Homework 6, due: Nov 24, 11:30 am

(1) The Cartesian coordinates of a point (x, y, z) in $\mathbb{R}^3$ can be expressed from the spherical coordinates as follows:

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$$

where $r \in [0, \infty), \theta \in [0, \pi], \phi \in [0, 2\pi]$. Let $\Omega_1 = \{(x, y, z) \in \mathbb{R}^3 : x, y, z > 0, 1 < \sqrt{x^2 + y^2 + z^2} < 2\}$, $\Omega_2 = \{(r, \theta, \phi) : r \in (1, 2), \theta \in (0, \pi/2), \phi \in (0, \pi/2)\}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be a Borel measurable non-negative function. Prove that

$$\int_{\Omega_1} f(x, y, z)\, dx\, dy\, dz =$$

$$\int_{\Omega_2} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

Using this, calculate the value of the integral $\int_{\Omega_1} yz\, dx\, dy\, dz$.

(2) Prove the general version of the theorem about the substitution: if $f : \mathbb{R}^n \to \mathbb{R}$ is a Lebesgue measurable function with $f \in L^1(\lambda)$ and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism where Ω is open then

$$\int_{G(\Omega)} f(x)\, d\lambda(x) = \int_{\Omega} f \circ G |det D_x G|\, d\lambda(x).$$