f^p_{\alpha+1}(\beta)$. This shows that $(s^{p'}_{\alpha+1}, A') \leq (s^p_{\alpha+1}, A)$. □

As a side result, let us mention that Hechler's forcing for adding a tower is σ-centered:

Corollary 3.4. *If $\lambda \leq \mathfrak{c}$, then $\mathbb{TOW}_\lambda$ is σ-centered.*

Proof. Since Mathias forcing with respect to a filter is always σ-centered (see the remark after Definition 2.1) and $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$ is densely embeddable into such a forcing by the above lemma, also $\mathbb{TOW}_{\alpha+1}/\mathbb{TOW}_\alpha$ is σ-centered.

So $\mathbb{TOW}_\lambda$ is a finite support iteration of σ-centered forcings of length at most $\mathfrak{c}$. As a matter of fact, the finite support iteration of σ-centered forcings of length strictly less than $\mathfrak{c}^+$ is σ-centered (the result was mentioned without proof in [Tal94, proof of Lemma 2]; for a proof, see [Bla11] or [Gui19, Lemma 5.3.8]). □

3.3 The filters are $\mathcal{B}$-Canjar

Finally, we show that Hechler's forcing $\mathbb{TOW}_\lambda$ preserves the unboundedness of certain unbounded families $\mathcal{B}$. More precisely, let V be the ground model over which we force with $\mathbb{TOW}_\lambda$, and let $\mathcal{B} \in V$ be an unbounded family of reals satisfying the closure property (1) from Theorem 2.6, i.e.,

$$\forall \mathcal{A} \subseteq \mathcal{B}\ (|\mathcal{A}| = \aleph_0 \rightarrow \exists f \in \mathcal{B}\ \forall g \in \mathcal{A}\ g \leq^* f);$$

we want to show that $\mathcal{B}$ is still unbounded in the extension by $\mathbb{TOW}_\lambda$. Since there always exists an unbounded family $\mathcal{B}$ of size $\mathfrak{b}$ satisfying the above closure property, $\mathbb{TOW}_\lambda$ does not increase the bounding number $\mathfrak{b}$ (for more details, see Section 5; in fact, we argue there that we even get $\mathfrak{b} = \omega_1$ whenever we force with $\mathbb{TOW}_\lambda$).

In Section 3.2, we have defined filters $\mathcal{F}_\alpha$ for $\alpha < \lambda$ and have shown that $\mathbb{TOW}_\lambda$ is equivalent to the finite support iteration of the Mathias forcings $\mathbb{M}(\mathcal{F}_\alpha)$. So we can finish the proof by showing that the filters $\mathcal{F}_\alpha$ are $\mathcal{B}$-Canjar (and $\mathbb{M}(\mathcal{F}_\alpha)$ therefore preserves the unboundedness of $\mathcal{B}$), and using Theorem 2.6 at limits. In fact, we show the following:

Theorem 3.5. *$\mathbb{TOW}_\lambda$ preserves the unboundedness of $\mathcal{B}$. More precisely,*

1. *$\mathbb{TOW}_\alpha$ preserves the unboundedness of $\mathcal{B}$ for each $\alpha \leq \lambda$,*

2. *$\mathcal{F}_\alpha$ is $\mathcal{B}$-Canjar for each $\alpha < \lambda$.*

Proof. We prove (1) and (2) by (simultaneous) induction on $\alpha < \lambda$. So suppose that (1) and (2) holds for all $\alpha' < \alpha$.

Proof of (1):

In case $\alpha = \alpha' + 1$ is a successor ordinal, use the fact that (1) holds for α' by induction, so $\mathcal{B}$ is unbounded in the extension by $\mathbb{TOW}_{\alpha'}$; recall that, by Lemma 3.3, $\mathbb{TOW}_\alpha = \mathbb{TOW}_{\alpha'} * \mathbb{M}(\mathcal{F}_{\alpha'})$; since (2) holds for α' by induction, $\mathbb{M}(\mathcal{F}_{\alpha'})$ preserves the unboundedness of $\mathcal{B}$, hence the same is true for $\mathbb{TOW}_\alpha$, as desired.

In case α is a limit ordinal, we use the fact that $\mathbb{TOW}_\alpha$ is the finite support iteration of c.c.c. forcings, as well as that (1) holds for each $\alpha' < \alpha$; so we can apply Theorem 2.6 to conclude (1) for α.

Proof of (2):

In case $\mathrm{cf}(\alpha) \leq \omega$, just note that $\mathcal{F}_\alpha$ is countably generated; so, by Lemma 2.4, $\mathcal{F}_\alpha$ is $\mathcal{B}$-Canjar, as desired.

In case $\mathrm{cf}(\alpha) > \omega$, we proceed as follows (this is going to be the main technical part of the proof): in order to show that $\mathcal{F}_\alpha$ is $\mathcal{B}$-Canjar, it is sufficient to establish the hypothesis of Lemma 2.5.

Let W be the extension of V by $\mathbb{TOW}_\alpha$; note that $\mathcal{F}_\alpha$ which is generated by the Frechét filter and $\{a_\beta \mid \beta < \alpha\}$ lies in W. Now observe that we have already proven (1) for α (without having used (2) for α), i.e., we know that $\mathcal{B}$ is unbounded in W.

