Homework 1, due: Sep 22, 11:30 am

(1) Prove that the set of intervals $\{[a, b) : a, b \in \mathbb{R}\}$ generates the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$.

(2) Let X, Y be sets, $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ be σ-algebras. If $f : X \to Y$ is a function let $f^{-1}(\mathcal{B}) = \{f^{-1}(S) : S \in \mathcal{B}\}$ and $f(\mathcal{A}) = \{f(S) : S \in \mathcal{A}\}$ and $f^*\mathcal{A} = \{S : f^{-1}(S) \in \mathcal{A}\}$. Decide whether the following statements are true or false. Justify your answer!

(a) for every f the set $f^{-1}(\mathcal{B})$ is a σ-algebra on X,
(b) for every f the set $f(\mathcal{A})$ is a σ-algebra on Y,
(c) for every f the set $f^*\mathcal{A}$ is a σ-algebra on Y,
(d) for every surjective f the set $f(\mathcal{A})$ is a σ-algebra on Y.

Solution. First we show that for every collection of sets $B_0, B_1, \dots \subset Y$ we have $f^{-1}(\bigcup_{i\in\mathbb{N}} B_i) = \bigcup_{i\in\mathbb{N}} f^{-1}(B_i)$. Indeed, for every $x \in X$ we have

$$x \in \bigcup_{i\in\mathbb{N}} f^{-1}(B_i) \iff f(x) \in \bigcup_{i\in\mathbb{N}} B_i \iff x \in f^{-1}(\bigcup_{i\in\mathbb{N}} B_i).$$

A similar calculation shows that $f^{-1}(Y \setminus B_0) = X \setminus f^{-1}(B_0)$.

(a) Yes. Obviously $\emptyset \in f^{-1}(\mathcal{B})$, we check that $f^{-1}(\mathcal{B})$ is closed under countable unions. Let $A_0, A_1, \dots$ a countable collection of sets in $f^{-1}(\mathcal{B})$. By definition there exist $B_0, B_1, \dots$ with $f^{-1}(B_i) = A_i$. Then, by the above observation we have $\bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i\in\mathbb{N}} f^{-1}(B_i) = f^{-1}(\bigcup_{i\in\mathbb{N}} B_i)$. But, as $\mathcal{B}$ is a σ algebra we get that $\bigcup_{i\in\mathbb{N}} B_i \in \mathcal{B}$, which shows that $f^{-1}(\mathcal{B})$ is closed under countable unions. The fact that it is closed under complements can be shown similarly.

(b) No. Let $f : \mathbb{R} \to \mathbb{R}$ be a constant map and $\mathcal{A} = \mathcal{B} = \mathcal{B}_{\mathbb{R}}$. Then $Y \notin f(\mathcal{A})$, so this family is not a σ-algebra (actually, any non-surjective function would be suitable).

(c) Yes. Again, $\emptyset \in f^*\mathcal{A}$, so we check that $f^*\mathcal{A}$ is closed under countable unions. Let $B_0, B_1, \dots f^*(\mathcal{A})$. Then, by definition $f^{-1}(B_i) \in \mathcal{A}$ for every $i \in \mathbb{N}$. Thus, using the above observation we get $f^{-1}(\bigcup_{i\in\mathbb{N}} B_i) = \bigcup_{i\in\mathbb{N}} f^{-1}(B_i)$ and the latter set is the element of $\mathcal{A}$, which shows that $\bigcup_{i\in\mathbb{N}} B_i \in f^*\mathcal{A}$ holds. The case of the complements can be proved similarly.

(d) No. Let $X = [-1, 1]$, $Y = [0, 1]$, $f(x) = x^2$, and $\mathcal{A} = \{\emptyset, [-1, 1], [0, 1], [-1, 0)\}$. Then f is surjective, $\mathcal{A}$ is a σ-algebra and $f(\mathcal{A}) = \{\emptyset, [0, 1], (0, 1]\}$, which is not a σ algebra, as $[0, 1] \setminus (0, 1] = \{0\} \notin f(\mathcal{A})$.

(3) Show that every closed subset of the reals is G_δ.

Solution. Let $F \subset \mathbb{R}$ be a closed set. Consider the set $G = \mathbb{R} \setminus F$. It is enough to express G as a countable union of closed sets (in other words, show that G is F_σ), since then F will be the intersection of the complements of these sets.

Clearly, G is open hence it can be expressed as a countable union of bounded open intervals (a_i, b_i) for $i \in \mathbb{N}$. For every i we can find a natural number N_i so large that $a_i + \frac{1}{N_i} < b_i - \frac{1}{N_i}$. It is not hard to see that for each i we have

$$(a_i, b_i) = \bigcup_{n \geq N_i} [a_i + \frac{1}{n}, b_i + \frac{1}{n}].$$

Thus, $G = \bigcup_{i\in\mathbb{N}}(a_i, b_i) = \bigcup_{i\in\mathbb{N}} \bigcup_{n\geq N_i} [a_i + \frac{1}{n}, b_i + \frac{1}{n}]$, showing that G is indeed an F_σ set.

(4) Prove that the cardinality of a σ-algebra is either finite or at least $\mathfrak{c}$.
Solution. Suppose that $\mathcal{A} \subset \mathcal{P}(X)$ is an infinite σ-algebra. Then there exists a pairwise distinct collection $A_1, A_2, \dots$ of sets in $\mathcal{A}$. For each $x \in X$ let

$$B_x = \bigcap_{x\in A_i} A_i \cap \bigcap_{x\notin A_i} A_i^c.$$

We claim that for every $x, y \in X$ either $B_x = B_y$, or $B_x \cap B_y = \emptyset$. Indeed, if $B_x \neq B_y$ then there exists some i with $x \in A_i$ and $y \notin A_i$ or $x \notin A_i$ and $y \in A_i$ (otherwise all sets in the two intersections would be the same). If $x \in A_i$ and $y \notin A_i$ (and similarly in the other case) then by definition $B_x \subset A_i$ while $B_y \subset A_i^c$, so these sets are disjoint.

Now we prove that the set $\{B_x : x \in X\}$ is infinite. Suppose the contrary. Clearly, $x \in B_x$ for every x, moreover, if $x \in A_i$ for some i, then $A_i \supset B_x$. Consequently, $A_i = \bigcup_{x\in A_i} B_x$, in other words, every A_i can be expressed as a union of the sets of type B_x. If the collection $\{B_x : x \in X\}$ was finite, the number of possible distinct unions would be finite as well, contradicting the assumption the sets (A_i) are pairwise different.

Thus, we can find $x_0, x_1, \dots$ with $B_{x_0}, B_{x_1}, \dots$ pairwise disjoint. Finally, to every subset A of $\mathbb{N}$ we assign the set $S_A = \bigcup_{k\in A} B_{x_k}$. Clearly, the sets S_A are all in $\mathcal{A}$, and as $|\mathcal{P}(\mathbb{N})| = \mathfrak{c}$ it is enough to show that the map $A \mapsto S_A$ is injective. But this is easy: if $A \neq A'$ then there is some $l \in A \setminus A'$ or $l \in A' \setminus A$. Let $l \in A \setminus A'$ (the other case is similar) then $x_l \in B_{x_l} \subset S_A$ but

$$B_{x_l} \cap S_{A'} = B_{x_l} \cap (\bigcup_{k\in A'} B_{x_k}) = \bigcup_{k\in A'} B_{x_l} \cap B_{x_k} = \emptyset.$$

So $x_l \in S_A \setminus S_{A'}$ which shows that $S_A \neq S_{A'}$, thus finishing the proof.