Now suppose that $\{X_n \mid n \in \omega\} \subseteq (\mathcal{F}_\alpha{}^{<\omega})^+$ is given. We will find $\{s_n \mid n \in \omega\}$ and V' with $V \subseteq V' \subseteq W$ such that Lemma 2.5(1)–(3) hold.

Since the X_n's are essentially reals, the forcing $\mathbb{TOW}_\alpha$ has the c.c.c., and $\mathrm{cf}(\alpha) > \omega$, we can fix $\gamma < \alpha$ such that $\{X_n \mid n \in \omega\}$ belongs to the extension of V by $\mathbb{TOW}_\gamma$; let V' be the extension by $\mathbb{TOW}_{\gamma+1}$; clearly, $V \subseteq V' \subseteq W$.

For each $n \in \omega$, we have $a_\gamma \setminus n \in \mathcal{F}_\alpha$ and $X_n \in (\mathcal{F}_\alpha{}^{<\omega})^+$; therefore, for each n, there exists an $s \in X_n$ such that $s \subseteq a_\gamma \setminus n$. The same holds in V' since $X_n \in V'$ for each n and $a_\gamma \in V'$. Since $\{X_n \mid n \in \omega\} \in V'$, we can pick a sequence $\{s_n \mid n \in \omega\} \in V'$ such that $s_n \in X_n$ and $s_n \subseteq a_\gamma \setminus n$ for every n.

It remains to show that Lemma 2.5(3) holds true. So fix $D \in [\omega]^\omega \cap V'$; we have to prove that each element of $\mathcal{F}_\alpha$ contains (as a subset) an s_n for some $n \in D$, i.e., that the following holds for each $\beta < \alpha$:

$$\forall k \in \omega \; \exists n \in D \;\; s_n \subseteq a_\beta \setminus k. \tag{2}$$

In case $\beta \leq \gamma$, this is easy: fix $k \in \omega$; recall that $a_\gamma \subseteq^* a_\beta$, so we can pick $n \geq k$ with $n \in D$ such that $a_\gamma \setminus n \subseteq a_\beta$; but then $s_n \subseteq a_\gamma \setminus n \subseteq a_\beta \setminus k$, as desired.

In case $\beta > \gamma$, we show (2) by induction on β: assume we have shown it for every $\beta' < \beta$; we will show it for β.

Fix $k \in \omega$, and work in the extension by $\mathbb{TOW}_\beta$ (note that D belongs to the extension by $\mathbb{TOW}_{\gamma+1}$, hence also to the extension by $\mathbb{TOW}_\beta$ due to $\beta \geq \gamma + 1$); observe that a_β is added in the step from β to $\beta + 1$, i.e., by the quotient forcing $\mathbb{TOW}_{\beta+1}/\mathbb{TOW}_\beta$ (which is equivalent to $\mathbb{M}(\mathcal{F}_\beta)$). We finish the proof by showing that the set

$$\{q \in \mathbb{TOW}_{\beta+1}/\mathbb{TOW}_\beta \mid \exists n \in D \ \ q \Vdash s_n \subseteq a_\beta \setminus k\}$$

is dense.

Let $p \in \mathbb{TOW}_{\beta+1}/\mathbb{TOW}_\beta$, so $p = (s, f)$ where f is a function with finite $\mathrm{dom}(f) \subseteq \beta$. Let $\beta' := \max(\mathrm{dom}(f))$, and note that $\beta' < \beta$. Moreover, let ℓ be the maximum of the errors between $a_{\beta'}$ and all the $a_{\beta''}$ with $\beta'' \in \mathrm{dom}(f)$, and let $L := \max(\ell, k, |s|)$. Use (2) for β' and L to pick $n \in D$ such that $s_n \subseteq a_{\beta'} \setminus L$; because L is above the errors between the lines, $s_n \subseteq a_{\beta''} \setminus L$ for each $\beta'' \in \mathrm{dom}(f)$. Now strengthen p by extending s first with 0's up to the beginning of s_n and then concatenate s_n (this is possible, because s_n is a subset of each $a_{\beta''}$ with $\beta'' \in \mathrm{dom}(f)$, i.e., in all the sets s has to become a subset of); now let s_q be this extension of s; then $q := (s_q, f)$ is a condition, $q \leq p$ and

$$q \Vdash s_n \subseteq a_\beta \setminus k,$$

as desired. □

4 Hechler's mad family forcing

In this section, we analyze Hechler's forcing from [Hec72] to add a mad family. Again, we start with some basic definitions.

For $a, b \in [\omega]^\omega$, we say that a and b are *almost disjoint* if $a \cap b$ is finite. Moreover, we say that $A \subseteq [\omega]^\omega$ is an *almost disjoint family* if a and a' are almost disjoint whenever $a, a' \in A$ with $a \neq a'$. An almost disjoint family A is *maximal* (called *mad family*) if for each $b \in [\omega]^\omega$ there exists $a \in A$ such that $|b \cap a| = \aleph_0$.

The definition of the forcing we are giving here is not exactly as in [Hec72], but it is easy to see that it is equivalent. Let λ be a regular uncountable cardinal.

Definition 4.1. $\mathbb{MAD}_\lambda$ is defined as follows: $p \in \mathbb{MAD}_\lambda$ if p is a function with finite domain, $\mathrm{dom}(p) \subseteq \lambda$, and for each $\alpha \in \mathrm{dom}(p)$, we have

$$p(\alpha) = (s^p_\alpha, h^p_\alpha) = (s_\alpha, h_\alpha),$$

where

1. $s_\alpha \in 2^{<\omega}$,
2. $\mathrm{dom}(h_\alpha) \subseteq \mathrm{dom}(p) \cap \alpha$,
3. $h_\alpha : \mathrm{dom}(h_\alpha) \to \omega$,
4. whenever $\beta \in \mathrm{dom}(h_\alpha)$, and $n \in \omega$ with $n \in \mathrm{dom}(s_\beta) \cap \mathrm{dom}(s_\alpha)$ and $n \geq h_\alpha(\beta)$, we have
$$s_\beta(n) = 0 \vee s_\alpha(n) = 0.$$

The order on $\mathbb{MAD}_\lambda$ is defined as follows: $q \leq p$ ("q is stronger than p") if

1. $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
2. and for each $\alpha \in \mathrm{dom}(p)$, we have
   (a) $s^p_\alpha \trianglelefteq s^q_\alpha$,
   (b) $\mathrm{dom}(h^p_\alpha) \subseteq \mathrm{dom}(h^q_\alpha)$ and $h^p_\alpha(\beta) \geq h^q_\alpha(\beta)$ for each $\beta \in \mathrm{dom}(h^p_\alpha)$.

Given a generic filter G for $\mathbb{MAD}_\lambda$, we define, for each $\alpha < \lambda$,

$$a_\alpha := \bigcup\{s^p_\alpha \mid p \in G \wedge \alpha \in \mathrm{dom}(p)\}.$$

This completes the definition of the forcing.

The generic object $\{a_\alpha \mid \alpha < \lambda\}$ added by $\mathbb{MAD}_\lambda$ is a mad family of size λ. For the proof we refer to [Hec72].

4.1 Complete subforcings

Let us start with a useful definition:

Definition 4.2. A condition $p \in \mathbb{MAD}_\lambda$ is called *full* if there exists an $N \in \omega$ such that for all $\alpha \in \mathrm{dom}(p)$

1. $|s^p_\alpha| = N$

2. $N > \max(\mathrm{rng}(h^p_\alpha))$.

3. $\forall \beta, \alpha \in \mathrm{dom}(p)$ with $\beta < \alpha$ it holds that $\beta \in \mathrm{dom}(h^p_\alpha)$.

The set of full conditions is dense:

Lemma 4.3. *For every condition $p \in \mathbb{Q}$ there exists a full condition q with $q \leq p$ and* $\mathrm{dom}(q) = \mathrm{dom}(p)$*. In particular the set of full conditions is dense in $\mathbb{MAD}_\lambda$.*

Proof. First extend p by defining $h_\alpha(\beta) := |s^p_\alpha|$ for every $\alpha, \beta \in \mathrm{dom}(p)$, with $\beta < \alpha$. It is easy to see, that this extension yields a condition which fulfills (3). Now let

$$N > \max(\mathrm{rng}(h^p_\alpha)), |s^p_\alpha|$$

for every $\alpha \in \mathrm{dom}(p)$. For every $\beta \in \mathrm{dom}(p)$ extend s^p_β with 0's to length N. It is easy to see that this is a condition and it is full. □

For any $C \subseteq \lambda$, let

$$\mathbb{MAD}_C = \{p \in \mathbb{MAD}_\lambda \mid \mathrm{dom}(p) \subseteq C\}.$$

In particular, for any $\alpha \leq \lambda$, we have $\mathbb{MAD}_\alpha = \{p \in \mathbb{MAD}_\lambda \mid \mathrm{dom}(p) \subseteq \alpha\}$.

Moreover, for $p \in \mathbb{MAD}_\lambda$, let $p \upharpoonright\upharpoonright C$ be the condition p' with $\mathrm{dom}(p') = \mathrm{dom}(p) \cap C$, and $s^{p'}_\alpha = s^p_\alpha$, and $h^{p'}_\alpha = h^p_\alpha \upharpoonright C$ for each $\alpha \in \mathrm{dom}(p')$. Clearly, $p \upharpoonright\upharpoonright C$ is a condition in $\mathbb{MAD}_C$. Note that if $C \subseteq \lambda$ is downward closed (i.e., if C is an ordinal), then $p \upharpoonright\upharpoonright C = p \upharpoonright C$.

Lemma 4.4. *Let $C \subseteq \alpha \leq \lambda$. Then $\mathbb{MAD}_C$ is a complete subforcing of $\mathbb{MAD}_\alpha$.*

Before proving the lemma, let us note that in Lemma 3.2 we only prove that $\mathbb{TOW}_\beta$ is a complete subforcing of $\mathbb{TOW}_\alpha$, whereas here we prove the more general version for arbitrary $C \subseteq \alpha$. For Section 4.2, we need again only the special case of $\beta < \alpha$; the more general version is needed in Section 4.3. In Section 3.3, when dealing with $\mathbb{TOW}_\lambda$, we do not need such a more general version, for the following reason: the filter $\mathcal{F}_{\gamma+1}$ is always countably generated (just because $\{a_\gamma \setminus n \mid n \in \omega\}$ is a basis, due to the fact that $a_\gamma \subseteq^* a_\beta$ for each $\beta < \gamma$), and so the analogue of the set $C \subseteq \alpha$ needed in Theorem 4.7 can be replaced by any upper bound which is a successor ordinal. This is not possible when dealing with $\mathbb{MAD}_\lambda$ since then $\mathcal{F}_\beta$ is never countably generated unless $\beta < \omega_1$.

Proof of Lemma 4.4. We first show that $\mathbb{MAD}_C \subseteq_{ic} \mathbb{MAD}_\alpha$: Let $p_0, p_1 \in \mathbb{MAD}_C$ and $q \in \mathbb{MAD}_\alpha$ with $q \leq p_0, p_1$. We have to show that there exists a condition $q' \in \mathbb{MAD}_C$ with $q' \leq p_0, p_1$. Let $q' := q \upharpoonright\upharpoonright C$. It is very easy to check that q' is as we wanted.

Let $p \in \mathbb{MAD}_\alpha$. We want to define a reduction of p to $\mathbb{MAD}_C$. Let $p' \leq p$ be a full condition as in Lemma 4.3. Let $\textsf{RED}(p) := p' \upharpoonright\upharpoonright C$.

Let $q \leq \textsf{RED}(p)$, $q \in \mathbb{MAD}_C$. We have to show that q is compatible with p. To show this, we define a witness r as follows: $\mathrm{dom}(r) := \mathrm{dom}(p') \cup \mathrm{dom}(q)$. For $\beta \in \mathrm{dom}(q)$ let $s^r_\beta := s^q_\beta$, and for $\beta \in \mathrm{dom}(q) \cap \mathrm{dom}(p')$ let $\mathrm{dom}(h^r_\beta) := \mathrm{dom}(h^q_\beta) \cup \mathrm{dom}(h^{p'}_\beta)$, $h^r_\beta(\beta') = \min(h^q_\beta(\beta'), h^{p'}_\beta(\beta'))$ for every $\beta' \in \mathrm{dom}(h^r_\beta)$, for $\beta \in \mathrm{dom}(q) \setminus \mathrm{dom}(p')$ let $h^r_\beta := h^q_\beta$. For $\beta \in \mathrm{dom}(p') \setminus \mathrm{dom}(q)$ let $s^r_\beta := s^{p'}_\beta$ and $h^r_\beta := h^{p'}_\beta$.

First we check that r is a condition:

It is very easy to check that s^r_β and h^r_β are well-defined with the right domains and ranges for all $\beta \in \mathrm{dom}(r)$.

Assume $\beta' \in \mathrm{dom}(h^r_\beta)$, $m \geq h^r_\beta(\beta')$ and $s^r_\beta(m) = 1$. We have to show that $s^r_{\beta'}(m) = 0$, if it is defined. We have three cases, depending on where β and β' are. *Case 1:* $\beta, \beta' \in \mathrm{dom}(q)$. So the requirement follows because q is a condition. *Case 2:* $\beta, \beta' \in \mathrm{dom}(p') \setminus \mathrm{dom}(q)$. In this case the requirement holds, because p' is a condition. *Case 3:* $\beta \in \mathrm{dom}(p') \setminus \mathrm{dom}(q)$ and $\beta' \in \mathrm{dom}(p') \cap \mathrm{dom}(q)$. So $\beta' \in \mathrm{dom}(h^{p'}_\beta) \setminus \mathrm{dom}(h^q_\beta)$. So for $m < |s^{p'}_\beta|$ this holds because that depends just on p'. For $m \geq |s^{p'}_\beta|$, only $s^r_{\beta'}(m)$ is defined, and we have nothing to show. Other combinations are not possible, because $\beta' \in \mathrm{dom}(h^r_\beta)$.

If $\beta \in \mathrm{dom}(q) \cap \mathrm{dom}(p')$ then $s^{p'}_\beta = s^{\textsf{RED}(p)}_\beta \subseteq s^q_\beta = s^r_\beta$, $\mathrm{dom}(h^{p'}_\beta), \mathrm{dom}(h^q_\beta) \subseteq \mathrm{dom}(h^r_\beta)$ and for $\beta' \in \mathrm{dom}(h^{p'}_\beta) \cap \mathrm{dom}(h^q_\beta)$, $h^r_\beta(\beta') \leq h^{p'}_\beta(\beta'), h^q_\beta(\beta')$, so r is a condition, which extends both q and p' (and therefore p). □

4.2 Iteration via filtered Mathias forcings

By Lemma 4.4, $\mathbb{MAD}_\beta$ is a complete subforcing of $\mathbb{MAD}_\alpha$ for each $\beta < \alpha \leq \lambda$. In particular, $\mathbb{MAD}_\alpha$ is a complete subforcing of $\mathbb{MAD}_{\alpha+1}$ for each $\alpha < \lambda$, so we can form the quotient $\mathbb{MAD}_{\alpha+1}/\mathbb{MAD}_\alpha$. Moreover, because conditions in $\mathbb{MAD}_\lambda$ have finite domain,

$$\mathbb{MAD}_\alpha = \bigcup_{\delta<\alpha} \mathbb{MAD}_\delta$$

for each limit ordinal $\alpha \leq \lambda$; in other words, $\mathbb{MAD}_\alpha$ is the direct limit of the forcings $\mathbb{MAD}_\delta$ for $\delta < \alpha$. So $\mathbb{MAD}_\lambda$ is forcing equivalent to the finite support iteration of the quotients $\mathbb{MAD}_{\alpha+1}/\mathbb{MAD}_\alpha$ for $\alpha < \lambda$.

Recall that $\mathbb{M}(\mathcal{F})$ denotes Mathias forcing with respect to the filter $\mathcal{F}$ (see Definition 2.1). We are now going to show that $\mathbb{MAD}_{\alpha+1}/\mathbb{MAD}_\alpha$ is forcing equivalent to $\mathbb{M}(\mathcal{F}_\alpha)$ for a filter $\mathcal{F}_\alpha$. Work in an extension by $\mathbb{MAD}_\alpha$, and note that, for each $\beta < \alpha$, a set a_β has been added by $\mathbb{MAD}_\alpha$. Let

$$\mathcal{F}_\alpha := \langle\{\omega \setminus a_\beta \mid \beta < \alpha\}\rangle_{\text{Frechét}},$$

i.e., $\mathcal{F}_\alpha$ is the filter generated by (the Frechét filter and) the complements of the members of the ad family $\{a_\beta \mid \beta < \alpha\}$ added by $\mathbb{MAD}_\alpha$.

The quotient $\mathbb{MAD}_{\alpha+1}/\mathbb{MAD}_\alpha$ adds the set a_α. The following lemma will provide a dense embedding from $\mathbb{MAD}_{\alpha+1}/\mathbb{MAD}_\alpha$ to $\mathbb{M}(\mathcal{F}_\alpha)$ which preserves (the finite appoximations of) the generic real a_α. Therefore, a_α is also the generic real for $\mathbb{M}(\mathcal{F}_\alpha)$. Recall that the generic real for $\mathbb{M}(\mathcal{F})$ is a pseudo-intersection of $\mathcal{F}$, and the definition of $\mathcal{F}_\alpha$ ensures that a pseudo-intersection of it is almost disjoint from a_β for each $\beta < \alpha$, as it is the case for the real a_α.

Lemma 4.5. *$\mathbb{MAD}_{\alpha+1}/\mathbb{MAD}_\alpha$ is densely embeddable into $\mathbb{M}(\mathcal{F}_\alpha)$.*

Proof. We work in a fixed extension by $\mathbb{MAD}_\alpha$. The embedding ι is defined as follows: $\iota(p) := (s^p_{\alpha+1}, A)$ where

$$A := \bigcap_{\beta \in \operatorname{dom}(h^p_{\alpha+1})} ((\omega \setminus a_\beta) \cup h(\beta)) \setminus |s|.$$

To see that it is a dense embedding, we have to check the following conditions:

1. For every condition $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha)$ exists a condition p such that $\iota(p) \leq (s, A)$.

2. If p and p' are incompatible, then so are $\iota(p)$ and $\iota(p')$.

3. If $p' \leq p$ then $\iota(p') \leq \iota(p)$.

To show (1): Let $(s, A) \in \mathbb{M}(\mathcal{F}_\alpha)$. Since $A \in \mathcal{F}_\alpha$, there exist finitely many $\{\beta_i \mid i < m\}$ and $N \in \omega$ such that $\bigcap_{i<m}(\omega \setminus a_{\beta_i}) \setminus N \subseteq A$. Extend s with 0's to $s_{\alpha+1}$ such that $|s_{\alpha+1}| = \max(|s|, N, \max\{h(\beta) \mid \beta \in \operatorname{dom}(h)\})$ and extend h with value $|s_{\alpha+1}|$ to $h_{\alpha+1}$ such that $\operatorname{dom}(h_{\alpha+1}) = \operatorname{dom}(h) \cup \{\beta_i \mid i < m\}$. Let $p := \{(\alpha + 1, (s_{\alpha+1}, h_{\alpha+1}))\} \cup \{(\beta, (\langle\rangle, \emptyset, \emptyset)) \mid \beta \in \operatorname{dom}(h_{\alpha+1})\}$.

$\iota(p) = ((s_{\alpha+1}, A'))$ where

$$A' = \bigcap_{\beta \in \mathrm{dom}(h_{\alpha+1})} ((\omega \setminus a_\beta) \cup h_{\alpha+1}(\beta)) \setminus |s_{\alpha+1}|.$$

It follows that $A' \overset{(*)}{=} \bigcap_{\beta \in \mathrm{dom}(h_{\alpha+1})} (\omega \setminus a_\beta) \setminus |s_{\alpha+1}| \subseteq \bigcap_{i<m} (\omega \setminus a_{\beta_i}) \setminus N \subseteq A$ (where (*) holds because $|s_{\alpha+1}| \geq h_{\alpha+1}(\beta)$ for every $\beta \in \mathrm{dom}(h_{\alpha+1})$). Therefore $s_{\alpha+1} \trianglerighteq s$, $A' \subseteq A$, and $s'(n) = 0$ for all $n \geq |s|$. So $\iota(p) = (s_{\alpha+1}, A') \leq (s, A)$.

We prove (2) by showing the contrapositive: Assume $\iota(p) = (s, A)$ and $\iota(p') = (s', A')$ are compatible. Let (t, B) be a witness. Define q as follows: $\mathrm{dom}(q) := \mathrm{dom}(p) \cup \mathrm{dom}(p')$, for every $\beta \in \mathrm{dom}(q)$ let $s^q_\beta := s^p_\beta \cup s^{p'}_\beta$, $\mathrm{dom}(h^q_\beta) := \mathrm{dom}(h^p_\beta) \cup \mathrm{dom}(h^{p'}_\beta)$ and for $\rho \in \mathrm{dom}(h^q_\beta)$ let $h^q_\beta(\rho) = \min(h^p_\beta(\rho), h^{p'}_\beta(\rho))$. It is easy to check, that q is a condition in the quotient and $q \leq p, p'$.

To show (3): Let $p' \leq p$. By definition it follows that $|s^{p'}_{\alpha+1}| \geq |s^p_{\alpha+1}|$ and $\mathrm{dom}(h^{p'}_{\alpha+1}) \supseteq \mathrm{dom}(h^p_{\alpha+1})$ and $h^{p'}_{\alpha+1}(\beta) \leq h^p_{\alpha+1}(\beta)$ for $\beta \in \mathrm{dom}(h^p_{\alpha+1})$; so

$$A' := \bigcap_{\beta \in \mathrm{dom}(h^{p'}_{\alpha+1})} ((\omega \setminus a_\beta) \cup h^{p'}_{\alpha+1}(\beta)) \setminus |s^{p'}_{\alpha+1}| \subseteq \bigcap_{\beta \in \mathrm{dom}(h^p_{\alpha+1})} ((\omega \setminus a_\beta) \cup h^p_{\alpha+1}(\beta)) \setminus |s| =: A.$$

Let $n \geq |s^p_{\alpha+1}|$ and $s^{p'}_{\alpha+1}(n) = 1$. We have to show that $n \in A$; fix $\beta \in \mathrm{dom}(h^p_{\alpha+1})$ and show that $n \in (\omega \setminus a_\beta) \cup h^p_{\alpha+1}(\beta)$. If $n < h^p_{\alpha+1}(\beta)$, this is clear. If $n \geq h^p_{\alpha+1}(\beta)$, we know that $s^{p'}_{\alpha+1}$ respects $h^p_{\alpha+1}$, and so $n \in \omega \setminus a_\beta$. So in both cases, $n \in (\omega \setminus a_\beta) \cup h^{p'(\beta)}_{\alpha+1}$. This shows that $(s^{p'}_{\alpha+1}, A') \leq (s, A)$. □

As a side result, let us mention that Hechler's forcing for adding a mad family is σ-centered:

Corollary 4.6. *If $\lambda \leq \mathfrak{c}$, then $\mathbb{MAD}_\lambda$ is σ-centered.*

Proof. The proof is completely analogous to the proof of Corollary 3.4. □

4.3 The filters are $\mathcal{B}$-Canjar

Finally, as we did in Section 3.3 for Hechler's tower forcing $\mathbb{TOW}_\lambda$, we show that Hechler's forcing $\mathbb{MAD}_\lambda$ preserves the unboundedness of certain unbounded families $\mathcal{B}$. More precisely, let V be the ground model over which we force with $\mathbb{MAD}_\lambda$, and let $\mathcal{B} \in V$ be an unbounded family of reals satisfying the closure property (1) from Theorem 2.6, i.e.,

$$\forall \mathcal{A} \subseteq \mathcal{B}\ (|\mathcal{A}| = \aleph_0 \to \exists f \in \mathcal{B}\ \forall g \in \mathcal{A}\ g \leq^* f);$$

we want to show that $\mathcal{B}$ is still unbounded in the extension by $\mathbb{MAD}_\lambda$. Since there always exists an unbounded family $\mathcal{B}$ of size $\mathfrak{b}$ satisfying the above closure property, $\mathbb{MAD}_\lambda$ does not increase the bounding number $\mathfrak{b}$ (for more details, see Section 5; in fact, we argue there that we even get $\mathfrak{b} = \omega_1$ whenever we force with $\mathbb{MAD}_\lambda$).

In Section 4.2, we have defined filters $\mathcal{F}_\alpha$ for $\alpha < \lambda$ and have shown that $\mathbb{MAD}_\lambda$ is equivalent to the finite support iteration of the Mathias forcings $\mathbb{M}(\mathcal{F}_\alpha)$. So we can finish the proof by showing that the filters $\mathcal{F}_\alpha$ are $\mathcal{B}$-Canjar (and $\mathbb{M}(\mathcal{F}_\alpha)$ therefore preserves the unboundedness of $\mathcal{B}$), and using Theorem 2.6 at limits. In fact, we show the following:

Theorem 4.7. *For every $\alpha \leq \lambda$, we have*

1. *$\mathbb{MAD}_\alpha$ preserves the unboundedness of $\mathcal{B}$,*
2. *$\mathcal{F}_\alpha$ is $\mathcal{B}$-Canjar.*

Proof. We prove (1) and (2) by (simultaneous) induction on $\alpha < \lambda$. So suppose that (1) and (2) holds for all $\alpha' < \alpha$.

Proof of (1):

In case $\alpha = \alpha' + 1$ is a successor ordinal, use the fact that (1) holds for α' by induction, so $\mathcal{B}$ is unbounded in the extension by $\mathbb{MAD}_{\alpha'}$; recall that, by Lemma 4.5, $\mathbb{MAD}_\alpha = \mathbb{MAD}_{\alpha'} * \mathbb{M}(\mathcal{F}_{\alpha'})$; since (2) holds for α' by induction, $\mathbb{M}(\mathcal{F}_{\alpha'})$ preserves the unboundedness of $\mathcal{B}$, hence the same is true for $\mathbb{MAD}_\alpha$, as desired.

In case α is a limit ordinal, we use the fact that $\mathbb{MAD}_\alpha$ is the finite support iteration of c.c.c. forcings, as well as that (1) holds for each $\alpha' < \alpha$; so we can apply Theorem 2.6 to conclude (1) for α.

Proof of (2):

We proceed as follows (this is going to be the main technical part of the proof): in order to show that $\mathcal{F}_\alpha$ is $\mathcal{B}$-Canjar, it is sufficient to establish the hypothesis of Lemma 2.5.

Let W be the extension of V by $\mathbb{MAD}_\alpha$; note that $\mathcal{F}_\alpha$ which is generated by the Frechét filter and $\{\omega \setminus a_\beta \mid \beta < \alpha\}$ lies in W. Now observe that we have already proven (1) for α (without having used (2) for α), i.e., we know that $\mathcal{B}$ is unbounded in W.

Now suppose that $\{X_n \mid n \in \omega\} \subseteq ({\mathcal{F}_\alpha}^{<\omega})^+$ is given. We will find $\{s_n \mid n \in \omega\}$ and V' with $V \subseteq V' \subseteq W$ such that Lemma 2.5(1)–(3) hold.

Since the X_n's are essentially reals, the forcing $\mathbb{MAD}_\alpha$ has the c.c.c., and the domains of its conditions are finite, we can pick a countable "support" $C \subseteq \alpha$, i.e., a set C such that $\{X_n \mid n \in \omega\}$ belongs to the extension by $\mathbb{MAD}_C$ (which is complete in $\mathbb{MAD}_\alpha$ by Lemma 4.4); enumerate C by $\{\gamma_\ell \mid \ell < \omega\}$ and let $c^\ell := \omega \setminus a_{\gamma_\ell}$ for each $\ell \in \omega$ (in case C is finite, let $c^\ell := \omega$ for any ℓ which has not been used – this is "necessary" if $\alpha < \omega$). Let V' be the extension by $\mathbb{MAD}_C$; clearly, $V \subseteq V' \subseteq W$.

For each $n \in \omega$, we have $\bigcap_{\ell \leq n} c^\ell \setminus n \in \mathcal{F}_\alpha$ and $X_n \in (\mathcal{F}_\alpha{}^{<\omega})^+$; therefore, for each n, there exists an $s \in X_n$ such that $s \subseteq \bigcap_{\ell \leq n} c^\ell \setminus n$. The same holds in V' since $X_n \in V'$ for each n and $c^\ell \in V'$ for each ℓ. Since $\{X_n \mid n \in \omega\} \in V'$ and $\{c^\ell \mid \ell \in \omega\} \in V'$, we can pick a sequence $\{s_n \mid n \in \omega\} \in V'$ such that $s_n \in X_n$ and $s_n \subseteq \bigcap_{\ell \leq n} c^\ell \setminus n$ for every n.

It remains to show that Lemma 2.5(3) holds true. So fix $D \in [\omega]^\omega \cap V'$; we have to prove that each element of $\mathcal{F}_\alpha$ contains (as a subset) an s_n for some $n \in D$, i.e., that the following holds for each finite sequence $\langle \beta_i \mid i < N \rangle \subseteq \alpha$:

$$\forall k \in \omega \; \exists n \in D \;\; s_n \subseteq \bigcap_{i<N} (\omega \setminus a_{\beta_i}) \setminus k. \tag{3}$$

We first observe that (3) holds in case that $\{\beta_i \mid i < N\} \subseteq C$: fix $k \in \omega$, and note that there is $m \in \omega$ such that for each $n \geq m$, we have

$$s_n \subseteq \bigcap_{\ell \leq n} c^\ell \setminus n \subseteq \bigcap_{i<N} (\omega \setminus a_{\beta_i}) \setminus k,$$

hence there is such an n in the infinite set D, as desired.

We now show (3) for arbitrary $\{\beta_i \mid i < N\} \subseteq \alpha$, using a genericity argument. Let $N_C := \{i \in N \mid \beta_i \in C\}$, and $N_{\alpha \setminus C} := \{i \in N \mid \beta_i \notin C\}$, so $N = N_C \cup N_{\alpha \setminus C}$.

Fix $k \in \omega$, and work in V', the extension by $\mathbb{MAD}_C$ (note that $D \in V'$); observe that the a_{β_i}'s for $i \in N_{\alpha \setminus C}$ are added by the quotient forcing $\mathbb{MAD}_\alpha / \mathbb{MAD}_C$. We finish the proof by showing that the set

$$\{q \in \mathbb{MAD}_\alpha / \mathbb{MAD}_C \mid \exists n \in D \;\; q \Vdash s_n \subseteq \bigcap_{i<N} (\omega \setminus a_{\beta_i}) \setminus k\}$$

is dense.

Let $p \in \mathbb{MAD}_\alpha / \mathbb{MAD}_C$; for $i \in N_{\alpha \setminus C}$, let $p(\beta_i) =: (s^{\beta_i}, h^{\beta_i})$. Let $L := \max(\{k\} \cup \{|s^{\beta_i}| \mid i \in N_{\alpha \setminus C}\})$. Since (3) holds for β_i's in C (as shown above), we can pick $n \in D$ such that

$$s_n \subseteq \bigcap_{i \in N_C} (\omega \setminus a_{\beta_i}) \setminus L.$$

Extend all s^{β_i} with $i \in N_{\alpha \setminus C}$ with 0's up to the maximum of s_n (recall that we can always extend with 0's, because this does not harm the almost disjointness); if we now extend p to q by extending the s^{β_i} with $i \in N_{\alpha \setminus C}$ in this way, we get that q forces 1's into $\omega \setminus a_{\beta_i}$ for all $i \in N_{\alpha \setminus C}$ at the "position" of s_n, and hence

$$q \Vdash s_n \subseteq \bigcap_{i \in N_{\alpha \setminus C}} (\omega \setminus a_{\beta_i}) \cap \bigcap_{i \in N_C} (\omega \setminus a_{\beta_i}) \setminus L \subseteq \bigcap_{i \in N} (\omega \setminus a_{\beta_i}) \setminus k,$$

as desired. □

5 Conclusion

In this last section, we present some facts about cardinal characteristics which easily follow from our analysis of $\mathbb{TOW}_\lambda$ and $\mathbb{MAD}_\lambda$.

It is easy to see that there exists always an unbounded scale of size $\mathfrak{b}$, i.e., an unbounded set $\mathcal{B} = \{f_i \mid i < \mathfrak{b}\}$ such that $f_i \leq^* f_j$ for $i \leq j$. Since $\mathfrak{b}$ is regular uncountable, such a scale always has the closure property (1) from Theorem 2.6. Therefore $\mathcal{B}$ remains unbounded in the extension by $\mathbb{TOW}_\lambda$ by Theorem 3.5. So if $V \models$ "$\mathfrak{b} = \kappa$", then $V^{\mathbb{TOW}_\lambda} \models$ "there exists an unbounded scale of size κ and there exists a tower of length λ". In particular, this implies that $V^{\mathbb{TOW}_\lambda} \models$ "$\mathfrak{b} \leq \kappa$". The same argument works for $\mathbb{MAD}_\lambda$, therefore $V^{\mathbb{MAD}_\lambda} \models$ "$\mathfrak{b} \leq \kappa$ and there exists an unbounded scale of size κ and a mad family of size λ".

Note that the above shows that $\mathfrak{b} = \omega_1$ in the extension by $\mathbb{TOW}_\lambda$ provided that $\mathfrak{b} = \omega_1$ holds true in the ground model. But in fact the following argument shows that no assumption about $\mathfrak{b}$ in the ground model is necessary for this conclusion. The forcing $\mathbb{TOW}_\lambda$ can be decomposed into $\mathbb{TOW}_{\omega_1} * (\mathbb{TOW}_\lambda / \mathbb{TOW}_{\omega_1})$. By Lemma 3.3, $\mathbb{TOW}_{\omega_1}$ is equivalent to an iteration of length ω_1 of Mathias forcings with respect to countably generated filters, therefore it is equivalent to the Cohen forcing which adds ω_1 many Cohen reals. Since these ω_1 many Cohen reals form an unbounded family, it follows that $V^{\mathbb{TOW}_{\omega_1}} \models$ "$\mathfrak{b} = \omega_1$". In $V^{\mathbb{TOW}_{\omega_1}}$, let $\mathcal{B}$ be an unbounded family of size ω_1 which has the closure property (1) from Theorem 2.6. The quotient $\mathbb{TOW}_\lambda / \mathbb{TOW}_{\omega_1}$ is equivalent to an iteration of Mathias forcings with respect to filters which are $\mathcal{B}$-Canjar (which follows as in the proof of Theorem 3.5), therefore $\mathcal{B}$ is unbounded in $V^{\mathbb{TOW}_\lambda}$, thus, using that $\mathfrak{t} \leq \mathfrak{b}$, we get the following:

Corollary 5.1. *Let λ be a regular uncountable cardinal. Then the following holds in $V^{\mathbb{TOW}_\lambda}$:*

1. $\mathfrak{t} = \mathfrak{b} = \omega_1$.

2. There exist towers[3] *of length ω_1 and of length λ.*

3. There exist unbounded scales of size ω_1 and of size $\mathfrak{b}^V$.

The analogous argument works for $\mathbb{MAD}_\lambda$, so we get the following:

Corollary 5.2. *Let λ be a regular uncountable cardinal. Then the following holds in $V^{\mathbb{MAD}_\lambda}$:*

1. $\mathfrak{t} = \mathfrak{b} = \omega_1$.

2. There exists a mad family of size λ.

3. There exist unbounded scales of size ω_1 and of size $\mathfrak{b}^V$.

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[3] Note that the generic object added by $\mathbb{TOW}_{\omega_1}$ is a tower of length ω_1 in $V^{\mathbb{TOW}_{\omega_1}}$, but it is clearly not a tower in $V^{\mathbb{TOW}_\lambda}$ any more.

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